

Mathematical Analysis of a Fractional SEIQHRV COVID-19 Model with Optimal Control and Public Health Interventions

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Abstract. This paper develops a seven-compartment fractional-order SEIQHRV model for COVID-19 transmission, with susceptible, exposed, infected, quarantined, hospitalized, recovered, and vaccinated classes and Caputo derivatives capturing memory effects. Three controls representing vaccination, quarantine-isolation, and clinical treatment are incorporated. Well-posedness, positivity, and boundedness are proved; the reproduction number, its sensitivity indices, a controlled threshold, and the endemic equilibrium are derived explicitly; and existence of optimal controls and the fractional Pontryagin Maximum Principle with exact costate equations are established. Simulations by a forward-backward sweep with an Adams-type scheme, verified by convergence and gradient tests, show that combined control is most effective, that vaccination and quarantine are complementary, and that smaller fractional orders yield flatter but more persistent epidemics.

1. INTRODUCTION

The emergence of SARS-CoV-2, the etiological agent of COVID-19, in late 2019 initiated one of the most consequential infectious disease events in modern history. The rapid global spread of the virus exposed fundamental gaps in pandemic preparedness and challenged healthcare systems worldwide [22,23]. Against this backdrop, mathematical modeling has served as an indispensable

Received: May 12, 2026.

2020 *Mathematics Subject Classification.* 34A08; 49J15; 49K15; 92D30.

Key words and phrases. COVID-19; SEIQHRV fractional model; Caputo derivative; vaccination; quarantine; optimal control; Pontryagin maximum principle.

tool for projecting epidemic trajectories, quantifying the impact of non-pharmaceutical interventions, and guiding vaccination rollout strategies [5, 17].

Classical compartmental models were among the first to be adapted for COVID-19 analysis. Standard susceptible-exposed-infected-recovered (SEIR) frameworks were extended to incorporate asymptomatic transmission [12], age structure [11], and spatial heterogeneity. Giordano et al. [11] introduced the SIDARTHE model with eight compartments distinguishing clinical severity and detection status, demonstrating that contact reduction measures could meaningfully alter the epidemic trajectory in Italy. Khan and Atangana [13] analyzed a fractional COVID-19 model and found that the fractional order significantly influenced the predicted epidemic dynamics.

Vaccination and quarantine are the two principal intervention pillars in COVID-19 control, yet their combined optimization within a fractional framework has not been fully addressed. Most existing fractional COVID-19 models either incorporate vaccination without quarantine [7, 20] or include quarantine without vaccination [21]. Furthermore, the separate tracking of hospitalized individuals as a distinct compartment, which is essential for healthcare capacity planning, is absent from many formulations. The present work fills this gap by developing a seven-compartment SEIQHRV model that explicitly includes quarantine (Q), hospitalization (H), and vaccination (V) compartments alongside the standard S, E, I, and R classes.

The use of fractional calculus in epidemic modeling is motivated by the observation that biological processes such as immune response, viral replication, and recovery from severe illness often exhibit non-Markovian dynamics in which the future state depends on the entire past history, not merely the current state [14, 18]. The Caputo fractional derivative, which requires only standard initial conditions and reduces to the classical integer-order derivative at $\alpha = 1$, is the most natural choice for such applications [4]. Fractional models have been shown to produce slower epidemic decay, higher peak infections, and modified stability properties relative to integer-order counterparts [6, 17, 20].

Optimal control theory provides a rigorous mathematical framework for designing intervention schedules that minimize disease burden subject to resource constraints [15, 19]. For COVID-19, optimal control analyses have evaluated the timing and intensity of lockdowns [11], social distancing mandates, testing campaigns, and vaccination programs. Extending these analyses to the fractional-order setting requires the fractional Pontryagin Maximum Principle [1, 3], which yields analogous optimality conditions involving right-sided Riemann-Liouville adjoints.

The present contribution makes the following advances. First, the fractional SEIQHRV model is formulated using Caputo derivatives, incorporating seven epidemiological states and three control inputs. Second, mathematical well-posedness is established through positivity and boundedness proofs. Third, $\mathcal{R}_0$ is derived analytically via the next-generation matrix, and the effect of vaccination on the reproductive threshold is quantified. Fourth, existence of optimal controls is proved. Fifth, necessary optimality conditions are derived and the control characterizations are expressed in closed form. Sixth, numerical simulations for multiple fractional orders demonstrate

the quantitative superiority of combined interventions and the influence of memory on epidemic persistence.

The paper is structured as follows. Section 2 reviews the fractional calculus background. Section 3 defines all notation. Section 4 presents the SEIQHRV model. Sections 5 and 6 address well-posedness and the reproduction number. Sections 7 and 8 cover the optimal control formulation and optimality conditions. Section 9 describes the numerical algorithm. Sections 10 and 11 present and discuss the simulation results. Section 12 provides the conclusion.

2. FRACTIONAL CALCULUS PRELIMINARIES

This section records the fractional calculus definitions and results needed throughout the paper, together with short proofs of the specific facts that are used in the analysis. For a comprehensive treatment see [9, 14, 18].

2.1. Riemann-Liouville Fractional Integral.

Definition 2.1. For $f \in L^1([0, T])$ and $\alpha > 0$, the Riemann-Liouville fractional integral of order α is

$$(I_t^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0. \quad (2.1)$$

2.2. Caputo Fractional Derivative.

Definition 2.2 ([4]). For $f \in C^1([0, T])$ and $\alpha \in (0, 1]$, the Caputo fractional derivative of order α is

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds = (I_t^{1-\alpha} f')(t). \quad (2.2)$$

Key properties:

- (1) **Linearity:** ${}^C D_t^\alpha (af + bg) = a {}^C D_t^\alpha f + b {}^C D_t^\alpha g$.
- (2) **Constant:** ${}^C D_t^\alpha c = 0$ for any constant c .
- (3) **Reduction:** When $\alpha = 1$, ${}^C D_t^1 f = f'$.
- (4) **Mittag-Leffler:** The equation ${}^C D_t^\alpha x = \lambda x$ has the solution $x(t) = x_0 E_\alpha(\lambda t^\alpha)$ where $E_\alpha(z) = \sum_{k=0}^{\infty} z^k / \Gamma(\alpha k + 1)$ (Lemma 2.3).
- (5) **Extremum Principle [16]:** If x attains a global minimum (resp. maximum) at t^* , then ${}^C D_t^\alpha x(t^*) \leq 0$ (resp. ≥ 0) (Lemma 2.4).

2.3. Elementary Calculations.

Lemma 2.1 (Power functions). For $p > -1$ and $\alpha \in (0, 1)$,

$$I_t^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)} t^{p+\alpha}, \quad {}^C D_t^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} t^{p-\alpha} \quad (p \geq 1). \quad (2.3)$$

Proof. Substituting $s = tr$ in (2.1) gives $I_t^\alpha t^p = \frac{t^{p+\alpha}}{\Gamma(\alpha)} \int_0^1 (1-r)^{\alpha-1} r^p dr = \frac{t^{p+\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha)\Gamma(p+1)}{\Gamma(p+\alpha+1)}$, using the Beta integral. For the Caputo derivative, $f'(s) = p s^{p-1}$, so by (2.2) and the first formula with $p-1$ in place of p and $1-\alpha$ in place of α , ${}^C D_t^\alpha t^p = p \frac{\Gamma(p)}{\Gamma(p+1-\alpha)} t^{p-\alpha} = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} t^{p-\alpha}$. $\square$

In particular ${}^C D_t^\alpha t = t^{1-\alpha}/\Gamma(2-\alpha)$, which shows that a function that grows linearly has a Caputo derivative that is *unbounded* as $t \rightarrow 0^+$ for $\alpha < 1$; this is the origin of the non-local, history-dependent behavior exploited in this paper.

Lemma 2.2 (Volterra form and Laplace transform). *Let $x \in C([0, T])$ with $x' \in L^1(0, T)$ and let $\alpha \in (0, 1]$.*

$$(1) \quad I_t^\alpha {}^C D_t^\alpha x = x(t) - x(0).$$

(2) *For continuous f , the initial value problem ${}^C D_t^\alpha x = f(t, x)$, $x(0) = x_0$, is equivalent to the Volterra integral equation*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds. \quad (2.4)$$

(3) *Whenever the transforms exist, $\mathcal{L}\{{}^C D_t^\alpha x\}(s) = s^\alpha X(s) - s^{\alpha-1}x(0)$.*

Proof. (i) By the semigroup property $I^\alpha I^{1-\alpha} = I^1$, we get $I^\alpha {}^C D^\alpha x = I^\alpha I^{1-\alpha} x' = I^1 x' = x - x(0)$. (ii) follows by applying I^α to both sides and using (i). (iii) Since $\mathcal{L}\{I^{1-\alpha} g\} = s^{\alpha-1} G(s)$ and $\mathcal{L}\{x'\} = sX - x(0)$, we obtain $\mathcal{L}\{{}^C D^\alpha x\} = s^{\alpha-1}(sX - x(0))$. $\square$

Lemma 2.3 (Linear fractional equation and Mittag-Leffler function). *For $c > 0$, $\Lambda \geq 0$, the problem ${}^C D_t^\alpha y = \Lambda - c y$, $y(0) = y_0$ has the unique solution*

$$y(t) = \frac{\Lambda}{c} + \left(y_0 - \frac{\Lambda}{c}\right) E_\alpha(-c t^\alpha). \quad (2.5)$$

Moreover, for $0 < \alpha \leq 1$ and $z \geq 0$: $0 < E_\alpha(-z) \leq 1$, $E_\alpha(-z)$ is decreasing in z , and

$$E_\alpha(-z) = \frac{1}{\Gamma(1-\alpha)z} + O(z^{-2}) \quad (z \rightarrow \infty, \alpha < 1), \quad E_\alpha(z) \sim \frac{1}{\alpha} \exp(z^{1/\alpha}) \quad (z \rightarrow +\infty). \quad (2.6)$$

Proof. Taking Laplace transforms with Lemma 2.2(iii), $s^\alpha Y - s^{\alpha-1}y_0 = \Lambda/s - cY$, hence

$$Y(s) = y_0 \frac{s^{\alpha-1}}{s^\alpha + c} + \frac{\Lambda}{s(s^\alpha + c)} = y_0 \frac{s^{\alpha-1}}{s^\alpha + c} + \frac{\Lambda}{c} \left(\frac{1}{s} - \frac{s^{\alpha-1}}{s^\alpha + c} \right).$$

Since $\mathcal{L}\{E_\alpha(-ct^\alpha)\} = s^{\alpha-1}/(s^\alpha + c)$ (termwise transform of the series), inversion gives (2.5). The monotonicity and positivity of $E_\alpha(-z)$ for $0 < \alpha \leq 1$ (complete monotonicity) and the asymptotic expansions (2.6) are classical [14, 18]. $\square$

Remark 2.1 (Algebraic versus exponential relaxation). *For the infected class near the disease-free state the relevant decay law is ${}^C D^\alpha I = -\theta I$ with $\theta = \mu + d_I + \gamma_I \approx 0.1020$ (see Section 6), whose solution is $I(0)E_\alpha(-\theta t^\alpha)$. By (2.6), this decays only like $t^{-\alpha}$ rather than $e^{-\theta t}$. Table 1 compares the two laws.*

