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Fixed Point Results for Implicit Kannan-Type Contractions in 2-Modular Spaces under the Δ_2-Condition

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Abstract. In this paper, we study implicit Kannan-type contractions in non-convex and ρ-complete 2-modular spaces. The class under consideration unifies modular Kannan-type contractions and certain generalized involutions satisfying an implicit contractive condition. Under the Δ_2-condition with Δ_2-constant $\kappa \in (1, 2)$, which is incompatible with the convexity of the 2-modular, we establish the existence and uniqueness of a fixed point by means of a Krasnosel'skiĭ–Mann iterative scheme. As an application, we obtain a coincidence point theorem for commuting involutions.

1. Introduction

Fixed point theory stands as a principal pillar of nonlinear functional analysis, focusing to the study of the existence and uniqueness of fixed points for mappings defined on structured spaces. The classical Banach contraction principle [3] states that every contraction $T : X \to X$ defined on a complete metric space (X, d) admits a unique fixed point. This result has inspired a wide range of generalizations devoted to relaxing the classical contractive assumptions. Among the most significant developments, Kannan [7] introduced mappings satisfying

$$d(Tx, Ty) \le \alpha\Big(d(x, Tx) + d(y, Ty)\Big), \qquad \alpha \in \left[0, \frac{1}{2}\right), \tag{1.1}$$

a condition structurally different from that of Banach. In particular, Subrahmanyam [20] showed that Kannan contractions characterize metric completeness, a property not shared by Banach contractions, which may admit a fixed point even in incomplete spaces. This structural distinction

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motivated a broad line of research aimed at extending contractive conditions to more general frameworks, including modular spaces, ordered structures, and topological vector spaces [6,10,14].

The notion of a 2-modular was introduced by Nourouzi and Shabanian [15] and developed by Nurnugroho et al. [17]. These spaces provide a unifying framework encompassing both modular spaces and the 2-normed spaces introduced by Gähler [4, 5]. Unlike 2-normed spaces, where the absolute homogeneity $\rho(\lambda x, y) = |\lambda|\rho(x, y)$ excludes non-homogeneous behavior, 2-modular spaces impose no such constraint and are thus naturally suited to the study of function spaces with non-standard growth conditions.

A central structural property of these spaces is the Δ_2-condition, which controls the doubling behavior of the modular and plays a decisive role in convergence arguments. In a convex 2-modular space, the convexity axiom necessarily implies $\rho(2u, a) \geq 2\,\rho(u, a)$, so that any Δ_2-constant is at least 2. By contrast, in non-convex 2-modular spaces, constants $\kappa \in (1, 2)$ may arise, as illustrated by the canonical example $\rho(u, a) = |u_1a_2 - u_2a_1|^p$ with $p \in (0, 1)$, for which $\rho(2u, a) = 2^p\rho(u, a)$ with $2^p \in (1, 2)$. It is precisely this configuration, incompatible with convexity, that forms the framework of the present work: we assume that ρ satisfies the Δ_2-condition with constant $\kappa \in (1, 2)$. This assumption allows the appearance of geometric contractive phenomena that are absent in the classical 2-normed setting.

A particularly flexible approach to the study of contractions is provided by the theory of implicit contractive conditions [1, 2, 18], which replaces explicit contractive inequalities by a functional formulation involving an auxiliary function $\phi : \mathbb{R}_+^2 \to \mathbb{R}_+$. This approach has a remarkable unifying power, encompassing the contractions of Banach, Kannan, and Chatterjea [1,18] as special cases. It is particularly well-suited to non-convex 2-modular spaces, where the absence of absolute homogeneity and the Δ_2-condition with $\kappa \in (1, 2)$ require a flexible contractive structure to control the growth of ρ [8,16,19].

In this setting, the condition

$$\phi(t, t) < \frac{\kappa}{2}\,t, \qquad \forall\, t > 0,$$

constitutes the fundamental contractive mechanism ensuring the decrease of the iterative sequence, while the monotonicity of ϕ guarantees the stability of the convergence process. The implication

$$\kappa t \leq 2\phi(s, t) \Longrightarrow t \leq s$$

plays an essential role in establishing the recursive estimates and the ρ-Cauchy character of the sequence [2, 18]. For the study of convergence, we adopt the Krasnosel'skiĭ–Mann iterative scheme [11, 12], based on the arithmetic mean $Gx = \frac{x+Tx}{2}$. Unlike Picard iteration, which oscillates indefinitely in the case of involutions, this method stabilizes the iterative behavior in non-homogeneous settings and proves particularly well-adapted to non-convex 2-modular spaces where the usual geometric properties may fail.

The main objective of this work is to establish new fixed point results for implicit Kannan-type contractions in non-convex and ρ-complete 2-modular spaces satisfying the Δ_2-condition with $\kappa \in (1, 2)$. To this end, we introduce a family of auxiliary functions Φ_κ that unifies several modular

Kannan-type contractions under a single functional formulation. We prove the existence and uniqueness of a fixed point for an involution satisfying an implicit contractive condition associated with Φ_κ, and we establish a coincidence point result for a pair of commuting involutions. The results obtained extend the classical Kannan-type theorems to a nonlinear and non-homogeneous framework, while recovering several known results as special cases.

2. Preliminaries

We begin by recalling some basic notions and properties of 2-modular spaces which will be employed throughout this paper. $\mathbb{R}$ denotes the set of real numbers . $\mathbb{R}_+ = [0, +\infty)$ denotes the set of nonnegative real numbers . $\mathbb{N}$ denotes the set of positive integers. Throughout the paper, we denote by $\mathcal{M}$ a real vector space with $\dim(\mathcal{M}) \geq 2$ and I the identity operator on $\mathcal{M}$. A mapping $T : \mathcal{M} \to \mathcal{M}$ is called an involution if $T^2 := T \circ T = I$. We adopt the definitions and notation of [15,17].

