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Generalized Ordered Nano-Topological Rough Approximations for Decision-Making and COVID-19 Risk Assessment

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Abstract. Rough set theory is widely used to deal with uncertainty in information systems, but its classical form mainly depends on equivalence relations, which limits its use in many practical situations involving ordered or partially related data. In this work, we introduce a generalized rough approximation model based on neighborhood structures and partial order relations within an ordered nano-topological setting. The proposed approach defines new approximation operators that describe increasing and decreasing behaviors in ordered spaces. We study their main properties and show that the model offers more accurate approximations and smaller boundary regions compared with several existing approaches. To illustrate its effectiveness, the framework is applied to COVID-19 risk assessment, where medical indicators naturally involve uncertainty and partial ordering. The obtained results indicate that the proposed method can improve the identification of important risk factors and support more reliable decision-making in complex environments.

1. Introduction

1.1. **Motivation and Challenges.** The theory of rough sets provides a formal mathematical framework for uncertainty modelling using lower and upper approximation operators. These operators show how unclear the characteristics of a target concept can be. The lower approximation is of things that do belong to a concept and the upper approximation is of things that may belong to it. The rough set theory introduced by Pawlak [1, 2] is a natural extension of the classical set theory, especially for intelligent systems working with incomplete, unclear or wrong information. In the

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original formulation of Pawlak, equivalence relations are used in a fundamental way and lead to the essential approximation spaces.

Classical rough set theory based on equivalence relations is of great theoretical importance and has wide practical applications. However, it often cannot adequately model complex real-world systems. In numerous practical scenarios, data objects are related through partial comparability, similarity, or ordering rather than strict equivalence. To overcome these limitations, several generalizations of rough sets have been proposed, including models based on reflexive relations [3], similarity relations [4], general binary relations [5–8], topological structures [9, 10], and covering-based approaches [11, 12]. Marei subsequently introduced neighborhood-based and topological extensions, which significantly enhanced the descriptive power of rough approximations [13, 14]. More recently, El-Sheikh et al. [15] introduced increasing and decreasing soft rough sets, which improve the accuracy of rough approximations by replacing partial orders with linear orders. These concepts were later extended to decision-making [16] and bitopological ordered spaces [16–19].

A significant step toward unifying the neighborhood- and relation-based approaches was made by Abd El-Monsef et al. [10] through the introduction of the notion of a $\jmath$-neighborhood space ($\jmath$-$\boldsymbol{NS}$). Their work generalized classical neighborhood spaces by incorporating binary relations and introduced two new types of neighborhood constructions, namely, intersection- and union-based neighborhoods. These constructions were based on Yao's [5,6] left and right neighborhood systems together with the minimal neighborhood concepts introduced by Allam et al. [20], giving rise to eight distinct types of neighborhoods induced by binary relations. Building on this framework, subsequent studies proposed numerous neighborhood structures, including basic neighborhoods [21–25] and initial-neighborhood models [26–29]. These contributions have significantly enriched Pawlak's rough set theory by embedding it within various topological structures associated with $\jmath$-$\boldsymbol{NS}$, thereby enhancing its flexibility and broadening its range of applications.

To further advance topological rough set theory, Abd El-Monsef et al. introduced a methodology for generating topologies from neighborhood spaces. Important developments based on this methodology include primal approximation spaces [30] and $\theta\beta$-approximation spaces, introduced by Nawar et al. in 2022 [31]. In that work, the authors introduced the notion of $\theta\beta$-open sets and developed the corresponding $\theta\beta$-rough approximations. They also presented two different approaches for generalizing rough sets based on $\theta\beta$-open sets and ideals. These results were further extended by El-Bably et al. [32] through the introduction of new theoretical concepts and computational algorithms, thereby strengthening the framework of $\theta\beta$-approximation spaces. Collectively, the works in [31,32] establish the original theoretical framework of $\theta\beta$-approximation spaces and clarify the role of ideals in their construction. Consequently, alternative formulations should be examined carefully to ensure their consistency with these foundational definitions and theoretical principles.

Topology provides a powerful mathematical framework for modeling uncertainty and structural complexity, as emphasized by Sierpiński [33]. Topological structures play a fundamental role in numerous applications, including fuzzy topological spaces [34], generalized rough multisets [35], topological rough sets [3], bipolar hypersoft sets [36, 37], bipolar soft topologies and fuzzy soft sets [38, 39], medical diagnosis and economic applications [40–42], and decision-making problems [43–45].

The concept of nano-topology was introduced by Lellis Thivagar and has since become an important development in topology and rough set theory. It is constructed from Pawlak's rough approximations, which are induced by equivalence relations. Although nano-topology has been successfully applied in many areas, its reliance on equivalence relations limits its applicability to more general data structures. To overcome this limitation, El-Bably and Abo-Tabl [40] introduced the more general notion of generalized nano-topological spaces.

In this paper, we further extend this framework by introducing new rough approximation operators. The proposed model aims to provide greater flexibility in modeling uncertainty while improving approximation accuracy. Compared with the models proposed by Yao [5], Allam et al. [20], Dai et al. [4], and El-Sayed et al. [26], our approach offers a more general and expressive framework. Furthermore, we introduce a new method for topological feature reduction based on partial order relations, which extends the classical equivalence relation model of Pawlak and provides greater modeling capability.

From a practical perspective, the case study conducted by El-Sayed et al. [26] identified several important COVID-19 risk indicators, including dyspnea, thoracic pain, non-productive cough, cephalalgia, elevated body temperature, and anosmia or ageusia. These findings highlight the need for flexible and expressive uncertainty models capable of handling heterogeneous and partially ordered medical data.

Pawlak's rough set theory has become one of the most influential mathematical tools in artificial intelligence and data analysis because of its solid theoretical foundation and intuitive approximation mechanisms. Nevertheless, as real-world applications become increasingly complex, more general approximation models are required. In particular, frameworks based on partial orders and general binary relations provide greater flexibility for modeling uncertainty in decision-making systems, medical diagnosis, and other complex application domains.

1.2. **Our Contributions.** In this paper, we propose a new generalization of Yao's rough set model by incorporating partial order relations into the approximation process. The main contributions of this paper are summarized as follows:

(1) We introduce new generalized rough approximation operators, called *increasing* and *decreasing right-approximations*, based on partial order relations. This formulation includes Pawlak's classical rough set model as a special case while significantly extending its scope of applicability.

(2) We present a comprehensive theoretical study of the proposed operators by establishing their fundamental properties and clarifying their relationships with existing rough set models, particularly those developed by Pawlak and Yao. Furthermore, we provide several illustrative examples and counterexamples to demonstrate these relationships and highlight the flexibility of the proposed framework.
(3) We extend the behavior of nano-topological structures to ordered approximation spaces and generalize the notions of nano-increasing and nano-decreasing accuracy measures within this new framework. Theoretical analysis establishes their fundamental properties, monotonic behavior, and descriptive effectiveness. The practical applicability of the proposed approach is further demonstrated through a medical case study on the analysis of COVID-19 risk factors.