Lemma 2.4 (Extremum principle). *Let $x \in C([0, T]) \cap C^1((0, T])$ with $x' \in L^1(0, T)$, $\alpha \in (0, 1)$. If $x(t^*) \leq x(s)$ for all $s \in [0, t^*]$, then ${}^C D_t^\alpha x(t^*) \leq 0$.*

Proof. Integrating by parts in (2.2) with $w(s) = x(s) - x(t^*)$,

$${}^C D_t^\alpha x(t^*) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{x(t^*) - x(0)}{(t^*)^\alpha} + \alpha \int_0^{t^*} \frac{x(t^*) - x(s)}{(t^* - s)^{\alpha+1}} ds \right].$$

TABLE 1. Relaxation of $I(t)/I(0)$ under ${}^C D^\alpha I = -\theta I$, $\theta = 0.1020$: exponential law ($\alpha = 1$) versus Mittag-Leffler law $E_\alpha(-\theta t^\alpha)$.

t (days)	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.7$
30	4.68×10^{-2}	1.43×10^{-1}	3.70×10^{-1}
60	2.19×10^{-3}	4.90×10^{-2}	2.39×10^{-1}
100	3.70×10^{-5}	2.32×10^{-2}	1.64×10^{-1}
150	2.25×10^{-7}	1.41×10^{-2}	1.20×10^{-1}

(The boundary term at $s = t^*$ vanishes because $w(s) = O(t^* - s)$.) Under the hypothesis, $x(t^*) - x(0) \leq 0$ and $x(t^*) - x(s) \leq 0$ for all s , so every term in brackets is nonpositive. $\square$

Lemma 2.5 (Linear comparison principle). *Let $c \geq 0$ and let $w \in C([0, T])$ satisfy ${}^C D_t^\alpha w + c w = g(t)$ with $g \in C([0, T])$, $g \geq 0$ and $w(0) \geq 0$. Then $w(t) \geq 0$ on $[0, T]$.*

Proof. By the variation-of-constants formula for linear Caputo equations [14, 18],

$$w(t) = w(0)E_\alpha(-ct^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-c(t-s)^\alpha) g(s) ds,$$

where $E_{\alpha,\alpha}(-z) > 0$ for $z \geq 0$, $0 < \alpha \leq 1$. All terms are nonnegative. $\square$

Lemma 2.6 (Fractional Gronwall inequality). *If $\varphi \in C([0, T])$, $\varphi \geq 0$, and $\varphi(t) \leq a + L I_t^\alpha \varphi(t)$ for constants $a, L \geq 0$, then $\varphi(t) \leq a E_\alpha(L t^\alpha)$.*

Proof. Iterating the hypothesis n times and using $I^{k\alpha} 1 = t^{k\alpha} / \Gamma(k\alpha + 1)$,

$$\varphi(t) \leq a \sum_{k=0}^{n-1} \frac{(L t^\alpha)^k}{\Gamma(k\alpha + 1)} + L^n I_t^{n\alpha} \varphi(t), \quad L^n I_t^{n\alpha} \varphi \leq \|\varphi\|_\infty \frac{(L t^\alpha)^n}{\Gamma(n\alpha + 1)} \rightarrow 0.$$

Letting $n \rightarrow \infty$ yields the claim. $\square$

2.4. Right-Sided Derivative and Integration by Parts. The adjoint equations involve the right-sided Riemann-Liouville derivative:

$${}_t D_T^\alpha f(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (s-t)^{-\alpha} f(s) ds, \quad \alpha \in (0, 1), \quad (2.7)$$

and the right-sided integral $({}_t I_T^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} f(s) ds$, so that ${}_t D_T^\alpha f = -\frac{d}{dt} {}_t I_T^{1-\alpha} f$.

Lemma 2.7 (Integration by parts and time reversal). *For $\varphi \in C^1([0, T])$ and $\psi \in C([0, T])$,*

$$\int_0^T \psi {}^C D_t^\alpha \varphi dt = \int_0^T \varphi {}_t D_T^\alpha \psi dt + \left[\varphi(t) ({}_t I_T^{1-\alpha} \psi)(t) \right]_{t=0}^{t=T}. \quad (2.8)$$

Moreover, if $\bar{\psi}(\tau) = \psi(T - \tau)$, then $({}_t D_T^\alpha \psi)(T - \tau) = ({}^{RL} D_\tau^\alpha \bar{\psi})(\tau)$, i.e. the right-sided derivative becomes an ordinary left-sided derivative in the reversed time $\tau = T - t$.

Proof. By (2.2) and Fubini's theorem, $\int_0^T \psi I^{1-\alpha} \varphi' dt = \int_0^T \varphi' {}_t I_T^{1-\alpha} \psi dt$. Integrating by parts, this equals $[\varphi {}_t I_T^{1-\alpha} \psi]_0^T - \int_0^T \varphi \frac{d}{dt} {}_t I_T^{1-\alpha} \psi dt$, and $-\frac{d}{dt} {}_t I_T^{1-\alpha} \psi = {}_t D_T^\alpha \psi$. For the reversal, substitute $s = T - \sigma$, $t = T - \tau$ in (2.7): the inner integral becomes $\int_0^\tau (\tau - \sigma)^{-\alpha} \bar{\psi}(\sigma) d\sigma$ and $d/dt = -d/d\tau$. $\square$

Note that when $\psi(T) = 0$ the right-sided Riemann-Liouville and Caputo derivatives coincide, so no distinction is needed for costates with $\lambda_i(T) = 0$.

3. NOTATION

4. SEIQHRV MODEL FORMULATION

4.1. Population Structure and Assumptions. The total population $N(t)$ is partitioned into seven epidemiological classes:

$$N(t) = S(t) + E(t) + I(t) + Q(t) + H(t) + R(t) + V(t). \quad (4.1)$$

Susceptible (S): individuals with no immunity, susceptible to infection upon contact with an infectious individual.

Exposed (E): individuals who have been infected but are still within the incubation period (typically 2–14 days for SARS-CoV-2). They are not yet symptomatic and are assumed to be sub-clinically infectious with a reduced transmission factor $\epsilon \in [0, 1]$.

Infected (I): individuals with confirmed symptomatic infection, capable of transmitting the virus to susceptible contacts.

Quarantined (Q): detected cases placed under isolation, either at home or in designated facilities. Quarantined individuals may recover naturally or require hospitalization.

Hospitalized (H): individuals with severe or critical disease admitted to clinical care. They may recover or experience disease-induced mortality.

Recovered (R): individuals who have cleared the infection and are assumed to have temporary immunity within the modeling horizon.

Vaccinated (V): individuals who have received a complete vaccination course, conferring reduced susceptibility $(1 - \kappa)$ to infection. Vaccine protection wanes at rate ω , returning individuals to the susceptible pool.

The following assumptions are adopted:

Assumption 4.1. *Transmission occurs through direct contact between susceptible (or vaccine-waned susceptible) individuals and either exposed (with factor ϵ) or infected individuals, following frequency-dependent (standard) incidence.*

Assumption 4.2. *Quarantined individuals have negligible further transmission due to strict isolation protocols. Hospitalized individuals are also assumed non-transmitting.*

Assumption 4.3. *Vaccination does not confer sterilizing immunity in all recipients; breakthrough infections occur at rate $(1 - \kappa)\beta$. Vaccine protection wanes at rate ω .*

Assumption 4.4. *Recovered individuals do not return to susceptibility within the study horizon $[0, T]$.*

TABLE 2. State variables, parameters, and controls used in the SEIQHRV model

Symbol	Description	Unit/Range
$S(t)$	Susceptible individuals	persons
$E(t)$	Exposed (incubating) individuals	persons
$I(t)$	Symptomatic infectious individuals	persons
$Q(t)$	Quarantined/isolated individuals	persons
$H(t)$	Hospitalized individuals	persons
$R(t)$	Recovered individuals	persons
$V(t)$	Vaccinated individuals	persons
$N(t)$	Total population	persons
Λ	Recruitment rate	persons/day
μ	Natural mortality rate	day ⁻¹
β	Effective contact rate	day ⁻¹
σ	Progression rate (exposed to infectious)	day ⁻¹
γ_I	Natural recovery rate (infected)	day ⁻¹
γ_H	Recovery rate (hospitalized)	day ⁻¹
γ_Q	Recovery rate (quarantined)	day ⁻¹
δ	Quarantine rate (infected to quarantined)	day ⁻¹
δ_H	Hospitalization rate (infected to hospitalized)	day ⁻¹
d_I	Disease-induced mortality (infected)	day ⁻¹
d_H	Disease-induced mortality (hospitalized)	day ⁻¹
κ	Vaccine efficacy	dimensionless
ω	Vaccine waning rate	day ⁻¹
ϕ	Breakthrough infection rate modifier	dimensionless
$u_1(t)$	Vaccination control	[0, 1]
$u_2(t)$	Quarantine-isolation control	[0, 1]
$u_3(t)$	Treatment (hospitalization) control	[0, 1]
α	Fractional order	(0, 1]
${}^C D_t^\alpha$	Caputo fractional derivative	operator
$\mathcal{R}_0$	Basic reproduction number	dimensionless
$\lambda_i(t)$	Adjoint costate variables	$i = 1, \dots, 7$
J	Objective functional	cost
T	Time horizon	days
A, B, C	Weights on I, H, Q in objective	cost
d_1, d_2, d_3	Control cost coefficients	cost

4.2. **Control Strategies.** Three time-varying, Lebesgue-measurable control functions are defined:

- $u_1(t) \in [0, 1]$: **Vaccination control.** The effective vaccination rate applied to susceptibles at time t is $\kappa u_1(t)$, transferring a fraction of susceptibles to the vaccinated class.

- $u_2(t) \in [0, 1]$: **Quarantine-isolation control**. This represents the intensity of contact tracing, testing, and mandatory isolation applied to the infected population. The effective quarantine rate is $\delta u_2(t)$.
- $u_3(t) \in [0, 1]$: **Clinical treatment control**. This governs the rate of clinical intervention and hospitalization of severe cases among the infected population, transferring individuals from I to H at rate $\delta_H u_3(t)$ with enhanced survival probability.

The admissible control set is

$$\mathcal{U} = \{(u_1, u_2, u_3) : u_i \in L^\infty([0, T]), 0 \leq u_i(t) \leq 1, \text{ a.e. on } [0, T], i = 1, 2, 3\}. \quad (4.2)$$

4.3. Fractional-Order SEIQHRV System. The Caputo fractional-order SEIQHRV model is:

$${}^C D_t^\alpha S = \Lambda - \lambda(t)S - \kappa u_1 S - \mu S + \omega V, \quad (4.3)$$

$${}^C D_t^\alpha E = \lambda(t)S + (1 - \kappa)\lambda(t)V - (\sigma + \mu)E, \quad (4.4)$$

$${}^C D_t^\alpha I = \sigma E - (\mu + d_I + \gamma_I + \delta u_2 + \delta_H u_3)I, \quad (4.5)$$

$${}^C D_t^\alpha Q = \delta u_2 I - (\mu + \gamma_Q + d_Q)Q, \quad (4.6)$$

$${}^C D_t^\alpha H = \delta_H u_3 I - (\mu + \gamma_H + d_H)H, \quad (4.7)$$

$${}^C D_t^\alpha R = \gamma_I I + \gamma_Q Q + \gamma_H H - \mu R, \quad (4.8)$$

$${}^C D_t^\alpha V = \kappa u_1 S - (1 - \kappa)\lambda(t)V - (\mu + \omega)V, \quad (4.9)$$

where the force of infection is

$$\lambda(t) = \frac{\beta(\epsilon E + I)}{N}, \quad (4.10)$$

with $\epsilon \in (0, 1)$ the relative transmissibility of exposed individuals, β the effective contact rate, and $N = N(t)$ the total live population. The system is supplemented with nonnegative initial conditions

$$S(0) = S_0, E(0) = E_0, I(0) = I_0, Q(0) = Q_0, H(0) = H_0, R(0) = R_0, V(0) = V_0. \quad (4.11)$$

Setting $\alpha = 1$ recovers the classical integer-order SEIQHRV model.