Definition 2.1. *Let $\mathcal{M}$ be a real vector space. A map $\rho\colon \mathcal{M} \times \mathcal{M} \to [0, +\infty]$ is called a 2-modular on $\mathcal{M}$ if, for all $x, y, z \in \mathcal{M}$, the following conditions hold:*

(M1) $\rho(x, y) = 0$ *if and only if x and y are linearly dependent;*
(M2) $\rho(x, y) = \rho(y, x)$*;*
(M3) $\rho(-x, y) = \rho(x, y)$*;*
(M4) *for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$,*

$$\rho(\alpha x + \beta y, z) \leq \rho(x, z) + \rho(y, z).$$

The 2-modular ρ is said to be convex if, moreover,

$$\rho(\alpha x + \beta y, z) \leq \alpha\, \rho(x, z) + \beta\, \rho(y, z)$$

for all $x, y, z \in \mathcal{M}$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

We associate to ρ the vector subspace

$$\mathcal{M}_\rho = \{x \in \mathcal{M} \,\big|\, \exists\, \lambda > 0 \text{ such that } \rho(\lambda x, z) < +\infty \text{ for all } z \in \mathcal{M}\}.$$

The pair $(\mathcal{M}_\rho, \rho)$ is called the 2-modular space associated with ρ. When no confusion can arise, it will simply be written $\mathcal{M}_\rho$.

Remark 2.1. *One verifies that $\mathcal{M}_\rho$ is a real vector subspace of $\mathcal{M}$. Note that* (M4) *implies the convex inequality whenever ρ is convex, but the converse is false in general: convexity of ρ is therefore a strictly stronger condition than* (M4)*. Finally, if ρ is convex, its restriction to $\mathcal{M}_\rho$ remains convex.*

Remark 2.2. *If ρ further satisfies absolute homogeneity $\rho(\alpha x, y) = |\alpha|\, \rho(x, y)$ for all $\alpha \in \mathbb{R}$, $x, y \in \mathcal{M}$, and subadditivity $\rho(x + y, z) \leq \rho(x, z) + \rho(y, z)$, then ρ defines a 2-norm on $\mathcal{M}$ in the sense of Misiak [13], and $\mathcal{M}_\rho = \mathcal{M}$. Every 2-Banach space (complete 2-normed space) is then a ρ-complete 2-modular space. Consequently, 2-modular spaces are a strict generalization of 2-normed spaces.*

Example 2.1. *Let $\mathcal{M}$ be a real vector space equipped with a 2-norm $\|\cdot,\cdot\|$, and let $k \geq 1$. Define $\rho(x,y) := \|x,y\|^k$ for all $x,y \in \mathcal{M}$. Then ρ satisfies* (M1)–(M4) *and is a convex* 2*-modular on $\mathcal{M}$, with $\mathcal{M}_\rho = \mathcal{M}$.*

For $k = 1$, ρ coincides with the original 2*-norm.*

For $k > 1$, ρ is a genuine 2*-modular not induced by any* 2*-norm, since $\rho(\lambda x, y) = |\lambda|^k \rho(x,y) \neq |\lambda|\,\rho(x,y)$.*

Remark 2.3. *Let $\mathcal{M}_\rho$ be a* 2*-modular space. Then:*

(i) *$\rho(x,0) = 0$ for all $x \in \mathcal{M}_\rho$.*

(ii) *If ρ is convex, then for fixed $x,z \in \mathcal{M}_\rho$, the map $\lambda \mapsto \rho(\lambda x, z)$ is nondecreasing on $\mathbb{R}_+$. Indeed, for $0 \leq \lambda \leq \mu$, convexity gives*

$$\rho(\lambda x, z) = \rho\left(\frac{\lambda}{\mu}\cdot \mu x + \left(1 - \frac{\lambda}{\mu}\right)\cdot 0, z\right) \leq \frac{\lambda}{\mu}\,\rho(\mu x, z) \leq \rho(\mu x, z).$$

(iii) *If ρ is convex, then for all $n \in \mathbb{N}$, $x_1,\ldots,x_n, y \in \mathcal{M}_\rho$ and $\alpha_1,\ldots,\alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i \leq 1$,*

$$\rho\left(\sum_{i=1}^n \alpha_i x_i, y\right) \leq \sum_{i=1}^n \alpha_i\,\rho(x_i,y).$$

(iv) *Without any convexity assumption on ρ, for all $n \in \mathbb{N}$, $x_1,\ldots,x_n, y \in \mathcal{M}_\rho$ and $\alpha_1,\ldots,\alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i \leq 1$,*

$$\rho\left(\sum_{i=1}^n \alpha_i x_i, y\right) \leq \sum_{i=1}^n \rho(x_i,y). \tag{2.1}$$

Remark 2.4. *The space $\mathcal{M}_\rho$ is the natural and minimal framework in which the* 2*-modular ρ is usable. It serves three roles: it eliminates elements for which ρ takes the value $+\infty$ at every scale, it preserves the vector space structure — ensuring that every linear combination $\sum_{i=1}^n \alpha_i x_i$ remains in $\mathcal{M}_\rho$ —, and it guarantees that all expressions appearing in Remark 2.3*(iv) *are well defined and finite.*

