International Journal of Analysis and Applications

Convergence Analysis of Altered Inertial Subgradient Extragradient Method for Solving Equilibrium and Fixed Point Problems in Hilbert Spaces

Fouzia Amir[1,*], Juraev Ilhom Ishoqovich[2], Namazova Nodira Djumayevna[3], Abdullaev Mekhriddin Junaydulloevich[4], Khamidov Sherali Shirinovich[2], Farxod Ixtiyarov[1]

[1]*Center for Research and Innovation, Asia International University, Yangiobod MFY, G'ijduvon street, House 74, Bukhara, Uzbekistan*

[2]*Department of Mathematics and Informatics, Bukhara State Pedagogical Institute, Bukhara, Uzbekistan*

[3]*Department of Social Sciences, Bukhara State Medical Institute, Bukhara, Uzbekistan*

[4]*Department of theory and methodology of sports, Bukhara State University, Bukhara, Uzbekistan*

**Corresponding author:* fouziaur.abbasi@oxu.uz

Abstract. This paper aims to derive weak convergence analysis of the subgradient extragradient method with an alternating inertial step for addressing equilibrium and fixed-point problems in real Hilbert spaces. Our approach incorporates altered inertial techniques into a Halpern-type method, extending the classical subgradient extragradient framework.

1. Introduction

In this paper, we investigate the problem of finding a fixed point of a specified nonlinear operator and equilibrium problems in real Hilbert spaces. The motivation for studying these problems arises from their practical significance in mathematical models representing systems with constraints, formulated either as equilibrium or fixed-point problems. These scenarios have wide-ranging applications in fields such as signal processing, composite minimization, optimal control, and image restoration, as extensively discussed in the literature [30–35].

Let $\mathcal{D}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{X}$, equipped with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Consider $\mathcal{L} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ as a bifunction such that $\mathcal{L}(x, x) = 0$ for all $x \in \mathcal{X}$. The equilibrium problem (EP) is to find $x^* \in \mathcal{X}$ satisfying:

$$\mathcal{L}(x^*, x) \geq 0, \text{ for all } x \in \mathcal{D}. \tag{1.1}$$

Received: May 14, 2026.

Key words and phrases. equilibrium problem; fixed point problem; extragradient method; pseudomonotone bifunction; strong convergence.

https://doi.org/10.28924/2291-8639-24-2026-221

ISSN: 2291-8639

Equilibrium problems (EP) encompass a wide range of mathematical challenges, including applications in non-cooperative game theory, saddle point problems, variational inequalities, fixed-point problems, and both scalar and vector optimization problems. The formulation of EP, as defined by (1.1), has gained recognition and application across various fields, such as poroelasticity in petroleum engineering [1], the study of porous materials [2,3], economic modeling [11,12], image reconstruction in image processing [13–15], telecommunication networks, public infrastructure systems such as roads [16, 17], and non-cooperative game theory, particularly in relation to Nash equilibria [18]. Frequently referred to as EP, it is also known as Ky Fan's inequality, acknowledging Fan's contributions to this area [19]. Before its formalization within this framework, the problem received limited attention. Nikaido identified Nash equilibria as solutions to EP [20], while Gwinner employed it as a tool for solving optimization and variational inequality problems [21], although neither proposed a distinct formal framework.

Flåm et al. [36] introduced a novel solution approach based on the auxiliary problem principle to address EP as described by (1.1). Tran et al. [37] further investigated this method, termed the proximal-like method, focusing on its convergence properties. In particular, Tran et al. [37] explored settings where the equilibrium bifunction possesses Lipschitz continuity and pseudomonotonicity. A notable technique in this context is the *extragradient method*, an iterative optimization strategy designed for variational inequalities and equilibrium problems, which introduces an additional step beyond the standard gradient method to enhance convergence rates.

We now turn to another important topic discussed in this paper: the fixed-point problem (FPP) for a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$. This problem is expressed as follows:

$$\text{Find } \tau^* \in \mathcal{X} \text{ such that } \mathcal{T}(\tau^*) = \tau^*. \tag{1.2}$$

The set of solutions to the FPP, known as the fixed-point set of $\mathcal{T}$, is denoted by $F(\mathcal{T})$. The primary objective of this paper is to address the following problem:

$$\tau^* \in \mathcal{X} \text{ such that } \mathcal{L}(\tau^*, x) \geq 0 \text{ for all } x \in \mathcal{X}, \text{ and } \tau^* \in F(\mathcal{T}). \tag{1.3}$$

The primary focus of this study is on applying extragradient-type methods to solve problem (1.3). Korpelevich [38] first introduced this approach for equilibrium problems in Euclidean spaces, which was subsequently extended by Quoc et al. [26]. Recently, these methods have been generalized to equilibrium problems in infinite-dimensional Hilbert spaces, building on the foundational work of [26]. Notable contributions in this area include [4–7] and related research.

In 2015, Vuong et al. [22] introduced a numerical method for addressing a variational inequality problem. The method seeks to identify a unique solution that lies within the intersection of the solution set of an equilibrium problem and the set of fixed points of a demicontraction mapping. The algorithm begins with an initial point $u_0 \in \mathcal{X}$ and proceeds iteratively, updating for all $n \in \mathbb{N}$ as follows:

$$\begin{cases} v_n = \arg\min_{y\in\mathcal{D}} \left\{\delta_n \mathcal{L}(u_n, y) + \frac{1}{2}\|y - u_n\|^2\right\}, \\ z_n = \arg\min_{y\in\mathcal{D}} \left\{\delta_n \mathcal{L}(v_n, y) + \frac{1}{2}\|y - u_n\|^2\right\}, \\ p_n = z_n - \xi_n \mathcal{F}(z_n), \\ u_{n+1} = (1-\eta_n)p_n + \eta_n \mathcal{T} p_n, \end{cases} \tag{1.4}$$

where $\{\xi_n\} \subset [0,1)$, $\{\eta_n\} \subset (0,1)$, and $\{\delta_n\} \subset (0,\infty)$. The convergence of these iterates is established through a combination of viscosity approximation and extragradient methods.

