Int. J. Anal. Appl. (2026), 24:255

International Journal of Analysis and Applications

Fuzzy and Hyperfuzzy Perspectives on Length, Mean, and Dot in IUP-Algebras

Apichaya Arwut, Kanyanat Phuangma, Kittiyapron Thuanpung, Aiyared Iampan*

Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand

**Corresponding author:* aiyared.ia@up.ac.th

Abstract. This paper studies fuzzy and hyperfuzzy perspectives on length, mean, and dot constructions in IUP-algebras. For a given hyperstructure over an IUP-algebra, three induced fuzzy structures are considered: the length obtained from the difference between the supremum and infimum values, the mean obtained from their average, and the dot obtained from their product. Based on these induced fuzzy structures, we introduce the notions of length fuzzy IUP-subalgebras, mean fuzzy IUP-subalgebras, and dot fuzzy IUP-subalgebras of types 1, 2, 3, and 4. Several basic properties and relationships among these types are investigated. In particular, we show that the type 3 case implies the type 1 case, the type 2 case implies the type 4 case, and the type 2 and type 3 cases coincide under the induced fuzzy structure. We also provide characterizations of length, mean, and dot fuzzy IUP-subalgebras in terms of upper level subsets, lower level subsets, and equal level subsets. Furthermore, we establish sufficient conditions connecting these induced fuzzy IUP-subalgebras with (i, j)-hyperfuzzy IUP-subalgebras through the infimum and supremum fuzzy structures. Examples are given to illustrate the results and to show that some converse implications do not hold in general.

1. Introduction

Independent UP-algebras, abbreviated as IUP-algebras [4], were introduced by Iampan et al. in 2022 as an algebraic structure related to logical algebra. In that work, the authors established the basic theory of IUP-algebras by defining IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals, together with homomorphisms and their basic properties. Since then, several studies have developed the theory of IUP-algebras in different directions. In 2023, Chanmanee et al. [2] studied direct products and weak direct products of IUP-algebras and investigated some properties of homomorphisms and anti-homomorphisms. In another work, Chanmanee et al. [1] examined external direct products in dual IUP-algebras. In 2024, Kuntama et al. [13] introduced fuzzy IUP-subalgebras, fuzzy IUP-ideals, fuzzy IUP-filters, and fuzzy strong IUP-ideals. Later, Suayngam et

Received: May 15, 2026.

2020 *Mathematics Subject Classification.* 03G25, 06F35, 08A72.

Key words and phrases. IUP-algebra; fuzzy IUP-subalgebra; hyperfuzzy IUP-subalgebra; length fuzzy IUP-subalgebra; mean fuzzy IUP-subalgebra; dot fuzzy IUP-subalgebra; level subset.

https://doi.org/10.28924/2291-8639-24-2026-255

ISSN: 2291-8639

al. [23] studied Fermatean fuzzy IUP-algebras, and Suayngam et al. [25] considered intuitionistic fuzzy IUP-algebras. Further developments were given in 2025, including neutrosophic IUP-algebras [21], Pythagorean fuzzy IUP-algebras [24], Pythagorean neutrosophic IUP-algebras [20], and intuitionistic neutrosophic IUP-algebras [19]. In addition, Phaeyai et al. [14] investigated (l, r)- and (r, l)-derivations in IUP-algebras, Surat et al. [26] studied hesitant fuzzy subalgebraic systems in IUP-algebras, and Khonyong et al. [12] considered extensions and contractions of fuzzy translations in IUP-algebras. More recently, Suayngam et al. [22] introduced transmitted, reflected, resonant, and dominant IUP-ideals, while Inthachot et al. [6] applied bipolar fuzzy sets to IUP-algebras. These studies show that IUP-algebras provide a useful setting for developing algebraic and fuzzy-type structures.

Fuzzy set theory has been widely applied to logical algebraic structures. For example, Jun and Song [9] applied fuzzy set theory to implicative ideals in BCK-algebras and showed how fuzzy membership functions can be used to study algebraic subsets. Later, Ghosh and Samanta [3] introduced hyperfuzzy sets and hyperfuzzy groups, where each element is assigned a nonempty subset of the unit interval instead of a single membership value. This idea was further developed in BCK/BCI-algebras. Song et al. [18] studied hyperfuzzy ideals in BCK/BCI-algebras, while Jun et al. [7] introduced hyperfuzzy subalgebras of BCK/BCI-algebras and investigated their related properties. In another related direction, Jun et al. [10] introduced length-fuzzy subalgebras in BCK/BCI-algebras by considering the length induced by a hyperfuzzy set, and Kang and Bordbar [11] later provided further results on length fuzzy subalgebras of BCK/BCI-algebras. Tacha et al. [27] studied length and mean fuzzy UP-subalgebras of UP-algebras. Rajesh et al. [16] investigated length and mean fuzzy ideals of Sheffer stroke Hilbert algebras, and Rajesh et al. [15] further studied length and mean fuzzy subalgebras in Sheffer stroke Hilbert algebras. More recently, Iampan et al. [5] considered characterizations of length and mean fuzzy ideals in Hilbert algebras over hyperstructures. These works indicate that hyperfuzzy structures and their induced fuzzy forms, especially length and mean, are useful tools for constructing and analyzing fuzzy-type substructures in logical algebraic systems.

Although length and mean fuzzy structures have been studied in some logical algebraic systems, their counterparts in IUP-algebras have not yet been investigated. Moreover, the dot of a hyperstructure, defined in this paper by combining the infimum and supremum values of a hyperfuzzy set through their product, has not been considered in this setting. Therefore, it is natural to develop these three induced fuzzy structures in the setting of IUP-algebras.

Motivated by the above works, this paper develops length, mean, and dot perspectives of hyperstructures in IUP-algebras. For a hyperstructure $(X, \tilde{f})$ over an IUP-algebra, we define three induced fuzzy structures, namely the length $\tilde{f}_{\mathrm{l}}$, the mean $\tilde{f}_{\mathrm{m}}$, and the dot $\tilde{f}_{\mathrm{d}}$. Based on these induced fuzzy structures, we introduce length fuzzy IUP-subalgebras, mean fuzzy IUP-subalgebras, and dot fuzzy IUP-subalgebras of types 1, 2, 3, and 4. We investigate their basic properties and establish characterizations in terms of upper level subsets, lower level subsets, and equal level subsets. We also study their relationships with (i, j)-hyperfuzzy IUP-subalgebras through the infimum and supremum fuzzy structures.

2. Preliminaries

In this section, we recall some basic notions and results on IUP-algebras that will be used in the subsequent sections. Since the length, mean, and dot of a hyperstructure are defined on an IUP-algebra, we first present the underlying algebraic framework. We also recall several classes of special subsets of IUP-algebras, including IUP-subalgebras, IUP-filters, IUP-ideals, strong IUP-ideals, transmitted IUP-ideals, reflected IUP-ideals, resonant IUP-ideals, and dominant IUP-ideals. These notions provide algebraic background for the fuzzy and hyperfuzzy structures considered in this paper.

Definition 2.1. *[4] An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called an IUP-algebra, where X is a nonempty set, $\cdot$ is a binary operation on X, and 0 is the constant of X, if it satisfies the following axioms:*

$$(\forall x \in X)(0 \cdot x = x), \tag{IUP-1}$$

$$(\forall x \in X)(x \cdot x = 0), \tag{IUP-2}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot (x \cdot z) = y \cdot z). \tag{IUP-3}$$

Throughout this paper, unless otherwise stated, X denotes an IUP-algebra $(X, \cdot, 0)$.

Example 2.1. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the Cayley table as follows:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	0	3	1	5	2
2	2	5	0	4	3	1
3	5	4	1	0	2	3
4	1	3	5	2	0	4
5	3	2	4	5	1	0

Then $X = (X, \cdot, 0)$ is an IUP-algebra.

We next recall some fundamental properties of IUP-algebras. These properties will be used in the proofs of later results.

Lemma 2.1. *[4] Let $X = (X, \cdot, 0)$ be an IUP-algebra. Then the following assertions hold:*

$$(\forall x, y \in X)((x \cdot 0) \cdot (x \cdot y) = y), \tag{2.1}$$

$$(\forall x \in X)((x \cdot 0) \cdot (x \cdot 0) = 0), \tag{2.2}$$

$$(\forall x, y \in X)((x \cdot y) \cdot 0 = y \cdot x), \tag{2.3}$$

$$(\forall x \in X)((x \cdot 0) \cdot 0 = x), \tag{2.4}$$

$$(\forall x, y \in X)(x \cdot ((x \cdot 0) \cdot y) = y), \tag{2.5}$$

$$(\forall x, y \in X)(((x \cdot 0) \cdot y) \cdot x = y \cdot 0), \tag{2.6}$$

$$(\forall x, y, z \in X)(x \cdot y = x \cdot z \Leftrightarrow y = z), \tag{2.7}$$

$$(\forall x, y \in X)(x \cdot y = 0 \Leftrightarrow x = y), \tag{2.8}$$

$$(\forall x \in X)(x \cdot 0 = 0 \Leftrightarrow x = 0), \tag{2.9}$$

$$(\forall x, y, z \in X)(y \cdot x = z \cdot x \Leftrightarrow y = z), \tag{2.10}$$

$$(\forall x, y \in X)(x \cdot y = y \Rightarrow x = 0), \tag{2.11}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot 0 = (z \cdot y) \cdot (z \cdot x)), \tag{2.12}$$

$$(\forall x, y, z \in X)(x \cdot y = 0 \Leftrightarrow (z \cdot x) \cdot (z \cdot y) = 0), \tag{2.13}$$

$$(\forall x, y, z \in X)(x \cdot y = 0 \Leftrightarrow (x \cdot z) \cdot (y \cdot z) = 0). \tag{2.14}$$

In particular, by (2.7) and (2.10), the left and right cancellation laws hold in X.

We first recall the original four classes of special subsets introduced for IUP-algebras, namely IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals.

Definition 2.2. *[4] A nonempty subset S of X is called*

(i) *an IUP-subalgebra of X if it satisfies the following condition:*

$$(\forall x, y \in S)(x \cdot y \in S),$$

(ii) *an IUP-filter of X if it satisfies the following conditions:*

$$0 \in S, \tag{2.15}$$

$$(\forall x, y \in X)(x \cdot y \in S,\ x \in S \Rightarrow y \in S),$$

(iii) *an IUP-ideal of X if it satisfies* (2.15) *and the following condition:*

$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in S,\ y \in S \Rightarrow x \cdot z \in S),$$

(iv) *a strong IUP-ideal of X if it satisfies the following condition:*

$$(\forall x, y \in X)(y \in S \Rightarrow x \cdot y \in S).$$

From axiom (IUP-2), we immediately obtain the following remark.

Remark 2.1. *Every IUP-subalgebra of X satisfies* (2.15)*. Indeed, if S is an IUP-subalgebra of X and $x \in S$, then $x \cdot x = 0 \in S$.*

Beyond the basic special subsets recalled above, Suayngam et al. [22] introduced four further types of IUP-ideals, namely transmitted IUP-ideals, reflected IUP-ideals, resonant IUP-ideals, and dominant IUP-ideals. These notions are defined by imposing additional closure-type conditions on nonempty subsets of an IUP-algebra. For completeness, we recall their definitions as follows.

Definition 2.3. *[22] A nonempty subset S of X is called* (*i*) *a transmitted IUP-ideal of X if it satisfies* (2.15) *and the following condition:*

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \in S, y \in S \Rightarrow x \cdot z \in S),$$

(*ii*) *a reflected IUP-ideal of X if it satisfies* (2.15) *and the following condition:*

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \in S, z \in S \Rightarrow x \cdot (y \cdot (y \cdot x)) \in S),$$

(*iii*) *a resonant IUP-ideal of X if it satisfies* (2.15) *and the following condition:*

$$(\forall x, y \in X)(x \cdot y \in S \Rightarrow x \cdot ((y \cdot x) \cdot x) \in S),$$

(*iv*) *a dominant IUP-ideal of X if it satisfies* (2.15) *and the following condition:*

$$(\forall x, y \in X)(x \cdot y \in S \Rightarrow (x \cdot (x \cdot y)) \cdot x \in S).$$

Let X be an IUP-algebra. Based on the relationships among special subsets of X described in [4, 22], we summarize in Figure 1 the inclusion relations among IUP-filters, IUP-subalgebras, IUP-ideals, strong IUP-ideals, transmitted IUP-ideals, reflected IUP-ideals, resonant IUP-ideals, and dominant IUP-ideals. Each arrow indicates that every subset of the source class belongs to the target class.

$$\begin{tikzcd}
\text{reflected IUP-ideal} & \text{resonant IUP-ideal} & \text{IUP-filter} \\
\text{dominant IUP-ideal} \arrow[u] \arrow[ur] & \text{IUP-ideal} \arrow[u] \arrow[ur] & \text{IUP-subalgebra} \arrow[u] \\
& \text{transmitted IUP-ideal} \arrow[ul] \arrow[u] \arrow[ur] & \\
& \text{strong IUP-ideal} (= X) \arrow[u] &
\end{tikzcd}$$

Figure 1. Relationships among eight classes of special subsets in IUP-algebras

The additional classes are included here to provide a broader view of special subsets in IUP-algebras. Although the main results of this paper focus on fuzzy IUP-subalgebras induced by hyperstructures, the remaining classes of special subsets may provide a basis for parallel future research on length, mean, and dot constructions in IUP-algebras.