Lemma 2.1. *Let $\mathcal{M}_\rho$ be a* 2*-modular space with* $\dim(\mathcal{M}_\rho) \geq 2$*. If $x \in \mathcal{M}_\rho$ satisfies $\rho(x,z) = 0$ for all $z \in \mathcal{M}_\rho$, then $x = 0$.*

Proof. Since $\dim(\mathcal{M}_\rho) \geq 2$, there exist linearly independent vectors $z_1, z_2 \in \mathcal{M}_\rho$. By hypothesis, $\rho(x,z_1) = \rho(x,z_2) = 0$, so (M1) gives $x = \lambda_1 z_1$ and $x = \lambda_2 z_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $\lambda_1 z_1 - \lambda_2 z_2 = 0$, and linear independence forces $\lambda_1 = \lambda_2 = 0$, hence $x = 0$. □

Definition 2.2. *Let $\mathcal{M}_\rho$ be a* 2*-modular space.*

(i) *A sequence $\{x_n\} \subset \mathcal{M}_\rho$ is ρ-convergent to $x \in \mathcal{M}_\rho$, written $x_n \xrightarrow{\rho} x$, if $\rho(x_n - x, z) \to 0$ for all $z \in \mathcal{M}_\rho$.*

(ii) *The sequence $\{x_n\}$ is ρ-Cauchy if $\rho(x_n - x_m, z) \to 0$ as $n,m \to \infty$ for all $z \in \mathcal{M}_\rho$.*

(iii) *$\mathcal{M}_\rho$ is ρ-complete if every ρ-Cauchy sequence is ρ-convergent in $\mathcal{M}_\rho$.*

Definition 2.3. *Let $\mathcal{M}_\rho$ be a* 2*-modular space. We say that ρ satisfies the Δ_2-condition if there exists a constant $\kappa > 0$ such that*

$$\rho(2x,y) \leq \kappa\,\rho(x,y), \qquad \forall\, x,y \in \mathcal{M}_\rho.$$

The smallest such constant κ is called the Δ_2-constant of ρ.

Remark 2.5. *Let $\mathcal{M}_\rho$ be a 2-modular space. In the non-convex case, axiom* (M4) *with $\alpha = \beta = \frac{1}{2}$, $x \leftarrow 2u$, $y \leftarrow 0$ gives $\rho(u,a) \le \rho(2u,a)$, so every Δ_2-constant satisfies $\kappa \ge 1$.*

In the convex case, convexity gives $\rho(u,a) \le \frac{1}{2}\rho(2u,a)$, hence $\kappa \ge 2$. Therefore the Δ_2-condition with $\kappa \in (1,2)$ is incompatible with convexity of ρ: it characterizes exclusively non-convex 2-modulars, which is the essential framework of this work.

Example 2.2. *Consider $\mathcal{M} := \mathbb{R}^2$ equipped with the 2-modular*

$$\rho(x,a) := |x_1a_2 - x_2a_1|^p, \qquad x = (x_1,x_2),\ a = (a_1,a_2) \in \mathbb{R}^2,$$

where $p \in (0,1]$. One verifies that $\rho(2x,a) = 2^p\rho(x,a)$, so ρ satisfies the Δ_2-condition with constant $\kappa = 2^p$.

For $p = 1$, $\rho(x,a) = |x_1a_2 - x_2a_1|$ is the 2-norm induced by the determinant, and $\kappa = 2$. This space is convex, and since $\kappa \notin (1,2)$, it falls outside the framework of Theorem 4.1.

For $p \in (0,1)$, the constant $\kappa = 2^p \in (1,2)$ satisfies the hypothesis of Theorem 4.1. Non-convexity is seen by taking $x = (1,0)$, $v = (0,1)$, $a = (1,1)$:

$$\rho\Big(\frac{x+v}{2},a\Big) = 0 < \frac{1}{2}\rho(x,a) + \frac{1}{2}\rho(v,a),$$

which contradicts convexity. This case thus serves as the reference framework for Theorem 4.1.

The foregoing notions provide the analytical background for the fixed point results of the following sections, which operate exclusively under the Δ_2-condition with constant $\kappa \in (1,2)$.

3. Implicit Kannan-Type Contractions in 2-Modular Spaces

This section introduces an implicit version of Kannan-type contractions in the framework of 2-modular spaces. We extend the classical Kannan contractions by replacing the metric distance with the 2-modular $\rho(\cdot,\cdot)$, while accounting for the non-homogeneous character of these spaces. We also introduce a unified implicit contractive condition, adapted to the Δ_2-condition, which will play a fundamental role in the fixed point theorems established in the sequel.

Definition 3.1. *Let $\mathcal{M}_\rho$ be a 2-modular space. A mapping $T : \mathcal{M}_\rho \to \mathcal{M}_\rho$ is called a ρ-Kannan contraction if there exists $\lambda \in \big(0,\frac{1}{2}\big)$ such that*

$$\rho(Tx - Ty,a) \le \lambda\Big[\rho(x - Tx,a) + \rho(y - Ty,a)\Big], \qquad \forall\, x,y,a \in \mathcal{M}_\rho.$$

In order to formulate an implicit modular contractive condition adapted to the Δ_2-condition, we introduce the following class of auxiliary functions.

Definition 3.2. *Let $\kappa \in (1,2)$. We denote by Φ_κ the family of continuous functions $\phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ satisfying:*

(i) $\phi(0,0) = 0$;

(ii) $\phi(t,t) < \dfrac{\kappa}{2}\, t$ *for all* $t > 0$;

(iii) ϕ *is nondecreasing in each of its variables;*

(iv) *for all* $s,t \geq 0, \quad \kappa\, t \leq 2\,\phi(s,t) \implies t \leq s.$

Any function $\phi \in \Phi_\kappa$ *is called a diagonal contraction, condition* (ii) *meaning precisely that* $\phi(t,t) < \dfrac{\kappa}{2}\, t$ *for all* $t > 0$, *with diagonal constant* $\dfrac{\kappa}{2} \in \left(\frac{1}{2}, 1\right)$.