In addition, we propose a new approach to the topological attribute reduction of information tables under partial order relations. Unlike Pawlak's equivalence-based reduction method, the proposed approach is applicable to non-equivalence structures and therefore provides a more general and flexible tool for the analysis of real-world data.

1.3. **Related Work.** Considerable research has been devoted to extending and enhancing rough set theory beyond its classical equivalence-based framework. Among the earliest contributions, Yao [5] introduced the use of general binary relations to relax the equivalence requirement in rough approximations. Subsequently, Abd El-Monsef et al. [10] proposed the notion of $\jmath$-neighborhood spaces ($\jmath$-$\boldsymbol{NS}$), which generalize classical neighborhood systems through binary relations. Their framework led to the development of eight distinct neighborhood types based on union- and intersection-based constructions, extending the neighborhood systems introduced by Yao and the minimal neighborhood concepts proposed by Allam et al.

Other important developments in this research direction include:

- *Basic*-neighborhood structures [22–25] and *initial*-neighborhood models [21, 27–29];
- The development of $\theta\beta$-approximation spaces [31, 32];
- The introduction of primal approximation spaces [30].

These theoretical developments have enabled a wide range of applications, particularly in medical diagnosis [23–25], economic analysis [22, 42], and related fields, thereby demonstrating the practical significance of the $\jmath$-$\boldsymbol{NS}$ framework. Furthermore, the integration of rough set models with ideals [46,47] and advanced topological constructions [4,20,26] has substantially strengthened the theoretical foundations of rough set theory.

Table 1 presents a comparative overview of several representative rough set models. It highlights their underlying relational structures and compares their capabilities for representing order information.

Table 1. Comparison of Selected Rough Set Approaches

Approach	Relation Type	Order Awareness	Reference
Classical Rough Sets	Equivalence	No	[1]
Yao's Rough Sets	Binary	No	[5]
Allam's Model	Binary	No	[20]
Proposed Approach	Binary + Partial Order	Full	—

The remainder of this paper is organized as follows. Section 2 reviews the fundamental concepts of rough set theory, increasing and decreasing sets, and nano-topological structures. Section 3 introduces the proposed ordered approximation spaces. In Section 4, the generalized rough approximation operators are developed and their fundamental properties are investigated. Section 5 is devoted to nano-topological structures in partially ordered settings together with the corresponding uncertainty measures. Finally, Section 6 concludes the paper and outlines several directions for future research.

2. Preliminaries

Definition 2.1. *[48] Let A and B be two sets. A binary relation $\preceq$ from A to B is defined as a subset of the Cartesian product $A \times B$, consisting of ordered pairs $(a,b) \in \preceq$ where $a \in A$ and $b \in B$.*

Note: *When $A = B$, we say $\preceq$ is a binary relation on A. The notation $a \preceq b$ denotes that $(a,b) \in \preceq$, meaning a is related to b under $\preceq$.*

Definition 2.2. *[5,6] Let $\preceq$ be a binary relation on χ. Then $\preceq$ is called:*

(1) *The equality relation on χ (denoted by $\blacktriangle$) if $\preceq = \{(x,x) : x \in \chi\}$;*
(2) *Reflexive if $x \preceq x$ for all $x \in \chi$;*
(3) *Symmetric if $x \preceq y$ implies $y \preceq x$ for all $x, y \in \chi$;*
(4) *Transitive if $x \preceq y$ and $y \preceq z$ imply $x \preceq z$ for all $x, y, z \in \chi$;*
(5) *An equivalence relation if it is reflexive, symmetric, and transitive;*
(6) *Antisymmetric if $x \preceq y$ and $y \preceq x$ imply $x = y$ for all $x, y \in \chi$.*

Definition 2.3. *[49,50] A binary relation $\preceq$ on χ is a partial order relation if it is reflexive, antisymmetric, and transitive. The pair $(\chi, \preceq)$ is called a partially ordered set (briefly, POS).*

Definition 2.4. *[49,50] A POS can be further classified as:*

- *A linearly ordered set (LOS) if for any $n, b \in \chi$, either $n \preceq b$, $b \preceq n$, or $n = b$;*
- *A directed ordered set (DOS) if for any $n, b \in \chi$, there exists $z \in \chi$ such that $z \preceq n$ and $z \preceq b$.*

Definition 2.5. *[51] A total directed order relation $\preceq$ on χ satisfies:*

(1) *Reflexivity;*
(2) *Transitivity;*
(3) *For any $n, b \in \chi$, there exists $z \in \chi$ with $z \preceq n$ and $z \preceq b$.*

The pair $(\chi, \preceq)$ is called a total directed ordered set (TDOS).

Definition 2.6. *[52] Let $(\chi, \preceq)$ be a POS, $\ell \in \chi$, and $\Lambda \subseteq \chi$. We define:*

(1) $i(\ell) = \{b \in \chi : \ell \preceq b\}$ *(increasing set of ℓ);*
(2) $d(\ell) = \{b \in \chi : b \preceq \ell\}$ *(decreasing set of ℓ);*
(3) $i(\Lambda) = \bigcup_{\ell \in \Lambda} i(\ell) = \{b \in \chi : \exists \ell \in \Lambda, \ell \preceq b\}$;
(4) $d(\Lambda) = \bigcup_{\ell \in \Lambda} d(\ell) = \{b \in \chi : \exists \ell \in \Lambda, b \preceq \ell\}$.

Λ is increasing if $i(\Lambda) = \Lambda$, and decreasing if $d(\Lambda) = \Lambda$.

Definition 2.7. *[1] Let χ be a non-empty finite universe and $\mathfrak{R}$ an equivalence relation on χ. The pair $(\chi, \mathfrak{R})$ is called an approximation space. For any $\Lambda \subseteq \chi$:*

(1) *The lower approximation of Λ is $L_{\mathfrak{R}}(\Lambda) = \bigcup\{\nu : \mathfrak{R}(\nu) \subseteq \Lambda\}$;*
(2) *The upper approximation of Λ is $U_{\mathfrak{R}}(\Lambda) = \bigcup\{\nu : \mathfrak{R}(\nu) \cap \Lambda \neq \emptyset\}$;*
(3) *The boundary region is $B_{\mathfrak{R}}(\Lambda) = U_{\mathfrak{R}}(\Lambda) \setminus L_{\mathfrak{R}}(\Lambda)$;*
(4) *The accuracy measure is $C_{\mathfrak{R}}(\Lambda) = \frac{|L_{\mathfrak{R}}(\Lambda)|}{|U_{\mathfrak{R}}(\Lambda)|}$.*

Λ is exact if $U_{\mathfrak{R}}(\Lambda) = L_{\mathfrak{R}}(\Lambda)$; otherwise, it is rough.