In 2017, Hieu [23] extended the Halpern subgradient extragradient method to address equilibrium problems involving pseudomonotone and Lipschitz-type continuous bifunctions. The method is defined by the following system of expressions:

$$\begin{cases} v_n = \arg\min_{y\in\mathcal{D}} \left\{\delta_n \mathcal{L}(u_n, y) + \frac{1}{2}\|y - u_n\|^2\right\} \\ z_n = \arg\min_{y\in\mathcal{X}_n} \left\{\delta_n \mathcal{L}(v_n, y) + \frac{1}{2}\|y - u_n\|^2\right\} \\ p_n = \xi_n u_0 + (1-\xi_n) z_n \\ u_{n+1} = (1-\eta_n)p_n + \eta_n \mathcal{T} p_n, \end{cases} \tag{1.5}$$

where $\mathcal{X}_n = \{v \in \mathcal{X} : \langle (u_n - \delta_n t_n) - v_n, v - v_n\rangle \le 0\}$ and $t_n \in \partial_2 \mathcal{L}(u_n, v_n)$. By integrating the subgradient extragradient method with Halpern's iteration, the authors established the strong convergence of the generated sequence, under the conditions that the sequences $\{\xi_n\}$ and $\{\eta_n\}$ satisfy:

$$\begin{cases} 0 < \xi_n < 1, \quad \lim_{n\to\infty} \xi_n = 0, \quad \sum_{n=1}^{\infty} \xi_n = +\infty \\ 0 < a \le \eta_n \le \frac{1-\kappa}{2}. \end{cases}$$

Hieu [28] presented a novel hybrid algorithm in 2019 that approximates a common solution x^* of the equilibrium problem (EPFP) by combining the extragradient approach with a modified Mann iteration. The algorithm is defined as follows:

$$\begin{cases} v_n = \arg\min_{y\in\mathcal{D}} \left\{\delta_n \mathcal{L}(u_n, y) + \frac{1}{2}\|y - u_n\|^2\right\}, \\ z_n = \arg\min_{y\in\mathcal{D}} \left\{\delta_n \mathcal{L}(v_n, y) + \frac{1}{2}\|y - u_n\|^2\right\}, \\ u_{n+1} = (1-\xi_n-\eta_n) z_n + \eta_n \mathcal{T} z_n, \end{cases} \tag{1.6}$$

where $\{\xi_n\} \subset [0,1]$, $\{\eta_n\} \subset [0,1]$, and $\{\delta_n\} \subset (0,+\infty)$ satisfy the following conditions:

$$\begin{cases} 0 < \underline{\delta} \le \delta_n < \bar{\delta} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}, \\ \lim_{n\to\infty} \xi_n = 0, \quad \sum_{n=1}^{\infty} \xi_n = +\infty, \\ \eta_n \in (a,b) \subset (0,1) \text{ with } b > a > 0. \end{cases}$$

In 2022, Tan et al. [29] presented an algorithm that demonstrates strong convergence of iterative sequences under specific conditions. Let $\wp_n > 0$, $\delta_1 > 0$, $\mu \in (0,1)$, and $\rho \in \left(0, \frac{2}{1+\mu}\right)$. Assume that

u_0 and u_1 are arbitrarily chosen elements in $\mathcal{X}$. For the iterates u_{n-1} and u_n, define $\wp_n$ as follows:

$$\wp_n = \begin{cases} \min\left\{\wp, \frac{\mu_n}{\|u_n - u_{n-1}\|}\right\}, & \text{if } u_n \neq u_{n-1}, \\ \wp, & \text{otherwise.} \end{cases} \tag{1.7}$$

Compute:

$$\begin{cases} \nu_n = u_n + \wp_n(u_n - u_{n-1}), \\ v_n = \arg\min_{y \in \mathcal{D}} \left\{\delta_n \mathcal{L}(\nu_n, y) + \frac{1}{2}\|\nu_n - y\|^2\right\}, \\ p_n = \arg\min_{y \in \mathcal{X}_n} \left\{\rho\delta_n \mathcal{L}(v_n, y) + \frac{1}{2}\|\nu_n - y\|^2\right\}, \\ \vartheta_n = (1 - \xi_n)p_n + \xi_n \mathcal{S}(u_n), \\ u_{n+1} = (1 - \eta_n)\mathcal{T}\vartheta_n + \eta_n p_n, \end{cases} \tag{1.8}$$

where $\mathcal{X}_n = \{z \in \mathcal{X} : \langle \nu_n - \delta_n \mathcal{U}_n - v_n, z - v_n\rangle \leq 0\}$ and $\mathcal{U}_n \in \partial_2 \mathcal{L}(\nu_n, v_n)$. Define δ_{n+1} as follows:

$$\delta_{n+1} = \begin{cases} \min\left\{\delta_n, \frac{\mu\|\nu_n - v_n\|^2 + \mu\|v_n - p_n\|^2}{2[\mathcal{L}(\nu_n, p_n) - \mathcal{L}(\nu_n, v_n) - \mathcal{L}(v_n, p_n)]}\right\}, & \text{if } \mathcal{L}(\nu_n, p_n) - \mathcal{L}(\nu_n, v_n) - \mathcal{L}(v_n, p_n) > 0, \\ \delta_n, & \text{otherwise.} \end{cases} \tag{1.9}$$

In 2023, Panyanak et al. [25] introduced an algorithm designed to solve both an equilibrium problem and a fixed-point problem simultaneously. The algorithm uses a monotone step size rule that does not depend on Lipschitz constants. It incorporates a Mann-type iteration for the fixed-point problem and a two-step extragradient method for the equilibrium problem. Let a, b be selected such that $0 < a, b < 1 - \rho$ and $0 < a, b < 1 - \eta_k$. Given $u_0, u_1 \in \mathcal{X}$, compute the following sequence:

$$\begin{cases} \nu_n = u_n + \wp_n(u_n - u_{n-1}), \\ v_n = \arg\min_{y \in \mathcal{D}} \left\{\delta_n \mathcal{L}(\nu_n, y) + \frac{1}{2}\|\nu_n - y\|^2\right\}, \\ p_n = \arg\min_{y \in \mathcal{X}_n} \left\{\rho\delta_n \mathcal{L}(v_n, y) + \frac{1}{2}\|\nu_n - y\|^2\right\}, \\ u_{n+1} = (1 - \varrho_n - \eta_k)p_n + \varrho_n \mathcal{T}(p_n), \quad n \in \mathbb{N}, \end{cases}$$

where $\mathcal{X}_n = \{z \in \mathcal{X} : \langle \nu_n - \delta_n \mathcal{U}_n - v_n, z - v_n\rangle \leq 0\}$, $\mathcal{U}_n \in \partial_2 \mathcal{L}(\nu_n, v_n)$, and $\wp_n$, δ_n are defined as in expressions (1.7) and (1.9), respectively.

Inspired by the works of [22, 23, 25, 28, 29], we have developed a new algorithm that efficiently tackles equilibrium and fixed-point problems using a subgradient extragradient-type method with inertial effects. Our research offers several significant contributions:

(a) We introduce a novel inertial self-adaptive subgradient extragradient algorithm that operates without requiring prior knowledge of the Lipschitz constant. Unlike methods with fixed step sizes, our algorithm permits variable step sizes δ_n, without the necessity for $\lim_{n\to\infty} \delta_n = 0$ or $\sum_{n=1}^{\infty} \delta_n = 0$, thus enhancing its flexibility.

(b) The algorithm features an inertial term that substantially improves convergence rates. This distinguishes our method from those discussed in [22, 23, 25, 28, 29], as it eliminates the need to compute the norm difference between consecutive iterates for determining the inertial factor θ_n.