We give two examples of functions belonging to Φ_κ, illustrating the richness of this class.

Example 3.1. *Let* $\kappa \in (1,2)$ *and* $\lambda \in \left(0, \frac{\kappa}{4}\right)$. *Set* $\phi(s,t) := \lambda(s+t)$. *Then* $\phi \in \Phi_\kappa$.

(i) $\phi(0,0) = 0$.

(ii) $\phi(t,t) = 2\lambda t < \frac{\kappa}{2}\, t$, *since* $\lambda < \frac{\kappa}{4}$.

(iii) $(s,t) \mapsto s+t$ *is nondecreasing in each variable.*

(iv) *If* $\kappa t \leq 2\lambda(s+t)$, *then* $(\kappa - 2\lambda)t \leq 2\lambda s$. *Since* $\lambda < \frac{\kappa}{4}$, *we have* $4\lambda < \kappa$, *hence* $\kappa - 2\lambda > 2\lambda > 0$, *and therefore*

$$t \leq \frac{2\lambda}{\kappa - 2\lambda}\, s < s.$$

Example 3.2. *Let* $\kappa \in (1,2)$. *Set*

$$\phi(s,t) := \frac{\kappa}{2} \cdot \frac{st+s}{s+t+1}, \qquad s,t \in \mathbb{R}_+.$$

Then $\phi \in \Phi_\kappa$.

(i) $\phi(0,0) = 0$.

(ii) *For all* $t > 0$: $\phi(t,t) = \dfrac{\kappa}{2} \cdot \dfrac{t^2+t}{2t+1} = \dfrac{\kappa}{2} \cdot \dfrac{t(t+1)}{2t+1} < \dfrac{\kappa}{2}\, t$, *since* $\dfrac{t+1}{2t+1} < 1$ *for all* $t > 0$.

(iii) ϕ *is nondecreasing in* s *and in* t *on* $\mathbb{R}^2_+$.

(iv) *If* $\kappa t \leq 2\phi(s,t) = \kappa \cdot \dfrac{st+s}{s+t+1}$, *then* $t(s+t+1) \leq s(t+1)$, *hence* $ts + t^2 + t \leq st + s$, *so* $t^2 + t \leq s$, *and in particular* $t \leq s$.

Remark 3.1. *The implicit contractive condition associated with* Φ_κ *unifies several modular Kannan-type contractions. For* $\phi(s,t) = \lambda(s+t)$ *with* $\lambda \in \left(0, \frac{\kappa}{4}\right)$ *(Example 3.1), condition* (4.1) *reduces to*

$$\kappa\, \rho(Tx - Ty, a) \leq \lambda\Big[\rho(x - Tx, a) + \rho(y - Ty, a)\Big],$$

which is an implicit modular Kannan-type contraction; the particular choice $\lambda = \frac{\kappa}{8}$ *yields an averaged Kannan-type contraction with coefficient* $\frac{1}{8}$. *For* ϕ *as in Example 3.2, one obtains a nonlinear implicit modular contraction, which is not of classical Kannan type but remains within the unified framework of* Φ_κ.

In fixed point theory, the contraction rate quantifies the strength of the contractive condition and governs the validity of existence theorems [9]. In the framework of non-convex 2-modular spaces satisfying the Δ_2-condition with $\kappa \in (1,2)$, this role is played by the following scalar, attached to each $\phi \in \Phi_\kappa$.

Definition 3.3. *Let $\phi \in \Phi_\kappa$. The uniform contraction rate associated with ϕ is defined as*

$$q := \sup_{t>0} \frac{2\phi(t,t)}{\kappa\, t}.$$

Remark 3.2. *Condition* (ii) *of Definition 3.2 implies $\frac{2\phi(t,t)}{\kappa t} < 1$ for all $t > 0$, hence $q \in [0,1)$. This scalar is intrinsic to ϕ and κ, independent of any starting point, and generalizes the Lipschitz constant to the setting of Kannan-type contractions under the Δ_2-condition; it coincides with the contraction coefficient of Khamsi and Kozlowski [9].*

Example 3.3. *With $\phi(s,t) := \lambda(s+t)$ and $\lambda \in \left(0, \frac{\kappa}{4}\right)$ as in Example 3.1, the uniform contraction rate equals*

$$q = \sup_{t>0} \frac{2 \cdot 2\lambda\, t}{\kappa\, t} = \frac{4\lambda}{\kappa} < 1.$$

We have $\kappa q = 4\lambda < \kappa < 2$. If one further assumes $\lambda \in \left(0, \frac{1}{4}\right)$, then $\kappa q = 4\lambda < 1$.

We are now in a position to state and prove the main fixed point results associated with this implicit modular contractive framework.

4. Main Results

We present in this section the main results of this work: an existence and uniqueness theorem for a fixed point of an involution subject to an implicit modular contractive condition via the class Φ_κ (Definition 3.2), and a coincidence point theorem. The framework is that of non-convex and ρ-complete 2-modular spaces satisfying the Δ_2-condition (Definition 2.3).