Proposition 2.1. *[1] Given an approximation space $(\chi, \mathfrak{R})$ and subsets $\Lambda, \Gamma \subseteq \chi$, the following properties hold:*

(1) $L_{\mathfrak{R}}(\Lambda) \subseteq \Lambda \subseteq U_{\mathfrak{R}}(\Lambda)$
(2) $L_{\mathfrak{R}}(\emptyset) = \emptyset$ *and* $U_{\mathfrak{R}}(\chi) = \chi$
(3) *If $\Lambda \subseteq \Gamma$, then $L_{\mathfrak{R}}(\Lambda) \subseteq L_{\mathfrak{R}}(\Gamma)$ and $U_{\mathfrak{R}}(\Lambda) \subseteq U_{\mathfrak{R}}(\Gamma)$*
(4) $L_{\mathfrak{R}}(\Lambda \cap \Gamma) \subseteq L_{\mathfrak{R}}(\Lambda) \cap L_{\mathfrak{R}}(\Gamma)$ *and* $L_{\mathfrak{R}}(\Lambda) \cup L_{\mathfrak{R}}(\Gamma) \subseteq L_{\mathfrak{R}}(\Lambda \cup \Gamma)$
(5) $U_{\mathfrak{R}}(\Lambda \cap \Gamma) \subseteq U_{\mathfrak{R}}(\Lambda) \cap U_{\mathfrak{R}}(\Gamma)$ *and* $U_{\mathfrak{R}}(\Lambda) \cup U_{\mathfrak{R}}(\Gamma) \subseteq U_{\mathfrak{R}}(\Lambda \cup \Gamma)$
(6) $L_{\mathfrak{R}}(L_{\mathfrak{R}}(\Lambda)) = L_{\mathfrak{R}}(\Lambda)$ *and* $U_{\mathfrak{R}}(U_{\mathfrak{R}}(\Lambda)) = U_{\mathfrak{R}}(\Lambda)$
(7) $U_{\mathfrak{R}}(\Lambda^c) = [L_{\mathfrak{R}}(\Lambda)]^c$ *and* $L_{\mathfrak{R}}(\Lambda^c) = [U_{\mathfrak{R}}(\Lambda)]^c$

Definition 2.8. *[53] Given a universe χ with equivalence relation $\mathfrak{R}$ and $\Lambda \subseteq \chi$, the nano topology on χ is defined as:*

$$\tau_{\mathfrak{R}}(\Lambda) = \{\chi, \emptyset, L_{\mathfrak{R}}(\Lambda), U_{\mathfrak{R}}(\Lambda), B_{\mathfrak{R}}(\Lambda)\}$$

This collection satisfies the topological axioms:

(1) $\chi, \emptyset \in \tau_{\mathfrak{R}}(\Lambda)$
(2) *Arbitrary unions of elements in $\tau_{\mathfrak{R}}(\Lambda)$ remain in $\tau_{\mathfrak{R}}(\Lambda)$*
(3) *Finite intersections of elements in $\tau_{\mathfrak{R}}(\Lambda)$ remain in $\tau_{\mathfrak{R}}(\Lambda)$*

The pair $(\chi, \tau_{\mathfrak{R}}(\Lambda))$ is called a nano topological space, with elements of $\tau_{\mathfrak{R}}(\Lambda)$ termed nano open sets and their complements nano closed sets.

Definition 2.9. *[54] An ordered approximation space (OAS) is a triple $(\chi, \mathfrak{R}, \preceq)$ where:*

- *χ is a universe set*
- *$\mathfrak{R}$ is an equivalence relation on χ*

- *$\preceq$ is a partial order relation on χ*

Definition 2.10. *[54] For an OAS $(\chi, \mathfrak{R}, \preceq)$ and $\Lambda \subseteq \chi$, the nano increasing topology is:*

$$\tau^I_{\mathfrak{R}}(\Lambda) = \{\chi, \emptyset, IL_{\mathfrak{R}}(\Lambda), IU_{\mathfrak{R}}(\Lambda), IB_{\mathfrak{R}}(\Lambda)\}$$

This collection forms a topology satisfying:

(1) $\chi, \emptyset \in \tau^I_{\mathfrak{R}}(\Lambda)$
(2) *Closed under arbitrary unions*
(3) *Closed under finite intersections*

Remark 2.1. *[54] The nano decreasing topology for $\Lambda \subseteq \chi$ in an OAS is:*

$$\tau^D_{\mathfrak{R}}(\Lambda) = \{\chi, \emptyset, DL_{\mathfrak{R}}(\Lambda), DU_{\mathfrak{R}}(\Lambda), DB_{\mathfrak{R}}(\Lambda)\}$$

Definition 2.11. *[54] For a nano increasing topological space $(\chi, \tau^I_{\mathfrak{R}_E}(\Lambda))$ with attribute set E, the nano increasing accuracy measure is:*

$$C^I_{\mathfrak{R}_E}(\Lambda) = \frac{|IL_{\mathfrak{R}_E}(\Lambda)|}{|IU_{\mathfrak{R}_E}(\Lambda)|}$$

where $0 \leq C^I_{\mathfrak{R}_E}(\Lambda) \leq 1$. When $IL_{\mathfrak{R}_E}(\Lambda) = IU_{\mathfrak{R}_E}(\Lambda)$, Λ is increasing exact; otherwise it is increasing rough.

Note: *The nano decreasing accuracy measure is defined analogously as:*

$$C^D_{\mathfrak{R}_E}(\Lambda) = \frac{|DL_{\mathfrak{R}_E}(\Lambda)|}{|DU_{\mathfrak{R}_E}(\Lambda)|}$$

Definition 2.12. *[55] An m-flou set ω in universe χ ($m \geq 2$) is an m-tuple $(\Pi_1, \Pi_2, \ldots, \Pi_m)$ where:*

- *$core(\omega) = \Pi_1$ is the core*
- *$hull(\omega) = \Pi_m$ is the hull*

Definition 2.13. *[54] For an OAS $(\chi, \mathfrak{R}, \preceq)$ with attribute set E and $\Lambda \subseteq \chi$, an m-nano flou set ω ($m \geq 2$) is an m-tuple where component inclusion $\{\hbar_i\} \subseteq \{\hbar_j\}$ holds when:*

$$mid\left(C^I_{\mathfrak{R}_{E\setminus\{\hbar_i\}}}(\Lambda), C^D_{\mathfrak{R}_{E\setminus\{\hbar_i\}}}(\Lambda)\right) < mid\left(C^I_{\mathfrak{R}_{E\setminus\{\hbar_j\}}}(\Lambda), C^D_{\mathfrak{R}_{E\setminus\{\hbar_j\}}}(\Lambda)\right)$$

for all $\hbar_i, \hbar_j \in E$, where:

- *$C^I_{\mathfrak{R}_{E\setminus\{\hbar_i\}}}(\Lambda)$ is the nano increasing accuracy after removing $\hbar_i$*
- *$C^D_{\mathfrak{R}_{E\setminus\{\hbar_i\}}}(\Lambda)$ is the nano decreasing accuracy after removing $\hbar_i$*

3. Generalized Rough Set Models Based on Binary Relations via Partial Orders

In this section, we present ordered approximation spaces as a generalization of Yao's approximation spaces, exploring their properties and providing illustrative counterexamples.