The paper is structured as follows: Section 2 reviews relevant preliminary results. Section 3 explores the convergence of the algorithms proposed in this study. Lastly, Section 4 provides a summary of the results.

2. Preliminaries

Let $\mathcal{D}$ be a nonempty, closed, convex subset of $\mathcal{X}$, where $\mathcal{X}$ is a real Hilbert space. We represent weak convergence by $u_n \rightharpoonup u$ and strong convergence by $u_n \to u$. For any $t_1, t_2 \in \mathcal{X}$ and $\phi \in \mathbb{R}$, the following properties hold:

(1) $\|t_1 + t_2\|^2 = \|t_1\|^2 + 2\langle t_1, t_2\rangle + \|t_2\|^2$,
(2) $\|t_1 + t_2\|^2 \leq \|t_1\|^2 + 2\langle t_2, t_1 + t_2\rangle$,
(3) $\|\phi t_1 + (1-\phi)t_2\|^2 = \phi\|t_1\|^2 + (1-\phi)\|t_2\|^2 - \phi(1-\phi)\|t_1 - t_2\|^2$.

Every point $u \in \mathcal{X}$ has a unique nearest point in $\mathcal{D}$, represented by $P_{\mathcal{D}}(u)$, such that

$$P_{\mathcal{D}}(u) = \arg\min_{v\in\mathcal{D}} \|u - v\|.$$

The metric projection of $\mathcal{X}$ onto $\mathcal{D}$ is denoted by $P_{\mathcal{D}}$. It fulfills the subsequent characteristic:

$$\langle u - P_{\mathcal{D}}(u), v - P_{\mathcal{D}}(u)\rangle \leq 0, \quad \forall v \in \mathcal{D}. \tag{2.1}$$

Definition 2.1. *Let $\mathcal{D}$ be a subset of a real Hilbert space $\mathcal{X}$, and let $\mathcal{J} : \mathcal{D} \to \mathbb{R}$ be a convex function.*

(1) *A normal cone at $t_1 \in \mathcal{D}$ is defined by $N_{\mathcal{D}}(t_1) = \{t_3 \in \mathcal{X} \mid \langle t_3, t_2 - t_1\rangle \leq 0$ for all $t_2 \in \mathcal{D}\}$.*
(2) *The subdifferential of $\mathcal{J}$ at $t_1 \in \mathcal{D}$ is defined by*

$$\partial\mathcal{J}(t_1) = \{t_3 \in \mathcal{X} \mid \mathcal{J}(t_2) - \mathcal{J}(t_1) \geq \langle t_3, t_2 - t_1\rangle \text{ for all } t_2 \in \mathcal{D}\}.$$

Definition 2.2. *[39] Let $\mathcal{L} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a bifunction defined on the set $\mathcal{D}$, and let $\gamma > 0$ be a positive constant. The bifunction $\mathcal{L}$ is said to be:*

(1) *strongly monotone if $\mathcal{L}(t_1, t_2) + \mathcal{L}(t_2, t_1) \leq -\gamma\|t_1 - t_2\|^2$ for all $t_1, t_2 \in \mathcal{D}$,*
(2) *monotone if $\mathcal{L}(t_1, t_2) + \mathcal{L}(t_2, t_1) \leq 0$ for all $t_1, t_2 \in \mathcal{D}$,*
(3) *strongly pseudomonotone if $\mathcal{L}(t_1, t_2) \geq 0 \implies \mathcal{L}(t_2, t_1) \leq -\gamma\|t_1 - t_2\|^2$ for all $t_1, t_2 \in \mathcal{D}$,*
(4) *pseudomonotone if $\mathcal{L}(t_1, t_2) \geq 0 \implies \mathcal{L}(t_2, t_1) \leq 0$ for all $t_1, t_2 \in \mathcal{D}$,*

Definition 2.3. *Let $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ be a mapping with $F(\mathcal{T}) \neq \emptyset$. The mapping $\mathcal{T}$ is classified as follows:*

(1) *nonexpansive if $\|\mathcal{T}(t_1) - \mathcal{T}(t_2)\| \leq \|t_1 - t_2\|$, $\forall\, t_1, t_2 \in \mathcal{D}$,*
(2) *quasi-nonexpansive if $\|\mathcal{T}(t_1) - t_3\| \leq \|t_1 - t_3\|$, $\forall\, t_3 \in F(\mathcal{T})$ and $t_1 \in \mathcal{D}$.*

Definition 2.4. *[40] If the following statement holds for $r_n \in \mathcal{X}$, we define a nonlinear operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ to have an empty fixed point set ($Fix(\mathcal{T}) \neq \emptyset$):*

$$\begin{cases} r_n \rightharpoonup r \\ (I - \mathcal{T})r_n \to 0, \end{cases} \tag{2.2}$$

which implies that $r \in Fix(\mathcal{T})$. Moreover, I stands for the identity operator. In these situations, we say that $I - \mathcal{T}$ is demiclosed at zero.

Lemma 2.1. [28] *Consider $\mathcal{J} : \mathcal{D} \to \mathbb{R}$ is a lower semicontinuous, convex and subdifferentiable function on $\mathcal{D}$. A minimizer of $\mathcal{J}$ is an element $t_1 \in \mathcal{D}$ if and only if*

$$0 \in \partial \mathcal{J}(t_1) + N_{\mathcal{D}}(t_1),$$

where $\partial \mathcal{J}(t_1)$ represents the subdifferential of $\mathcal{J}$ at t_1, and $N_{\mathcal{D}}(t_1)$ denotes the normal cone to $\mathcal{D}$ at t_1.

Lemma 2.2 ([24]). *Let $\{\xi_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \xi_n = \infty$, and let $\{b_n\}$ be a sequence of nonnegative real numbers. Take $\{c_n\}$ to be a real number sequence. Assume that for all $n \geq 1$,*

$$b_{n+1} \leq (1 - \xi_n) b_n + \xi_n c_n.$$

If every subsequence $\{b_{n_k}\}$ of $\{b_n\}$ satisfies

$$\liminf_{k\to\infty} (b_{n_k+1} - b_{n_k}) \geq 0,$$

then $\limsup_{k\to\infty} c_{n_k} \leq 0$, which implies $\lim_{n\to\infty} b_n = 0$.

Lemma 2.3. *[39] For all $u \in \mathcal{X}$, $v \in \mathcal{D}$, and $\delta > 0$, the following inequality holds:*

$$\delta \left[\mathcal{J}(v) - f(prox_{\delta\mathcal{J}}(u)) \right] \geq \langle u - prox_{\delta\mathcal{J}}(u), v - prox_{\delta\mathcal{J}}(u) \rangle.$$

3. Main Results

In this section, we introduce a new algorithm designed to find the common solution to the given problem. Our approach integrates both monotonic and non-monotonic step size strategies within an inertial subgradient extragradient framework. We begin by selecting two points, u_0 and u_1, from the set $\mathcal{D}$. The algorithm then generates a sequence of iterates $\{u_n\}$ according to the steps provided in Algorithm 3.1. To support the convergence analysis, we impose the following assumptions at the outset.