3. Fuzzy and Hyperfuzzy Structures in IUP-Algebras

In this section, we recall the notions of fuzzy sets, hyperfuzzy sets, and hyperstructures. We also introduce four types of fuzzy IUP-subalgebras that will be used to define length, mean, and dot fuzzy IUP-subalgebras in the subsequent sections.

Definition 3.1. *[28] Let X be a nonempty set. A mapping $f : X \to [0,1]$ is called a fuzzy set in X (or a fuzzy subset of X), where $[0,1]$ is the unit interval of the real line. An ordered pair (X, f) is called a fuzzy structure in X. A fuzzy structure (X, f) in X is said to be constant if the fuzzy set f is constant.*

Definition 3.2. *[3] Let X be a nonempty set. A mapping $\tilde{f} : X \to \tilde{P}([0,1])$ is called a hyperfuzzy set over X, where $\tilde{P}([0,1])$ denotes the family of all nonempty subsets of $[0,1]$. An ordered pair $(X, \tilde{f})$ is called a hyperstructure over X.*

Definition 3.3. *[8] Given a hyperstructure $(X, \tilde{f})$ over a nonempty set X, we define two fuzzy structures $(X, \tilde{f}_{\inf})$ and $(X, \tilde{f}_{\sup})$ in X as follows:*

$$\tilde{f}_{\inf} : X \to [0,1], \quad x \mapsto \inf \tilde{f}(x),$$
$$\tilde{f}_{\sup} : X \to [0,1], \quad x \mapsto \sup \tilde{f}(x).$$

We now formalize these four types of fuzzy IUP-subalgebras. These notions are the basic fuzzy structures used later to study length, mean, and dot constructions.

Definition 3.4. *A fuzzy structure (X, f) in X is called*

(1) a fuzzy IUP-subalgebra of X with type 1 (briefly, a 1-fuzzy IUP-subalgebra of X) if

$$(\forall x, y \in X)(f(x \cdot y) \geq \min\{f(x), f(y)\}); \tag{3.1}$$

(2) a fuzzy IUP-subalgebra of X with type 2 (briefly, a 2-fuzzy IUP-subalgebra of X) if

$$(\forall x, y \in X)(f(x \cdot y) \leq \min\{f(x), f(y)\}); \tag{3.2}$$

(3) a fuzzy IUP-subalgebra of X with type 3 (briefly, a 3-fuzzy IUP-subalgebra of X) if

$$(\forall x, y \in X)(f(x \cdot y) \geq \max\{f(x), f(y)\}); \tag{3.3}$$

(4) a fuzzy IUP-subalgebra of X with type 4 (briefly, a 4-fuzzy IUP-subalgebra of X) if

$$(\forall x, y \in X)(f(x \cdot y) \leq \max\{f(x), f(y)\}). \tag{3.4}$$

Proposition 3.1. *If (X, f) is a k-fuzzy IUP-subalgebra of X for $k = 1, 3$, then*

$$(\forall x \in X)(f(0) \geq f(x)).$$

Proof. Assume first that (X, f) is a 1-fuzzy IUP-subalgebra of X. Then, for all $x \in X$,

$$f(0) = f(x \cdot x) \geq \min\{f(x), f(x)\} = f(x)$$

by (IUP-2) and (3.1).

Assume next that (X, f) is a 3-fuzzy IUP-subalgebra of X. Then, for all $x \in X$,

$$f(0) = f(x \cdot x) \geq \max\{f(x), f(x)\} = f(x)$$

by (IUP-2) and (3.3). Therefore, $f(0) \geq f(x)$ for all $x \in X$. □

Proposition 3.2. *If (X, f) is a k-fuzzy IUP-subalgebra of X for $k = 2, 4$, then*

$$(\forall x \in X)(f(0) \leq f(x)).$$

Proof. Assume first that (X, f) is a 2-fuzzy IUP-subalgebra of X. Then, for all $x \in X$,

$$f(0) = f(x \cdot x) \leq \min\{f(x), f(x)\} = f(x)$$

by (IUP-2) and (3.2).

Assume next that (X, f) is a 4-fuzzy IUP-subalgebra of X. Then, for all $x \in X$,

$$f(0) = f(x \cdot x) \leq \max\{f(x), f(x)\} = f(x)$$

by (IUP-2) and (3.4). Therefore, $f(0) \leq f(x)$ for all $x \in X$. □

Theorem 3.1. *Every 3-fuzzy IUP-subalgebra of X is a 1-fuzzy IUP-subalgebra.*

Proof. Assume that (X, f) is a 3-fuzzy IUP-subalgebra of X. Let $x, y \in X$. Then

$$f(x \cdot y) \geq \max\{f(x), f(y)\} \geq \min\{f(x), f(y)\}.$$

Hence, (X, f) is a 1-fuzzy IUP-subalgebra of X. □

The following example shows that the converse of Theorem 3.1 is not true.

Example 3.1. *Consider an IUP-algebra $X = \{0, 1, 2, 3, 4, 5\}$ with the binary operation $\cdot$ given as follows:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	4	2	3	1
2	4	2	0	1	5	3
3	3	4	5	0	1	2
4	2	3	1	5	0	4
5	1	5	3	4	2	0

Let (X, f) be a fuzzy structure in X in which f is given as follows:

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.6 & 0.6 & 0.2 & 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

Then (X, f) is a 1-fuzzy IUP-subalgebra of X by direct verification. However,

$$f(0 \cdot 4) = 0.2 \ngeq 0.6 = \max\{0.6, 0.2\} = \max\{f(0), f(4)\}.$$

Thus (X, f) is not a 3-fuzzy IUP-subalgebra of X.

Theorem 3.2. *Every 2-fuzzy IUP-subalgebra of X is a 4-fuzzy IUP-subalgebra.*

Proof. Assume that (X, f) is a 2-fuzzy IUP-subalgebra of X. Let $x, y \in X$. Then

$$f(x \cdot y) \leq \min\{f(x), f(y)\} \leq \max\{f(x), f(y)\}.$$

Hence, (X, f) is a 4-fuzzy IUP-subalgebra of X. □

The following example shows that the converse of Theorem 3.2 is not true.

Example 3.2. *Consider the IUP-algebra $X = \{0,1,2,3,4,5\}$ in Example 3.1. Let (X, f) be a fuzzy structure in X in which f is given as follows:*

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.5 & 0.5 & 0.6 & 0.6 & 0.6 & 0.5 \end{pmatrix}.$$

Then (X, f) is a 4-fuzzy IUP-subalgebra of X by direct verification. However,

$$f(0 \cdot 3) = f(3) = 0.6 \nleq 0.5 = \min\{f(0), f(3)\}.$$

Thus (X, f) is not a 2-fuzzy IUP-subalgebra of X.

Theorem 3.3. *A fuzzy structure (X, f) in X is a 2-fuzzy IUP-subalgebra of X if and only if it is constant.*

Proof. Assume that (X, f) is a 2-fuzzy IUP-subalgebra of X. Then, by Proposition 3.2, we have $f(0) \leq f(x)$ for all $x \in X$. By (IUP-1), we have

$$f(x) = f(0 \cdot x) \leq \min\{f(0), f(x)\} = f(0)$$

for all $x \in X$. Thus $f(x) = f(0)$ for all $x \in X$, and hence f is constant.

Conversely, assume that (X, f) is constant. Then $f(x) = f(0)$ for all $x \in X$. Let $x, y \in X$. Then

$$f(x \cdot y) = f(0) = \min\{f(0), f(0)\} \leq \min\{f(x), f(y)\}.$$

Therefore, (X, f) is a 2-fuzzy IUP-subalgebra of X. □

Theorem 3.4. *A fuzzy structure (X, f) in X is a 3-fuzzy IUP-subalgebra of X if and only if it is constant.*

Proof. Assume that (X, f) is a 3-fuzzy IUP-subalgebra of X. Then, by Proposition 3.1, we have $f(0) \geq f(x)$ for all $x \in X$. By (IUP-1), we have

$$f(x) = f(0 \cdot x) \geq \max\{f(0), f(x)\} = f(0)$$

for all $x \in X$. Thus $f(x) = f(0)$ for all $x \in X$, and hence f is constant.

Conversely, assume that (X, f) is constant. Then $f(x) = f(0)$ for all $x \in X$. Let $x, y \in X$. Then

$$f(x \cdot y) = f(0) = \max\{f(0), f(0)\} \geq \max\{f(x), f(y)\}.$$

Therefore, (X, f) is a 3-fuzzy IUP-subalgebra of X. □

By Theorems 3.3 and 3.4, 2-fuzzy IUP-subalgebras, 3-fuzzy IUP-subalgebras, and constant fuzzy structures coincide.

Definition 3.5. *For any $i, j \in \{1,2,3,4\}$, a hyperstructure $(X, \tilde{f})$ over X is called an (i, j)-hyperfuzzy IUP-subalgebra of X if the fuzzy structure $(X, \tilde{f}_{\inf})$ is an i-fuzzy IUP-subalgebra of X and the fuzzy structure $(X, \tilde{f}_{\sup})$ is a j-fuzzy IUP-subalgebra of X.*

Definition 3.6. *[17] Let (X, f) be a fuzzy structure in X. For any $t \in [0,1]$, the sets*

$$U(f;t) = \{x \in X \mid f(x) \geq t\}, \quad L(f;t) = \{x \in X \mid f(x) \leq t\}, \quad E(f;t) = \{x \in X \mid f(x) = t\}$$

are called the upper t-level subset, lower t-level subset, and equal t-level subset of f, respectively.

4. Length of a Hyperstructure in IUP-Algebras

In this section, we introduce the notion of the length of a hyperstructure in IUP-algebras. We define length fuzzy IUP-subalgebras and investigate some related properties. We also establish relationships between length fuzzy IUP-subalgebras and hyperfuzzy IUP-subalgebras. Moreover, we study the connections between length fuzzy IUP-subalgebras and the corresponding upper level subsets, lower level subsets, and equal level subsets.

Definition 4.1. *Given a hyperstructure $(X, \tilde{f})$ over X, we define a fuzzy structure $(X, \tilde{f}_l)$ in X by*

$$\tilde{f}_l : X \to [0,1], \quad x \mapsto \tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x).$$

The fuzzy set $\tilde{f}_l$ is called the length of $\tilde{f}$.

Definition 4.2. *A hyperstructure $(X, \tilde{f})$ over X is called a length 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, and 4-fuzzy) IUP-subalgebra of X if the fuzzy structure $(X, \tilde{f}_l)$ is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, and 4-fuzzy) IUP-subalgebra of X.*

Example 4.1. *Consider the IUP-algebra $X = \{0,1,2,3,4,5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.1,0.2) \cup [0.4,0.7) & (0.2,0.8] & [0.4,0.6) & (0.2,0.4] & [0.6,0.8] & [0.2,0.5) \cup [0.7,0.8) \end{pmatrix}.$$

Then the length of $\tilde{f}$ is given by

$$\tilde{f}_l = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.6 & 0.6 & 0.2 & 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

By direct verification, $(X, \tilde{f}_l)$ is a 1-fuzzy IUP-subalgebra of X. Hence, $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X.

Proposition 4.1. *If $(X, \tilde{f})$ is a length k-fuzzy IUP-subalgebra of X for $k = 1, 3$, then*

$$(\forall x \in X)(\tilde{f}_l(0) \geq \tilde{f}_l(x)).$$

Proof. It follows directly from Proposition 3.1, applied to the fuzzy structure $(X, \tilde{f}_l)$. □

Proposition 4.2. *If $(X, \tilde{f})$ is a length k-fuzzy IUP-subalgebra of X for $k = 2, 4$, then*

$$(\forall x \in X)(\tilde{f}_l(0) \leq \tilde{f}_l(x)).$$

Proof. It follows directly from Proposition 3.2, applied to the fuzzy structure $(X, \tilde{f}_l)$. □

Theorem 4.1. *Every length 3-fuzzy IUP-subalgebra of X is a length 1-fuzzy IUP-subalgebra.*

Proof. It follows directly from Theorem 3.1. □

Theorem 4.2. *Every length* 2*-fuzzy IUP-subalgebra of* X *is a length* 4*-fuzzy IUP-subalgebra.*

Proof. It follows directly from Theorem 3.2. □

Theorem 4.3. *The length* 2*-fuzzy IUP-subalgebras and the length* 3*-fuzzy IUP-subalgebras of* X *coincide.*

Proof. It follows directly from Theorems 3.3 and 3.4, applied to the fuzzy structure $(X, \tilde{f}_1)$. □

Theorem 4.4. *Let* S *be an IUP-subalgebra of* X*, and let* $(X, \tilde{f})$ *be a hyperstructure over* X*. Suppose that there exist* $a, b \in [0, 1]$ *such that*

$$\tilde{f}_1(x) = \begin{cases} a & \text{if } x \in S, \\ b & \text{if } x \notin S. \end{cases}$$

If $b \leq a$*, then* $(X, \tilde{f})$ *is a length* 1*-fuzzy IUP-subalgebra of* X*. If* $a \leq b$*, then* $(X, \tilde{f})$ *is a length* 4*-fuzzy IUP-subalgebra of* X*.*

Proof. Assume first that $b \leq a$. Let $x, y \in X$.