3.1. **Generalized Yao's Rough Sets.**

Definition 3.1. *[5] For a binary relation $\mathfrak{R}$ on universe χ, the right neighborhood of $\nu \in \chi$ is:*

$$\mathfrak{R}_r(\nu) = \{\beta \in \chi : \nu \mathfrak{R} \beta\}$$

Definition 3.2. *[5] For $\mathfrak{R}$ on χ and $\Lambda \subseteq \chi$, the right approximations are:*

$$L_r(\Lambda) = \{\nu \in \chi : \mathfrak{R}_r(\nu) \subseteq \Lambda\}$$

$$U_r(\Lambda) = \{\nu \in \chi : \mathfrak{R}_r(\nu) \cap \Lambda \neq \emptyset\}$$

Definition 3.3. *[5] For $\mathfrak{R}$ on χ and $\emptyset \neq \Lambda \subseteq \chi$, define:*

$$B_r(\Lambda) = U_r(\Lambda) \setminus L_r(\Lambda)$$

$$K_r(\Lambda) = \frac{|L_r(\Lambda)|}{|U_r(\Lambda)|}$$

Note: When $K_r(\Lambda) = 1$, then $B_r(\Lambda) = \emptyset$ and Λ is right-exact; otherwise it is right-rough.

Now, by incorporating partial order relations through the notion of an ordered approximation space (OAS), we propose a novel generalization of Yao's rough set model [5,6].

Definition 3.4. *Let $(\chi, \mathfrak{R}, \preceq)$ be an ordered approximation space (OAS), and let $\Lambda \subseteq \chi$. Then the following approximation operators are defined:*

(1) *Increasing right-lower: $IL_r(\Lambda) = \bigcup\{i(\nu) : \mathfrak{R}_r(\nu) \subseteq \Lambda\} \cap \Lambda$*
(2) *Decreasing right-lower: $DL_r(\Lambda) = \bigcup\{d(\nu) : \mathfrak{R}_r(\nu) \subseteq \Lambda\} \cap \Lambda$*
(3) *Increasing right-upper: $IU_r(\Lambda) = [DL_r(\Lambda^c)]^c$*
(4) *Decreasing right-upper: $DU_r(\Lambda) = [IL_r(\Lambda^c)]^c$*
(5) *Increasing right-boundary: $IB_r(\Lambda) = IU_r(\Lambda) \setminus IL_r(\Lambda)$*
(6) *Decreasing right-boundary: $DB_r(\Lambda) = DU_r(\Lambda) \setminus DL_r(\Lambda)$*

Proposition 3.1. *Let $(\chi, \mathfrak{R}, \preceq)$ be an OAS, and let $\Lambda, \Gamma \subseteq \chi$. Then the following properties hold (the corresponding results for the decreasing approximation operators are analogous):*

(1) $IL_r(\Lambda) \subseteq \Lambda \subseteq IU_r(\Lambda)$ *(analogous for decreasing case)*
(2) $IL_r(\emptyset) = \emptyset$ *and* $IU_r(\chi) = \chi$ *(analogous for decreasing case)*
(3) $\Lambda \subseteq \Gamma \Rightarrow IL_r(\Lambda) \subseteq IL_r(\Gamma)$ *(analogous for decreasing case)*
(4) $\Lambda \subseteq \Gamma \Rightarrow IU_r(\Lambda) \subseteq IU_r(\Gamma)$ *(analogous for decreasing case)*
(5) $IL_r(\Lambda \cap \Gamma) \subseteq IL_r(\Lambda) \cap IL_r(\Gamma)$ *(analogous for decreasing case)*
(6) $IL_r(\Lambda) \cup IL_r(\Gamma) \subseteq IL_r(\Lambda \cup \Gamma)$ *(analogous for decreasing case)*
(7) $IU_r(\Lambda \cap \Gamma) \subseteq IU_r(\Lambda) \cap IU_r(\Gamma)$ *(analogous for decreasing case)*
(8) $IU_r(\Lambda) \cup IU_r(\Gamma) \subseteq IU_r(\Lambda \cup \Gamma)$ *(analogous for decreasing case)*
(9) $IL_r(IL_r(\Lambda)) = IL_r(\Lambda)$ *(analogous for decreasing case)*
(10) $IU_r(IU_r(\Lambda)) = IU_r(\Lambda)$ *(analogous for decreasing case)*
(11) $IL_r(IU_r(\Lambda)) \subseteq IU_r(\Lambda)$ *(analogous for decreasing case)*

(12) $IL_r(\Lambda) \subseteq IU_r(IL_r(\Lambda))$ *(analogous for decreasing case)*

(13) $IU_r(\Lambda^c) = [DL_r(\Lambda)]^c$ *(analogous for decreasing case)*

Proof. We prove selected properties (complete proofs follow similarly):

1. From Definition 3.4, $IL_r(\Lambda) \subseteq \Lambda$ and $DL_r(\Lambda) \subseteq \Lambda$. Then $\Lambda^c \subseteq [DL_r(\Lambda)]^c = IU_r(\Lambda^c)$, hence $\Lambda \subseteq IU_r(\Lambda)$.

3. $IL_r(\Lambda) = \bigcup\{i(\nu) : \mathfrak{R}_r(\nu) \subseteq \Lambda\} \cap \Lambda \subseteq \bigcup\{i(\nu) : \mathfrak{R}_r(\nu) \subseteq \Gamma\} \cap \Gamma = IL_r(\Gamma)$

4. If $\Lambda \subseteq \Gamma$, then $\Gamma^c \subseteq \Lambda^c$, so $DL_r(\Gamma^c) \subseteq DL_r(\Lambda^c)$. Thus $[DL_r(\Lambda^c)]^c \subseteq [DL_r(\Gamma^c)]^c$.