Assumption 3.1. *Let $\mathcal{X}$ be a real Hilbert space and $\mathcal{D}$ a nonempty, closed, and convex subset of $\mathcal{X}$. We adopt the following assumptions throughout this investigation:*

(c1) *The bifunction $\mathcal{L}$ is pseudomonotone and Lipschitz continuous on $\mathcal{D}$.*
(c2) *For each fixed $x \in \mathcal{X}$, the function $\mathcal{L}(x; \cdot)$ is convex and subdifferentiable on $\mathcal{X}$.*
(c3) *The operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is quasi-nonexpansive mapping.*
(c4) *The set $\Gamma = EP(\mathcal{L}, \mathcal{D}) \cap F(\mathcal{T})$ is nonempty.*
(c5) $\mu \in (0,1)$, $\theta \in \left(\frac{1-\mu}{4}, \frac{1-\mu}{2} \right)$, $\Phi \in (\mu, 1-2\theta)$, $0 < \tau_n < \min\left\{ \frac{1-\Phi-2\theta}{2\theta}, \frac{1-\theta}{1+\theta} \right\}$.

Algorithm 3.1. ***Initialization:*** *Let $u_0, u_1 \in \mathcal{X}$ be chosen arbitrarily, while $n \geq 1$.*

STEP 1: *Given the iterates u_{n-1} and u_n, for $n \geq 1$, compute*

$$t_n = \begin{cases} u_n, & n = even, \\ u_n + \theta(u_n - u_{n-1}), & n = odd. \end{cases} \tag{3.1}$$

STEP 2: *Compute*

$$\begin{cases} v_n = \arg\min_{y \in \mathcal{D}} \left\{ \delta_n \mathcal{L}(t_n, y) + \frac{1}{2}\|t_n - y\|^2 \right\} \\ z_n = \arg\min_{y \in \mathcal{X}_n} \left\{ \delta_n \mathcal{L}(v_n, y) + \frac{1}{2}\|t_n - y\|^2 \right\}, \\ u_{n+1} = (1 - \tau_n) z_n + \tau_n \mathcal{T} z_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.2}$$

where $\mathcal{X}_n = \{z \in \mathcal{X} : \langle t_n - \delta_n \mathcal{U}_n - v_n, z - v_n \rangle \leq 0\}$, and $\mathcal{U}_n \in \partial_2 \mathcal{L}(t_n, v_n)$.

STEP 3: *Update δ_{n+1} by*

$$\delta_{n+1} = \begin{cases} \min\left\{ \delta_n, \frac{\mu\|t_n - v_n\|^2 + \mu\|v_n - z_n\|^2}{2[\mathcal{L}(t_n, z_n) - \mathcal{L}(t_n, v_n) - \mathcal{L}(v_n, z_n)]} \right\} & \text{if } \mathcal{L}(t_n, z_n) - \mathcal{L}(t_n, v_n) - \mathcal{L}(v_n, z_n) > 0, \\ \delta_n, & \text{otherwise.} \end{cases} \tag{3.3}$$

Lemma 3.1. *The sequence $\{\delta_n\}$ produced by the update rule* (3.3) *in Algorithm 3.1 is non-increasing and satisfies*

$$\delta_1 \geq \delta_n \geq \min\left\{ \frac{\mu}{2\max\{c^1, c^2\}}, \delta_1 \right\}.$$

Proof. From equation (3.3), it is clear that the sequence $\{\delta_n\}$ is non-increasing. Furthermore, applying the Lipschitz condition to $\mathcal{L}$ yields the following inequality:

$$\begin{aligned} \frac{\mu}{2} \frac{\|v_n - y_n\|^2 + \|y_n - q_n\|^2}{\mathcal{L}(v_n, q_n) - \mathcal{L}(v_n, y_n) - \mathcal{L}(y_n, q_n)} &\geq \frac{\mu}{2} \frac{\|v_n - y_n\|^2 + \|y_n - q_n\|^2}{c^1\|v_n - y_n\|^2 + c^2\|q_n - y_n\|^2} \\ &\geq \frac{\mu}{2\max\{c^1, c^2\}}. \end{aligned}$$

Thus, the sequence $\{\delta_n\}$ is non-increasing and bounded below. Furthermore, we have $\lim_{n\to\infty} \delta_n = \delta$. ☐

The following lemma is essential for demonstrating the boundedness of an iterative sequence and plays a pivotal role in establishing the convergence of the sequence to a common solution.

Lemma 3.2. *Consider $\{z_n\}$, $\{t_n\}$, and $\{v_n\}$ be sequences generated by Algorithm 3.1. Then, the following inequality holds, for every $\hat{u} \in \Gamma$*

$$\|z_n - \hat{u}\|^2 \leq \|t_n - \hat{u}\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|t_n - v_n\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|z_n - v_n\|^2. \tag{3.4}$$

Proof. By applying Lemma 2.1, we have:

$$0 \in \partial_2\left(\delta_n \mathcal{L}(v_n, \cdot) + \frac{1}{2}\|t_n - \cdot\|^2\right)(z_n) + N_{\mathcal{X}_n}(z_n),$$

which implies there exist $w \in \partial \mathcal{L}(v_n, z_n)$ and $\bar{w} \in N_{\mathcal{X}_n}(z_n)$ such that:

$$\delta_n w + z_n - t_n + \bar{w} = 0.$$

Hence, we have:

$$\langle t_n - z_n, y - z_n\rangle = \delta_n\langle w, y - z_n\rangle + \langle \bar{w}, y - z_n\rangle, \quad \forall y \in \mathcal{X}_n.$$

Since $\bar{w} \in N_{\mathcal{X}_n}(z_n)$, for every $y \in \mathcal{X}_n$, $\langle \bar{w}, y - z_n\rangle \le 0$. Therefore:

$$\delta_n\langle w, y - z_n\rangle \ge \langle t_n - z_n, y - z_n\rangle, \quad \forall y \in \mathcal{X}_n. \tag{3.5}$$

Given $w \in \partial\mathcal{L}(v_n, z_n)$, and using the definition of the subdifferential, we obtain:

$$\mathcal{L}(v_n, y) - \mathcal{L}(v_n, z_n) \ge \langle w, y - z_n\rangle, \quad \forall y \in \mathcal{X}. \tag{3.6}$$

When we combine the inequalities (3.5) and (3.6), we get:

$$\delta_n\left(\mathcal{L}(v_n, y) - \mathcal{L}(v_n, z_n)\right) \ge \langle t_n - z_n, y - z_n\rangle, \quad \forall y \in \mathcal{X}_n. \tag{3.7}$$