If $x, y \in S$, then $x \cdot y \in S$ because S is an IUP-subalgebra of X. Hence,

$$\tilde{f}_1(x \cdot y) = a = \min\{\tilde{f}_1(x), \tilde{f}_1(y)\}.$$

If at least one of x, y is not in S, then

$$\min\{\tilde{f}_1(x), \tilde{f}_1(y)\} = b.$$

Since $\tilde{f}_1(x \cdot y) \in \{a, b\}$ and $b \leq a$, we have

$$\tilde{f}_1(x \cdot y) \geq b = \min\{\tilde{f}_1(x), \tilde{f}_1(y)\}.$$

Thus $(X, \tilde{f}_1)$ is a 1-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X.

Assume next that $a \leq b$. Let $x, y \in X$.

If $x, y \in S$, then $x \cdot y \in S$, and hence

$$\tilde{f}_1(x \cdot y) = a = \max\{\tilde{f}_1(x), \tilde{f}_1(y)\}.$$

If at least one of x, y is not in S, then

$$\max\{\tilde{f}_1(x), \tilde{f}_1(y)\} = b.$$

Since $\tilde{f}_1(x \cdot y) \in \{a, b\}$ and $a \leq b$, we have

$$\tilde{f}_1(x \cdot y) \leq b = \max\{\tilde{f}_1(x), \tilde{f}_1(y)\}.$$

Thus $(X, \tilde{f}_1)$ is a 4-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X. □

Example 4.2. *Consider the IUP-algebra $X = \{0,1,2,3,4,5\}$ in Example 3.1. Then $S = \{0,1,5\}$ is an IUP-subalgebra of X.*

Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.2, 0.9) & [0.3, 1] & (0.2, 0.5) & [0.4, 0.7] & [0.5, 0.8) & (0.1, 0.8) \end{pmatrix}.$$

Then

$$\tilde{f}_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.7 & 0.7 & 0.3 & 0.3 & 0.3 & 0.7 \end{pmatrix}.$$

By Theorem 4.4, $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X. However,

$$\tilde{f}_1(0 \cdot 4) = 0.3 \ngeq 0.7 = \max\{\tilde{f}_1(0), \tilde{f}_1(4)\}.$$

Thus $(X, \tilde{f})$ is not a length 3-fuzzy IUP-subalgebra of X.

Next, let $(X, \tilde{f})$ be a hyperstructure over X given by

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ (0.3, 0.8] & [0.2, 0.7) & (0.3, 0.9) & [0.1, 0.7] & (0.4, 1] & (0.1, 0.6) \end{pmatrix}.$$

Then

$$\tilde{f}_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.5 & 0.5 & 0.6 & 0.6 & 0.6 & 0.5 \end{pmatrix}.$$

By Theorem 4.4, $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X. However,

$$\tilde{f}_1(0 \cdot 3) = \tilde{f}_1(3) = 0.6 \nleq 0.5 = \min\{\tilde{f}_1(0), \tilde{f}_1(3)\}.$$

Thus $(X, \tilde{f})$ is not a length 2-fuzzy IUP-subalgebra of X.

Theorem 4.5. *A hyperstructure $(X, \tilde{f})$ over X is a length 1-fuzzy IUP-subalgebra of X if and only if $U(\tilde{f}_1; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $U(\tilde{f}_1; t) \neq \emptyset$.*

Proof. Assume that $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X. Let $t \in [0,1]$ be such that $U(\tilde{f}_1; t) \neq \emptyset$, and let $x, y \in U(\tilde{f}_1; t)$. Then $\tilde{f}_1(x) \geq t$ and $\tilde{f}_1(y) \geq t$. Hence,

$$\tilde{f}_1(x \cdot y) \geq \min\{\tilde{f}_1(x), \tilde{f}_1(y)\} \geq t.$$

Thus $x \cdot y \in U(\tilde{f}_1; t)$, and so $U(\tilde{f}_1; t)$ is an IUP-subalgebra of X.

Conversely, assume that $U(\tilde{f}_1; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $U(\tilde{f}_1; t) \neq \emptyset$. Let $x, y \in X$ and put

$$t = \min\{\tilde{f}_1(x), \tilde{f}_1(y)\}.$$

Then $x, y \in U(\tilde{f}_1; t)$. By assumption, $x \cdot y \in U(\tilde{f}_1; t)$. Therefore,

$$\tilde{f}_1(x \cdot y) \geq t = \min\{\tilde{f}_1(x), \tilde{f}_1(y)\}.$$

Hence, $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X. □

Corollary 4.1. *If $(X, \tilde{f})$ is a length 3-fuzzy IUP-subalgebra of X, then $U(\tilde{f_1}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $U(\tilde{f_1}; t) \neq \emptyset$.*

Proof. It follows from Theorems 4.1 and 4.5. □

Example 4.3. *Consider the IUP-algebra $X = \{0,1,2,3,4,5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.2, 0.3) \cup [0.7, 0.9) & (0.5, 0.7] & [0.2, 0.4] & [0.1, 0.3] \cup (0.4, 0.6] & (0.7, 0.9] & [0.6, 0.8] \end{pmatrix}.$$

Then

$$\tilde{f_1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.7 & 0.2 & 0.2 & 0.5 & 0.2 & 0.2 \end{pmatrix}.$$

We have

$$U(\tilde{f_1}; t) = \begin{cases} \emptyset & \text{if } t \in (0.7, 1], \\ \{0\} & \text{if } t \in (0.5, 0.7], \\ \{0, 3\} & \text{if } t \in (0.2, 0.5], \\ X & \text{if } t \in [0, 0.2]. \end{cases}$$

Hence, $U(\tilde{f_1}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $U(\tilde{f_1}; t) \neq \emptyset$. However,

$$\tilde{f_1}(0 \cdot 5) = \tilde{f_1}(5) = 0.2 \ngeq 0.7 = \max\{\tilde{f_1}(0), \tilde{f_1}(5)\}.$$

Thus $(X, \tilde{f})$ is not a length 3-fuzzy IUP-subalgebra of X.

Theorem 4.6. *A hyperstructure $(X, \tilde{f})$ over X is a length 4-fuzzy IUP-subalgebra of X if and only if $L(\tilde{f_1}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $L(\tilde{f_1}; t) \neq \emptyset$.*

Proof. Assume that $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X. Let $t \in [0,1]$ be such that $L(\tilde{f_1}; t) \neq \emptyset$, and let $x, y \in L(\tilde{f_1}; t)$. Then $\tilde{f_1}(x) \leq t$ and $\tilde{f_1}(y) \leq t$. Hence,

$$\tilde{f_1}(x \cdot y) \leq \max\{\tilde{f_1}(x), \tilde{f_1}(y)\} \leq t.$$

Thus $x \cdot y \in L(\tilde{f_1}; t)$, and so $L(\tilde{f_1}; t)$ is an IUP-subalgebra of X.

Conversely, assume that $L(\tilde{f_1}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $L(\tilde{f_1}; t) \neq \emptyset$. Let $x, y \in X$ and put

$$t = \max\{\tilde{f_1}(x), \tilde{f_1}(y)\}.$$

Then $x, y \in L(\tilde{f_1}; t)$. By assumption, $x \cdot y \in L(\tilde{f_1}; t)$. Therefore,

$$\tilde{f_1}(x \cdot y) \leq t = \max\{\tilde{f_1}(x), \tilde{f_1}(y)\}.$$

Hence, $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X. □

Corollary 4.2. *If $(X, \tilde{f})$ is a length 2-fuzzy IUP-subalgebra of X, then $L(\tilde{f_1}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $L(\tilde{f_1}; t) \neq \emptyset$.*

Proof. It follows from Theorems 4.2 and 4.6. □

Example 4.4. *Consider the IUP-algebra $X = \{0,1,2,3,4,5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.2, 0.4) & (0.1, 0.9] & [0.1, 0.9) & [0.1, 0.7] & (0.1, 0.9] & [0.2, 1] \end{pmatrix}.$$

Then

$$\tilde{f}_l = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.2 & 0.8 & 0.8 & 0.6 & 0.8 & 0.8 \end{pmatrix}.$$

We have

$$L(\tilde{f}_l; t) = \begin{cases} X & \text{if } t \in [0.8, 1], \\ \{0, 3\} & \text{if } t \in [0.6, 0.8), \\ \{0\} & \text{if } t \in [0.2, 0.6), \\ \emptyset & \text{if } t \in [0, 0.2). \end{cases}$$

Hence, $L(\tilde{f}_l; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $L(\tilde{f}_l; t) \neq \emptyset$. However,

$$\tilde{f}_l(0 \cdot 3) = \tilde{f}_l(3) = 0.6 \nleq 0.2 = \min\{\tilde{f}_l(0), \tilde{f}_l(3)\}.$$

Thus $(X, \tilde{f})$ is not a length 2-fuzzy IUP-subalgebra of X.

Theorem 4.7. *For $k \in \{2, 3\}$, a hyperstructure $(X, \tilde{f})$ over X is a length k-fuzzy IUP-subalgebra of X if and only if*

$$E(\tilde{f}_l; \tilde{f}_l(0)) = X.$$

Proof. Let $k \in \{2, 3\}$. By Theorems 3.3 and 3.4, the fuzzy structure $(X, \tilde{f}_l)$ is a k-fuzzy IUP-subalgebra of X if and only if $\tilde{f}_l$ is constant. This is equivalent to

$$\tilde{f}_l(x) = \tilde{f}_l(0)$$

for all $x \in X$, which is equivalent to

$$E(\tilde{f}_l; \tilde{f}_l(0)) = X.$$

Therefore, the result follows. □

Theorem 4.8. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 1-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\inf})$ is constant, we have $\tilde{f}_{\inf}(x) = \tilde{f}_{\inf}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\sup})$ is a 1-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\sup}(x \cdot y) \geq \min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.$$

Thus

$$\begin{aligned} \tilde{f}_l(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(x \cdot y) \\ &= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(0) \end{aligned}$$

$$\begin{aligned}
&\geq \min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\} - \tilde{f}_{\inf}(0) \\
&= \min\{\tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y) - \tilde{f}_{\inf}(y)\} \\
&= \min\{\tilde{f}_{l}(x), \tilde{f}_{l}(y)\}.
\end{aligned}$$

Therefore, $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X. □

Corollary 4.3. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 3-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorems 3.1 and 4.8. □

Corollary 4.4. *For $i \in \{2, 3\}$ and $j \in \{1, 3\}$, every (i, j)-hyperfuzzy IUP-subalgebra of X is a length 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorem 4.8 and Corollary 4.3. □

Example 4.5. *Consider the IUP-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.2, 0.8] & (0.4, 1] & (0.3, 0.6] & [0.2, 0.5) & [0.5, 0.8) & [0.3, 0.9) \end{pmatrix}.$$

Then

$$\tilde{f}_{l} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.6 & 0.6 & 0.3 & 0.3 & 0.3 & 0.6 \end{pmatrix}.$$

Thus $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X. Since

$$\tilde{f}_{\inf} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.2 & 0.4 & 0.3 & 0.2 & 0.5 & 0.3 \end{pmatrix},$$

we have that $\tilde{f}_{\inf}$ is not constant. By Theorems 3.3 and 3.4, $(X, \tilde{f}_{\inf})$ is not a 2-fuzzy or 3-fuzzy IUP-subalgebra of X. Hence, $(X, \tilde{f})$ is not an (i, j)-hyperfuzzy IUP-subalgebra of X for $i \in \{2, 3\}$ and $j \in \{1, 3\}$.

Theorem 4.9. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 4-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\inf})$ is constant, we have $\tilde{f}_{\inf}(x) = \tilde{f}_{\inf}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\sup})$ is a 4-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\sup}(x \cdot y) \leq \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.$$

Thus

$$\begin{aligned}
\tilde{f}_{l}(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(x \cdot y) \\
&= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(0) \\
&\leq \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\} - \tilde{f}_{\inf}(0) \\
&= \max\{\tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y) - \tilde{f}_{\inf}(y)\}
\end{aligned}$$

$$= \max\{\tilde{f}_l(x), \tilde{f}_l(y)\}.$$

Therefore, $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X. □

Corollary 4.5. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 2-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorems 3.2 and 4.9. □

Corollary 4.6. *For $i \in \{2, 3\}$ and $j \in \{2, 4\}$, every (i, j)-hyperfuzzy IUP-subalgebra of X is a length 4-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorem 4.9 and Corollary 4.5. □

Example 4.6. *Consider the IUP-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.1, 0.5) & (0.4, 0.8] & [0, 0.7] & [0.2, 0.9] & [0.1, 0.8) & (0.2, 0.6) \end{pmatrix}.$$

Then

$$\tilde{f}_l = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.4 & 0.4 & 0.7 & 0.7 & 0.7 & 0.4 \end{pmatrix}.$$

Thus $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X. Since

$$\tilde{f}_{\inf} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.1 & 0.4 & 0 & 0.2 & 0.1 & 0.2 \end{pmatrix},$$

we have that $\tilde{f}_{\inf}$ is not constant. By Theorems 3.3 and 3.4, $(X, \tilde{f}_{\inf})$ is not a 2-fuzzy or 3-fuzzy IUP-subalgebra of X. Hence, $(X, \tilde{f})$ is not an (i, j)-hyperfuzzy IUP-subalgebra of X for $i \in \{2, 3\}$ and $j \in \{2, 4\}$.