5. $IL_r(\Lambda \cap \Gamma) = \bigcup\{i(\nu) : \mathfrak{R}_r(\nu) \subseteq (\Lambda \cap \Gamma)\} \cap (\Lambda \cap \Gamma) \subseteq \bigcup\{i(\nu) : \mathfrak{R}_r(\nu) \subseteq \Lambda\} \cap \Lambda = IL_r(\Lambda)$. Similarly for Γ.

9. Let $\Gamma = IL_r(\Lambda)$. For $y \in \Gamma = \bigcup\{i(\nu) : \mathfrak{R}_r(\nu) \subseteq \Lambda\} \cap \Lambda$, we have $y \in i(\nu) \cap \Lambda \subseteq i(\nu)$, so $y \in i(\nu) \cap \Gamma$. Thus $y \in IL_r(\Gamma)$ and $IL_r(\Lambda) \subseteq IL_r(IL_r(\Lambda))$. Combined with property (1), we get equality.

13. $[DL_r(\Lambda)]^c = [IU_r(\Lambda^c)^c]^c = IU_r(\Lambda^c)$.

Analogous proofs hold for the decreasing cases.

□

The converse of properties (11) and (12) in Proposition 3.1 does not hold in general, as demonstrated by the following example:

Example 3.1. *Let $\chi = \{1, 2, \dots, 8\}$ with relations:*

$$\mathfrak{R} = \blacktriangle \cup \{(1,4),(4,1),(1,5),(5,1),(4,5),(5,4),(3,6),(6,3)\}$$
$$\preceq = \blacktriangle \cup \{(1,3),(1,6),(4,3),(7,3),(4,6),(5,6),(7,6)\}$$

(1) *For $\omega = \{1,2,7\} \subseteq \chi$:*

$$IU_r(\omega) = [DL_r(\omega^c)]^c = [\{3,4,5,6,8\}]^c = \{1,2,7\}$$
$$IL_r(IU_r(\omega)) = \{2,3,6,7\} \cap \{1,2,7\} = \{2,7\}$$
$$\Rightarrow IU_r(\omega) \nsubseteq IL_r(IU_r(\omega))$$

For $\omega = \{1,3\} \subseteq \chi$:

$$DU_r(\omega) = [IL_r(\omega^c)]^c = [\{2,6,7,8\}]^c = \{1,3,4,5\}$$
$$DL_r(DU_r(\omega)) = \{1,4,5\} \cap \{1,3,4,5\} = \{1,4,5\}$$
$$\Rightarrow DU_r(\omega) \nsubseteq DL_r(DU_r(\omega))$$

(2) *For $\Lambda = \{1,2,3,7\} \subseteq \chi$:*

$$IL_r(\Lambda) = \{2,3,7\}$$
$$IU_r(IL_r(\Lambda)) = [DL_r(\{2,3,7\}^c)]^c$$
$$= [\{1,4,5,8\} \cap \{1,4,5,6,8\}]^c = \{2,3,6,7\}$$
$$\Rightarrow IU_r(IL_r(\Lambda)) \nsubseteq IL_r(\Lambda)$$

For $\Lambda = \{1,3,4,6,7\} \subseteq \chi$*:*

$$\begin{aligned} DL_r(\Lambda) &= \{1,3,4,6,7\} \\ DU_r(DL_r(\Lambda)) &= [IL_r(\{1,3,4,6,7\}^c)]^c \\ &= [\{2,8\} \cap \{2,5,8\}]^c = \{1,3,4,5,6,7\} \\ &\Rightarrow DU_r(DL_r(\Lambda)) \nsubseteq DL_r(\Lambda) \end{aligned}$$

Definition 3.5. *Let* $(\chi, \mathfrak{R}, \leq)$ *be an OAS, and let* $\Lambda \subseteq \chi$*. The increasing right-accuracy measure of* $\emptyset \neq \Lambda \subseteq \chi$ *is defined by:*

$$K_r^I(\Lambda) = \frac{|IL_r(\Lambda)|}{|IU_r(\Lambda)|}$$

where $0 \leq K_r^I(\Lambda) \leq 1$*. If* $IL_r(\Lambda) = IU_r(\Lambda)$*, then* Λ *is increasing right-exact; otherwise it is increasing right-rough.*

Note: *The decreasing right-accuracy measure is:*

$$K_r^D(\Lambda) = \frac{|DL_r(\Lambda)|}{|DU_r(\Lambda)|}$$

Theorem 3.1. *Let* $(\chi, \mathfrak{R}, \leq)$ *be an OAS, and let* $\Lambda \subseteq \chi$*. Then the following statements hold:*

(1) $L_r(\Lambda) \subseteq IL_r(\Lambda)$
(2) $L_r(\Lambda) \subseteq DL_r(\Lambda)$
(3) $IU_r(\Lambda) \subseteq U_r(\Lambda)$
(4) $DU_r(\Lambda) \subseteq U_r(\Lambda)$
(5) $DB_r(\Lambda) \subseteq U_r(\Lambda)$
(6) $IB_r(\Lambda) \subseteq U_r(\Lambda)$
(7) $K_r(\Lambda) \leq K_r^I(\Lambda)$
(8) $K_r(\Lambda) \leq K_r^D(\Lambda)$

Proof. We prove only (1), since the proofs of the remaining statements are analogous.

$$\begin{aligned} L_r(\Lambda) &= \{\nu \in \chi : \mathfrak{R}_r(\nu) \subseteq \Lambda\} \\ &\subseteq \bigcup \{i(\nu) : \mathfrak{R}_r(\nu) \subseteq \Lambda\} \cap \Lambda = IL_r(\Lambda) \end{aligned}$$

□

Example 3.2. *Let* $\chi = \{\rho, \delta, \sigma, \varsigma\}$ *with relations:*

$$\begin{aligned} \mathfrak{R} &= \blacktriangle \cup \{(\rho,\sigma),(\sigma,\rho)\} \\ \leq &= \blacktriangle \cup \{(\rho,\delta),(\delta,\varsigma),(\rho,\varsigma)\} \end{aligned}$$

Neighborhoods and relations:

$$\begin{array}{lll} i(\rho) = \{\rho,\delta,\varsigma\} & d(\rho) = \{\rho\} & \mathfrak{R}_r(\rho) = \{\rho,\sigma\} \\ i(\delta) = \{\delta,\varsigma\} & d(\delta) = \{\rho,\delta\} & \mathfrak{R}_r(\delta) = \{\delta\} \end{array}$$

$$i(\sigma) = \{\sigma\} \qquad d(\sigma) = \{\sigma\} \qquad \mathfrak{R}_r(\sigma) = \{\rho, \sigma\}$$

$$i(\varsigma) = \{\varsigma\} \qquad d(\varsigma) = \{\rho, \delta, \varsigma\} \qquad \mathfrak{R}_r(\varsigma) = \{\varsigma\}$$

For $\Lambda = \{\rho, \varsigma\}$*:*

$$DL_r(\Lambda) = \Lambda$$

$$L_r(\Lambda) = \{\varsigma\} \subsetneq \Lambda$$

This shows the reverse inclusions in Theorem 3.1 need not hold.