4.4. Conservation Law and Time-Unit Consistency. Adding the seven equations (4.3)–(4.9), the transfer terms cancel in pairs:

$$\begin{aligned} {}^C D_t^\alpha N &= \Lambda + \underbrace{(-\lambda S + \lambda S)}_{=0} + \underbrace{((1 - \kappa)\lambda V - (1 - \kappa)\lambda V)}_{=0} + \underbrace{(-\kappa u_1 S + \kappa u_1 S)}_{=0} + \underbrace{(\omega V - \omega V)}_{=0} + \underbrace{(\sigma E - \sigma E)}_{=0} \\ &\quad + \underbrace{(-\delta u_2 I + \delta u_2 I)}_{=0} + \underbrace{(-\delta_H u_3 I + \delta_H u_3 I)}_{=0} + \underbrace{(-\gamma_I I + \gamma_I I)}_{=0} + \underbrace{(-\gamma_Q Q + \gamma_Q Q)}_{=0} + \underbrace{(-\gamma_H H + \gamma_H H)}_{=0} \\ &\quad - \mu(S + E + I + Q + H + R + V) - d_I I - d_Q Q - d_H H, \end{aligned} \quad (4.12)$$

that is,

$${}^C D_t^\alpha N = \Lambda - \mu N - d_I I - d_Q Q - d_H H. \quad (4.13)$$

The population therefore changes only through recruitment, natural death and disease-induced death, as required for a closed compartmental model.

Remark 4.1 (Dimensional consistency). *The operator ${}^C D_t^\alpha$ has dimension $[time]^{-\alpha}$, so each rate constant in (4.3)–(4.9) must be read in units of $day^{-\alpha}$. A rate r calibrated for the classical model ($\alpha = 1, day^{-1}$) is rescaled to r^α if dimensional consistency is to be enforced. In the simulations of Section 10 the numerical values of Table 8 are kept fixed while α is varied, so differences between fractional orders reflect the combined effect of memory and of the implied rate scale; the qualitative conclusions concerning memory are not affected by this convention.*

5. WELL-POSEDNESS, POSITIVITY AND BOUNDEDNESS

Write the system compactly as ${}^C D_t^\alpha \mathbf{x} = \mathbf{f}(\mathbf{x}, \mathbf{u}(t))$ with $\mathbf{x} = (S, E, I, Q, H, R, V)^\top$, and let $\|\cdot\|_1$ denote the ℓ^1 -norm.

5.1. Lipschitz Continuity.

Lemma 5.1 (Explicit Lipschitz constant). *For every $\mathbf{u} \in [0, 1]^3$, the map $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \mathbf{u})$ is Lipschitz on $\mathbb{R}_+^7 \setminus \{0\}$ with respect to $\|\cdot\|_1$, with constant*

$$L = L_{\text{lin}} + \beta(4 - 2\kappa), \quad L_{\text{lin}} = \max\{2\kappa + \mu, 2\sigma + \mu, \mu + d_I + 2(\gamma_I + \delta + \delta_H), \quad (5.1)$$

$$\mu + 2\gamma_Q + d_Q, \mu + 2\gamma_H + d_H, \mu, \mu + 2\omega\}. \quad (5.2)$$

Proof. The ℓ^1 -Lipschitz constant on a convex set is bounded by the largest absolute column sum of the Jacobian. The linear part of $\mathbf{f}$ contributes, column by column, $\kappa u_1 + \mu + \kappa u_1$ (column S : the outflow $-(\kappa u_1 + \mu)$ in the S -row and the inflow κu_1 in the V -row), $(\sigma + \mu) + \sigma$ (E), $(\mu + d_I + \gamma_I + \delta u_2 + \delta_H u_3) + \delta u_2 + \delta_H u_3 + \gamma_I$ (I), $(\mu + \gamma_Q + d_Q) + \gamma_Q$ (Q), $(\mu + \gamma_H + d_H) + \gamma_H$ (H), μ (R) and $\omega + (\mu + \omega)$ (V); their maximum is L_{lin} . For the incidence terms put $\lambda = \beta(\epsilon E + I)/N$ with $N = \sum_j x_j$. Then

$$\begin{aligned} \frac{\partial(\lambda S)}{\partial S} &= \lambda \left(1 - \frac{S}{N}\right), \quad \frac{\partial(\lambda S)}{\partial E} = \frac{S}{N} (\beta\epsilon - \lambda), \quad \frac{\partial(\lambda S)}{\partial I} = \frac{S}{N} (\beta - \lambda), \\ \frac{\partial(\lambda S)}{\partial x_j} &= -\frac{S}{N} \lambda \quad (x_j \in \{Q, H, R, V\}), \end{aligned}$$

and since $0 \leq \lambda \leq \beta$ and $S \leq N$ every entry lies in $[-\beta, \beta]$; the same holds for λV with $V \leq N$. The terms λS , $(1 - \kappa)\lambda V$ enter the S -, E -, and V -equations with coefficients (-1) ; $(+1, 1 - \kappa)$; $(-(1 - \kappa))$, so the incidence terms add at most $\beta[1 + (2 - \kappa) + (1 - \kappa)] = \beta(4 - 2\kappa)$ to any column sum. $\square$

For the parameters of Table 8, $L_{\text{lin}} = 2\kappa + \mu = 1.7000$, $\beta(4 - 2\kappa) = 0.8050$, hence $L = 2.505$ ($days^{-\alpha}$). As a numerical check, the largest ℓ^1 column sum of the Jacobian over 3000 random states and controls was $1.94 \leq L$.

5.2. Positivity.

Theorem 5.1 (Positivity). *Under nonnegative initial conditions (4.11) (with $\sum_i x_i(0) > 0$) and admissible controls $(u_1, u_2, u_3) \in \mathcal{U}$, the solution of (4.3)-(4.9) satisfies $S, E, I, Q, H, R, V \geq 0$ on its interval of existence.*

Proof. Step 1: quasi-positivity. Whenever $\mathbf{x} \geq 0$ and $x_i = 0$, the i -th component of $\mathbf{f}$ is nonnegative:

$$\begin{aligned} f_S &= \Lambda + \omega V \geq \Lambda > 0, & f_E &= \lambda (S + (1 - \kappa)V) \geq 0, & f_I &= \sigma E \geq 0, \\ f_Q &= \delta u_2 I \geq 0, & f_H &= \delta_H u_3 I \geq 0, & f_R &= \gamma_I I + \gamma_Q Q + \gamma_H H \geq 0, \\ f_V &= \kappa u_1 S \geq 0. \end{aligned}$$

Step 2: perturbed problem. For $\varepsilon > 0$ let $\mathbf{x}^\varepsilon$ solve ${}^C D_t^\alpha \mathbf{x}^\varepsilon = \mathbf{f}(\mathbf{x}^\varepsilon, \mathbf{u}) + \varepsilon \mathbf{1}$, $\mathbf{x}^\varepsilon(0) = \mathbf{x}_0 + \varepsilon \mathbf{1}$. Suppose some component vanishes, and let $t^* > 0$ be the first time at which $x_i^\varepsilon(t^*) = 0$. Then $x_i^\varepsilon(t^*) \leq x_i^\varepsilon(s)$ for all $s \leq t^*$ and the other components are still nonnegative at t^* , so Lemma 2.4 gives ${}^C D_t^\alpha x_i^\varepsilon(t^*) \leq 0$. But by Step 1, ${}^C D_t^\alpha x_i^\varepsilon(t^*) = f_i(\mathbf{x}^\varepsilon(t^*), \mathbf{u}) + \varepsilon \geq \varepsilon > 0$, a contradiction. Hence $\mathbf{x}^\varepsilon > 0$.

Step 3: limit $\varepsilon \rightarrow 0$. The difference $\varphi = \|\mathbf{x}^\varepsilon - \mathbf{x}\|_1$ satisfies, by (2.4) and Lemma 5.1, $\varphi(t) \leq 7\varepsilon(1 + T^\alpha/\Gamma(\alpha + 1)) + L I^\alpha \varphi(t)$, so Lemma 2.6 yields $\varphi \leq 7\varepsilon(1 + T^\alpha/\Gamma(\alpha + 1))E_\alpha(LT^\alpha) \rightarrow 0$. Hence $\mathbf{x}^\varepsilon \rightarrow \mathbf{x}$ uniformly and $\mathbf{x} \geq 0$. $\square$

5.3. Boundedness.

Theorem 5.2 (Boundedness and Invariant Region). *Let $d_{\max} = \max\{d_I, d_Q, d_H\}$ and $c = \mu + d_{\max}$. The total population satisfies, for all $t \in [0, T]$,*

$$\frac{\Lambda}{c} + \left(N_0 - \frac{\Lambda}{c}\right)E_\alpha(-c t^\alpha) \leq N(t) \leq \frac{\Lambda}{\mu} + \left(N_0 - \frac{\Lambda}{\mu}\right)E_\alpha(-\mu t^\alpha) \leq \max\left\{N_0, \frac{\Lambda}{\mu}\right\}, \quad (5.3)$$

so in particular $N(t) \geq N_{\min} := \min\{N_0, \Lambda/c\} > 0$, and the region

$$\Omega = \{(S, E, I, Q, H, R, V) \in \mathbb{R}_+^7 : N \leq \Lambda/\mu\} \quad (5.4)$$

is positively invariant.

Proof. Upper bound. By (4.13), ${}^C D_t^\alpha N = \Lambda - \mu N - g_1$ with $g_1 = d_I I + d_Q Q + d_H H \geq 0$ (Theorem 5.1). Let y solve ${}^C D_t^\alpha y = \Lambda - \mu y$, $y(0) = N_0$, i.e. (2.5) with $c = \mu$. Then $w = y - N$ satisfies ${}^C D_t^\alpha w = -\mu w + g_1$, $w(0) = 0$, and Lemma 2.5 gives $w \geq 0$. Since $0 < E_\alpha(-\mu t^\alpha) \leq 1$, $y(t) \leq \max\{N_0, \Lambda/\mu\}$.

Lower bound. Let z solve ${}^C D_t^\alpha z = \Lambda - cz$, $z(0) = N_0$. Then $w = N - z$ satisfies ${}^C D_t^\alpha w = -cw + g_2$ with $g_2 = d_{\max}N - (d_I I + d_Q Q + d_H H) \geq d_{\max}(S + E + R + V) \geq 0$, $w(0) = 0$, so $w \geq 0$. $\square$

For the baseline parameters ($\Lambda = 5$, $\mu = 1/27375$, $N_0 = 10,070$, $d_{\max} = 0.01$) one has $\Lambda/\mu = 136,875$ and $N_{\min} = \Lambda/c \approx 498$. At $T = 180$ days, (5.3) gives $2070 \leq N(T) \leq 10,901$ for $\alpha = 1$, $3862 \leq N(T) \leq 10,585$ for $\alpha = 0.9$ and $6975 \leq N(T) \leq 10,263$ for $\alpha = 0.7$.