Theorem 4.10. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 4-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\sup})$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\inf})$ is a 4-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\inf}(x \cdot y) \leq \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.$$

Thus

$$\begin{aligned} \tilde{f}_l(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(x \cdot y) \\ &= \tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(x \cdot y) \\ &\geq \tilde{f}_{\sup}(0) - \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\} \\ &= \min\{\tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(y)\} \\ &= \min\{\tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y) - \tilde{f}_{\inf}(y)\} \\ &= \min\{\tilde{f}_l(x), \tilde{f}_l(y)\}. \end{aligned}$$

Therefore, $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X. □

Corollary 4.7. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 2-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorems 3.2 and 4.10. □

Corollary 4.8. *For $i \in \{2, 4\}$ and $j \in \{2, 3\}$, every (i, j)-hyperfuzzy IUP-subalgebra of X is a length 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorem 4.10 and Corollary 4.7. □

Example 4.7. *Consider the IUP-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.4, 0.9) & [0.1, 0.6] & (0.7, 0.8] & (0.4, 0.5) & [0.2, 0.3] & [0, 0.5) \end{pmatrix}.$$

Then

$$\tilde{f}_l = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.5 & 0.5 & 0.1 & 0.1 & 0.1 & 0.5 \end{pmatrix}.$$

Thus $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X. Since

$$\tilde{f}_{\sup} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.9 & 0.6 & 0.8 & 0.5 & 0.3 & 0.5 \end{pmatrix},$$

we have that $\tilde{f}_{\sup}$ is not constant. By Theorems 3.3 and 3.4, $(X, \tilde{f}_{\sup})$ is not a 2-fuzzy or 3-fuzzy IUP-subalgebra of X. Hence, $(X, \tilde{f})$ is not an (i, j)-hyperfuzzy IUP-subalgebra of X for $i \in \{2, 4\}$ and $j \in \{2, 3\}$.

Theorem 4.11. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 1-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\sup})$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\inf})$ is a 1-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\inf}(x \cdot y) \geq \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.$$

Thus

$$\begin{aligned} \tilde{f}_l(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(x \cdot y) \\ &= \tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(x \cdot y) \\ &\leq \tilde{f}_{\sup}(0) - \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\} \\ &= \max\{\tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(y)\} \\ &= \max\{\tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y) - \tilde{f}_{\inf}(y)\} \\ &= \max\{\tilde{f}_l(x), \tilde{f}_l(y)\}. \end{aligned}$$

Therefore, $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X. □

Corollary 4.9. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 3-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorems 3.1 and 4.11. ◻

Corollary 4.10. *For $i \in \{1, 3\}$ and $j \in \{2, 3\}$, every (i, j)-hyperfuzzy IUP-subalgebra of X is a length 4-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorem 4.11 and Corollary 4.9. ◻

Example 4.8. *Consider the IUP-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0, 0.3) & (0.7, 1] & [0.5, 0.9] & [0.2, 0.6) & [0.4, 0.8) & [0.5, 0.8) \end{pmatrix}.$$

Then

$$\tilde{f}_l = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.3 & 0.3 & 0.4 & 0.4 & 0.4 & 0.3 \end{pmatrix}.$$

Thus $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X. Since

$$\tilde{f}_{\sup} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.3 & 1 & 0.9 & 0.6 & 0.8 & 0.8 \end{pmatrix},$$

we have that $\tilde{f}_{\sup}$ is not constant. By Theorems 3.3 and 3.4, $(X, \tilde{f}_{\sup})$ is not a 2-fuzzy or 3-fuzzy IUP-subalgebra of X. Hence, $(X, \tilde{f})$ is not an (i, j)-hyperfuzzy IUP-subalgebra of X for $i \in \{1, 3\}$ and $j \in \{2, 3\}$.

Theorem 4.12. *If $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\inf}$ is constant, then $(X, \tilde{f})$ is a $(k, 1)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X and $\tilde{f}_{\inf}$ is constant. Since $\tilde{f}_{\inf}$ is constant, $(X, \tilde{f}_{\inf})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Let $x, y \in X$. Then

$$\begin{aligned} \tilde{f}_{\sup}(x \cdot y) &= \tilde{f}_l(x \cdot y) + \tilde{f}_{\inf}(x \cdot y) \\ &= \tilde{f}_l(x \cdot y) + \tilde{f}_{\inf}(0) \\ &\geq \min\{\tilde{f}_l(x), \tilde{f}_l(y)\} + \tilde{f}_{\inf}(0) \\ &= \min\{\tilde{f}_l(x) + \tilde{f}_{\inf}(x), \tilde{f}_l(y) + \tilde{f}_{\inf}(y)\} \\ &= \min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}. \end{aligned}$$

Thus $(X, \tilde{f}_{\sup})$ is a 1-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(k, 1)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. ◻

Corollary 4.11. *If $(X, \tilde{f})$ is a length 3-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\inf}$ is constant, then $(X, \tilde{f})$ is a $(k, 1)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 4.1 and 4.12. ◻

Theorem 4.13. *If $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\inf}$ is constant, then $(X, \tilde{f})$ is a $(k, 4)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X and $\tilde{f}_{\inf}$ is constant. Since $\tilde{f}_{\inf}$ is constant, $(X, \tilde{f}_{\inf})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Let $x, y \in X$. Then

$$\begin{aligned}
\tilde{f}_{\sup}(x \cdot y) &= \tilde{f}_{l}(x \cdot y) + \tilde{f}_{\inf}(x \cdot y) \\
&= \tilde{f}_{l}(x \cdot y) + \tilde{f}_{\inf}(0) \\
&\leq \max\{\tilde{f}_{l}(x), \tilde{f}_{l}(y)\} + \tilde{f}_{\inf}(0) \\
&= \max\{\tilde{f}_{l}(x) + \tilde{f}_{\inf}(x), \tilde{f}_{l}(y) + \tilde{f}_{\inf}(y)\} \\
&= \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.
\end{aligned}$$

Thus $(X, \tilde{f}_{\sup})$ is a 4-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(k, 4)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. □

Corollary 4.12. *If $(X, \tilde{f})$ is a length 2-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\inf}$ is constant, then $(X, \tilde{f})$ is a $(k, 4)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 4.2 and 4.13. □

Theorem 4.14. *If $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\sup}$ is constant, then $(X, \tilde{f})$ is a $(4, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \tilde{f})$ is a length 1-fuzzy IUP-subalgebra of X and $\tilde{f}_{\sup}$ is constant. Since $\tilde{f}_{\sup}$ is constant, $(X, \tilde{f}_{\sup})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Let $x, y \in X$. Then

$$\begin{aligned}
\tilde{f}_{\inf}(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{l}(x \cdot y) \\
&= \tilde{f}_{\sup}(0) - \tilde{f}_{l}(x \cdot y) \\
&\leq \tilde{f}_{\sup}(0) - \min\{\tilde{f}_{l}(x), \tilde{f}_{l}(y)\} \\
&= \max\{\tilde{f}_{\sup}(0) - \tilde{f}_{l}(x), \tilde{f}_{\sup}(0) - \tilde{f}_{l}(y)\} \\
&= \max\{\tilde{f}_{\sup}(x) - \tilde{f}_{l}(x), \tilde{f}_{\sup}(y) - \tilde{f}_{l}(y)\} \\
&= \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.
\end{aligned}$$

Thus $(X, \tilde{f}_{\inf})$ is a 4-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(4, k)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. □

Corollary 4.13. *If $(X, \tilde{f})$ is a length 3-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\sup}$ is constant, then $(X, \tilde{f})$ is a $(4, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 4.1 and 4.14. □

Theorem 4.15. *If $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\sup}$ is constant, then $(X, \tilde{f})$ is a $(1, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \tilde{f})$ is a length 4-fuzzy IUP-subalgebra of X and $\tilde{f}_{\text{sup}}$ is constant. Since $\tilde{f}_{\text{sup}}$ is constant, $(X, \tilde{f}_{\text{sup}})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Let $x, y \in X$. Then

$$\begin{aligned}\tilde{f}_{\text{inf}}(x \cdot y) &= \tilde{f}_{\text{sup}}(x \cdot y) - \tilde{f}_{\text{l}}(x \cdot y) \\ &= \tilde{f}_{\text{sup}}(0) - \tilde{f}_{\text{l}}(x \cdot y) \\ &\geq \tilde{f}_{\text{sup}}(0) - \max\{\tilde{f}_{\text{l}}(x), \tilde{f}_{\text{l}}(y)\} \\ &= \min\{\tilde{f}_{\text{sup}}(0) - \tilde{f}_{\text{l}}(x), \tilde{f}_{\text{sup}}(0) - \tilde{f}_{\text{l}}(y)\} \\ &= \min\{\tilde{f}_{\text{sup}}(x) - \tilde{f}_{\text{l}}(x), \tilde{f}_{\text{sup}}(y) - \tilde{f}_{\text{l}}(y)\} \\ &= \min\{\tilde{f}_{\text{inf}}(x), \tilde{f}_{\text{inf}}(y)\}.\end{aligned}$$

Thus $(X, \tilde{f}_{\text{inf}})$ is a 1-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(1, k)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. □

Corollary 4.14. *If $(X, \tilde{f})$ is a length 2-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\text{sup}}$ is constant, then $(X, \tilde{f})$ is a $(1, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 4.2 and 4.15. □

5. Mean of a Hyperstructure in IUP-Algebras

In this section, we introduce the notion of the mean of a hyperstructure in IUP-algebras. We define mean fuzzy IUP-subalgebras and investigate some related properties. We also establish relationships between mean fuzzy IUP-subalgebras and hyperfuzzy IUP-subalgebras. Moreover, we study the connections between mean fuzzy IUP-subalgebras and the corresponding upper level subsets, lower level subsets, and equal level subsets.

Definition 5.1. *Given a hyperstructure $(X, \tilde{f})$ over X, we define a fuzzy structure $(X, \tilde{f}_{\text{m}})$ in X by*

$$\tilde{f}_{\text{m}} : \mathrm{X} \to [0, 1], \quad x \mapsto \frac{\tilde{f}_{\text{sup}}(x) + \tilde{f}_{\text{inf}}(x)}{2}.$$

The fuzzy set $\tilde{f}_{\text{m}}$ is called the mean of $\tilde{f}$.

Definition 5.2. *A hyperstructure $(X, \tilde{f})$ over X is called a mean 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, and 4-fuzzy) IUP-subalgebra of X if the fuzzy structure $(X, \tilde{f}_{\text{m}})$ is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, and 4-fuzzy) IUP-subalgebra of X.*

Example 5.1. *Consider the IUP-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.7, 0.9) & (0.6, 1] & [0.1, 0.4) \cup [0.6, 0.9) & [0.4, 0.6] & \{0.5\} & [0.6, 0.7] \cup (0.8, 1] \end{pmatrix}.$$

Then the mean of $\tilde{f}$ is given by

$$\tilde{f}_{\text{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.8 & 0.8 & 0.5 & 0.5 & 0.5 & 0.8 \end{pmatrix}.$$

By direct verification, $(X, \tilde{f}_{\mathrm{m}})$ *is a* 1*-fuzzy IUP-subalgebra of* X*. Hence,* $(X, \tilde{f})$ *is a mean* 1*-fuzzy IUP-subalgebra of* X*.*

Proposition 5.1. *If* $(X, \tilde{f})$ *is a mean* k*-fuzzy IUP-subalgebra of* X *for* $k = 1, 3$*, then*

$$(\forall x \in X)(\tilde{f}_{\mathrm{m}}(0) \geq \tilde{f}_{\mathrm{m}}(x)).$$

Proof. It follows directly from Proposition 3.1, applied to the fuzzy structure $(X, \tilde{f}_{\mathrm{m}})$. □

Proposition 5.2. *If* $(X, \tilde{f})$ *is a mean* k*-fuzzy IUP-subalgebra of* X *for* $k = 2, 4$*, then*

$$(\forall x \in X)(\tilde{f}_{\mathrm{m}}(0) \leq \tilde{f}_{\mathrm{m}}(x)).$$

Proof. It follows directly from Proposition 3.2, applied to the fuzzy structure $(X, \tilde{f}_{\mathrm{m}})$. □

Theorem 5.1. *Every mean* 3*-fuzzy IUP-subalgebra of* X *is a mean* 1*-fuzzy IUP-subalgebra.*

Proof. It follows directly from Theorem 3.1. □

Theorem 5.2. *Every mean* 2*-fuzzy IUP-subalgebra of* X *is a mean* 4*-fuzzy IUP-subalgebra.*

Proof. It follows directly from Theorem 3.2. □

Theorem 5.3. *The mean* 2*-fuzzy IUP-subalgebras and the mean* 3*-fuzzy IUP-subalgebras of* X *coincide.*

Proof. It follows directly from Theorems 3.3 and 3.4, applied to the fuzzy structure $(X, \tilde{f}_{\mathrm{m}})$. □

Theorem 5.4. *Let* S *be an IUP-subalgebra of* X*, and let* $(X, \tilde{f})$ *be a hyperstructure over* X*. Suppose that there exist* $a, b \in [0, 1]$ *such that*

$$\tilde{f}_{\mathrm{m}}(x) = \begin{cases} a & \text{if } x \in S, \\ b & \text{if } x \notin S. \end{cases}$$

If $b \leq a$*, then* $(X, \tilde{f})$ *is a mean* 1*-fuzzy IUP-subalgebra of* X*. If* $a \leq b$*, then* $(X, \tilde{f})$ *is a mean* 4*-fuzzy IUP-subalgebra of* X*.*

Proof. Assume first that $b \leq a$. Let $x, y \in X$.