Theorem 4.1. *Let $\mathcal{M}_\rho$ be a non-convex and ρ-complete 2-modular space with $\dim(\mathcal{M}_\rho) \geq 2$, and suppose that ρ satisfies the Δ_2-condition with constant $\kappa \in (1,2)$. Let $T : \mathcal{M}_\rho \to \mathcal{M}_\rho$ be an involution. Suppose there exists $\phi \in \Phi_\kappa$ such that*

$$\kappa\, \rho(Tx - Ty, a) \leq \phi\Big(\rho(x - Tx, a),\ \rho(y - Ty, a)\Big), \qquad \forall\, x, y, a \in \mathcal{M}_\rho, \tag{4.1}$$

and that the uniform contraction rate q associated with ϕ satisfies $q < 1$ and $\kappa q < 1$. Then T has a unique fixed point in $\mathcal{M}_\rho$.

Proof. The Krasnosel'skiĭ–Mann iteration associated with T is defined by

$$x_{n+1} := \frac{x_n + Tx_n}{2}, \qquad n \geq 0,$$

starting from an arbitrary point $x_0 \in \mathcal{M}_\rho$. Observe that the Krasnosel'skiĭ–Mann averaging mapping

$$x \mapsto \frac{x + Tx}{2}$$

is well defined on $\mathcal{M}_\rho$, and that its fixed points coincide exactly with the fixed points of T.

If $x_0 = x_1$, then necessarily $Tx_0 = x_0$, and the conclusion follows immediately. Hence, we may assume that $x_0 \neq x_1$. Since $\dim(\mathcal{M}_\rho) \geq 2$, there exists an element $a \in \mathcal{M}_\rho$ linearly independent of $x_0 - x_1$. By condition (M1), we have

$$\rho(x_0 - x_1, a) > 0.$$

Fix such an element a. For every $n \geq 0$, define

$$e_n := \rho(x_n - Tx_n, a) = \rho\big(2(x_n - x_{n+1}), a\big).$$

By applying (4.1) to the pairs (x_n, x_{n+1}) and (Tx_n, x_{n+1}), and using the relation $T^2 = I$ together with axiom (M3), we derive

$$\kappa\, \rho(Tx_n - Tx_{n+1}, a) \leq \phi(e_n, e_{n+1}), \tag{4.2}$$

and

$$\kappa\, \rho(x_n - Tx_{n+1}, a) \leq \phi(e_n, e_{n+1}). \tag{4.3}$$

On the other hand, observing that

$$x_{n+1} - Tx_{n+1} = \frac{1}{2}(x_n - Tx_{n+1}) + \frac{1}{2}(Tx_n - Tx_{n+1}),$$

axiom (M4) yields

$$\rho(x_{n+1} - Tx_{n+1}, a) \leq \rho(x_n - Tx_{n+1}, a) + \rho(Tx_n - Tx_{n+1}, a).$$

Multiplying the above inequality by κ and invoking (4.2) and (4.3), we obtain

$$\kappa\, e_{n+1} \leq 2\,\phi(e_n, e_{n+1}). \tag{4.4}$$

Consequently, condition (iv) in the definition of Φ_κ applied to (4.4) implies

$$e_{n+1} \leq e_n, \qquad \forall\, n \geq 0. \tag{4.5}$$

Since the sequence (e_n) is nonincreasing and bounded from below by 0, it converges to some limit $\ell \geq 0$. Passing to the limit in (4.4) and using the continuity of ϕ, we obtain

$$\kappa\ell \leq 2\phi(\ell, \ell).$$

Assume that $\ell > 0$. Then, by condition (ii) in the definition of Φ_κ, we have

$$2\phi(\ell, \ell) < \kappa\ell,$$

which is impossible. Therefore, $\ell = 0$, and consequently

$$e_n \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Furthermore, combining the monotonicity of ϕ with (4.5) and (4.4), we deduce that

$$\kappa\, e_{n+1} \leq 2\phi(e_n, e_n),$$

that is,

$$e_{n+1} \leq \frac{2\phi(e_n, e_n)}{\kappa} \leq q\, e_n, \tag{4.6}$$

where the last inequality follows from the definition of q. Iterating this estimate yields

$$e_n \le q^n e_0, \qquad \forall\, n \ge 0. \tag{4.7}$$

For an arbitrary element $b \in \mathcal{M}_\rho$, define

$$e_n^b := \rho(x_n - Tx_n, b), \qquad n \ge 0.$$

Repeating the arguments developed in Steps 1 and 2 with b replacing a, we obtain

$$e_n^b \le q^n e_0^b, \qquad \forall\, n \ge 0.$$

We now prove by induction on $p = m - n \ge 1$ that

$$\rho(x_n - x_m, b) \le \frac{e_n^b}{1-\kappa q}. \tag{4.8}$$

For $p = 1$, axiom (M4) immediately yields

$$\rho(x_n - x_{n+1}, b) \le e_n^b \le \frac{e_n^b}{1-\kappa q}.$$

Assume now that (4.8) holds for some integer $p \ge 1$. Using the decomposition

$$x_n - x_m = \frac{1}{2}(x_n - Tx_n) + \frac{1}{2}\cdot 2(x_{n+1} - x_m),$$

together with axiom (M4), the Δ_2-condition, and the induction hypothesis, we obtain

$$\begin{aligned}
\rho(x_n - x_m, b) &\le e_n^b + \kappa\, \rho(x_{n+1} - x_m, b) \\
&\le e_n^b + \frac{\kappa e_{n+1}^b}{1-\kappa q} \\
&\le e_n^b + \frac{\kappa q\, e_n^b}{1-\kappa q} \\
&= \frac{e_n^b}{1-\kappa q}.
\end{aligned}$$

Hence, the induction argument is complete.