By changing y with $\hat{u}$ into inequality (3.7), we acquire:

$$\delta_n\left(\mathcal{L}(v_n, \hat{u}) - \mathcal{L}(v_n, z_n)\right) \ge \langle t_n - z_n, \hat{u} - z_n\rangle. \tag{3.8}$$

Given that $\hat{u} \in EP(\mathcal{L}, \mathcal{D})$, we have $\mathcal{L}(\hat{u}, v_n) \ge 0$. By the pseudomonotonicity of the bifunction $\mathcal{L}$, we deduce that $\mathcal{L}(v_n, \hat{u}) \le 0$. Therefore, from inequality (3.8), we conclude:

$$\langle t_n - z_n, z_n - \hat{u}\rangle \ge \delta_n\mathcal{L}(v_n, z_n). \tag{3.9}$$

From the definition of the half-space $\mathcal{X}_n$, we derive:

$$\delta_n\langle \mathcal{U}_n, z_n - v_n\rangle \ge \langle t_n - v_n, z_n - v_n\rangle. \tag{3.10}$$

Since $\mathcal{U}_n \in \partial\mathcal{L}(t_n, v_n)$, we attain:

$$\mathcal{L}(t_n, y) - \mathcal{L}(t_n, v_n) \ge \langle \mathcal{U}_n, y - v_n\rangle, \quad \forall y \in \mathcal{X}. \tag{3.11}$$

If we replace $y = z_n$ in inequality (3.11), we obtain:

$$\mathcal{L}(t_n, z_n) - \mathcal{L}(t_n, v_n) \ge \langle \mathcal{U}_n, z_n - v_n\rangle. \tag{3.12}$$

Combining (3.10) and (3.12), we achieve:

$$\delta_n\left(\mathcal{L}(t_n, z_n) - \mathcal{L}(t_n, v_n)\right) \ge \langle t_n - v_n, z_n - v_n\rangle. \tag{3.13}$$

Starting from the definition of δ_{n+1}, we derive the following inequality:

$$\delta_{n+1}\left(\mathcal{L}(t_n, z_n) - \mathcal{L}(t_n, v_n) - \mathcal{L}(v_n, z_n)\right) \le \frac{\mu}{2}\left(\|t_n - v_n\|^2 + \|z_n - v_n\|^2\right), \tag{3.14}$$

which can be equivalently expressed as:

$$\delta_n\left(\mathcal{L}(t_n, z_n) - \mathcal{L}(t_n, v_n) - \mathcal{L}(v_n, z_n)\right) \le \frac{\delta_n}{\delta_{n+1}}\frac{\mu}{2}\left(\|t_n - v_n\|^2 + \|z_n - v_n\|^2\right). \tag{3.15}$$

By substituting (3.15) into (3.13), we obtain:

$$\langle t_n - v_n, z_n - v_n\rangle \le \delta_n\mathcal{L}(v_n, z_n) + \frac{\delta_n}{\delta_{n+1}}\frac{\mu}{2}\left(\|t_n - v_n\|^2 + \|z_n - v_n\|^2\right). \tag{3.16}$$

Summing (3.9) and (3.16), we get:

$$\langle t_n - v_n, z_n - v_n \rangle \le \langle t_n - z_n, z_n - \hat{u} \rangle + \frac{\delta_n}{\delta_{n+1}} \frac{\mu}{2} \left(\|t_n - v_n\|^2 + \|z_n - v_n\|^2 \right). \tag{3.17}$$

Additionally, the following equality exist:

$$2\langle t_n - z_n, z_n - \hat{u} \rangle = \|t_n - \hat{u}\|^2 - \|z_n - t_n\|^2 - \|z_n - \hat{u}\|^2, \tag{3.18}$$

and

$$2\langle t_n - v_n, z_n - v_n \rangle = \|t_n - v_n\|^2 + \|z_n - v_n\|^2 - \|t_n - z_n\|^2. \tag{3.19}$$

By substituting expressions (3.18) and (3.19) into (3.17), we deduce:

$$\begin{aligned} &\|t_n - v_n\|^2 + \|z_n - v_n\|^2 - \|t_n - z_n\|^2 \\ &\quad \le \left(\|t_n - \hat{u}\|^2 - \|z_n - t_n\|^2 - \|z_n - \hat{u}\|^2 \right) \\ &\qquad + \frac{\delta_n}{\delta_{n+1}} \mu \left(\|t_n - v_n\|^2 + \|z_n - v_n\|^2 \right), \end{aligned}$$

which is equivalent to:

$$\|z_n - \hat{u}\|^2 \le \|t_n - \hat{u}\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right) \|t_n - v_n\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right) \|z_n - v_n\|^2. \tag{3.20}$$

□

The key lemma employed to demonstrate the boundedness of an iterative sequence is given below. This theorem establishes the boundedness of the sequence.

Lemma 3.3. *Assuming the validity of Assumption 3.1, let $\{u_n\}$ be a sequence produced by Algorithm 3.1. Then, $\{u_n\}$ is a bounded sequence.*

Proof. Recall that δ_n is monotonically decreasing. Based on Lemma 3.1, $\{\delta_n\}$ is convergent, with a limit of $\delta \ge \min\left\{\frac{\mu}{2\max\{c^1, c^2\}}, \delta_1\right\} > 0$. Since $\{\delta_n\}$ is monotonically decreasing, we have $\delta_n \ge \delta\ \forall n \ge 1$. Let Φ be a fixed number in the interval $(\mu, 1 - 2\theta)$. Moreover, since $\lim_{n\to\infty} \dfrac{\delta_n \mu}{\delta_{n+1}} = \mu < \Phi$, there exists a natural number N, such that $\dfrac{\delta_n \mu}{\delta_{n+1}} < \Phi, \forall n \ge N$. For any $n \ge \mathbb{N}$, we thus have

$$\delta_n \ge \delta,\ \frac{\delta_n \mu}{\delta_{n+1}} < \Phi. \tag{3.21}$$

Plugging (3.21) in (3.20), we have for all $n \ge N$

$$\|z_n - \hat{u}\|^2 \le \|t_n - \hat{u}\|^2 - (1 - \Phi) \|t_n - v_n\|^2 - (1 - \Phi) \|z_n - v_n\|^2. \tag{3.22}$$

Definition of $\{u_{n+1}\}$ and (3.22) implies that

$$\begin{aligned}\|u_{n+1}-\hat{u}\|^2 &= \|(1-\tau_n)z_n+\tau_n\mathcal{T}z_n-\hat{u}\|^2\\ &= \|\tau_n(\mathcal{T}z_n-\hat{u})+(1-\tau_n)(z_n-\hat{u})\|^2\\ &= \tau_n\|\mathcal{T}z_n-\hat{u}\|^2+(1-\tau_n)\|(z_n-\hat{u})\|^2-\tau_n(1-\tau_n)\|\mathcal{T}z_n-z_n\|^2\\ &\le \tau_n\|z_n-\hat{u}\|^2+(1-\tau_n)\|(z_n-\hat{u})\|^2-\tau_n(1-\tau_n)\|\mathcal{T}z_n-z_n\|^2\\ &= \|z_n-\hat{u}\|^2-\tau_n(1-\tau_n)\|\mathcal{T}z_n-z_n\|^2\end{aligned} \tag{3.23}$$