If $x, y \in S$, then $x \cdot y \in S$ because S is an IUP-subalgebra of X. Hence,

$$\tilde{f}_{\mathrm{m}}(x \cdot y) = a = \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

If at least one of x, y is not in S, then

$$\min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} = b.$$

Since $\tilde{f}_{\mathrm{m}}(x \cdot y) \in \{a, b\}$ and $b \leq a$, we have

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \geq b = \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

Thus $(X, \tilde{f}_{\mathrm{m}})$ is a 1-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X.

Assume next that $a \leq b$. Let $x, y \in X$.

If $x, y \in S$, then $x \cdot y \in S$, and hence

$$\tilde{f}_{\mathrm{m}}(x \cdot y) = a = \max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

If at least one of x, y is not in S, then

$$\max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} = b.$$

Since $\tilde{f}_{\mathrm{m}}(x \cdot y) \in \{a, b\}$ and $a \leq b$, we have

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \leq b = \max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

Thus $(X, \tilde{f}_{\mathrm{m}})$ is a 4-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X. □

Example 5.2. *Consider the IUP-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 3.1. Then $S = \{0, 1, 5\}$ is an IUP-subalgebra of X.*

Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.5, 0.9) & [0.6, 0.8) & [0.1, 0.5) & (0.2, 0.4] & [0.1, 0.5] & (0.4, 1] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.7 & 0.7 & 0.3 & 0.3 & 0.3 & 0.7 \end{pmatrix}.$$

By Theorem 5.4, $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X. However,

$$\tilde{f}_{\mathrm{m}}(1 \cdot 3) = \tilde{f}_{\mathrm{m}}(2) = 0.3 \not\geq 0.7 = \max\{\tilde{f}_{\mathrm{m}}(1), \tilde{f}_{\mathrm{m}}(3)\}.$$

Thus $(X, \tilde{f})$ is not a mean 3-fuzzy IUP-subalgebra of X.

Next, let $(X, \tilde{f})$ be a hyperstructure over X given by

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.1, 0.3] & [0, 0.4] & [0.3, 0.9] & [0.4, 0.8] & [0.5, 0.7] & [0.1, 0.3] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.2 & 0.2 & 0.6 & 0.6 & 0.6 & 0.2 \end{pmatrix}.$$

By Theorem 5.4, $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X. However,

$$\tilde{f}_{\mathrm{m}}(0 \cdot 3) = \tilde{f}_{\mathrm{m}}(3) = 0.6 \not\leq 0.2 = \min\{\tilde{f}_{\mathrm{m}}(0), \tilde{f}_{\mathrm{m}}(3)\}.$$

Thus $(X, \tilde{f})$ is not a mean 2-fuzzy IUP-subalgebra of X.

Theorem 5.5. *A hyperstructure $(X, \tilde{f})$ over X is a mean 1-fuzzy IUP-subalgebra of X if and only if $U(\tilde{f}_{\mathrm{m}}; t)$ is an IUP-subalgebra of X for all $t \in [0, 1]$ with $U(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$.*

Proof. Assume that $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X. Let $t \in [0,1]$ be such that $U(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$, and let $x, y \in U(\tilde{f}_{\mathrm{m}}; t)$. Then $\tilde{f}_{\mathrm{m}}(x) \geq t$ and $\tilde{f}_{\mathrm{m}}(y) \geq t$. Hence,

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \geq \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} \geq t.$$

Thus $x \cdot y \in U(\tilde{f}_{\mathrm{m}}; t)$, and so $U(\tilde{f}_{\mathrm{m}}; t)$ is an IUP-subalgebra of X.

Conversely, assume that $U(\tilde{f}_{\mathrm{m}}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $U(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$. Let $x, y \in X$ and put

$$t = \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

Then $x, y \in U(\tilde{f}_{\mathrm{m}}; t)$. By assumption, $x \cdot y \in U(\tilde{f}_{\mathrm{m}}; t)$. Therefore,

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \geq t = \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

Hence, $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X. □

Corollary 5.1. *If $(X, \tilde{f})$ is a mean 3-fuzzy IUP-subalgebra of X, then $U(\tilde{f}_{\mathrm{m}}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $U(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$.*

Proof. It follows from Theorems 5.1 and 5.5. □

Example 5.3. *Consider the IUP-algebra $X = \{0,1,2,3,4,5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.8, 0.9] & [0.2, 0.4] & [0.1, 0.2] \cup (0.4, 0.5] & (0.5, 0.7) & (0, 0.6] & [0.3, 0.3] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.85 & 0.3 & 0.3 & 0.6 & 0.3 & 0.3 \end{pmatrix}.$$

We have

$$U(\tilde{f}_{\mathrm{m}}; t) = \begin{cases} \emptyset & \text{if } t \in (0.85, 1], \\ \{0\} & \text{if } t \in (0.6, 0.85], \\ \{0, 3\} & \text{if } t \in (0.3, 0.6], \\ X & \text{if } t \in [0, 0.3]. \end{cases}$$

Hence, $U(\tilde{f}_{\mathrm{m}}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $U(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$. However,

$$\tilde{f}_{\mathrm{m}}(0 \cdot 2) = \tilde{f}_{\mathrm{m}}(2) = 0.3 \ngeq 0.85 = \max\{\tilde{f}_{\mathrm{m}}(0), \tilde{f}_{\mathrm{m}}(2)\}.$$

Thus $(X, \tilde{f})$ is not a mean 3-fuzzy IUP-subalgebra of X.

Theorem 5.6. *A hyperstructure $(X, \tilde{f})$ over X is a mean 4-fuzzy IUP-subalgebra of X if and only if $L(\tilde{f}_{\mathrm{m}}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $L(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$.*

Proof. Assume that $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X. Let $t \in [0,1]$ be such that $L(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$, and let $x, y \in L(\tilde{f}_{\mathrm{m}}; t)$. Then $\tilde{f}_{\mathrm{m}}(x) \leq t$ and $\tilde{f}_{\mathrm{m}}(y) \leq t$. Hence,

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \leq \max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} \leq t.$$

Thus $x \cdot y \in L(\tilde{f}_{\mathrm{m}}; t)$, and so $L(\tilde{f}_{\mathrm{m}}; t)$ is an IUP-subalgebra of X.

Conversely, assume that $L(\tilde{f}_{\mathrm{m}}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $L(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$. Let $x, y \in X$ and put

$$t = \max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

Then $x, y \in L(\tilde{f}_{\mathrm{m}}; t)$. By assumption, $x \cdot y \in L(\tilde{f}_{\mathrm{m}}; t)$. Therefore,

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \leq t = \max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

Hence, $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X. □

Corollary 5.2. *If $(X, \tilde{f})$ is a mean 2-fuzzy IUP-subalgebra of X, then $L(\tilde{f}_{\mathrm{m}}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $L(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$.*

Proof. It follows from Theorems 5.2 and 5.6. □

Example 5.4. *Consider the IUP-algebra $X = \{0,1,2,3,4,5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.1, 0.3) & [0.7, 0.9] & (0.6, 1) & [0.3, 0.5] & [0.8, 0.8] & [0.6, 1] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.2 & 0.8 & 0.8 & 0.4 & 0.8 & 0.8 \end{pmatrix}.$$

We have

$$L(\tilde{f}_{\mathrm{m}}; t) = \begin{cases} X & \text{if } t \in [0.8, 1], \\ \{0, 3\} & \text{if } t \in [0.4, 0.8), \\ \{0\} & \text{if } t \in [0.2, 0.4), \\ \emptyset & \text{if } t \in [0, 0.2). \end{cases}$$

Hence, $L(\tilde{f}_{\mathrm{m}}; t)$ is an IUP-subalgebra of X for all $t \in [0,1]$ with $L(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$. However,

$$\tilde{f}_{\mathrm{m}}(0 \cdot 2) = \tilde{f}_{\mathrm{m}}(2) = 0.8 \nleq 0.2 = \min\{\tilde{f}_{\mathrm{m}}(0), \tilde{f}_{\mathrm{m}}(2)\}.$$

Thus $(X, \tilde{f})$ is not a mean 2-fuzzy IUP-subalgebra of X.

Theorem 5.7. *For $k \in \{2, 3\}$, a hyperstructure $(X, \tilde{f})$ over X is a mean k-fuzzy IUP-subalgebra of X if and only if*

$$E(\tilde{f}_{\mathrm{m}}; \tilde{f}_{\mathrm{m}}(0)) = X.$$

Proof. Let $k \in \{2,3\}$. By Theorems 3.3 and 3.4, the fuzzy structure $(X, \tilde{f}_{\mathrm{m}})$ is a k-fuzzy IUP-subalgebra of X if and only if $\tilde{f}_{\mathrm{m}}$ is constant. This is equivalent to

$$\tilde{f}_{\mathrm{m}}(x) = \tilde{f}_{\mathrm{m}}(0)$$

for all $x \in X$, which is equivalent to

$$E(\tilde{f}_{\mathrm{m}}; \tilde{f}_{\mathrm{m}}(0)) = X.$$

Therefore, the result follows. □

Theorem 5.8. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 1-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\inf})$ is constant, we have $\tilde{f}_{\inf}(x) = \tilde{f}_{\inf}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\sup})$ is a 1-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\sup}(x \cdot y) \geq \min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.$$

Thus

$$\begin{aligned}
\tilde{f}_{\mathrm{m}}(x \cdot y) &= \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(x \cdot y)}{2} \\
&= \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(0)}{2} \\
&\geq \frac{\min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\} + \tilde{f}_{\inf}(0)}{2} \\
&= \min\left\{\frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2}, \frac{\tilde{f}_{\sup}(y) + \tilde{f}_{\inf}(y)}{2}\right\} \\
&= \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.
\end{aligned}$$

Therefore, $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X. □

Corollary 5.3. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 3-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorems 3.1 and 5.8. □

Corollary 5.4. *For $i \in \{2,3\}$ and $j \in \{1,3\}$, every (i, j)-hyperfuzzy IUP-subalgebra of X is a mean 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorem 5.8 and Corollary 5.3. □

Example 5.5. *Consider the IUP-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.4, 0.9) & (0.5, 0.8] & [0.1, 0.9] & [0.4, 0.6) & [0.2, 0.8) & [0.6, 0.7) \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.65 & 0.65 & 0.5 & 0.5 & 0.5 & 0.65 \end{pmatrix}.$$

Thus $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X. Since

$$\tilde{f}_{\inf} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.4 & 0.5 & 0.1 & 0.4 & 0.2 & 0.6 \end{pmatrix},$$

we have that $\tilde{f}_{\inf}$ is not constant. By Theorems 3.3 and 3.4, $(X, \tilde{f}_{\inf})$ is not a 2-fuzzy or 3-fuzzy IUP-subalgebra of X. Hence, $(X, \tilde{f})$ is not an (i, j)-hyperfuzzy IUP-subalgebra of X for $i \in \{2, 3\}$ and $j \in \{1, 3\}$.