5.4. Global Existence, Uniqueness and Continuous Dependence.

Theorem 5.3 (Well-posedness). *For every $\mathbf{u} \in \mathcal{U}$ and every initial state $\mathbf{x}_0 \in \mathbb{R}_+^7$ with $N_0 > 0$, system (4.3)-(4.9) has a unique solution $\mathbf{x} \in C([0, T]; \Omega')$, where $\Omega' = \Omega \cap \{N \geq N_{\min}\}$. If $\mathbf{x}, \mathbf{y}$ are two solutions with initial data $\mathbf{x}_0, \mathbf{y}_0$ and the same control, then*

$$\|\mathbf{x}(t) - \mathbf{y}(t)\|_1 \leq \|\mathbf{x}_0 - \mathbf{y}_0\|_1 E_\alpha(L t^\alpha), \quad t \in [0, T]. \quad (5.5)$$

Proof. On $\{N > 0\}$ the right-hand side is C^1 in $\mathbf{x}$, measurable and bounded in t , so the Caputo analogue of the Picard-Lindelöf theorem [9] gives a unique local solution. By Theorems 5.1 and 5.2 it remains in the compact set Ω' , on which $N \geq N_{\min} > 0$, so it cannot blow up or reach the singularity $N = 0$ and extends to $[0, T]$. For (5.5), subtract the Volterra equations (2.4) for $\mathbf{x}$ and $\mathbf{y}$, apply Lemma 5.1 to obtain $\varphi \leq \|\mathbf{x}_0 - \mathbf{y}_0\|_1 + L I^\alpha \varphi$ and invoke Lemma 2.6. $\square$

Remark 5.1. *The bounded invariant region Ω' ensures that the state trajectories remain confined to a biologically meaningful compact set, a property that is essential for proving existence of optimal controls (Section 7). Estimate (5.5) shows that, for $\alpha < 1$, sensitivity to initial data grows at the Mittag-Leffler rate $E_\alpha(L t^\alpha) \sim \alpha^{-1} \exp(L^{1/\alpha} t)$.*

6. EQUILIBRIA, BASIC REPRODUCTION NUMBER AND STABILITY

6.1. Disease-Free Equilibrium. With $u_1 = u_2 = u_3 = 0$, setting the right-hand sides of (4.3)-(4.9) to zero and requiring $E = I = 0$ (hence $\lambda = 0$) gives $V' = -(\mu + \omega)V = 0$, so $V = 0$, and $Q = H = 0$ from (4.6)-(4.7), $R = 0$ from (4.8), and $\Lambda - \mu S = 0$ from (4.3). The disease-free equilibrium (DFE) is

$$\mathcal{E}_0 = (S^0, 0, 0, 0, 0, 0, 0), \quad S^0 = \frac{\Lambda}{\mu} \quad (= 136,875 \text{ for Table 8}). \quad (6.1)$$

For the controlled case with constant controls the DFE is computed in Section 6.6.

6.2. Next-Generation Matrix. Following the next-generation matrix method [10], the infected compartments are $\mathbf{x} = (E, I)^\top$. Write $\theta := \mu + d_I + \gamma_I$. At the DFE, $S^0/N^0 = 1$ and the new-infection and transition matrices are

$$\mathbf{F} = \begin{pmatrix} \epsilon\beta & \beta \\ 0 & 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \sigma + \mu & 0 \\ -\sigma & \theta \end{pmatrix}. \quad (6.2)$$

Since $\det \mathbf{V} = (\sigma + \mu)\theta$,

$$\mathbf{V}^{-1} = \frac{1}{(\sigma + \mu)\theta} \begin{pmatrix} \theta & 0 \\ \sigma & \sigma + \mu \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma + \mu} & 0 \\ \frac{\sigma}{(\sigma + \mu)\theta} & \frac{1}{\theta} \end{pmatrix}. \quad (6.3)$$

Theorem 6.1 (Basic Reproduction Number). *The basic reproduction number for the uncontrolled SEIQHRV model is*

$$\mathcal{R}_0 = \frac{\beta\sigma}{(\sigma + \mu)(\mu + d_I + \gamma_I)} + \frac{\epsilon\beta}{\sigma + \mu} = \frac{\beta}{\sigma + \mu} \left(\epsilon + \frac{\sigma}{\theta} \right). \quad (6.4)$$

Proof. Multiplying (6.2) by (6.3),

$$\mathbf{K} = \mathbf{FV}^{-1} = \begin{pmatrix} \frac{\epsilon\beta}{\sigma + \mu} + \frac{\beta\sigma}{(\sigma + \mu)\theta} & \frac{\beta}{\theta} \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of this upper-triangular matrix are its diagonal entries, so $\mathcal{R}_0 = \rho(\mathbf{K}) = K_{11}$. $\square$

Remark 6.1 (Interpretation and numerical value). Formula (6.4) is a sum of two infection pathways: an individual spends on average $1/(\sigma + \mu)$ days in E , infecting at rate $\epsilon\beta$, and with probability $\sigma/(\sigma + \mu)$ proceeds to I , where it spends $1/\theta$ days infecting at rate β . For the parameters of Table 8, $\sigma + \mu = 0.2000365$ and $\theta = 0.1020365$, so

$$\mathcal{R}_0 = \underbrace{\frac{0.35 \times 0.2}{0.2000365 \times 0.1020365}}_{3.4295} + \underbrace{\frac{0.30 \times 0.35}{0.2000365}}_{0.5249} = 3.9544,$$

i.e. about 87% of secondary infections originate from the symptomatic class. The threshold $\mathcal{R}_0$ does not depend on α .

6.3. Local Stability of the DFE. Ordering the states as (S, E, I, Q, H, R, V) , the Jacobian at $\mathcal{E}_0$ with $\mathbf{u} = 0$ is

$$\mathbf{J}_0 = \begin{pmatrix} -\mu & -\epsilon\beta & -\beta & 0 & 0 & 0 & \omega \\ 0 & \epsilon\beta - (\sigma + \mu) & \beta & 0 & 0 & 0 & 0 \\ 0 & \sigma & -\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta_Q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta_H & 0 & 0 \\ 0 & 0 & \gamma_I & \gamma_Q & \gamma_H & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(\mu + \omega) \end{pmatrix}, \quad (6.5)$$

with $\theta_Q = \mu + \gamma_Q + d_Q$, $\theta_H = \mu + \gamma_H + d_H$. After a permutation, $\mathbf{J}_0$ is block triangular, so its spectrum consists of the real numbers $-\mu, -\mu, -\theta_Q, -\theta_H, -(\mu + \omega)$ together with the two eigenvalues of the infection block

$$\mathbf{J}_{\text{inf}} = \begin{pmatrix} \epsilon\beta - (\sigma + \mu) & \beta \\ \sigma & -\theta \end{pmatrix}, \quad \text{tr } \mathbf{J}_{\text{inf}} = \epsilon\beta - (\sigma + \mu) - \theta, \quad \det \mathbf{J}_{\text{inf}} = (\sigma + \mu)\theta(1 - \mathcal{R}_0). \quad (6.6)$$

The determinant identity follows from $(\sigma + \mu)\theta - \epsilon\beta\theta - \beta\sigma = (\sigma + \mu)\theta(1 - \mathcal{R}_0)$, which is (6.4) multiplied by $(\sigma + \mu)\theta$.

Theorem 6.2 (Local stability of the DFE). For every $\alpha \in (0, 1]$, the DFE $\mathcal{E}_0$ of the uncontrolled system is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.

Proof. For fractional systems ${}^C D^\alpha \mathbf{x} = \mathbf{f}(\mathbf{x})$, an equilibrium is locally asymptotically stable if all eigenvalues ζ of the Jacobian satisfy $|\arg \zeta| > \alpha\pi/2$ (Matignon's criterion; see [2]), and unstable if some eigenvalue satisfies $|\arg \zeta| < \alpha\pi/2$. The five real eigenvalues above are negative ($|\arg| = \pi$). If $\mathcal{R}_0 < 1$, then $\epsilon\beta < (\sigma + \mu)\mathcal{R}_0 < \sigma + \mu$, so $\text{tr } \mathbf{J}_{\text{inf}} < 0$, and $\det \mathbf{J}_{\text{inf}} > 0$ by (6.6); hence both eigenvalues

have negative real part and $|\arg \zeta| > \pi/2 \geq \alpha\pi/2$. If $\mathcal{R}_0 > 1$, then $\det \mathbf{J}_{\text{inf}} < 0$, so $\mathbf{J}_{\text{inf}}$ has a positive real eigenvalue ($\arg = 0 < \alpha\pi/2$). $\square$

For the baseline parameters, $\text{tr } \mathbf{J}_{\text{inf}} = -0.19707$ and $\det \mathbf{J}_{\text{inf}} = 0.2000365 \times 0.1020365 \times (1 - 3.9544) = -0.06030$, so

$$\zeta_{\pm} = \frac{1}{2}(\text{tr} \pm \sqrt{\text{tr}^2 - 4 \det}) = \frac{1}{2}(-0.19707 \pm 0.52920) = +0.16606, -0.36313, \quad (6.7)$$

and the DFE is unstable, consistent with the epidemic observed in the uncontrolled simulations.

6.4. Early Growth Under Memory. Near the DFE the infection block obeys ${}^C D^\alpha \mathbf{x} = \mathbf{J}_{\text{inf}} \mathbf{x}$, whose solution is $\mathbf{x}(t) = E_\alpha(\mathbf{J}_{\text{inf}} t^\alpha) \mathbf{x}_0$ and is dominated by the positive eigenvalue ζ_+ : $x(t) \approx c E_\alpha(\zeta_+ t^\alpha)$. By the second relation in (2.6), for large $\zeta_+ t^\alpha$,

$$E_\alpha(\zeta_+ t^\alpha) \sim \frac{1}{\alpha} \exp(\zeta_+^{1/\alpha} t), \quad r_{\text{eff}}(\alpha) = \zeta_+^{1/\alpha}, \quad (6.8)$$

so a Caputo model with $\zeta_+ < 1$ grows *more slowly* than the integer-order model. Numerically, $r_{\text{eff}} = 0.1661, 0.1360, 0.1060, 0.0769 \text{ day}^{-1}$ for $\alpha = 1.0, 0.9, 0.8, 0.7$, i.e. doubling times of 4.2, 5.1, 6.5, 9.0 days. This explains analytically why the uncontrolled epidemic peak is delayed as α decreases (Section 10), and (2.6) explains the long algebraic tail of the epidemic (Remark 2.1).