In particular,

$$\begin{aligned}\|u_{2n+2}-\hat{u}\|^2 \le{}& \|t_{2n+1}-\hat{u}\|^2-(1-\Phi)\,\|t_{2n+1}-v_{2n+1}\|^2\\ &-(1-\Phi)\,\|z_{2n+1}-v_{2n+1}\|^2-\tau_{2n+1}(1-\tau_{2n+1})\|\mathcal{T}z_{2n+1}-z_{2n+1}\|^2\end{aligned} \tag{3.24}$$

Equation (3.1) provides us the following:

$$\begin{aligned}\|t_{2n+1}-\hat{u}\|^2 &= \|(u_{2n+1}+\theta(u_{2n+1}-u_{2n}))-\hat{u}\|^2\\ &\le (1+\theta)\|u_{2n+1}-\hat{u}\|^2-\theta\|u_{2n}-\hat{u}\|^2+\theta(1+\theta)\|u_{2n+1}-u_{2n}\|^2.\end{aligned} \tag{3.25}$$

Using (3.23) as an additional special case, we have:

$$\begin{aligned}\|u_{2n+1}-\hat{u}\|^2 \le{}& \|u_{2n}-\hat{u}\|^2-(1-\Phi)\,\|u_{2n}-v_{2n}\|^2-(1-\Phi)\,\|z_{2n}-v_{2n}\|^2\\ &-\tau_{2n}(1-\tau_{2n})\|\mathcal{T}z_{2n}-z_{2n}\|^2\\ \le{}& \|u_{2n}-\hat{u}\|^2-\frac{1-\Phi}{2}\|u_{2n}-z_{2n}\|^2-\tau_{2n}(1-\tau_{2n})\|\mathcal{T}z_{2n}-z_{2n}\|^2.\end{aligned} \tag{3.26}$$

By putting (3.26) into (3.25), we obtain:

$$\begin{aligned}\|t_{2n+1}-\hat{u}\|^2 \le{}& \|u_{2n}-\hat{u}\|^2-\frac{(1+\theta)\,(1-\Phi)}{2}\|u_{2n}-v_{2n}\|^2\\ &-\tau_{2n}(1-\tau_{2n})\,(1+\theta)\,\|\mathcal{T}z_{2n}-z_{2n}\|^2+\theta(1+\theta)\|u_{2n+1}-u_{2n}\|^2.\end{aligned} \tag{3.27}$$

By putting (3.27) into (3.24), we attain:

$$\begin{aligned}\|u_{2n+2}-\hat{u}\|^2 \le{}& \|u_{2n}-\hat{u}\|^2-\frac{(1+\theta)\,(1-\Phi)}{2}\|u_{2n}-v_{2n}\|^2\\ &-\tau_{2n}(1-\tau_{2n})\,(1+\theta)\,\|\mathcal{T}z_{2n}-z_{2n}\|^2+\theta(1+\theta)\|u_{2n+1}-u_{2n}\|^2\\ &-(1-\Phi)\,\|t_{2n+1}-v_{2n+1}\|^2-(1-\Phi)\,\|z_{2n+1}-v_{2n+1}\|^2\\ &-\tau_{2n+1}(1-\tau_{2n+1})\|\mathcal{T}z_{2n+1}-z_{2n+1}\|^2.\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}\|u_{2n+1}-u_{2n}\|^2 &= \|(1-\tau_{2n})z_{2n}+\tau_{2n}\mathcal{T}z_{2n}-u_{2n}\|^2\\ &= \|z_{2n}-u_{2n}+\tau_{2n}(\mathcal{T}z_{2n}-z_{2n})\|^2\\ &= \|z_{2n}-u_{2n}\|^2+\tau_{2n}^2\|\mathcal{T}z_{2n}-z_{2n}\|^2+2\tau_{2n}\langle z_{2n}-u_{2n},\mathcal{T}z_{2n}-u_{2n}\rangle\\ &\le \|z_{2n}-u_{2n}\|^2+\tau_{2n}^2\|\mathcal{T}z_{2n}-z_{2n}\|^2+\tau_{2n}(\|z_{2n}-u_{2n}\|^2+\|\mathcal{T}z_{2n}-z_{2n}\|^2)\\ &= (1+\tau_{2n})\|z_{2n}-u_{2n}\|^2+\tau_{2n}(\tau_{2n}+1)\|\mathcal{T}z_{2n}-z_{2n}\|^2.\end{aligned} \tag{3.29}$$

Combining (3.28) with last inequality yields:

$$\begin{aligned}\|u_{2n+2}-\hat{u}\|^2 \le \|u_{2n}-\hat{u}\|^2 &- \Big[\frac{(1+\theta)\,(1-\Phi)}{2} - \theta(1+\theta)(1+\tau_{2n})\Big]\|z_{2n}-u_{2n}\|^2 \\ &- \Big[\tau_{2n}(1-\tau_{2n})(1+\theta) - \theta(1+\theta)\tau_{2n}(\tau_{2n}+1)\Big]\|\mathcal{T}z_{2n}-z_{2n}\|^2 \\ &- (1-\Phi)\,\|t_{2n+1}-v_{2n+1}\|^2 - (1-\Phi)\,\|z_{2n+1}-v_{2n+1}\|^2 \\ &- \tau_{2n+1}(1-\tau_{2n+1})\|\mathcal{T}z_{2n+1}-z_{2n+1}\|^2.\end{aligned} \tag{3.30}$$

By using the conditions on τ_n and θ, we get:

$$\|u_{2n+2}-\hat{u}\|^2 \le \|u_{2n}-\hat{u}\|^2. \tag{3.31}$$

Thus, by (3.31), we conclude that the sequence $\{u_{2n}\}$ is bounded, which completes the proof of Lemma 3.3. □

The main result of our work is the following theorem, which demonstrates that the sequence obtained through the proposed method weakly converges to a unique solution of the problem (1.3). Now let us prove the main theorem of this paper.