Theorem 5.9. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 4-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\inf})$ is constant, we have $\tilde{f}_{\inf}(x) = \tilde{f}_{\inf}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\sup})$ is a 4-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\sup}(x \cdot y) \leq \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.$$

Thus

$$\begin{aligned} \tilde{f}_{\mathrm{m}}(x \cdot y) &= \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(x \cdot y)}{2} \\ &= \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(0)}{2} \\ &\leq \frac{\max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\} + \tilde{f}_{\inf}(0)}{2} \\ &= \max\left\{\frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2}, \frac{\tilde{f}_{\sup}(y) + \tilde{f}_{\inf}(y)}{2}\right\} \\ &= \max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}. \end{aligned}$$

Therefore, $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X. □

Corollary 5.5. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 2-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorems 3.2 and 5.9. □

Corollary 5.6. *For $i \in \{2, 3\}$ and $j \in \{2, 4\}$, every (i, j)-hyperfuzzy IUP-subalgebra of X is a mean 4-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorem 5.9 and Corollary 5.5. □

Example 5.6. *Consider the IUP-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.2, 0.4) & (0, 0.6] & [0.3, 0.5] & [0.2, 0.6) & (0.1, 0.7] & [0.1, 0.5) \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.3 & 0.3 & 0.4 & 0.4 & 0.4 & 0.3 \end{pmatrix}.$$

Thus $(X, \tilde{f})$ *is a mean* 4*-fuzzy IUP-subalgebra of* X*. Since*

$$\tilde{f}_{\inf} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.2 & 0 & 0.3 & 0.2 & 0.1 & 0.1 \end{pmatrix},$$

we have that $\tilde{f}_{\inf}$ *is not constant. By Theorems 3.3 and 3.4,* $(X, \tilde{f}_{\inf})$ *is not a* 2*-fuzzy or* 3*-fuzzy IUP-subalgebra of* X*. Hence,* $(X, \tilde{f})$ *is not an* (i, j)*-hyperfuzzy IUP-subalgebra of* X *for* $i \in \{2, 3\}$ *and* $j \in \{2, 4\}$*.*

Theorem 5.10. *If* $(X, \tilde{f})$ *is a hyperstructure over* X *such that* $(X, \tilde{f}_{\sup})$ *is constant and* $(X, \tilde{f}_{\inf})$ *is a* 4*-fuzzy IUP-subalgebra of* X*, then* $(X, \tilde{f})$ *is a mean* 4*-fuzzy IUP-subalgebra of* X*.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\sup})$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\inf})$ is a 4-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\inf}(x \cdot y) \leq \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.$$

Thus

$$\begin{aligned} \tilde{f}_{\mathrm{m}}(x \cdot y) &= \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(x \cdot y)}{2} \\ &= \frac{\tilde{f}_{\sup}(0) + \tilde{f}_{\inf}(x \cdot y)}{2} \\ &\leq \max\left\{\frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2}, \frac{\tilde{f}_{\sup}(y) + \tilde{f}_{\inf}(y)}{2}\right\} \\ &= \max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}. \end{aligned}$$

Therefore, $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X. □

Corollary 5.7. *If* $(X, \tilde{f})$ *is a hyperstructure over* X *such that* $(X, \tilde{f}_{\sup})$ *is constant and* $(X, \tilde{f}_{\inf})$ *is a* 2*-fuzzy IUP-subalgebra of* X*, then* $(X, \tilde{f})$ *is a mean* 4*-fuzzy IUP-subalgebra of* X*.*

Proof. It follows from Theorems 3.2 and 5.10. □

Corollary 5.8. *For* $i \in \{2, 4\}$ *and* $j \in \{2, 3\}$*, every* (i, j)*-hyperfuzzy IUP-subalgebra of* X *is a mean* 4*-fuzzy IUP-subalgebra of* X*.*

Proof. It follows from Theorem 5.10 and Corollary 5.7. □

Example 5.7. *Consider the IUP-algebra* $X = \{0, 1, 2, 3, 4, 5\}$ *in Example 3.1. Let* $(X, \tilde{f})$ *be a hyperstructure over* X *in which* $\tilde{f}$ *is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.5, 0.6) & (0.3, 0.8) & [0.5, 0.9] & [0.4, 1) & [0.7, 0.7] & [0.4, 0.7] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.55 & 0.55 & 0.7 & 0.7 & 0.7 & 0.55 \end{pmatrix}.$$

Thus $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X. Since

$$\tilde{f}_{\sup} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.6 & 0.8 & 0.9 & 1 & 0.7 & 0.7 \end{pmatrix},$$

we have that $\tilde{f}_{\sup}$ is not constant. By Theorems 3.3 and 3.4, $(X, \tilde{f}_{\sup})$ is not a 2-fuzzy or 3-fuzzy IUP-subalgebra of X. Hence, $(X, \tilde{f})$ is not an (i, j)-hyperfuzzy IUP-subalgebra of X for $i \in \{2, 4\}$ and $j \in \{2, 3\}$.

Theorem 5.11. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 1-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\sup})$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\inf})$ is a 1-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\inf}(x \cdot y) \geq \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.$$

Thus

$$\begin{aligned} \tilde{f}_{\mathrm{m}}(x \cdot y) &= \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(x \cdot y)}{2} \\ &= \frac{\tilde{f}_{\sup}(0) + \tilde{f}_{\inf}(x \cdot y)}{2} \\ &\geq \min\left\{\frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2}, \frac{\tilde{f}_{\sup}(y) + \tilde{f}_{\inf}(y)}{2}\right\} \\ &= \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}. \end{aligned}$$

Therefore, $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X. □

Corollary 5.9. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 3-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorems 3.1 and 5.11. □

Corollary 5.10. *For $i \in \{1, 3\}$ and $j \in \{2, 3\}$, every (i, j)-hyperfuzzy IUP-subalgebra of X is a mean 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorem 5.11 and Corollary 5.9. □

Example 5.8. *Consider the IUP-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ (0.4, 0.8) & (0.2, 1) & [0, 0.4] & (0.1, 0.3] & [0.2, 0.2] & [0.3, 0.9] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.6 & 0.6 & 0.2 & 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

Thus $(X, \tilde{f})$ *is a mean* 1*-fuzzy IUP-subalgebra of* X*. Since*

$$\tilde{f}_{\sup} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.8 & 1 & 0.4 & 0.3 & 0.2 & 0.9 \end{pmatrix},$$

we have that $\tilde{f}_{\sup}$ *is not constant. By Theorems 3.3 and 3.4,* $(X, \tilde{f}_{\sup})$ *is not a* 2*-fuzzy or* 3*-fuzzy IUP-subalgebra of* X*. Hence,* $(X, \tilde{f})$ *is not an* (i, j)*-hyperfuzzy IUP-subalgebra of* X *for* $i \in \{1, 3\}$ *and* $j \in \{2, 3\}$*.*

We next provide sufficient conditions under which a mean fuzzy IUP-subalgebra induces an (i, j)-hyperfuzzy IUP-subalgebra through the infimum and supremum fuzzy structures.

Theorem 5.12. *If* $(X, \tilde{f})$ *is a mean* 1*-fuzzy IUP-subalgebra of* X *in which* $\tilde{f}_{\inf}$ *is constant, then* $(X, \tilde{f})$ *is a* $(k, 1)$*-hyperfuzzy IUP-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*.*

Proof. Assume that $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X and $\tilde{f}_{\inf}$ is constant. Since $\tilde{f}_{\inf}$ is constant, $(X, \tilde{f}_{\inf})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Let $x, y \in X$. Then

$$\begin{aligned} \tilde{f}_{\sup}(x \cdot y) &= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\inf}(x \cdot y) \\ &= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\inf}(0) \\ &\geq 2\min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} - \tilde{f}_{\inf}(0) \\ &= \min\{2\tilde{f}_{\mathrm{m}}(x) - \tilde{f}_{\inf}(x), 2\tilde{f}_{\mathrm{m}}(y) - \tilde{f}_{\inf}(y)\} \\ &= \min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}. \end{aligned}$$

Thus $(X, \tilde{f}_{\sup})$ is a 1-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(k, 1)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. □

Corollary 5.11. *If* $(X, \tilde{f})$ *is a mean* 3*-fuzzy IUP-subalgebra of* X *in which* $\tilde{f}_{\inf}$ *is constant, then* $(X, \tilde{f})$ *is a* $(k, 1)$*-hyperfuzzy IUP-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*.*

Proof. It follows from Theorems 5.1 and 5.12. □

Theorem 5.13. *If* $(X, \tilde{f})$ *is a mean* 4*-fuzzy IUP-subalgebra of* X *in which* $\tilde{f}_{\inf}$ *is constant, then* $(X, \tilde{f})$ *is a* $(k, 4)$*-hyperfuzzy IUP-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*.*

Proof. Assume that $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X and $\tilde{f}_{\inf}$ is constant. Since $\tilde{f}_{\inf}$ is constant, $(X, \tilde{f}_{\inf})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Let $x, y \in X$. Then

$$\begin{aligned} \tilde{f}_{\sup}(x \cdot y) &= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\inf}(x \cdot y) \\ &= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\inf}(0) \\ &\leq 2\max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} - \tilde{f}_{\inf}(0) \\ &= \max\{2\tilde{f}_{\mathrm{m}}(x) - \tilde{f}_{\inf}(x), 2\tilde{f}_{\mathrm{m}}(y) - \tilde{f}_{\inf}(y)\} \end{aligned}$$

$$= \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.$$

Thus $(X, \tilde{f}_{\sup})$ is a 4-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(k, 4)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. □

Corollary 5.12. *If $(X, \tilde{f})$ is a mean 2-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\inf}$ is constant, then $(X, \tilde{f})$ is a $(k, 4)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 5.2 and 5.13. □

Theorem 5.14. *If $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\sup}$ is constant, then $(X, \tilde{f})$ is a $(4, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \tilde{f})$ is a mean 4-fuzzy IUP-subalgebra of X and $\tilde{f}_{\sup}$ is constant. Since $\tilde{f}_{\sup}$ is constant, $(X, \tilde{f}_{\sup})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Let $x, y \in X$. Then

$$\begin{aligned}
\tilde{f}_{\inf}(x \cdot y) &= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\sup}(x \cdot y) \\
&= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\sup}(0) \\
&\leq 2\max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} - \tilde{f}_{\sup}(0) \\
&= \max\{2\tilde{f}_{\mathrm{m}}(x) - \tilde{f}_{\sup}(x), 2\tilde{f}_{\mathrm{m}}(y) - \tilde{f}_{\sup}(y)\} \\
&= \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.
\end{aligned}$$

Thus $(X, \tilde{f}_{\inf})$ is a 4-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(4, k)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. □

Corollary 5.13. *If $(X, \tilde{f})$ is a mean 2-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\sup}$ is constant, then $(X, \tilde{f})$ is a $(4, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 5.2 and 5.14. □

Theorem 5.15. *If $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\sup}$ is constant, then $(X, \tilde{f})$ is a $(1, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \tilde{f})$ is a mean 1-fuzzy IUP-subalgebra of X and $\tilde{f}_{\sup}$ is constant. Since $\tilde{f}_{\sup}$ is constant, $(X, \tilde{f}_{\sup})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Let $x, y \in X$. Then

$$\begin{aligned}
\tilde{f}_{\inf}(x \cdot y) &= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\sup}(x \cdot y) \\
&= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\sup}(0) \\
&\geq 2\min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} - \tilde{f}_{\sup}(0) \\
&= \min\{2\tilde{f}_{\mathrm{m}}(x) - \tilde{f}_{\sup}(x), 2\tilde{f}_{\mathrm{m}}(y) - \tilde{f}_{\sup}(y)\} \\
&= \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.
\end{aligned}$$

Thus $(X, \tilde{f}_{\inf})$ is a 1-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(1, k)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. □

Corollary 5.14. *If $(X, \tilde{f})$ is a mean 3-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\sup}$ is constant, then $(X, \tilde{f})$ is a $(1, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 5.1 and 5.15. □

6. Dot of a Hyperstructure in IUP-Algebras

In this section, we introduce the notion of the dot of a hyperstructure in IUP-algebras. We define dot fuzzy IUP-subalgebras and investigate some related properties. We also establish relationships between dot fuzzy IUP-subalgebras and hyperfuzzy IUP-subalgebras. Moreover, we study the connections between dot fuzzy IUP-subalgebras and the corresponding upper level subsets, lower level subsets, and equal level subsets.

Definition 6.1. *Given a hyperstructure $(X, \tilde{f})$ over X, we define a fuzzy structure $(X, \tilde{f}_{\mathrm{d}})$ in X by*

$$\tilde{f}_{\mathrm{d}} : X \to [0, 1], \quad x \mapsto \tilde{f}_{\sup}(x)\tilde{f}_{\inf}(x).$$

The fuzzy set $\tilde{f}_{\mathrm{d}}$ is called the dot of $\tilde{f}$.

Definition 6.2. *A hyperstructure $(X, \tilde{f})$ over X is called a dot 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, and 4-fuzzy) IUP-subalgebra of X if the fuzzy structure $(X, \tilde{f}_{\mathrm{d}})$ is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, and 4-fuzzy) IUP-subalgebra of X.*

Example 6.1. *Consider the IUP-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.2, 0.4) \cup [0.5, 0.8) & (0.16, 1] & [0.3, 0.4] & [0.12, 1] & [0.2, 0.6] & [0.4, 0.4] \end{pmatrix}.$$

Then the dot of $\tilde{f}$ is given by

$$\tilde{f}_{\mathrm{d}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.16 & 0.16 & 0.12 & 0.12 & 0.12 & 0.16 \end{pmatrix}.$$

By direct verification, $(X, \tilde{f}_{\mathrm{d}})$ is a 1-fuzzy IUP-subalgebra of X. Hence, $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X.