6.5. Sensitivity of $\mathcal{R}_0$. The normalized forward sensitivity index of $\mathcal{R}_0$ with respect to a parameter p is $Y_p = \frac{\partial \mathcal{R}_0}{\partial p} \frac{p}{\mathcal{R}_0}$. Differentiating (6.4) gives

$$\begin{aligned} \frac{\partial \mathcal{R}_0}{\partial \beta} &= \frac{\mathcal{R}_0}{\beta}, & \frac{\partial \mathcal{R}_0}{\partial \epsilon} &= \frac{\beta}{\sigma + \mu}, & \frac{\partial \mathcal{R}_0}{\partial \sigma} &= \frac{\beta(\mu/\theta - \epsilon)}{(\sigma + \mu)^2}, \\ \frac{\partial \mathcal{R}_0}{\partial \gamma_I} &= \frac{\partial \mathcal{R}_0}{\partial d_I} = -\frac{\beta\sigma}{(\sigma + \mu)\theta^2}, & \frac{\partial \mathcal{R}_0}{\partial \mu} &= -\frac{\beta\sigma(\theta + \sigma + \mu)}{(\sigma + \mu)^2\theta^2} - \frac{\epsilon\beta}{(\sigma + \mu)^2}. \end{aligned} \quad (6.9)$$

The resulting indices at the baseline parameters are listed in Table 3. They show that $\mathcal{R}_0$ is proportional to β ($Y_\beta = +1$: a 10% reduction of contacts lowers $\mathcal{R}_0$ by 10%) and is next most sensitive to the recovery rate γ_I ; the effect of demographic turnover μ is negligible on this time scale.

TABLE 3. Normalized forward sensitivity indices Y_p of $\mathcal{R}_0 = 3.9544$.

Parameter p	Value	$\partial \mathcal{R}_0 / \partial p$	Y_p
β	0.35	11.298	+1.0000
γ_I	0.10	-33.611	-0.8500
ϵ	0.30	1.750	+0.1327
σ	0.20	-2.621	-0.1326
d_I	0.002	-33.611	-0.0170
μ	3.65×10^{-5}	-53.379	-0.0005

6.6. Controlled Reproduction Number and Intervention Thresholds. Consider constant controls $(u_1, u_2, u_3) \in [0, 1]^3$. At a disease-free state the V - and S -equations give $V = \kappa u_1 S / (\mu + \omega)$ and $S[\mu + \kappa u_1 \mu / (\mu + \omega)] = \Lambda$, so with $m := \mu + \omega$,

$$S_c^0 = \frac{\Lambda m}{\mu(m + \kappa u_1)}, \quad V_c^0 = \frac{\Lambda \kappa u_1}{\mu(m + \kappa u_1)}, \quad N^0 = \frac{\Lambda}{\mu}. \quad (6.10)$$

The new-infection matrix now carries the factor $s(u_1) = (S_c^0 + (1 - \kappa)V_c^0)/N^0$, and the transition matrix has $\theta_u = \theta + \delta u_2 + \delta_H u_3$ in place of θ .

Proposition 6.1 (Controlled reproduction number). *For constant controls,*

$$\mathcal{R}_c(u_1, u_2, u_3) = \left(1 - \frac{\kappa^2 u_1}{\mu + \omega + \kappa u_1}\right) \frac{\beta}{\sigma + \mu} \left(\epsilon + \frac{\sigma}{\theta + \delta u_2 + \delta_H u_3}\right). \quad (6.11)$$

Equivalently, $\mathcal{R}_c = (1 - \kappa p) \mathcal{R}_0$ when $u_2 = u_3 = 0$, where $p = V_c^0/N^0 = \kappa u_1 / (m + \kappa u_1)$ is the vaccinated fraction.

Proof. By (6.10), $s = \frac{m + (1 - \kappa)\kappa u_1}{m + \kappa u_1} = 1 - \frac{\kappa^2 u_1}{m + \kappa u_1} = 1 - \kappa p$. Repeating the computation of Theorem 6.1 with $\mathbf{F} \rightarrow s\mathbf{F}$ and $\theta \rightarrow \theta_u$ yields (6.11). $\square$

TABLE 4. Controlled reproduction number (6.11) for constant controls (baseline parameters).

(u_1, u_2, u_3)	$(0, 0, 0)$	$(1, 0, 0)$	$(0, 1, 0)$
$\mathcal{R}_c$	3.954	0.615	1.222
(u_1, u_2, u_3)	$(0, 0, 1)$	$(0.5, 0.5, 0.5)$	$(1, 1, 1)$
$\mathcal{R}_c$	2.257	0.245	0.172

Thresholds. Setting $\mathcal{R}_c < 1$ with a single active control:

- *Quarantine alone:* $\delta u_2 > \frac{\beta \sigma}{\sigma + \mu - \epsilon \beta} - \theta$ (requires $\sigma + \mu > \epsilon \beta$), i.e. $u_2 > 1.586$. Since $u_2 \leq 1$, quarantine alone can never bring $\mathcal{R}_c$ below one with these parameters ($\mathcal{R}_c(0, 1, 0) = 1.222$).
- *Treatment alone:* $u_3 > 6.35 > 1$, likewise impossible.
- *Vaccination alone:* with $a = 1 - 1/\mathcal{R}_0 = 0.7471$ the condition $\kappa^2 u_1 / (m + \kappa u_1) > a$ is equivalent to $u_1 > \frac{a m}{\kappa(\kappa - a)} = 0.0478$, a vaccination rate $\kappa u_1 \approx 0.041 \text{ day}^{-1}$.

In terms of coverage, $(1 - \kappa p)\mathcal{R}_0 < 1$ is equivalent to the *herd-immunity threshold*

$$p > p_c := \frac{1}{\kappa} \left(1 - \frac{1}{\mathcal{R}_0}\right) = 0.879, \quad (6.12)$$

which coincides with $p = \kappa u_1 / (m + \kappa u_1)$ at the critical rate $u_1 = 0.0478$ computed above, confirming the consistency of the two derivations. Note that (6.12) is not $1 - 1/(\kappa \mathcal{R}_0)$; the latter would give 70% and under-estimate the required coverage for an imperfect vaccine.

6.7. Endemic Equilibrium.

Theorem 6.3 (Endemic equilibrium). *For $u_1 = u_2 = u_3 = 0$ the system has no equilibrium with $E + I > 0$ if $\mathcal{R}_0 \leq 1$, and exactly one if $\mathcal{R}_0 > 1$, namely $V^* = Q^* = H^* = 0$ and*

$$\lambda^* = \frac{(\sigma + \mu)(\mathcal{R}_0 - 1)}{c_1}, \quad c_1 = 1 + \frac{\sigma}{\theta} + \frac{\gamma_I \sigma}{\theta \mu}, \quad S^* = \frac{\Lambda}{\lambda^* + \mu}, \quad E^* = \frac{\lambda^* S^*}{\sigma + \mu}, \quad I^* = \frac{\sigma E^*}{\theta}, \quad R^* = \frac{\gamma_I I^*}{\mu}. \quad (6.13)$$

Proof. At an equilibrium the V -, Q - and H -equations read $-[(1 - \kappa)\lambda + \mu + \omega]V = 0$, $-\theta_Q Q = 0$, $-\theta_H H = 0$, hence $V = Q = H = 0$. Writing λ^* for the force of infection, (4.3) gives $S^* = \Lambda/(\lambda^* + \mu)$, (4.4) gives $E^* = \lambda^* S^*/(\sigma + \mu)$, (4.5) gives $I^* = \sigma E^*/\theta$ and (4.8) gives $R^* = \gamma_I I^*/\mu$. Put $c_2 = \epsilon + \sigma/\theta$, so $\epsilon E^* + I^* = c_2 E^*$ and $N^* = S^* + c'_1 E^*$ with $c'_1 = 1 + \sigma/\theta + \gamma_I \sigma/(\theta \mu) = c_1$. The consistency condition $\lambda^* N^* = \beta c_2 E^*$ becomes, after dividing by $\lambda^* \Lambda/(\lambda^* + \mu) > 0$,

$$1 + \frac{\lambda^* c_1}{\sigma + \mu} = \frac{\beta c_2}{\sigma + \mu} = \mathcal{R}_0,$$

which has the unique solution (6.13), positive if and only if $\mathcal{R}_0 > 1$. $\square$

For the baseline parameters $c_1 = 5368.7$, $\lambda^* = 1.1008 \times 10^{-4} \text{ day}^{-1}$, $S^* = 34,104$, $E^* = 18.8$, $I^* = 36.8$, $R^* = 100,702$ and $N^* = 134,861$ (the residual of (4.3)–(4.9) at this point is 1.3×10^{-15}). Because $\mu^{-1} = 75$ years, this equilibrium is approached only on demographic time scales, so within the 180-day horizon the epidemic is a transient wave rather than an approach to the endemic state.

7. OPTIMAL CONTROL FORMULATION

The objective functional penalizes infection burden and control costs:

$$J(u_1, u_2, u_3) = \int_0^T \left[A I(t) + B H(t) + C Q(t) + \frac{d_1}{2} u_1^2 + \frac{d_2}{2} u_2^2 + \frac{d_3}{2} u_3^2 \right] dt, \quad (7.1)$$

with $A, B, C > 0$ the infection-burden weights and $d_1, d_2, d_3 > 0$ the control-cost weights. The optimal control problem is

$$\min_{(u_1, u_2, u_3) \in \mathcal{U}} J(u_1, u_2, u_3) \quad \text{subject to (4.3)–(4.9) and (4.11)}. \quad (7.2)$$

7.1. Structure of the State Equations. The right-hand side of the state system is *affine in the control*:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = \mathbf{f}_0(\mathbf{x}) + u_1 \mathbf{g}_1(\mathbf{x}) + u_2 \mathbf{g}_2(\mathbf{x}) + u_3 \mathbf{g}_3(\mathbf{x}), \quad (7.3)$$

with $\mathbf{g}_1 = \kappa S(-1, 0, 0, 0, 0, 0, 1)^\top$, $\mathbf{g}_2 = \delta I(0, 0, -1, 1, 0, 0, 0)^\top$, $\mathbf{g}_3 = \delta_H I(0, 0, -1, 0, 1, 0, 0)^\top$. By Theorems 5.2 and 5.3, all states lie in the compact set Ω' and hence $\|\mathbf{f}\| \leq M := \sup_{\Omega' \times [0, 1]^3} \|\mathbf{f}\|_1 < \infty$.

7.2. Existence of Optimal Controls.

Theorem 7.1 (Existence of Optimal Controls). *There exists $(u_1^*, u_2^*, u_3^*) \in \mathcal{U}$ such that $J(u_1^*, u_2^*, u_3^*) = \min_{\mathcal{U}} J$.*

Proof. Step 1 (finite infimum). $\mathcal{U}$ is nonempty and, by Theorem 5.3, every admissible control produces a bounded state, so $0 \leq J(\mathbf{u}) \leq T[(A + B + C) \max\{N_0, \Lambda/\mu\} + \frac{1}{2}(d_1 + d_2 + d_3)]$. Let $m = \inf_{\mathcal{U}} J \in [0, \infty)$ and choose $\mathbf{u}^k \in \mathcal{U}$ with $J(\mathbf{u}^k) \rightarrow m$; let $\mathbf{x}^k$ be the corresponding states.