Theorem 3.1. *Let $\mathcal{D}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{X}$. Assume that $\mathcal{L}$ is a pseudomonotone Lipschitz continuous bifunction and that $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ is a quasi-nonexpansive mapping. If Assumption 3.1 is satisfied, then the sequence $\{u_{2n}\}$ converges weakly to the solution set.*

Proof. Since $\{u_{2n}\}$ is a bounded sequence, there exists a subsequence $\{u_{2n_k}\}$ such that $u_{2n_k} \rightharpoonup \hat{u}$. Now, we have to show that $\hat{u}\in\Gamma$. We can infer from (3.30) and the convergence of $\{\|u_{2n}-\bar{u}\|\}$ that:

$$\begin{aligned}&\|u_{2n}-\hat{u}\|\to 0,\ \|u_{2n}-v_{2n}\|\to 0,\\ &\|\mathcal{T}z_{2n}-u_{2n}\|\to 0,\ \|z_{2n}-v_{2n}\|\to 0,\\ &\|\mathcal{T}z_{2n+1}-t_{2n+1}\|\to 0,\ \|z_{2n+1}-v_{2n+1}\|\to 0,\ \textit{as } n\to\infty.\end{aligned} \tag{3.32}$$

By (3.32), we obtain:

$$\|z_{2n}-u_n\| \le \|z_{2n}-v_{2n}\| + \|v_{2n}-u_{2n}\| \to 0,\ \textit{as } n\to\infty \tag{3.33}$$

Furthermore, (3.32) and (3.33) make us to reach:

$$\begin{aligned}\|\mathcal{T}z_{2n}-z_{2n}\| &= \|\mathcal{T}z_{2n}-u_{2n}+u_{2n}-z_{2n}\| \\ &\le \|\mathcal{T}z_{2n}-u_{2n}\| + \|u_{2n}-z_{2n}\| \to 0,\ \textit{as } n\to\infty,\end{aligned} \tag{3.34}$$

Using (3.33) as well as $u_{2n_k}\rightharpoonup\hat{u}$, we can obtain:

$$z_{2n_k}\rightharpoonup\hat{u}. \tag{3.35}$$

Given the demicloseness of $\mathcal{T}$ at zero, the definitions of 3.34, (3.34) and (3.35) suggest:

$$\hat{u}\in Fix(\mathcal{T}). \tag{3.36}$$

Moreover, we show that $u \in \mathrm{EP}(\mathcal{L}, \mathcal{X})$. From equations (3.8) and (3.14), we obtain:

$$\begin{aligned}\delta_{2n_k}\mathcal{L}(v_{2n_k}, \hat{u}) \geq& \delta_{2n_k}\mathcal{L}(v_{2n_k}, z_{2n_k}) + \langle t_{2n_k} - z_{2n_k}, \hat{u} - z_{2n_k}\rangle \\ \geq& \langle t_{2n_k} - v_{2n_k}, z_{2n_k} - v_{2n_k}\rangle + \langle t_{2n_k} - z_{2n_k}, \hat{u} - z_{2n_k}\rangle \\ &- \frac{\delta_{2n_k}}{\delta_{2n_k+1}}\frac{\mu}{2}\left(\|t_{2n_k} - v_{2n_k}\|^2 + \|z_{2n_k} - v_{2n_k}\|^2\right).\end{aligned} \tag{3.37}$$

Using (3.32) and $t_{2n_k} = u_{2n_k}$ and letting $k \to +\infty$ in (3.37), we get that the right hand side of (3.37) tends to 0. Furthermore, $v_{2n_k} \rightharpoonup \hat{u}$ by (3.32) and $\lim_{k\to\infty} \delta_{2n_k} = \delta > 0$ together implies that:

$$0 \leq \limsup_{k\to\infty} \mathcal{L}(v_{2n_k}, y) = \mathcal{L}(\hat{u}, y), \quad \forall y \in \mathcal{X}.$$

Thus, $\hat{u} \in \mathrm{EP}(\mathcal{L}, \mathcal{X})$. Hence, $\hat{u} \in \Gamma$. □

4. Conclusion

In summary, the proposed study introduces methods that are particularly relevant for solving equilibrium problems involving a pseudomonotone and Lipschitz-type bifunction in a real Hilbert space. The algorithm utilizes variable step sizes, which are updated at each iteration based on previous iterates. A notable advantage of these algorithms is their independence from prior knowledge of Lipschitz-type constants and the elimination of the need for a line search procedure.

Author Contributions: All authors contributed equally to this work.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] G. Long, S. Liu, G. Xu, S.W. Wong, H. Chen, et al., A Perforation-Erosion Model for Hydraulic-Fracturing Applications, SPE Prod. Oper. 33 (2018), 770–783. https://doi.org/10.2118/174959-PA.

[2] B. Xiao, W. Wang, X. Zhang, G. Long, J. Fan, et al., A Novel Fractal Solution for Permeability and Kozeny-Carman Constant of Fibrous Porous Media Made up of Solid Particles and Porous Fibers, Powder Technol. 349 (2019), 92–98. https://doi.org/10.1016/j.powtec.2019.03.028.

[3] B. Xiao, X. Zhang, G. Jiang, G. Long, W. Wang, et al., Kozeny–Carman Constant for Gas Flow through Fibrous Porous Media by Fractal-Monte Carlo Simulations, Fractals 27 (2019), 1950062. https://doi.org/10.1142/S0218348X19500622.

[4] V. Dadashi, O.S. Iyiola, Y. Shehu, The Subgradient Extragradient Method for Pseudomonotone Equilibrium Problems, Optimization 69 (2019), 901–923. https://doi.org/10.1080/02331934.2019.1625899.

[5] D. Van Hieu, A. Gibali, Strong Convergence of Inertial Algorithms for Solving Equilibrium Problems, Optim. Lett. 14 (2019), 1817–1843. https://doi.org/10.1007/s11590-019-01479-w.

[6] D. Van Hieu, P.K. Quy, L. Van Vy, Explicit Iterative Algorithms for Solving Equilibrium Problems, Calcolo 56 (2019), 11. https://doi.org/10.1007/s10092-019-0308-5.

[7] Y. Shehu, O.S. Iyiola, D.V. Thong, N.T.C. Van, An Inertial Subgradient Extragradient Algorithm Extended to Pseudomonotone Equilibrium Problems, Math. Methods Oper. Res. 93 (2020), 213–242. https://doi.org/10.1007/s00186-020-00730-w.

[8] D.V. Thong, P. Cholamjiak, M.T. Rassias, Y.J. Cho, Strong Convergence of Inertial Subgradient Extragradient Algorithm for Solving Pseudomonotone Equilibrium Problems, Optim. Lett. 16 (2021), 545–573. https://doi.org/10.1007/s11590-021-01734-z.

[9] Y. Censor, A. Gibali, S. Reich, The Subgradient Extragradient Method for Solving Variational Inequalities in Hilbert Space, J. Optim. Theory Appl. 148 (2010), 318–335. https://doi.org/10.1007/s10957-010-9757-3.

[10] Y. Censor, A. Gibali, S. Reich, Strong Convergence of Subgradient Extragradient Methods for the Variational Inequality Problem in Hilbert Space, Optim. Methods Softw. 26 (2011), 827–845. https://doi.org/10.1080/10556788.2010.551536.