Proposition 6.1. *If $(X, \tilde{f})$ is a dot k-fuzzy IUP-subalgebra of X for $k = 1, 3$, then*

$$(\forall x \in X)(\tilde{f}_{\mathrm{d}}(0) \geq \tilde{f}_{\mathrm{d}}(x)).$$

Proof. It follows directly from Proposition 3.1, applied to the fuzzy structure $(X, \tilde{f}_{\mathrm{d}})$. □

Proposition 6.2. *If $(X, \tilde{f})$ is a dot k-fuzzy IUP-subalgebra of X for $k = 2, 4$, then*

$$(\forall x \in X)(\tilde{f}_{\mathrm{d}}(0) \leq \tilde{f}_{\mathrm{d}}(x)).$$

Proof. It follows directly from Proposition 3.2, applied to the fuzzy structure $(X, \tilde{f}_{\mathrm{d}})$. □

Theorem 6.1. *Every dot 3-fuzzy IUP-subalgebra of X is a dot 1-fuzzy IUP-subalgebra.*

Proof. It follows directly from Theorem 3.1. □

Theorem 6.2. *Every dot* 2*-fuzzy IUP-subalgebra of* X *is a dot* 4*-fuzzy IUP-subalgebra.*

Proof. It follows directly from Theorem 3.2. □

Theorem 6.3. *The dot* 2*-fuzzy IUP-subalgebras and the dot* 3*-fuzzy IUP-subalgebras of* X *coincide.*

Proof. It follows directly from Theorems 3.3 and 3.4, applied to the fuzzy structure $(X, \tilde{f}_{\mathrm{d}})$. □

Theorem 6.4. *Let* S *be an IUP-subalgebra of* X*, and let* $(X, \tilde{f})$ *be a hyperstructure over* X*. Suppose that there exist* $a, b \in [0, 1]$ *such that*

$$\tilde{f}_{\mathrm{d}}(x) = \begin{cases} a & \text{if } x \in S, \\ b & \text{if } x \notin S. \end{cases}$$

If $b \leq a$*, then* $(X, \tilde{f})$ *is a dot* 1*-fuzzy IUP-subalgebra of* X*. If* $a \leq b$*, then* $(X, \tilde{f})$ *is a dot* 4*-fuzzy IUP-subalgebra of* X*.*

Proof. Assume first that $b \leq a$. Let $x, y \in X$.

If $x, y \in S$, then $x \cdot y \in S$ because S is an IUP-subalgebra of X. Hence,

$$\tilde{f}_{\mathrm{d}}(x \cdot y) = a = \min\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}.$$

If at least one of x, y is not in S, then

$$\min\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\} = b.$$

Since $\tilde{f}_{\mathrm{d}}(x \cdot y) \in \{a, b\}$ and $b \leq a$, we have

$$\tilde{f}_{\mathrm{d}}(x \cdot y) \geq b = \min\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}.$$

Thus $(X, \tilde{f}_{\mathrm{d}})$ is a 1-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X.

Assume next that $a \leq b$. Let $x, y \in X$.

If $x, y \in S$, then $x \cdot y \in S$, and hence

$$\tilde{f}_{\mathrm{d}}(x \cdot y) = a = \max\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}.$$

If at least one of x, y is not in S, then

$$\max\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\} = b.$$

Since $\tilde{f}_{\mathrm{d}}(x \cdot y) \in \{a, b\}$ and $a \leq b$, we have

$$\tilde{f}_{\mathrm{d}}(x \cdot y) \leq b = \max\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}.$$

Thus $(X, \tilde{f}_{\mathrm{d}})$ is a 4-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X. □

Example 6.2. *Consider the IUP-algebra X in Example 3.1. Then $S = \{0, 1, 5\}$ is an IUP-subalgebra of X.*

Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.6, 1] & [0.6, 1] & [0.2, 1] & [0.2, 1] & [0.2, 1] & [0.6, 1] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{d}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.6 & 0.6 & 0.2 & 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

By Theorem 6.4, $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X. However,

$$\tilde{f}_{\mathrm{d}}(0 \cdot 3) = \tilde{f}_{\mathrm{d}}(3) = 0.2 \ngeq 0.6 = \max\{\tilde{f}_{\mathrm{d}}(0), \tilde{f}_{\mathrm{d}}(3)\}.$$

Thus $(X, \tilde{f})$ is not a dot 3-fuzzy IUP-subalgebra of X.

Next, let $(X, \tilde{f})$ be a hyperstructure over X given by

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.08, 1] & [0.08, 1] & [0.12, 1] & [0.12, 1] & [0.12, 1] & [0.08, 1] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{d}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.08 & 0.08 & 0.12 & 0.12 & 0.12 & 0.08 \end{pmatrix}.$$

By Theorem 6.4, $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X. However,

$$\tilde{f}_{\mathrm{d}}(0 \cdot 4) = \tilde{f}_{\mathrm{d}}(4) = 0.12 \nleq 0.08 = \min\{\tilde{f}_{\mathrm{d}}(0), \tilde{f}_{\mathrm{d}}(4)\}.$$

Thus $(X, \tilde{f})$ is not a dot 2-fuzzy IUP-subalgebra of X.

Theorem 6.5. *A hyperstructure $(X, \tilde{f})$ over X is a dot 1-fuzzy IUP-subalgebra of X if and only if $U(\tilde{f}_{\mathrm{d}}; t)$ is an IUP-subalgebra of X for all $t \in [0, 1]$ with $U(\tilde{f}_{\mathrm{d}}; t) \neq \emptyset$.*

Proof. Assume that $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X. Let $t \in [0, 1]$ be such that $U(\tilde{f}_{\mathrm{d}}; t) \neq \emptyset$, and let $x, y \in U(\tilde{f}_{\mathrm{d}}; t)$. Then $\tilde{f}_{\mathrm{d}}(x) \geq t$ and $\tilde{f}_{\mathrm{d}}(y) \geq t$. Hence,

$$\tilde{f}_{\mathrm{d}}(x \cdot y) \geq \min\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\} \geq t.$$

Thus $x \cdot y \in U(\tilde{f}_{\mathrm{d}}; t)$, and so $U(\tilde{f}_{\mathrm{d}}; t)$ is an IUP-subalgebra of X.

Conversely, assume that $U(\tilde{f}_{\mathrm{d}}; t)$ is an IUP-subalgebra of X for all $t \in [0, 1]$ with $U(\tilde{f}_{\mathrm{d}}; t) \neq \emptyset$. Let $x, y \in X$ and put

$$t = \min\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}.$$

Then $x, y \in U(\tilde{f}_{\mathrm{d}}; t)$. By assumption, $x \cdot y \in U(\tilde{f}_{\mathrm{d}}; t)$. Therefore,

$$\tilde{f}_{\mathrm{d}}(x \cdot y) \geq t = \min\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}.$$

Hence, $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X. □

Corollary 6.1. *If $(X, \tilde{f})$ is a dot 3-fuzzy IUP-subalgebra of X, then $U(\tilde{f}_{\mathrm{d}}; t)$ is an IUP-subalgebra of X for all $t \in [0, 1]$ with $U(\tilde{f}_{\mathrm{d}}; t) \neq \emptyset$.*

Proof. It follows from Theorems 6.1 and 6.5. □

Example 6.3. *Consider the IUP-algebra X in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.3, 0.8] & [0.2, 0.9] & [0.2, 0.9] & [0.3, 0.7] & [0.2, 0.9] & [0.18, 1] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{d}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.24 & 0.18 & 0.18 & 0.21 & 0.18 & 0.18 \end{pmatrix}.$$

We have

$$U(\tilde{f}_{\mathrm{d}}; t) = \begin{cases} \emptyset & \text{if } t \in (0.24, 1], \\ \{0\} & \text{if } t \in (0.21, 0.24], \\ \{0, 3\} & \text{if } t \in (0.18, 0.21], \\ X & \text{if } t \in [0, 0.18]. \end{cases}$$

Hence, $U(\tilde{f}_{\mathrm{d}}; t)$ is an IUP-subalgebra of X for all $t \in [0, 1]$ with $U(\tilde{f}_{\mathrm{d}}; t) \neq \emptyset$. However,

$$\tilde{f}_{\mathrm{d}}(0 \cdot 4) = \tilde{f}_{\mathrm{d}}(4) = 0.18 \ngeq 0.24 = \max\{\tilde{f}_{\mathrm{d}}(0), \tilde{f}_{\mathrm{d}}(4)\}.$$

Thus $(X, \tilde{f})$ is not a dot 3-fuzzy IUP-subalgebra of X.

Theorem 6.6. *A hyperstructure $(X, \tilde{f})$ over X is a dot 4-fuzzy IUP-subalgebra of X if and only if $L(\tilde{f}_{\mathrm{d}}; t)$ is an IUP-subalgebra of X for all $t \in [0, 1]$ with $L(\tilde{f}_{\mathrm{d}}; t) \neq \emptyset$.*

Proof. Assume that $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X. Let $t \in [0, 1]$ be such that $L(\tilde{f}_{\mathrm{d}}; t) \neq \emptyset$, and let $x, y \in L(\tilde{f}_{\mathrm{d}}; t)$. Then $\tilde{f}_{\mathrm{d}}(x) \leq t$ and $\tilde{f}_{\mathrm{d}}(y) \leq t$. Hence,

$$\tilde{f}_{\mathrm{d}}(x \cdot y) \leq \max\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\} \leq t.$$

Thus $x \cdot y \in L(\tilde{f}_{\mathrm{d}}; t)$, and so $L(\tilde{f}_{\mathrm{d}}; t)$ is an IUP-subalgebra of X.

Conversely, assume that $L(\tilde{f}_{\mathrm{d}}; t)$ is an IUP-subalgebra of X for all $t \in [0, 1]$ with $L(\tilde{f}_{\mathrm{d}}; t) \neq \emptyset$. Let $x, y \in X$ and put

$$t = \max\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}.$$

Then $x, y \in L(\tilde{f}_{\mathrm{d}}; t)$. By assumption, $x \cdot y \in L(\tilde{f}_{\mathrm{d}}; t)$. Therefore,

$$\tilde{f}_{\mathrm{d}}(x \cdot y) \leq t = \max\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}.$$

Hence, $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X. □

Corollary 6.2. *If $(X, \tilde{f})$ is a dot 2-fuzzy IUP-subalgebra of X, then $L(\tilde{f}_{\mathrm{d}}; t)$ is an IUP-subalgebra of X for all $t \in [0, 1]$ with $L(\tilde{f}_{\mathrm{d}}; t) \neq \emptyset$.*

Proof. It follows from Theorems 6.2 and 6.6. □

Example 6.4. *Consider the IUP-algebra X in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.3, 0.8] & [0.48, 1] & [0.48, 1] & [0.32, 1] & [0.48, 1] & [0.48, 1] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{d}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.24 & 0.48 & 0.48 & 0.32 & 0.48 & 0.48 \end{pmatrix}.$$

We have

$$L(\tilde{f}_{\mathrm{d}}; t) = \begin{cases} X & \text{if } t \in [0.48, 1], \\ \{0, 3\} & \text{if } t \in [0.32, 0.48), \\ \{0\} & \text{if } t \in [0.24, 0.32), \\ \emptyset & \text{if } t \in [0, 0.24). \end{cases}$$

Hence, $L(\tilde{f}_{\mathrm{d}}; t)$ is an IUP-subalgebra of X for all $t \in [0, 1]$ with $L(\tilde{f}_{\mathrm{d}}; t) \neq \emptyset$. However,

$$\tilde{f}_{\mathrm{d}}(0 \cdot 3) = \tilde{f}_{\mathrm{d}}(3) = 0.32 \nleq 0.24 = \min\{\tilde{f}_{\mathrm{d}}(0), \tilde{f}_{\mathrm{d}}(3)\}.$$

Thus $(X, \tilde{f})$ is not a dot 2-fuzzy IUP-subalgebra of X.