Step 2 (compactness). By (2.4) and $\|\mathbf{f}\| \leq M$, for $0 \leq t_1 < t_2 \leq T$,

$$\|\mathbf{x}^k(t_2) - \mathbf{x}^k(t_1)\|_1 \leq \frac{M}{\Gamma(\alpha)} \left[\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds + \int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) ds \right] \leq \frac{2M}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha,$$

so $\{\mathbf{x}^k\}$ is uniformly bounded and equi-Hölder, and by the Arzelà-Ascoli theorem a subsequence converges uniformly to some $\bar{\mathbf{x}}$. Since $\mathcal{U}$ is bounded, closed and convex in L^∞ , the Banach-Alaoglu theorem gives a further subsequence with $\mathbf{u}^k \rightharpoonup^* \bar{\mathbf{u}} \in \mathcal{U}$ (and weakly in L^2).

Step 3 ($\bar{\mathbf{x}}$ is the state of $\bar{\mathbf{u}}$). Using (7.3),

$$\int_0^t (t-s)^{\alpha-1} \mathbf{g}_i(\mathbf{x}^k) u_i^k ds = \int_0^t (t-s)^{\alpha-1} [\mathbf{g}_i(\mathbf{x}^k) - \mathbf{g}_i(\bar{\mathbf{x}})] u_i^k ds + \int_0^t (t-s)^{\alpha-1} \mathbf{g}_i(\bar{\mathbf{x}}) u_i^k ds.$$

The first term tends to zero because $\mathbf{g}_i$ is Lipschitz on Ω' , $|u_i^k| \leq 1$ and $\mathbf{x}^k \rightarrow \bar{\mathbf{x}}$ uniformly; the second tends to $\int_0^t (t-s)^{\alpha-1} \mathbf{g}_i(\bar{\mathbf{x}}) \bar{u}_i ds$ by weak-* convergence, because $s \mapsto (t-s)^{\alpha-1} \mathbf{g}_i(\bar{\mathbf{x}}(s)) \in L^1$. The terms involving $\mathbf{f}_0$ converge uniformly. Passing to the limit in (2.4) shows $\bar{\mathbf{x}} = \mathbf{x}(\bar{\mathbf{u}})$.

Step 4 (lower semicontinuity). The state-dependent part of J converges because $\mathbf{x}^k \rightarrow \bar{\mathbf{x}}$ uniformly, while $\int_0^T u_i^2 dt$ is weakly lower semicontinuous in L^2 . Hence $J(\bar{\mathbf{u}}) \leq \liminf_k J(\mathbf{u}^k) = m$, so $\bar{\mathbf{u}}$ is a minimizer. $\square$

8. FRACTIONAL PONTYAGIN MAXIMUM PRINCIPLE

8.1. Hamiltonian. Let $\lambda = (\lambda_1, \dots, \lambda_7)$ be the adjoint variables associated with (S, E, I, Q, H, R, V) . The Hamiltonian is

$$\begin{aligned} \mathcal{H} = & AI + BH + CQ + \frac{d_1}{2} u_1^2 + \frac{d_2}{2} u_2^2 + \frac{d_3}{2} u_3^2 \\ & + \lambda_1 [\Lambda - \lambda(t)S - \kappa u_1 S - \mu S + \omega V] \\ & + \lambda_2 [\lambda(t)S + (1 - \kappa)\lambda(t)V - (\sigma + \mu)E] \\ & + \lambda_3 [\sigma E - (\mu + d_I + \gamma_I + \delta u_2 + \delta_H u_3)I] \\ & + \lambda_4 [\delta u_2 I - (\mu + \gamma_Q + d_Q)Q] \\ & + \lambda_5 [\delta_H u_3 I - (\mu + \gamma_H + d_H)H] \\ & + \lambda_6 [\gamma_I I + \gamma_Q Q + \gamma_H H - \mu R] \\ & + \lambda_7 [\kappa u_1 S - (1 - \kappa)\lambda(t)V - (\mu + \omega)V]. \end{aligned} \tag{8.1}$$

Here $\lambda(t) = \beta(\epsilon E + I)/N$ is the force of infection (4.10), which depends on *all* state variables through $N = S + E + I + Q + H + R + V$.

Theorem 8.1 (Fractional Pontryagin Maximum Principle). *Let $(u_1^*, u_2^*, u_3^*) \in \mathcal{U}$ be optimal with state $\mathbf{x}^*(t)$. There exist adjoint functions $\lambda_i \in C([0, T])$, $i = 1, \dots, 7$, satisfying*

$${}_t D_T^\alpha \lambda_i = + \frac{\partial \mathcal{H}}{\partial x_i}, \quad \lambda_i(T) = 0, \quad i = 1, \dots, 7, \quad (8.2)$$

and the optimal controls are

$$u_1^*(t) = \min \left(1, \max \left(0, \frac{(\lambda_1 - \lambda_7) \kappa S^*}{d_1} \right) \right), \quad (8.3)$$

$$u_2^*(t) = \min \left(1, \max \left(0, \frac{(\lambda_3 - \lambda_4) \delta I^*}{d_2} \right) \right), \quad (8.4)$$

$$u_3^*(t) = \min \left(1, \max \left(0, \frac{(\lambda_3 - \lambda_5) \delta_H I^*}{d_3} \right) \right). \quad (8.5)$$

Remark 8.1 (Sign convention). *The sign in (8.2) is + because the right-sided derivative reduces to $-d/dt$ when $\alpha = 1$: ${}_t D_T^1 \lambda = -\lambda'$, so (8.2) becomes the classical costate equation $\lambda'_i = -\partial \mathcal{H} / \partial x_i$. (Writing $-\partial \mathcal{H} / \partial x_i$ on the right would produce $\lambda'_i = +\partial \mathcal{H} / \partial x_i$, an exponentially unstable backward system.)*

8.2. Derivation.

Proof of Theorem 8.1. Write the state system as ${}_t D^\alpha \mathbf{x} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ and the running cost as $L(\mathbf{x}, \mathbf{u})$, so that $\mathcal{H} = L + \lambda^\top \mathbf{f}$. Let $\mathbf{w} \in \mathcal{U}$ be arbitrary, $\mathbf{v} = \mathbf{w} - \mathbf{u}^*$, and $\mathbf{u}^\epsilon = \mathbf{u}^* + \epsilon \mathbf{v} \in \mathcal{U}$ for $\epsilon \in (0, 1]$ (convexity).

Step 1 (state variation). By the Lipschitz estimate of Lemma 5.1 and Lemma 2.6, the states satisfy $\|\mathbf{x}^\epsilon - \mathbf{x}^*\|_\infty = O(\epsilon)$, and $\xi = \lim_{\epsilon \rightarrow 0} (\mathbf{x}^\epsilon - \mathbf{x}^*) / \epsilon$ solves the linearized problem

$${}_t D_t^\alpha \xi = \mathbf{f}_x \xi + \mathbf{f}_u \mathbf{v}, \quad \xi(0) = 0, \quad (8.6)$$

with $\mathbf{f}_x, \mathbf{f}_u$ evaluated along $(\mathbf{x}^*, \mathbf{u}^*)$.

Step 2 (first variation of J). Optimality of $\mathbf{u}^*$ gives

$$0 \leq \lim_{\epsilon \rightarrow 0^+} \frac{J(\mathbf{u}^\epsilon) - J(\mathbf{u}^*)}{\epsilon} = \int_0^T (L_x \xi + L_u \mathbf{v}) dt, \quad (8.7)$$

where $L_x = (0, 0, A, C, B, 0, 0)$ and $L_u = (d_1 u_1^*, d_2 u_2^*, d_3 u_3^*)$.

Step 3 (adjoint). Let λ solve (8.2), i.e. ${}_t D_T^\alpha \lambda = L_x^\top + \mathbf{f}_x^\top \lambda$, $\lambda(T) = 0$ (a linear terminal value problem, solvable by time reversal, Lemma 2.7). Multiply (8.6) by $\lambda^\top$ and integrate, using (2.8); the boundary term $[\xi^\top {}_t I_T^{1-\alpha} \lambda]_0^T$ vanishes because $\xi(0) = 0$ and $({}_t I_T^{1-\alpha} \lambda)(T) = 0$. Thus

$$\int_0^T \lambda^\top (\mathbf{f}_x \xi + \mathbf{f}_u \mathbf{v}) dt = \int_0^T \xi^\top (L_x^\top + \mathbf{f}_x^\top \lambda) dt \implies \int_0^T \lambda^\top \mathbf{f}_u \mathbf{v} dt = \int_0^T L_x \xi dt.$$

Substituting into (8.7),

$$0 \leq \int_0^T (L_u + \lambda^\top \mathbf{f}_u) \mathbf{v} dt = \int_0^T \frac{\partial \mathcal{H}}{\partial \mathbf{u}} (\mathbf{w} - \mathbf{u}^*) dt. \quad (8.8)$$

Step 4 (pointwise minimization). Since $\mathbf{w} \in \mathcal{U}$ is arbitrary, (8.8) implies $\partial_{u_i} \mathcal{H}(t) (w_i - u_i^*(t)) \geq 0$ for a.e. t and all $w_i \in [0, 1]$. From (8.1),

$$\frac{\partial \mathcal{H}}{\partial u_1} = d_1 u_1 - \kappa S(\lambda_1 - \lambda_7), \quad \frac{\partial \mathcal{H}}{\partial u_2} = d_2 u_2 - \delta I(\lambda_3 - \lambda_4), \quad \frac{\partial \mathcal{H}}{\partial u_3} = d_3 u_3 - \delta_H I(\lambda_3 - \lambda_5), \quad (8.9)$$

and $\partial^2 \mathcal{H} / \partial u_i^2 = d_i > 0$, so $\mathcal{H}$ is strictly convex in each u_i . The pointwise minimizer over $[0, 1]$ of a strictly convex quadratic is the projection of its unconstrained root $\bar{u}_i$ onto $[0, 1]$, with $\bar{u}_1 = \kappa S(\lambda_1 - \lambda_7) / d_1$, $\bar{u}_2 = \delta I(\lambda_3 - \lambda_4) / d_2$, $\bar{u}_3 = \delta_H I(\lambda_3 - \lambda_5) / d_3$, which is (8.3)-(8.5). $\square$

8.3. Explicit Adjoint Equations. Differentiating (8.1) with respect to each state requires $\partial \lambda / \partial x_j$ for $\lambda = \beta(\epsilon E + I) / N$:

$$\frac{\partial \lambda}{\partial S} = -\frac{\lambda}{N}, \quad \frac{\partial \lambda}{\partial E} = \frac{\beta \epsilon - \lambda}{N}, \quad \frac{\partial \lambda}{\partial I} = \frac{\beta - \lambda}{N}, \quad \frac{\partial \lambda}{\partial x_j} = -\frac{\lambda}{N}, \quad x_j \in \{Q, H, R, V\}. \quad (8.10)$$