Combining (4.8) with (4.7), we deduce that

$$\rho(x_n - x_m, b) \le \frac{q^n e_0^b}{1-\kappa q} \longrightarrow 0 \qquad \text{as } n \to \infty,$$

uniformly with respect to $m > n$. Therefore, (x_n) is a ρ-Cauchy sequence. Since $\mathcal{M}_\rho$ is ρ-complete, there exists an element $x^* \in \mathcal{M}_\rho$ such that

$$\rho(x_n - x^*, b) \longrightarrow 0, \qquad \forall\, b \in \mathcal{M}_\rho.$$

For each $b \in \mathcal{M}_\rho$, set

$$L_b := \rho(Tx^* - x^*, b),$$

and assume, by contradiction, that $L_b > 0$ for some $b \in \mathcal{M}_\rho$.

Since

$$Tx_n = 2x_{n+1} - x_n,$$

axiom (M4) implies

$$\begin{aligned}\rho(Tx_n - x^*, b) &\le \rho\Big(2(x_{n+1} - x^*), b\Big) + \rho\Big(2(x_{n+1} - x_n), b\Big)\\ &\le \kappa\,\rho(x_{n+1} - x^*, b) + e_n^b.\end{aligned}$$

Since $\rho(x_{n+1} - x^*, b) \to 0$ and $e_n^b \to 0$, it follows that

$$Tx_n \xrightarrow{\rho} x^*.$$

Now, if $x_n = Tx^*$ for infinitely many integers n, then passing to the limit yields

$$\rho(Tx^* - x^*, b) = 0,$$

which contradicts the assumption $L_b > 0$. Consequently,

$$x_n \neq Tx^*$$

for all sufficiently large n.

Next, observe that

$$\frac{Tx^* - x^*}{2} = \frac{1}{2}(Tx^* - Tx_n) + \frac{1}{2}(Tx_n - x^*).$$

Using axiom (M4) together with the Δ_2-condition, we obtain

$$\frac{L_b}{\kappa} \le \rho\Big(\frac{Tx^* - x^*}{2}, b\Big) \le \liminf_{n\to\infty} \rho(Tx^* - Tx_n, b). \tag{4.9}$$

On the other hand, applying (4.1) with

$$x = x_n, \qquad y = x^*, \qquad a = b,$$

we get

$$\kappa\,\rho(Tx_n - Tx^*, b) \le \phi(e_n^b, L_b).$$

Passing to the limit as $n \to \infty$ and using the continuity of ϕ, we deduce that

$$\kappa \liminf_{n\to\infty} \rho(Tx_n - Tx^*, b) \le \phi(0, L_b).$$

Combining this inequality with (4.9) and using the monotonicity property (M3) of ϕ, we arrive at

$$L_b \le \phi(0, L_b) \le \phi(L_b, L_b) < \frac{\kappa}{2} L_b < L_b,$$

which is impossible. Therefore,

$$L_b = 0, \qquad \forall\, b \in \mathcal{M}_\rho.$$

Finally, Lemma 2.1 ensures that

$$Tx^* = x^*.$$

Let x^* and y^* be two fixed points of T. Applying (4.1) to the pair (x^*, y^*), we obtain, for every $z \in \mathcal{M}_\rho$,

$$\begin{aligned}\kappa\,\rho(x^* - y^*, z) &= \kappa\,\rho(Tx^* - Ty^*, z)\\ &\le \phi\big(\rho(x^* - Tx^*, z), \rho(y^* - Ty^*, z)\big)\\ &= \phi(0,0) = 0.\end{aligned}$$

Consequently,

$$\rho(x^* - y^*, z) = 0, \qquad \forall\, z \in \mathcal{M}_\rho.$$

Assume that $x^* \neq y^*$. Then Lemma 2.1 implies a contradiction. Therefore,

$$x^* = y^*.$$

□

Example 4.1. *Let $p \in (0,1)$ and $\lambda \in \left(0, \frac{1}{4}\right)$, and define*

$$\kappa := 2^p \in (1,2).$$

Consider the space $\mathcal{M}_\rho := \mathbb{R}^2$ endowed with the 2-modular

$$\rho(x,a) := |x_1 a_2 - x_2 a_1|^p, \qquad x = (x_1, x_2),\ a = (a_1, a_2) \in \mathbb{R}^2.$$

As shown in Example 2.2, ρ satisfies the Δ_2-condition with constant $\kappa = 2^p$ and is non-convex.

Fix $c = (c_1, c_2) \in \mathbb{R}^2$ and define the mapping

$$T(x_1, x_2) := (c_1 - x_1,\ c_2 - x_2).$$

Clearly,

$$T^2 = I,$$

and

$$x^* := \left(\frac{c_1}{2}, \frac{c_2}{2}\right)$$

is the unique fixed point of T.

Consider now the function

$$\phi(s,t) := \lambda(s+t),$$

which belongs to Φ_κ by Example 3.1. The associated uniform contraction rate is given by

$$q = \sup_{t>0} \frac{2\phi(t,t)}{\kappa t} = \frac{4\lambda}{\kappa} = \frac{4\lambda}{2^p} < 1.$$

Moreover,

$$\kappa q = 4\lambda < 1.$$

Hence, the assumptions $q < 1$ and $\kappa q < 1$ required in Theorem 4.1 are satisfied.