Theorem 6.7. *For $k \in \{2, 3\}$, a hyperstructure $(X, \tilde{f})$ over X is a dot k-fuzzy IUP-subalgebra of X if and only if*

$$E(\tilde{f}_{\mathrm{d}}; \tilde{f}_{\mathrm{d}}(0)) = X.$$

Proof. Let $k \in \{2, 3\}$. By Theorems 3.3 and 3.4, the fuzzy structure $(X, \tilde{f}_{\mathrm{d}})$ is a k-fuzzy IUP-subalgebra of X if and only if $\tilde{f}_{\mathrm{d}}$ is constant. This is equivalent to

$$\tilde{f}_{\mathrm{d}}(x) = \tilde{f}_{\mathrm{d}}(0)$$

for all $x \in X$, which is equivalent to

$$E(\tilde{f}_{\mathrm{d}}; \tilde{f}_{\mathrm{d}}(0)) = X.$$

Therefore, the result follows. □

Theorem 6.8. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 1-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\inf})$ is constant, we have $\tilde{f}_{\inf}(x) = \tilde{f}_{\inf}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\sup})$ is a 1-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\sup}(x \cdot y) \geq \min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.$$

Thus

$$\begin{aligned} \tilde{f}_{\mathrm{d}}(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y)\tilde{f}_{\inf}(x \cdot y) \\ &= \tilde{f}_{\sup}(x \cdot y)\tilde{f}_{\inf}(0) \end{aligned}$$

$$\begin{aligned}
&\geq \min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}\tilde{f}_{\inf}(0) \\
&= \min\{\tilde{f}_{\sup}(x)\tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y)\tilde{f}_{\inf}(y)\} \\
&= \min\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}.
\end{aligned}$$

Therefore, $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X. □

Corollary 6.3. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 3-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorems 3.1 and 6.8. □

Corollary 6.4. *For $i \in \{2, 3\}$ and $j \in \{1, 3\}$, every (i, j)-hyperfuzzy IUP-subalgebra of X is a dot 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorem 6.8 and Corollary 6.3. □

Example 6.5. *Consider the IUP-algebra X in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.2, 0.8] & [0.16, 1] & [0.3, 0.4] & [0.2, 0.6] & [0.12, 1] & [0.16, 1] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{d}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.16 & 0.16 & 0.12 & 0.12 & 0.12 & 0.16 \end{pmatrix}.$$

Thus $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X. Since

$$\tilde{f}_{\inf} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.2 & 0.16 & 0.3 & 0.2 & 0.12 & 0.16 \end{pmatrix},$$

we have that $\tilde{f}_{\inf}$ is not constant. By Theorems 3.3 and 3.4, $(X, \tilde{f}_{\inf})$ is not a 2-fuzzy or 3-fuzzy IUP-subalgebra of X. Hence, $(X, \tilde{f})$ is not an (i, j)-hyperfuzzy IUP-subalgebra of X for $i \in \{2, 3\}$ and $j \in \{1, 3\}$.

Theorem 6.9. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 4-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\inf})$ is constant, we have $\tilde{f}_{\inf}(x) = \tilde{f}_{\inf}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\sup})$ is a 4-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\sup}(x \cdot y) \leq \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.$$

Thus

$$\begin{aligned}
\tilde{f}_{\mathrm{d}}(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y)\tilde{f}_{\inf}(x \cdot y) \\
&= \tilde{f}_{\sup}(x \cdot y)\tilde{f}_{\inf}(0) \\
&\leq \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}\tilde{f}_{\inf}(0) \\
&= \max\{\tilde{f}_{\sup}(x)\tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y)\tilde{f}_{\inf}(y)\}
\end{aligned}$$

$$= \max\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}.$$

Therefore, $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X. □

Corollary 6.5. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\inf})$ is constant and $(X, \tilde{f}_{\sup})$ is a 2-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorems 3.2 and 6.9. □

Corollary 6.6. *For $i \in \{2, 3\}$ and $j \in \{2, 4\}$, every (i, j)-hyperfuzzy IUP-subalgebra of X is a dot 4-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorem 6.9 and Corollary 6.5. □

Example 6.6. *Consider the IUP-algebra X in Example 3.1. Let $(X, \tilde{f})$ be a hyperstructure over X in which $\tilde{f}$ is given by*

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0.3, 0.5] & [0.15, 1] & [0.3, 0.6] & [0.2, 0.9] & [0.18, 1] & [0.15, 1] \end{pmatrix}.$$

Then

$$\tilde{f}_{\mathrm{d}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.15 & 0.15 & 0.18 & 0.18 & 0.18 & 0.15 \end{pmatrix}.$$

Thus $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X. Since

$$\tilde{f}_{\inf} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.3 & 0.15 & 0.3 & 0.2 & 0.18 & 0.15 \end{pmatrix},$$

we have that $\tilde{f}_{\inf}$ is not constant. By Theorems 3.3 and 3.4, $(X, \tilde{f}_{\inf})$ is not a 2-fuzzy or 3-fuzzy IUP-subalgebra of X. Hence, $(X, \tilde{f})$ is not an (i, j)-hyperfuzzy IUP-subalgebra of X for $i \in \{2, 3\}$ and $j \in \{2, 4\}$.

Theorem 6.10. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 4-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\sup})$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\inf})$ is a 4-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\inf}(x \cdot y) \leq \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.$$

Thus

$$\begin{aligned} \tilde{f}_{\mathrm{d}}(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y)\tilde{f}_{\inf}(x \cdot y) \\ &= \tilde{f}_{\sup}(0)\tilde{f}_{\inf}(x \cdot y) \\ &\leq \tilde{f}_{\sup}(0)\max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\} \\ &= \max\{\tilde{f}_{\sup}(x)\tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y)\tilde{f}_{\inf}(y)\} \\ &= \max\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}. \end{aligned}$$

Therefore, $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X. □

Corollary 6.7. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 2-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorems 3.2 and 6.10. □

Corollary 6.8. *For $i \in \{2, 4\}$ and $j \in \{2, 3\}$, every (i, j)-hyperfuzzy IUP-subalgebra of X is a dot 4-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorem 6.10 and Corollary 6.7. □

Theorem 6.11. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 1-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \tilde{f}_{\sup})$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in X$. Since $(X, \tilde{f}_{\inf})$ is a 1-fuzzy IUP-subalgebra of X,

$$\tilde{f}_{\inf}(x \cdot y) \geq \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.$$

Thus

$$\begin{aligned} \tilde{f}_{\mathrm{d}}(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y)\tilde{f}_{\inf}(x \cdot y) \\ &= \tilde{f}_{\sup}(0)\tilde{f}_{\inf}(x \cdot y) \\ &\geq \tilde{f}_{\sup}(0) \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\} \\ &= \min\{\tilde{f}_{\sup}(x)\tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y)\tilde{f}_{\inf}(y)\} \\ &= \min\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}. \end{aligned}$$

Therefore, $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X. □

Corollary 6.9. *If $(X, \tilde{f})$ is a hyperstructure over X such that $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 3-fuzzy IUP-subalgebra of X, then $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorems 3.1 and 6.11. □

Corollary 6.10. *For $i \in \{1, 3\}$ and $j \in \{2, 3\}$, every (i, j)-hyperfuzzy IUP-subalgebra of X is a dot 1-fuzzy IUP-subalgebra of X.*

Proof. It follows from Theorem 6.11 and Corollary 6.9. □

Theorem 6.12. *If $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\inf}$ is nonzero constant, then $(X, \tilde{f})$ is a $(k, 1)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X and $\tilde{f}_{\inf}$ is nonzero constant. Since $\tilde{f}_{\inf}$ is constant, $(X, \tilde{f}_{\inf})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\inf}(0) > 0$, for all $x \in X$ we have

$$\tilde{f}_{\sup}(x) = \frac{\tilde{f}_{\mathrm{d}}(x)}{\tilde{f}_{\inf}(0)}.$$

Let $x, y \in X$. Then

$$\begin{aligned}\tilde{f}_{\sup}(x \cdot y) &= \frac{\tilde{f}_{\mathrm{d}}(x \cdot y)}{\tilde{f}_{\inf}(0)} \\ &\geq \frac{\min\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}}{\tilde{f}_{\inf}(0)} \\ &= \min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.\end{aligned}$$

Thus $(X, \tilde{f}_{\sup})$ is a 1-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(k, 1)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. □

Corollary 6.11. *If $(X, \tilde{f})$ is a dot 3-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\inf}$ is nonzero constant, then $(X, \tilde{f})$ is a $(k, 1)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 6.1 and 6.12. □

Theorem 6.13. *If $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\inf}$ is nonzero constant, then $(X, \tilde{f})$ is a $(k, 4)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X and $\tilde{f}_{\inf}$ is nonzero constant. Since $\tilde{f}_{\inf}$ is constant, $(X, \tilde{f}_{\inf})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\inf}(0) > 0$, for all $x \in X$ we have

$$\tilde{f}_{\sup}(x) = \frac{\tilde{f}_{\mathrm{d}}(x)}{\tilde{f}_{\inf}(0)}.$$

Let $x, y \in X$. Then

$$\begin{aligned}\tilde{f}_{\sup}(x \cdot y) &= \frac{\tilde{f}_{\mathrm{d}}(x \cdot y)}{\tilde{f}_{\inf}(0)} \\ &\leq \frac{\max\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}}{\tilde{f}_{\inf}(0)} \\ &= \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.\end{aligned}$$

Thus $(X, \tilde{f}_{\sup})$ is a 4-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(k, 4)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. □

Corollary 6.12. *If $(X, \tilde{f})$ is a dot 2-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\inf}$ is nonzero constant, then $(X, \tilde{f})$ is a $(k, 4)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 6.2 and 6.13. □

Theorem 6.14. *If $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\sup}$ is nonzero constant, then $(X, \tilde{f})$ is a $(4, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \tilde{f})$ is a dot 4-fuzzy IUP-subalgebra of X and $\tilde{f}_{\sup}$ is nonzero constant. Since $\tilde{f}_{\sup}$ is constant, $(X, \tilde{f}_{\sup})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\sup}(0) > 0$, for all $x \in X$ we have

$$\tilde{f}_{\inf}(x) = \frac{\tilde{f}_{\mathrm{d}}(x)}{\tilde{f}_{\sup}(0)}.$$

Let $x, y \in X$. Then

$$\begin{aligned}\tilde{f}_{\inf}(x \cdot y) &= \frac{\tilde{f}_{\mathrm{d}}(x \cdot y)}{\tilde{f}_{\sup}(0)} \\ &\leq \frac{\max\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}}{\tilde{f}_{\sup}(0)} \\ &= \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.\end{aligned}$$

Thus $(X, \tilde{f}_{\inf})$ is a 4-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(4, k)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. □

Corollary 6.13. *If $(X, \tilde{f})$ is a dot 2-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\sup}$ is nonzero constant, then $(X, \tilde{f})$ is a $(4, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 6.2 and 6.14. □

Theorem 6.15. *If $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\sup}$ is nonzero constant, then $(X, \tilde{f})$ is a $(1, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \tilde{f})$ is a dot 1-fuzzy IUP-subalgebra of X and $\tilde{f}_{\sup}$ is nonzero constant. Since $\tilde{f}_{\sup}$ is constant, $(X, \tilde{f}_{\sup})$ is a k-fuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\sup}(0) > 0$, for all $x \in X$ we have

$$\tilde{f}_{\inf}(x) = \frac{\tilde{f}_{\mathrm{d}}(x)}{\tilde{f}_{\sup}(0)}.$$

Let $x, y \in X$. Then

$$\begin{aligned}\tilde{f}_{\inf}(x \cdot y) &= \frac{\tilde{f}_{\mathrm{d}}(x \cdot y)}{\tilde{f}_{\sup}(0)} \\ &\geq \frac{\min\{\tilde{f}_{\mathrm{d}}(x), \tilde{f}_{\mathrm{d}}(y)\}}{\tilde{f}_{\sup}(0)} \\ &= \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.\end{aligned}$$

Thus $(X, \tilde{f}_{\inf})$ is a 1-fuzzy IUP-subalgebra of X. Therefore, $(X, \tilde{f})$ is a $(1, k)$-hyperfuzzy IUP-subalgebra of X for all $k \in \{1, 2, 3, 4\}$. □

Corollary 6.14. *If $(X, \tilde{f})$ is a dot 3-fuzzy IUP-subalgebra of X in which $\tilde{f}_{\sup}$ is nonzero constant, then $(X, \tilde{f})$ is a $(1, k)$-hyperfuzzy IUP-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 6.1 and 6.15. □

7. Conclusion

In this paper, we introduced and studied length, mean, and dot constructions of hyperstructures in IUP-algebras. From a hyperstructure $(X, \tilde{f})$, we defined three induced fuzzy structures, namely the length $\tilde{f}_{\mathrm{l}}$, the mean $\tilde{f}_{\mathrm{m}}$, and the dot $\tilde{f}_{\mathrm{d}}$. These induced fuzzy structures were then used to define length fuzzy IUP-subalgebras, mean fuzzy IUP-subalgebras, and dot fuzzy IUP-subalgebras of types 1, 2, 3, and 4.

We investigated several basic properties of these three classes of induced fuzzy IUP-subalgebras. In each case, the type 3 structure implies the type 1 structure, and the type 2 structure implies the type 4 structure. Moreover, the type 2 and type 3 structures coincide because they are both equivalent to the corresponding induced fuzzy structure being constant.

We also established characterizations of length, mean, and dot fuzzy IUP-subalgebras by using level subsets. More precisely, the type 1 case was characterized by upper level subsets, the type 4 case by lower level subsets, and the type 2 and type 3 cases by equal level subsets determined by the value at the zero element. These results provide useful criteria for checking whether a hyperstructure induces a fuzzy IUP-subalgebra of a given type.

Finally, we studied the relationships between length, mean, and dot fuzzy IUP-subalgebras and (i, j)-hyperfuzzy IUP-subalgebras. Several sufficient conditions were obtained by imposing suitable assumptions on the infimum and supremum fuzzy structures, such as constancy or nonzero constancy in the dot case. The examples presented throughout the paper show that the converse of some implications does not hold in general. These results extend the study of hyperfuzzy structures in logical algebraic systems and provide a framework for further investigation of induced fuzzy structures in IUP-algebras.

Acknowledgment: This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2027, FF70).

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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