Lemma 3.1. *If* $\mathfrak{R}$ *is a binary relation with* $\preceq = \blacktriangle$ *on* χ*, then equality holds in Theorem 3.1.*

Proof. Immediate from the definitions when $\preceq$ is the identity relation. □

4. Generalized Increasing and Decreasing Rough Sets

Proposition 4.1. *Let* $\mathfrak{R}$ *be a reflexive relation on a total directed ordered set (TDOS)* χ*. Then for any* $\Lambda \subseteq \chi$*:*

(1) $i(\Lambda) = \chi$ *(and* $d(\Lambda) = \chi$*)*
(2) $IL_r(\Lambda) = \Lambda = IU_r(\Lambda)$ *and* $DL_r(\Lambda) = \Lambda = DU_r(\Lambda)$

Proof. We prove each statement separately:

(1) For $i(\Lambda) = \chi$, we consider cases based on the cardinality of χ:
  (a) Case $|\chi| = 1$: Trivial since $i(\Lambda) = \chi$ for $\Lambda \subseteq \chi$.
  (b) Case $|\chi| = 2$: Let $\chi = \{\nu, \zeta\}$. For any non-empty $\Lambda \subseteq \chi$, since χ is TDOS, there exists $z \preceq \nu$ and $z \preceq \zeta$. Thus all elements are comparable, making $i(\Lambda) = \chi$.
  (c) Case $|\chi| = 3$: Let $\chi = \{\nu, \zeta, \varsigma\}$. The TDOS property ensures that for any $\Lambda \subseteq \chi$, all elements are comparable, hence $i(\Lambda) = \chi$.
  (d) Case $|\chi| > 3$: The TDOS property guarantees that for any subset Λ with $|\Lambda| > 3$, all elements are comparable, thus $i(\Lambda) = \chi$.

  The proof for $d(\Lambda) = \chi$ is analogous.

(2) For the approximation operators:

$$\begin{aligned} IL_r(\Lambda) &= \bigcup \{i(\beta) : \beta \in \chi, \mathfrak{R}_r(\beta) \subseteq \Lambda\} \cap \Lambda \\ &= \chi \cap \Lambda = \Lambda \quad \text{(since } i(\beta) = \chi \text{ by part 1)} \end{aligned}$$

Similarly, $DL_r(\Lambda) = \Lambda$.

For the upper approximations:

$$IU_r(\Lambda) = [DL_r(\Lambda^c)]^c = (\Lambda^c)^c = \Lambda$$

$$DU_r(\Lambda) = [IL_r(\Lambda^c)]^c = (\Lambda^c)^c = \Lambda$$

□

Remark: Proposition 4.1 does not hold in general for:

(1) Linearly ordered sets (LOS)
(2) Directed ordered sets (DOS)

Example 4.1. *Let $\chi = \{\rho, \delta, \sigma, \varsigma\}$ with relations:*

$$\mathfrak{R} = \blacktriangle \cup \{(\rho,\delta),(\delta,\rho),(\delta,\sigma),(\sigma,\delta)\}$$
$$\preceq = \blacktriangle \cup \{(\rho,\delta),(\rho,\sigma),(\rho,\varsigma),(\delta,\sigma),(\delta,\varsigma),(\sigma,\varsigma)\}$$

Here $\mathfrak{R}$ is reflexive and χ is an LOS. However:

- $i(\sigma) = \{\sigma, \varsigma\} \subsetneq \chi$
- $IL_r(\{\rho,\varsigma\}) = \{\varsigma\} \subsetneq \{\rho,\varsigma\}$
- $IU_r(\{\rho,\delta,\varsigma\}) = \chi \supsetneq \{\rho,\delta,\varsigma\}$

Example 4.2. *Using the same χ and $\mathfrak{R}$ from Example 4.1, but with:*

$$\preceq = \blacktriangle \cup \{(\rho,\delta),(\rho,\sigma),(\rho,\varsigma),(\delta,\rho),(\delta,\sigma),(\delta,\varsigma),(\sigma,\rho),(\sigma,\delta)\}$$

Now χ is a DOS and we observe:

- $d(\varsigma) = \{\rho,\delta,\varsigma\} \subsetneq \chi$
- $DL_r(\{\sigma\}) = \emptyset \subsetneq \{\sigma\}$
- $DU_r(\{\rho,\delta\}) = \{\rho,\delta,\sigma\} \supsetneq \{\rho,\delta\}$

5. Generalized Nano Ordered Topological Spaces

In this section, we introduce generalized nano-topological structures in ordered approximation spaces, with particular emphasis on the increasing and decreasing cases. We establish their fundamental topological properties and subsequently demonstrate the applicability of the proposed framework through a case study on COVID-19 risk analysis.

5.1. **Topological Structures.**

Definition 5.1. *Let χ be a finite universe, and let $\Lambda \subseteq \chi$. Denote by $IL_r(\Lambda)$ and $IU_r(\Lambda)$ the increasing right lower and upper approximations of Λ, respectively. Define the collection*

$$\tau_r^{In} = \Big\{\chi, \emptyset, IL_r(\Lambda), IU_r(\Lambda), IB_r(\Lambda)\Big\},$$

where the increasing boundary region is defined by

$$IB_r(\Lambda) = IU_r(\Lambda) \setminus IL_r(\Lambda).$$

Then, the family τ_r^{In} forms a topology on χ, called the generalized nano increasing topology*, or simply the In*-topology*.*

The topology τ_r^{In} satisfies the following axioms:

(1) The empty set and the whole universe belong to τ_r^{In}, that is, $\emptyset, \chi \in \tau_r^{In}$;
(2) τ_r^{In} is closed under arbitrary unions;
(3) τ_r^{In} is closed under finite intersections.

Remark 5.1. *The generalized nano decreasing topology (Dn-topology) is defined as:*

$$\tau_r^{Dn} = \{\chi, \emptyset, DL_r(\Lambda), DU_r(\Lambda), DB_r(\Lambda)\}$$

Lemma 5.1. *For $\Lambda \subseteq \chi$, the bases for these topologies are:*

$$\pi_r^{In} = \{\chi, IL_r(\Lambda), IB_r(\Lambda)\}$$
$$\pi_r^{Dn} = \{\chi, DL_r(\Lambda), DB_r(\Lambda)\}$$

Example 5.1. *Using Example 3.2 with $\Lambda = \{\rho, \sigma\}$:*

$$\tau_r^{In} = \{\chi, \emptyset, \{\sigma\}, \{\rho, \delta, \sigma\}, \{\rho, \delta\}\}$$
$$\tau_r^{Dn} = \{\chi, \emptyset, \{\rho\}, \{\rho, \delta, \sigma\}, \{\delta, \sigma\}\}$$

5.2. **COVID-19 Case Study.** The COVID-19 pandemic has created unprecedented problems for public health systems all over the world. This shows how important it is to have reliable and easy-to-understand diagnostic and decision-support tools. In this subsection, we demonstrate the practical efficacy of the proposed generalized nano-ordered topological framework in pinpointing influential factors related to COVID-19 infection.