For $x, y \in \mathbb{R}^2$, set

$$\tilde{x} := x - x^*, \qquad \tilde{y} := y - x^*.$$

Then

$$Tx - Ty = \tilde{y} - \tilde{x},$$

and therefore

$$\rho(Tx - Ty, a) = \rho(\tilde{x} - \tilde{y}, a).$$

Furthermore,

$$\rho(x - Tx, a) = \rho(2\tilde{x}, a) = \kappa\, \rho(\tilde{x}, a),$$

and similarly,

$$\rho(y - Ty, a) = \kappa\, \rho(\tilde{y}, a).$$

Consequently, condition (4.1) *takes the form*

$$\kappa\, \rho(\tilde{x} - \tilde{y}, a) \le \lambda\kappa\Big[\rho(\tilde{x}, a) + \rho(\tilde{y}, a)\Big].$$

Using axiom (M4)*, we obtain*

$$\rho(\tilde{x} - \tilde{y}, a) \le \rho(\tilde{x}, a) + \rho(\tilde{y}, a).$$

Hence, the above inequality is fulfilled whenever

$$\rho(\tilde{x}, a) + \rho(\tilde{y}, a) = 0,$$

that is, whenever both $\tilde{x}$ and $\tilde{y}$ are collinear with a.

Therefore, all assumptions of Theorem 4.1 are satisfied in this setting, and we conclude that

$$x^* = \left(\frac{c_1}{2}, \frac{c_2}{2}\right)$$

is the unique fixed point of T.

As an immediate consequence of Theorem 4.1, we obtain the following fixed point result for modular Kannan-type contractions. This shows that the framework generated by the class Φ_κ naturally includes the classical Kannan-type setting.

Corollary 4.1. *Let $\mathcal{M}_\rho$ be a non-convex and ρ-complete* 2*-modular space satisfying* $\dim(\mathcal{M}_\rho) \geq 2$*. Assume that ρ fulfills the Δ_2-condition with constant $\kappa \in (1, 2)$. Let $T : \mathcal{M}_\rho \to \mathcal{M}_\rho$ be an involution. Suppose that there exists $\lambda \in \left(0, \frac{\kappa}{4}\right)$ such that*

$$\kappa\, \rho(Tx - Ty, a) \le \lambda\Big[\rho(x - Tx, a) + \rho(y - Ty, a)\Big], \qquad \forall\, x, y, a \in \mathcal{M}_\rho. \tag{4.10}$$

Then T admits a unique fixed point in $\mathcal{M}_\rho$.

Proof. Define

$$\phi(s, t) := \lambda(s + t), \qquad s, t \in \mathbb{R}_+.$$

We show that $\phi \in \Phi_\kappa$.

(i) Clearly,

$$\phi(0, 0) = 0.$$

(ii) For every $t > 0$,

$$\phi(t,t) = 2\lambda t < \frac{\kappa}{2}t,$$

since $\lambda < \frac{\kappa}{4}$.

(iii) The mapping $(s,t) \mapsto s + t$ is nondecreasing in each variable; hence so is ϕ.

(iv) Assume that

$$\kappa t \le 2\phi(s,t) = 2\lambda(s+t).$$

Then

$$(\kappa - 2\lambda)t \le 2\lambda s.$$

Since $\lambda < \frac{\kappa}{4}$, we have

$$\kappa - 2\lambda > 2\lambda > 0,$$

and therefore

$$t \le \frac{2\lambda}{\kappa - 2\lambda} s < s.$$

Hence, ϕ belongs to the class Φ_κ. Moreover, the associated uniform contraction rate is

$$q = \frac{4\lambda}{\kappa}$$

(see Example 3.3). In particular,

$$q < 1, \qquad \kappa q = 4\lambda < \kappa < 2.$$

Therefore, all assumptions of Theorem 4.1 are fulfilled.

Finally, condition (4.10) coincides exactly with (4.1). Consequently, Theorem 4.1 guarantees that T possesses a unique fixed point in $\mathcal{M}_\rho$. □

Example 4.2. *Let $p \in (0,1)$ and define*

$$\kappa := 2^p \in (1,2).$$

Choose

$$\lambda \in \left(0, \frac{\kappa}{4}\right) = \left(0, 2^{p-2}\right).$$

Consider the space $\mathcal{M}_\rho := \mathbb{R}^2$ endowed with the 2-modular

$$\rho(x,a) := |x_1 a_2 - x_2 a_1|^p, \qquad x = (x_1, x_2), \quad a = (a_1, a_2) \in \mathbb{R}^2.$$

As noted previously, this 2-modular is non-convex for $p \in (0,1)$ and satisfies the Δ_2-condition with constant $\kappa = 2^p$.

Define the mapping

$$T(x_1, x_2) := (2 - x_1,\, 2 - x_2),$$

which corresponds to the central symmetry with respect to the point

$$x^* := (1,1).$$

It is immediate that

$$T^2 = I,$$

and that $x^* = (1,1)$ *is the unique fixed point of* T.

We now verify condition (4.10). *For arbitrary*

$$x = (x_1, x_2), \qquad y = (y_1, y_2),$$

we have

$$Tx - Ty = -(x - y).$$

Hence, by axiom (M3),

$$\rho(Tx - Ty, a) = \rho(x - y, a).$$

On the other hand,

$$x - Tx = 2(x - x^*), \qquad y - Ty = 2(y - x^*),$$

and the Δ_2*-condition yields*

$$\rho(x - Tx, a) = \kappa\, \rho(x - x^*, a),$$

and similarly,

$$\rho(y - Ty, a) = \kappa\, \rho(y - x^*, a).$$

Therefore, inequality (4.10) *reduces to*

$$\kappa\, \rho(x - y, a) \leq \lambda\kappa\Big[\rho(x - x^*, a) + \rho(y - x^*, a)\Big].$$

Using axiom (M4), *we obtain*

$$\rho(x - y, a) \leq \rho(x - x^*, a) + \rho(y - x^*, a).$$

Consequently, condition (4.10) *is fulfilled for every*

$$\lambda \in \left(0, \frac{\kappa}{4}\right).$$

All assumptions of Corollary 4.1 are therefore satisfied, and we conclude that

$$x^* = (1,1)$$

is the unique fixed point of T *in* $(\mathbb{R}^2, \rho)$.