[11] K.J. Arrow, G. Debreu, Existence of an Equilibrium for a Competitive Economy, Econometrica 22 (1954), 265. https://doi.org/10.2307/1907353.

[12] A.A. Cournot, Recherches sur les Principes Mathématiques de la Théorie des Richesses, L. Hachette, New York, (1838).

[13] S. Wernecke, M. D'Addario, Maximum Entropy Image Reconstruction, IEEE Trans. Comput. C-26 (1977), 351–364. https://doi.org/10.1109/TC.1977.1674845.

[14] D. Kundur, D. Hatzinakos, Blind Image Deconvolution, IEEE Signal Process. Mag. 13 (1996), 43–64. https://doi.org/10.1109/79.489268.

[15] S. Gharehkhani, E. Sadeghinezhad, S.N. Kazi, H. Yarmand, A. Badarudin, et al., Basic Effects of Pulp Refining on Fiber Properties—A Review, Carbohydr. Polym. 115 (2015), 785–803. https://doi.org/10.1016/j.carbpol.2014.08.047.

[16] A. Nagurney, Network Economics: A Variational Inequality Approach, Kluwer Academic Publishers, 1993.

[17] M. Patriksson, Traffic Assignment Problem: Models and Methods, Dover Publications, 2015.

[18] J. Nash, Non-Cooperative Games, Ann. Math. 54 (1951), 286. https://doi.org/10.2307/1969529.

[19] K. Fan, A Minimax Inequality and Applications, in: O. Shisha, (ed) Inequalities III, Academic Press, New York, (1972).

[20] H. Nikaidô, K. Isoda, Note on Non-Cooperative Convex Game, Pac. J. Math. 5 (1955), 807–815. https://doi.org/10.2140/pjm.1955.5.807.

[21] J. Gwinner, On the Penalty Method for Constrained Variational Inequalities, in: J.-B. Hiriart-Urruty, W. Oettli, J. Stoer, (eds) Optimization - Theory and Algorithms, Marcel Dekker, New York, pp. 197–211, (1981).

[22] P.T. Vuong, J.J. Strodiot, V.H. Nguyen, On Extragradient-Viscosity Methods for Solving Equilibrium and Fixed Point Problems in a Hilbert Space, Optimization 64 (2013), 429–451. https://doi.org/10.1080/02331934.2012.759327.

[23] D. Van Hieu, Halpern Subgradient Extragradient Method Extended to Equilibrium Problems, Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 111 (2016), 823–840. https://doi.org/10.1007/s13398-016-0328-9.

[24] S. Saejung, P. Yotkaew, Approximation of Zeros of Inverse Strongly Monotone Operators in Banach Spaces, Nonlinear Anal. Theory Methods Appl. 75 (2012), 742–750. https://doi.org/10.1016/j.na.2011.09.005.

[25] B. Panyanak, C. Khunpanuk, N. Pholasa, N. Pakkaranang, Dynamical Inertial Extragradient Techniques for Solving Equilibrium and Fixed-Point Problems in Real Hilbert Spaces, J. Inequal. Appl. 2023 (2023), 7. https://doi.org/10.1186/s13660-023-02912-6.

[26] D. Quoc Tran, M. Le Dung, V.H. Nguyen, Extragradient Algorithms Extended to Equilibrium Problems, Optimization 57 (2008), 749–776. https://doi.org/10.1080/02331930601122876.

[27] P.T. Harker, J.S. Pang, A Damped-Newton Method for the Linear Complementarity Problems, in: E.L. Allgower, K. Georg, (eds) Computational Solution of Nonlinear Systems of Equations, Lectures in Applied Mathematics, Vol 26, American Mathematical Society, Providence, RI, pp. 265–284, (1990).

[28] D.V. Hieu, Strong Convergence of a New Hybrid Algorithm for Fixed Point Problems and Equilibrium Problems, Math. Model. Anal. 24 (2018), 1–19. https://doi.org/10.3846/mma.2019.001.

[29] B. Tan, G.Y. Cho, J.C. Yao, Accelerated Inertial Subgradient Extragradient Algorithms with Non-monotonic Step Sizes for Equilibrium Problems and Fixed Point Problems, J. Nonlinear Var. Anal. 6 (2022), 89–122. https://doi.org/10.23952/jnva.6.2022.1.06.

[30] X. Qin, N.T. An, Smoothing Algorithms for Computing the Projection onto a Minkowski Sum of Convex Sets, Comput. Optim. Appl. 74 (2019), 821–850. https://doi.org/10.1007/s10589-019-00124-7.

[31] N.T. An, N.M. Nam, X. Qin, Solving k-Center Problems Involving Sets Based on Optimization Techniques, J. Glob. Optim. 76 (2019), 189–209. https://doi.org/10.1007/s10898-019-00834-6.

[32] D.R. Sahu, J.C. Yao, M. Verma, K.K. Shukla, Convergence Rate Analysis of Proximal Gradient Methods with Applications to Composite Minimization Problems, Optimization 70 (2020), 75–100. https://doi.org/10.1080/02331934.2019.1702040.

[33] H. Iiduka, I. Yamada, A Subgradient-Type Method for the Equilibrium Problem over the Fixed Point Set and Its Applications, Optimization 58 (2009), 251–261. https://doi.org/10.1080/02331930701762829.

[34] P.-E. Maingé, A Hybrid Extragradient-Viscosity Method for Monotone Operators and Fixed Point Problems, SIAM J. Control Optim. 47 (2008), 1499–1515. https://doi.org/10.1137/060675319.

[35] P.E. Mainge, A. Moudafi, Coupling Viscosity Methods with the Extragradient Algorithm for Solving Equilibrium Problems, J. Nonlinear Convex Anal. 9 (2008), 283–294.

[36] S.D. Flåm, A.S. Antipin, Equilibrium Programming Using Proximal-like Algorithms, Math. Program. 78 (1996), 29–41. https://doi.org/10.1007/BF02614504.

[37] D. Quoc Tran, M. Le Dung, V.H. Nguyen, Extragradient Algorithms Extended to Equilibrium Problems, Optimization 57 (2008), 749–776. https://doi.org/10.1080/02331930601122876.

[38] G.M. Korpelevich, An Extragradient Method for Finding Saddle Points and Other Problems, Ekon Mat. Metody 12 (1976), 747–756. https://cir.nii.ac.jp/crid/1571698600143951616.

[39] M. Bianchi, S. Schaible, Generalized Monotone Bifunctions and Equilibrium Problems, J. Optim. Theory Appl. 90 (1996), 31–43. https://doi.org/10.1007/bf02192244.

[40] H. Xu, Viscosity Approximation Methods for Nonexpansive Mappings, J. Math. Anal. Appl. 298 (2004), 279–291. https://doi.org/10.1016/j.jmaa.2004.04.059.