Let $\lambda_S = \lambda(t) = \beta(\epsilon E^* + I^*) / N^*$, define the *incidence-weighted costate difference*

$$\Phi = (\lambda_2 - \lambda_1) S^* + (1 - \kappa)(\lambda_2 - \lambda_7) V^*, \quad (8.11)$$

so that $\mathcal{H}$ contains the term $\lambda_S \Phi$, and denote $\theta_I = \mu + d_I + \gamma_I + \delta u_2^* + \delta_H u_3^*$, $\theta_Q = \mu + \gamma_Q + d_Q$, $\theta_H = \mu + \gamma_H + d_H$. The adjoint system (8.2) then reads

$${}_t D_T^\alpha \lambda_1 = \lambda_S (\lambda_2 - \lambda_1) - \frac{\lambda_S \Phi}{N^*} - (\kappa u_1^* + \mu) \lambda_1 + \kappa u_1^* \lambda_7, \quad (8.12)$$

$${}_t D_T^\alpha \lambda_2 = \frac{\Phi (\beta \epsilon - \lambda_S)}{N^*} - (\sigma + \mu) \lambda_2 + \sigma \lambda_3, \quad (8.13)$$

$${}_t D_T^\alpha \lambda_3 = A + \frac{\Phi (\beta - \lambda_S)}{N^*} - \theta_I \lambda_3 + \delta u_2^* \lambda_4 + \delta_H u_3^* \lambda_5 + \gamma_I \lambda_6, \quad (8.14)$$

$${}_t D_T^\alpha \lambda_4 = C - \frac{\lambda_S \Phi}{N^*} - \theta_Q \lambda_4 + \gamma_Q \lambda_6, \quad (8.15)$$

$${}_t D_T^\alpha \lambda_5 = B - \frac{\lambda_S \Phi}{N^*} - \theta_H \lambda_5 + \gamma_H \lambda_6, \quad (8.16)$$

$${}_t D_T^\alpha \lambda_6 = -\frac{\lambda_S \Phi}{N^*} - \mu \lambda_6, \quad (8.17)$$

$${}_t D_T^\alpha \lambda_7 = (1 - \kappa) \lambda_S (\lambda_2 - \lambda_7) - \frac{\lambda_S \Phi}{N^*} + \omega \lambda_1 - (\mu + \omega) \lambda_7, \quad (8.18)$$

with terminal conditions $\lambda_i(T) = 0$ for $i = 1, \dots, 7$. The terms containing $\lambda_S \Phi / N^*$ and $\Phi(\beta \epsilon - \lambda_S) / N^*$ arise from the dependence of the incidence on the total population N via (8.10); they are often neglected by treating N as constant. We retain them, so that (8.12)–(8.18) are the exact costate equations. They were checked against symbolic differentiation of (8.1) (agreement to machine precision at random states) and, more importantly, against finite differences of J (Section 9).

8.4. Behavior of the Costates and Controls Near the Terminal Time. Let $\tau = T - t$ and $\bar{\lambda}_i(\tau) = \lambda_i(T - \tau)$. By Lemma 2.7, (8.2) becomes the *forward* Caputo problem ${}^C D_\tau^\alpha \bar{\lambda}_i = g_i(\tau)$, $\bar{\lambda}_i(0) = 0$, with $g_i = \partial \mathcal{H} / \partial x_i$ evaluated at $t = T - \tau$, i.e. $\bar{\lambda} = I_\tau^\alpha \mathbf{g}$.

Proposition 8.1 (Terminal behavior). *As $t \rightarrow T^-$,*

$$\lambda_3(t) \sim \frac{A(T-t)^\alpha}{\Gamma(\alpha+1)}, \quad \lambda_4(t) \sim \frac{C(T-t)^\alpha}{\Gamma(\alpha+1)}, \quad \lambda_5(t) \sim \frac{B(T-t)^\alpha}{\Gamma(\alpha+1)}, \quad (8.19)$$

while $\lambda_1, \lambda_2, \lambda_6, \lambda_7 = O((T-t)^{2\alpha})$. Consequently all three optimal controls vanish at $t = T$; $u_1^* = O((T-t)^{2\alpha})$, $u_2^*(t) \sim \frac{\delta I^*(T)(A-C)}{d_2 \Gamma(\alpha+1)}(T-t)^\alpha > 0$ for $A > C$, and if $B > A$ then $u_3^* \equiv 0$ on a left neighborhood of T .

Proof. At $\tau = 0$ all costates vanish, so $g_3(0) = A$, $g_4(0) = C$, $g_5(0) = B$ and $g_i(0) = 0$ for $i \in \{1, 2, 6, 7\}$ (every term in (8.12), (8.13), (8.17), (8.18) is proportional to a costate). Continuity of g_i and Lemma 2.1 give $\bar{\lambda}_i = I^\alpha g_i = g_i(0)\tau^\alpha / \Gamma(\alpha+1) (1 + o(1))$ for $i = 3, 4, 5$; for the other indices $g_i = O(\tau^\alpha)$, so $\bar{\lambda}_i = O(\tau^{2\alpha})$. The statements on the controls follow by inserting these expansions into (8.3)-(8.5): for u_3^* the unconstrained root is $\sim \delta_H I^*(A-B)\tau^\alpha / (d_3 \Gamma(\alpha+1)) < 0$. $\square$

TABLE 5. Numerical check of Proposition 8.1: $\lambda_i(t) \Gamma(\alpha+1) / (T-t)^\alpha$ at $T-t = 0.5$ days (predicted limits $A = 1, C = 0.5, B = 2$).

α	$\lambda_3 (\rightarrow 1)$	$\lambda_4 (\rightarrow 0.5)$	$\lambda_5 (\rightarrow 2)$
1.0	0.976	0.490	1.960
0.9	0.970	0.487	1.951
0.7	0.957	0.482	1.931

Since $B = 2 > A = 1$ in Table 8, moving a patient from I to H is more expensive per day than leaving him in I , which explains why the treatment control is essentially inactive in the simulations below; with $B = 0.5 < A$ the same solver gives $\max u_3^* \approx 0.20$.

9. NUMERICAL ALGORITHM AND VERIFICATION

The optimality system, comprising the forward state equations (4.3)-(4.9), the backward adjoint equations (8.12)-(8.18), and the control formulas (8.3)-(8.5), is solved by a forward-backward sweep (FBS) algorithm. Each forward or backward pass uses the Adams-Bashforth-Moulton (ABM) predictor-corrector for Caputo fractional differential equations [8].

9.1. Derivation of the Predictor-Corrector Scheme. On a uniform grid $t_n = nh, h = T/N$, write (2.4) at t_{n+1} and split the integral over the cells $[t_j, t_{j+1}]$:

$$\mathbf{y}(t_{n+1}) = \mathbf{y}_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} \mathbf{f}(s, \mathbf{y}(s)) ds.$$

Predictor. Replacing $\mathbf{f}$ on each cell by the constant $\mathbf{f}_j = \mathbf{f}(t_j, \mathbf{y}_j)$ and using $\int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\alpha-1} ds = \frac{h^\alpha}{\alpha} [(n+1-j)^\alpha - (n-j)^\alpha]$,

$$\mathbf{y}_{n+1}^P = \mathbf{y}_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^n [(n-j+1)^\alpha - (n-j)^\alpha] \mathbf{f}(t_j, \mathbf{y}_j). \quad (9.1)$$

Corrector. Replacing $\mathbf{f}$ by its piecewise-linear interpolant through the nodes, with the unknown $\mathbf{f}_{n+1}$ approximated by $\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}^P)$, and integrating the weights exactly,

$$\mathbf{y}_{n+1} = \mathbf{y}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left[\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}^P) + \sum_{j=0}^n b_{j,n+1} \mathbf{f}(t_j, \mathbf{y}_j) \right], \quad (9.2)$$

with weights

$$b_{0,n+1} = n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, \quad b_{j,n+1} = (n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1}, \quad 1 \leq j \leq n. \quad (9.3)$$

(The first weight differs from the generic expression because the interpolant on the first cell has a different end-point contribution.) For sufficiently smooth $\mathbf{f}$ the global error satisfies

$$\max_{0 \leq j \leq N} \|\mathbf{y}(t_j) - \mathbf{y}_j\| = O(h^p), \quad p = \min\{2, 1 + \alpha\} \quad (9.4)$$

[8]. The scheme has $O(N^2)$ cost because every step uses the whole history $\{\mathbf{f}_j\}$; for $N = 720$ and seven equations each pass needs about 1.8×10^6 weighted additions.

Backward pass. With $\tau = T - t$ and $\bar{\lambda}_i(\tau) = \lambda_i(T - \tau)$, Lemma 2.7 turns (8.2) into the forward problem ${}^C D_\tau^\alpha \bar{\lambda} = \mathbf{g}(\tau)$, $\bar{\lambda}(0) = 0$, with $\mathbf{g}(\tau)$ given by the right-hand sides of (8.12)–(8.18) evaluated at the state and controls at time $T - \tau$. The same scheme (9.1)–(9.2) is applied and the result is reversed.

Algorithm 9.1 (Forward-backward sweep). 1. Input: T, N, α , tolerance ε_{tol} , relaxation $\rho \in (0, 1]$, initial conditions, parameters.

2. Initialize: $u_1^{(0)} = u_2^{(0)} = u_3^{(0)} \equiv 0$.
3. Forward: solve (4.3)–(4.9) via ABM.
4. Backward: solve (8.12)–(8.18) via ABM in the reversed time $\tau = T - t$.
5. Update: compute $u_{i,\text{new}}$ from (8.3)–(8.5); set $u_i^{(k+1)} = (1 - \rho)u_i^{(k)} + \rho u_{i,\text{new}}$.
6. Stop if $\max_i \|u_{i,\text{new}} - u_i^{(k)}\|_\infty < \varepsilon_{\text{tol}}$; otherwise return to Step 3.
7. Output: optimal states, adjoints, controls.

We use $N = 720$ ($h = 0.25$ day), $\rho = 0.2$ and $\varepsilon_{\text{tol}} = 10^{-6}$; the sweep converged in 66, 65, 65 and 64 iterations for $\alpha = 1.0, 0.9, 0.8, 0.7$. (A relaxation of $\rho = 0.5$ did not converge for $\alpha = 1$.) States are clipped at zero after each step to suppress round-off.

9.2. Verification.

(a) Convergence of the ABM scheme. For the test problem ${}^C D^\alpha y = -y$, $y(0) = 1$, whose exact solution is $E_\alpha(-t^\alpha)$ (Lemma 2.3), the error at $T = 5$ and the observed order of convergence are given in Table 6. The observed orders agree with (9.4): 2.0, 1.9, 1.8, 1.7 for $\alpha = 1.0, 0.9, 0.8, 0.7$.

TABLE 6. Absolute error at $T = 5$ for ${}^C D^\alpha y = -y$ and experimental order of convergence (EOC, last refinement).

α	$N = 20$	40	80	160	320	EOC	theory $\min\{2, 1 + \alpha\}$
1.0	4.37×10^{-4}	9.71×10^{-5}	2.30×10^{-5}	5.62×10^{-6}	1.39×10^{-6}	2.02	2.0
0.9	7.52×10^{-4}	1.83×10^{-4}	4.70×10^{-5}	1.24×10^{-5}	3.31×10^{-6}	1.91	1.9
0.8	1.03×10^{-3}	2.66×10^{-4}	7.27×10^{-5}	2.04×10^{-5}	5.81×10^{-6}	1.81	1.8
0.7	1.27×10^{-3}	3.44×10^{-4}	9.91×10^{-5}	2.95×10^{-5}	8.92×10^{-6}	1.73	1.7

(b) Integer-order limit. For $\alpha = 1$ and $\mathbf{u} = 0$, the ABM solution with $N = 720$ agrees with a high-accuracy Runge-Kutta solution (DOP853, relative tolerance 10^{-11}): the largest absolute deviations over $[0, 180]$ are 1.36 (S), 0.38 (E), 0.47 (I) and 0.80 (R) persons, i.e. 1.7×10^{-4} of the infection peak.