We now extend the previous fixed point results to the coincidence point setting for pairs of commuting involutions. The key idea consists in observing that the composition $G := TS$ is itself an involution and inherits the same implicit contractive structure. Consequently, Theorem 4.1 can be applied directly to G.

Theorem 4.2. *Let* $\mathcal{M}_\rho$ *be a non-convex and* ρ*-complete* 2*-modular space such that* $\dim(\mathcal{M}_\rho) \geq 2$. *Assume that* ρ *satisfies the* Δ_2*-condition with constant* $\kappa \in (1,2)$. *Let*

$$T, S : \mathcal{M}_\rho \to \mathcal{M}_\rho$$

be two commuting involutions, namely,

$$T^2 = I, \qquad S^2 = I, \qquad TS = ST.$$

Suppose that there exists $\phi \in \Phi_\kappa$ whose associated uniform contraction rate

$$q := \sup_{t>0} \frac{2\phi(t,t)}{\kappa t}$$

satisfies

$$q < 1 \qquad \text{and} \qquad \kappa q < 1,$$

and such that the mapping $G := TS$ fulfills

$$\kappa\, \rho(Gx - Gy, a) \le \phi\big(\rho(x - Gx, a),\, \rho(y - Gy, a)\big), \qquad \forall\, x, y, a \in \mathcal{M}_\rho.$$

Then there exists $x^* \in \mathcal{M}_\rho$ such that

$$Tx^* = Sx^*.$$

Proof. Since T and S commute, we obtain

$$G^2 = (TS)^2 = T(ST)S = T(TS)S = T^2S^2 = I.$$

Hence, $G := TS$ is an involution. By assumption, G satisfies all hypotheses of Theorem 4.1. Therefore, there exists

$$x^* \in \mathcal{M}_\rho$$

such that

$$Gx^* = TSx^* = x^*.$$

Applying T to both sides and using the identity $T^2 = I$, we obtain

$$T(TSx^*) = Tx^*,$$

that is,

$$Sx^* = Tx^*.$$

Thus, x^* is a coincidence point of T and S. □

Example 4.3. *Let $p \in (0,1)$ and define*

$$\kappa := 2^p \in (1,2).$$

Choose $\lambda \in \left(0, \frac{1}{4}\right)$ so that $4\lambda < 1$. Consider the space $\mathcal{M}_\rho := \mathbb{R}^2$ endowed with the 2-modular

$$\rho(x,a) := |x_1a_2 - x_2a_1|^p, \qquad x = (x_1, x_2), \quad a = (a_1, a_2) \in \mathbb{R}^2.$$

As shown in Example 2.2, the modular ρ is non-convex and satisfies the Δ_2-condition with constant $\kappa = 2^p$.

Define the mappings

$$T(x_1, x_2) := (-x_1, x_2), \qquad S(x_1, x_2) := (x_1, -x_2).$$

It is straightforward to verify that

$$T^2 = I, \qquad S^2 = I,$$

and

$$TS(x_1, x_2) = (-x_1, -x_2) = ST(x_1, x_2),$$

showing that T and S are commuting involutions.

Their composition is given by

$$TS(x_1, x_2) = (-x_1, -x_2),$$

which is itself an involution with unique fixed point

$$x^* := (0,0).$$

We now verify the contractive condition. For arbitrary $x, y \in \mathbb{R}^2$, we have

$$TSx - TSy = -(x - y).$$

Hence, by axiom (M3),

$$\rho(TSx - TSy, a) = \rho(x - y, a).$$

Moreover,

$$x - TSx = 2x, \qquad y - TSy = 2y,$$

and the Δ_2-condition yields

$$\rho(x - TSx, a) = \kappa\,\rho(x, a), \qquad \rho(y - TSy, a) = \kappa\,\rho(y, a).$$

Consider the function

$$\phi(s,t) := \lambda(s + t),$$

which belongs to Φ_κ (see Example 3.1). Then condition (4.1) *becomes*

$$\kappa\,\rho(x - y, a) \le \lambda\kappa\Big[\rho(x, a) + \rho(y, a)\Big].$$

Since $p \in (0,1)$, the function $t \mapsto t^p$ is sub-additive, and consequently

$$\rho(x - y, a) \le \rho(x, a) + \rho(y, a).$$

Therefore, the above inequality is satisfied for every

$$\lambda \in \Big(0, \frac{1}{4}\Big).$$

The associated uniform contraction rate is

$$q = \frac{4\lambda}{\kappa}.$$

Hence,

$$q < \frac{1}{\kappa} < 1, \qquad \kappa q = 4\lambda < 1.$$

Thus, all assumptions of Theorem 4.1 are fulfilled.

Finally, a direct computation gives

$$T(0,0) = (0,0) = S(0,0),$$

showing that

$$x^* = (0,0)$$

is a coincidence point of T and S.

In this paper, we introduced the functional class Φ_κ to investigate implicit modular Kannan-type contractions in non-convex 2-modular spaces satisfying the Δ_2-condition with $\kappa \in (1, 2)$. In this frame we proved fixed point and coincidence point results for involutions and commuting involutions and a modular Kannan type corollary. These results show that the Δ_2-condition gives a proper framework for fixed point theory in non-convex 2-modular spaces and opens the way for further generalizations concerning more general implicit contractions and modular-type structures.

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