5.2.1. *Data Description.* The dataset utilized in this study is derived from [26], which condensed an original dataset of 1000 patients into 10 representative cases for illustrative analysis. There are six clinical symptoms for each patient record. These symptoms are divided into two groups: serious and common.

- **Serious symptoms**: a_1 (difficulty breathing), a_2 (chest pain), and a_3 (headache);
- **Common symptoms**: a_4 (dry cough), a_5 (high temperature), and a_6 (losing your sense of taste or smell).

The decision attribute d tells us if the patient has been confirmed to have COVID-19. Table 2 shows some of the original symptom data that was used in the study.

Table 2. COVID-19 symptom dataset (adapted from [26])

Patient	**Serious Symptoms**			**Common Symptoms**			**Decision**
	a_1	a_2	a_3	a_4	a_5	a_6	d
P_1	yes	yes	no	yes	yes	yes	yes
P_2	yes	yes	yes	yes	yes	yes	yes
P_3	yes	yes	no	yes	no	yes	no
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
P_{10}	no	no	yes	yes	yes	no	yes

TABLE 3. Encoded COVID-19 data (2=yes, 1=no)

Patient	a_1	a_2	a_3	a_4	a_5	a_6	d
P_1	2	2	1	2	2	2	2
P_2	2	2	2	2	2	2	2
P_3	2	2	1	2	1	2	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
P_{10}	1	1	2	2	2	1	2

5.2.2. *Analysis Approach.* The proposed analysis follows a structured methodology designed to highlight the advantages of generalized nano ordered topologies in medical decision-making:

(1) Construction of ordered approximation spaces derived from the clinical symptom data;
(2) Application of generalized nano-topological structures to perform effective feature (attribute) reduction;
(3) Identification of core symptom attributes that contribute most significantly to accurate COVID-19 diagnosis;
(4) Derivation of sharper and more informative decision boundaries compared with those obtained using classical rough set approaches.

The detailed procedures for attribute reduction and the comparative evaluation of approximation accuracy are presented in the subsequent sections.

6. COVID-19 CASE STUDY ANALYSIS

6.1. **Data Preparation.** Based on the symptom data summarized in Table 3, we extract the symptom profiles associated with each patient. For patient P_i, the set $V(P_i)$ denotes the collection of symptoms observed in that case:

$$\begin{aligned} V(P_1) &= \{a_1, a_2, a_4, a_5, a_6\}, \\ V(P_2) &= \{a_1, a_2, a_3, a_4, a_5, a_6\}, \\ &\vdots \\ V(P_{10}) &= \{a_3, a_4, a_5\}. \end{aligned}$$

Remark. Patients P_7 and P_8 exhibit identical symptom profiles while being associated with different diagnostic outcomes. This inconsistency necessitates special consideration in the subsequent analysis.

6.2. **Approach 1: Exclusion of the Inconsistent Patient P_8.**

6.2.1. *Initial Setup.* We define a binary relation $\mathfrak{R}$ on the set of patients together with a partial order relation $\preceq$, both induced by symptom inclusion. Specifically, for any two patients P_i and P_j, we define:

$$(P_i, P_j) \in \mathfrak{R} \iff (P_i, P_j) \in \preceq \iff V(P_i) \subseteq V(P_j).$$

This relation captures the natural ordering of patients according to the containment of their symptom sets and serves as the foundation for constructing the ordered approximation space used in the subsequent analysis.

6.2.2. *Approximation Results.* For infected patients $\Lambda = \{P_1, P_2, P_6, P_7, P_9, P_{10}\}$:

- Increasing approximations:

$$IL_r(\Lambda) = \{P_1, P_2, P_6, P_7, P_{10}\}$$
$$IU_r(\Lambda) = \chi \setminus \{P_8\}$$
$$IB_r(\Lambda) = \{P_3, P_4, P_5, P_9\}$$

- Decreasing approximations:

$$DL_r(\Lambda) = \Lambda$$
$$DU_r(\Lambda) = \chi \setminus \{P_8\}$$
$$DB_r(\Lambda) = \{P_3, P_4, P_5\}$$

6.2.3. *Topological Structures.*

- Increasing topology base: $\pi_r^{In} = \{\chi \setminus \{P_8\}, IL_r(\Lambda), IB_r(\Lambda)\}$
- Decreasing topology base: $\pi_r^{Dn} = \{\chi \setminus \{P_8\}, \Lambda, DB_r(\Lambda)\}$

6.3. **Attribute Reduction Analysis.** We systematically remove each attribute and analyze the impact:

Table 4. Attribute Removal Impact

Removed Attribute	**Key Changes**	**Resulting Topology**
a_1 (Breathing)	$P_6 = P_{10}$ removed	Reduced universe $\chi \setminus \{P_8, P_{10}\}$
a_2 (Chest pain)	No patient merges	Similar structure
a_3 (Headache)	$P_1 = P_2$, $P_7 = P_{10}$ removed	$\chi \setminus \{P_2, P_8, P_{10}\}$
a_4 (Dry cough)	$P_4 = P_5$ removed	$\chi \setminus \{P_5, P_8\}$
a_5 (Fever)	$P_1 = P_3$	Two cases analyzed
a_6 (Taste/smell)	$P_3 = P_5$, P_9 empty	$\chi \setminus \{P_5, P_8, P_9\}$

6.4. **Key Findings.**

(1) **Core Symptoms**: Attributes a_1 (breathing difficulty) and a_6 (loss of taste/smell) cause the most significant changes when removed, suggesting their diagnostic importance.
(2) **Consistency**: The topological structures remain stable across attribute removals, with boundary regions consistently identifying uncertain cases.
(3) **Diagnostic Power**: The increasing and decreasing approximations provide complementary views of infection likelihood, with:
    - Increasing focus on symptom presence
    - Decreasing focus on symptom absence
(4) **Decision Boundaries**: The boundary regions consistently identify patients $\{P_3, P_4, P_5\}$ as borderline cases requiring further examination.

6.5. **Non-Infected Case Analysis.** For non-infected patients $\Lambda^c = \{P_3, P_4, P_5, P_8\}$, the analysis yields symmetric results, confirming the method's consistency.

7. Alternative Analysis: Removing Inconsistent Patient P_7

7.1. **Initial Setup.** For infected patients $\Lambda = \{P_1, P_2, P_6, P_9, P_{10}\}$ after removing P_7:

- Increasing approximations:

$$IL_r(\Lambda) = \{P_1, P_2, P_6, P_{10}\}$$
$$IU_r(\Lambda) = \chi \setminus \{P_7\}$$
$$IB_r(\Lambda) = \{P_3, P_4, P_5, P_8, P_9\}$$

- Decreasing approximations:

$$DL_r(\Lambda) = \Lambda$$
$$DU_r(\Lambda) = \chi \setminus \{P_7\}$$
$$DB_r(\Lambda) = \{P_3, P_4, P_5, P_8\}$$

7.2. **Attribute Impact Analysis.**

Table 5. Attribute Removal Effects (Removing P_7)

Removed Attribute	**Key Changes**	**Mid Accuracy**
None (Original)	Baseline case	0.500
a_1 (Breathing)	Reduced boundary region	0.4375
a_2 (Chest pain)	No structural changes	0.500
a_3 (Dry cough)	Significant reduction	0.357
a_4 (Headache)	Improved accuracy	0.5625
a_5 (Fever)	Two subcases analyzed	0.530
a_6 (Taste/smell)	Best performance	0.570

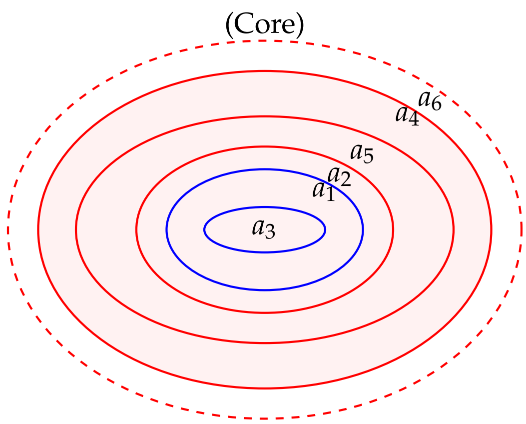
7.3. **Comparative Analysis.** Combining results from both approaches (removing P_7 vs P_8):

Table 6. Combined Accuracy Measures

Attribute	Average Mid Accuracy	Rank
a_3 (Dry cough)	0.4285	Most significant
a_1 (Breathing)	0.500	
a_2 (Chest pain)	0.555	
a_5 (Fever)	0.5618	
a_4 (Headache)	0.625	
a_6 (Taste/smell)	0.642	Least significant

7.4. **Key Findings.**

- **Core Diagnostic Attribute**: a_3 (Dry cough) emerges as the most significant factor (core of the m^r-nano flou set)
- **Stable Indicators**: Attributes a_4, a_5, a_6 show consistent high accuracy across both approaches
- **Decision Boundaries**: The boundary regions consistently identify patients $\{P_3, P_4, P_5, P_8, P_9\}$ as uncertain cases
- **Method Robustness**: Both approaches yield similar attribute rankings, validating the methodology



7.5. **Algorithm Summary.**

(1) **Data Preparation**: Create information table with patients and attributes
(2) **Approximation**: Calculate increasing/decreasing rough approximations
(3) **Topology Construction**: Build nano topologies for original data
(4) **Attribute Reduction**: Systematically remove each attribute and repeat analysis
(5) **Flou Set Generation**: Compare accuracy measures to identify core attributes

Remark 7.1. *The analysis demonstrates that:*

- *Dry cough (a_3) is the most critical diagnostic indicator*
- *The methodology is robust to different handling of inconsistent cases*

- *The topological approach provides stable decision boundaries*

8. Conclusion and Clinical Insights

8.1. **Key Observations.** The analysis of the COVID-19 symptom dataset yields several clinically significant insights that underscore the complexity of the disease and the need for sophisticated uncertainty modeling approaches.

(1) **Symptom Variability**:

The results reveal substantial variability in symptom manifestation among patients. In particular:

- Patient P_9 was diagnosed as COVID-19 positive despite exhibiting only loss of taste or smell (a_6);
- Patient P_3 exhibited the same symptom but was diagnosed as COVID-19 negative;
- This inconsistency indicates that the symptom a_6 alone cannot serve as a reliable diagnostic indicator and must be interpreted in conjunction with other clinical features.

(2) **Identification of Critical Symptoms**:

The attribute reduction process consistently identified headache (a_3) as the most influential diagnostic factor:

- The symptom a_3 emerged as the core attribute within the m^r-nano flow set;
- This finding highlights the diagnostic significance of headache when analyzed within an ordered nano-topological framework.

8.2. **Theoretical Contributions.** From a theoretical perspective, this work advances rough set theory in several important directions:

(1) **Novel Conceptual Framework**:

- We present the first systematic formulation of a generalized version of Yao's rough sets based on partial order relations;
- Increasing and decreasing right approximation operators are introduced to characterize ordered uncertainty;
- Proposition 4.1 establishes the fundamental properties of these operators under reflexive relations in TDOS.

(2) **Methodological Advancements**:

- The proposed framework achieves a significant reduction in boundary regions;
- Decision accuracy is enhanced through the integration of neighborhood-based and topological techniques;
- The approach extends Pawlak's equivalence-based model, enabling the analysis of complex real-world data that are not governed by equivalence relations.

(3) **Practical Implementation**:

- A complete and implementable algorithm is developed for real-world decision-making applications;

- The effectiveness of the proposed methodology is demonstrated through a medical diagnosis case study.

8.3. **Clinical Applications.** The proposed generalized nano ordered topological framework offers several practical advantages for COVID-19 diagnosis and related clinical decision-support systems:

- It accommodates both qualitative and quantitative clinical data within a unified and consistent modeling framework;
- Key diagnostic factors are systematically identified through topological attribute reduction;
- Clinical decision-making is supported by robust topological and order-theoretic analysis;
- The framework is particularly effective in challenging scenarios involving:
  - Inconsistent or overlapping symptom presentations;
  - Partial, uncertain, or conflicting clinical information;
  - Complex dependencies among clinical attributes.

8.4. **Future Directions.** Several promising research directions may be pursued to further extend the scope and impact of this work:

- Investigating *nano bi-ordered topological spaces* and their associated rough approximation operators;
- Applying the proposed framework to a broader range of medical decision-making problems beyond COVID-19;
- Integrating ordered nano-topological models with machine learning and other data-driven techniques;
- Developing real-time clinical decision-support systems based on the proposed methodology.

Availability of Data and Materials: The data supporting this study are publicly available in the previously published work cited as Reference [26]. No new datasets were generated during this study.

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