(c) Gradient test of the adjoint system. By (8.8), the directional derivative of J at an admissible control $\mathbf{u}$ in direction $\mathbf{v}$ equals $\int_0^T \partial_{\mathbf{u}} \mathcal{H} \cdot \mathbf{v} dt$ evaluated with the costates (8.12)-(8.18). We compared this quantity with a central finite difference of J ($\varepsilon = 10^{-4}$) at the interior controls $u_1 = 0.3 + 0.1 \sin(t/20)$, $u_2 = 0.4 + 0.1 \cos(t/30)$, $u_3 = 0.2$ in the direction $v_1 = \sin(t/15)e^{-t/150}$, $v_2 = \frac{1}{2} \cos(t/25)$, $v_3 = \sin(t/40)$. Table 7 shows agreement to better than 0.3%, the residual being the discretization error of the (continuous) adjoint, whereas a costate system with the wrong sign in (8.2) produces values of order 10^{15} – 10^{22} . This test confirms both the exact costate equations and the control characterization.

TABLE 7. Adjoint-based directional derivative of J versus finite differences.

α	adjoint gradient	finite difference	relative error
1.0	478.90	479.50	1.3×10^{-3}
0.9	457.07	458.36	2.8×10^{-3}
0.7	611.61	613.03	2.3×10^{-3}

10. NUMERICAL SIMULATIONS

10.1. Parameter Values and Initial Conditions. Table 8 lists the parameter values used in all simulations. Initial conditions represent an early outbreak in a largely susceptible population:

$$S_0 = 10,000, E_0 = 50, I_0 = 20, Q_0 = 0, H_0 = 0, R_0 = 0, V_0 = 0.$$

The time horizon is $T = 180$ days with $N = 720$ steps ($h = 0.25$ day).

TABLE 8. Parameter values for COVID-19 SEIQHRV simulations

Parameter	Value	Rationale
Λ	5.0	daily recruitment
μ	$1/(75 \times 365)$	natural mortality
β	0.35	COVID-19 contact rate
ϵ	0.30	sub-clinical transmissibility
σ	1/5	incubation rate (5 days)
γ_I	1/10	recovery (10 days, mild)
γ_H	1/14	hospital recovery (14 days)
γ_Q	1/12	quarantine recovery (12 days)
δ	0.40	quarantine rate (per infected)
δ_H	0.10	hospitalization rate
d_I	0.002	IFR for infectious class
d_Q	0.001	mortality in quarantine
d_H	0.010	hospital case fatality
κ	0.85	vaccine efficacy
ω	1/180	vaccine waning (6 months)
ϕ	0.15	breakthrough modifier ($= 1 - \kappa$)
A, B, C	1.0, 2.0, 0.5	burden weights
d_1, d_2, d_3	60, 40, 50	control costs

10.2. **Simulation Scenarios.** Each of four fractional orders $\alpha \in \{1.0, 0.9, 0.8, 0.7\}$ is simulated under:

- (1) **No control:** $u_1 = u_2 = u_3 = 0$.
- (2) **Vaccination only:** $u_2 = u_3 = 0$ and u_1 optimized.
- (3) **Quarantine only:** $u_1 = u_3 = 0$ and u_2 optimized.
- (4) **Treatment only:** $u_1 = u_2 = 0$ and u_3 optimized.
- (5) **Combined optimal:** all three controls optimized.

In scenarios 2–4 the sweep of Algorithm 9.1 is run with the inactive controls frozen at zero, so each single strategy is itself optimal within its class and the comparison with the combined strategy is fair. The control bound $u_i \leq 1$ corresponds, for vaccination, to a maximal rate $\kappa = 0.85 \text{ day}^{-1}$ of vaccinating susceptibles, an optimistic capacity that the framework can restrict by lowering the upper bound.

10.3. **Results.** Table 9 summarizes the effect of the fractional order on the uncontrolled and optimally controlled epidemic, Table 10 compares the objective values of the five strategies, and Table 11 gives sampled trajectories of the combined control scenario.

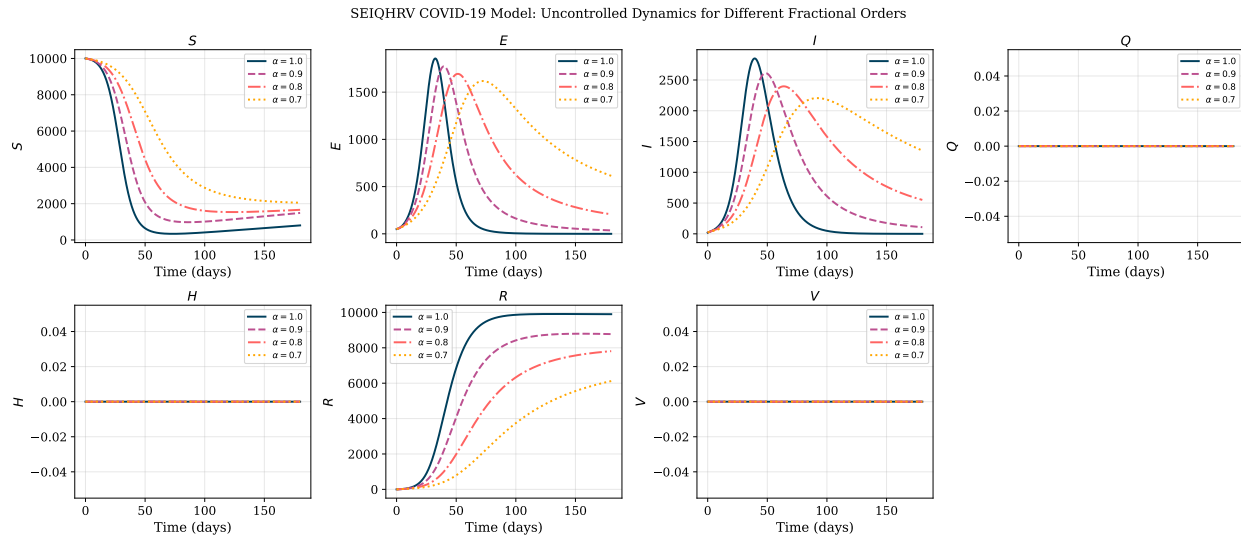


FIGURE 1. Uncontrolled SEIQRHV dynamics for $\alpha \in \{1.0, 0.9, 0.8, 0.7\}$. Smaller α delays and lowers the infection peak but prolongs the epidemic, leaving a long tail (e.g. $I(180) = 1353$ for $\alpha = 0.7$ versus 0.16 for $\alpha = 1$), consistent with the Mittag-Leffler growth law (6.8) and the algebraic relaxation of Remark 2.1.

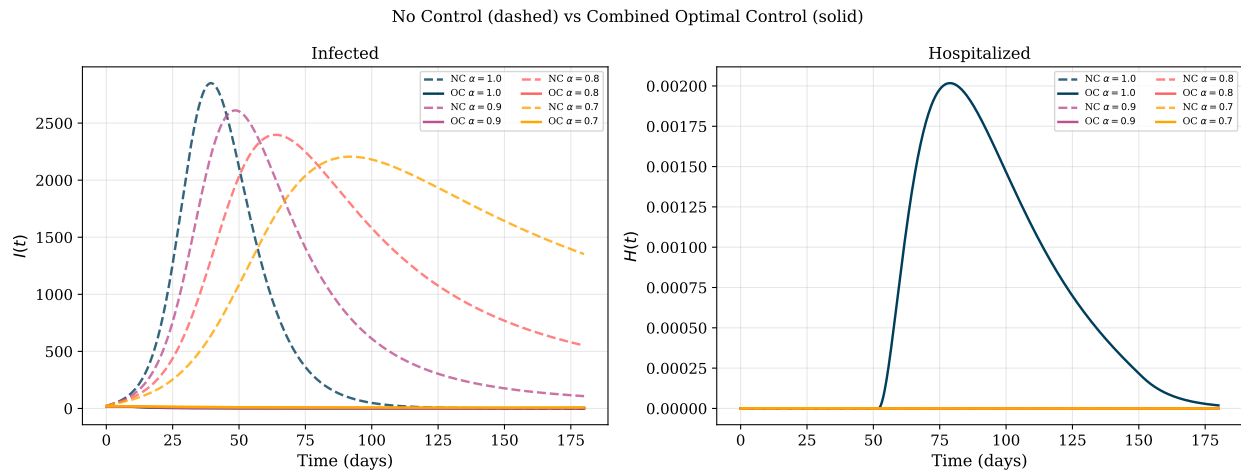


FIGURE 2. Infected (I) and hospitalized (H) populations under no control versus combined optimal control, for all fractional orders. Combined control suppresses the outbreak at its start; hospitalization remains negligible because the optimal treatment control is essentially inactive (see text).

11. RESULTS AND DISCUSSION

11.1. Memory Effects and Fractional Order. A central finding of this study is the measurable influence of the fractional order α on epidemic dynamics (Table 9). In the uncontrolled scenario, reducing α from 1.0 to 0.7 *lowers* the infection peak by 22.6% ($2850 \rightarrow 2206$) but *delays* it from day 39.5 to day 92, leaves $I(180) = 1353$ instead of 0.16 infected individuals, and increases the

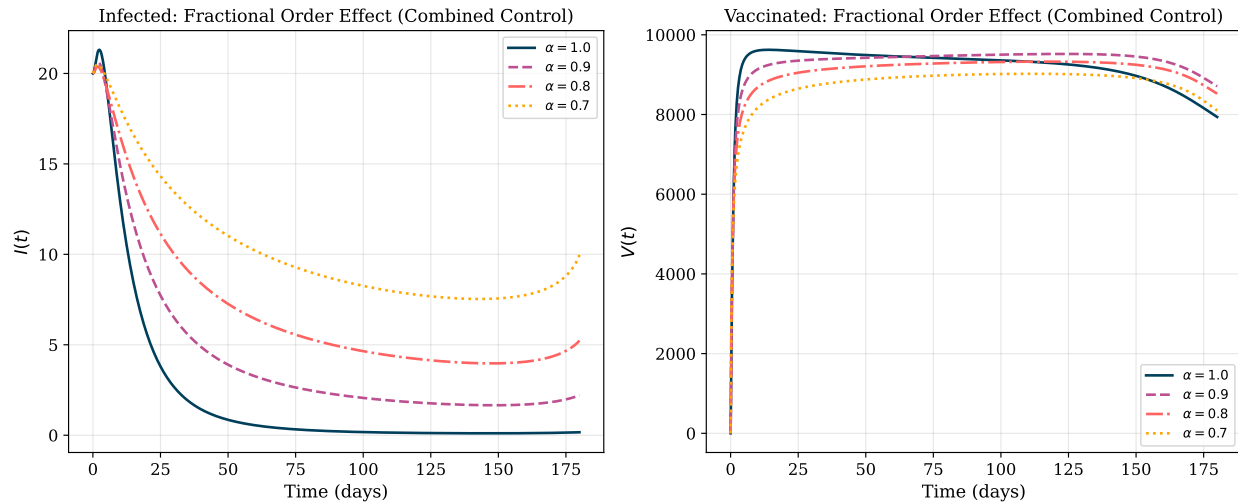


FIGURE 3. Fractional-order sensitivity: infected and vaccinated populations for $\alpha \in \{1.0, 0.9, 0.8, 0.7\}$ under combined control. Lower α leaves a higher residual level of infection and requires more sustained intervention.

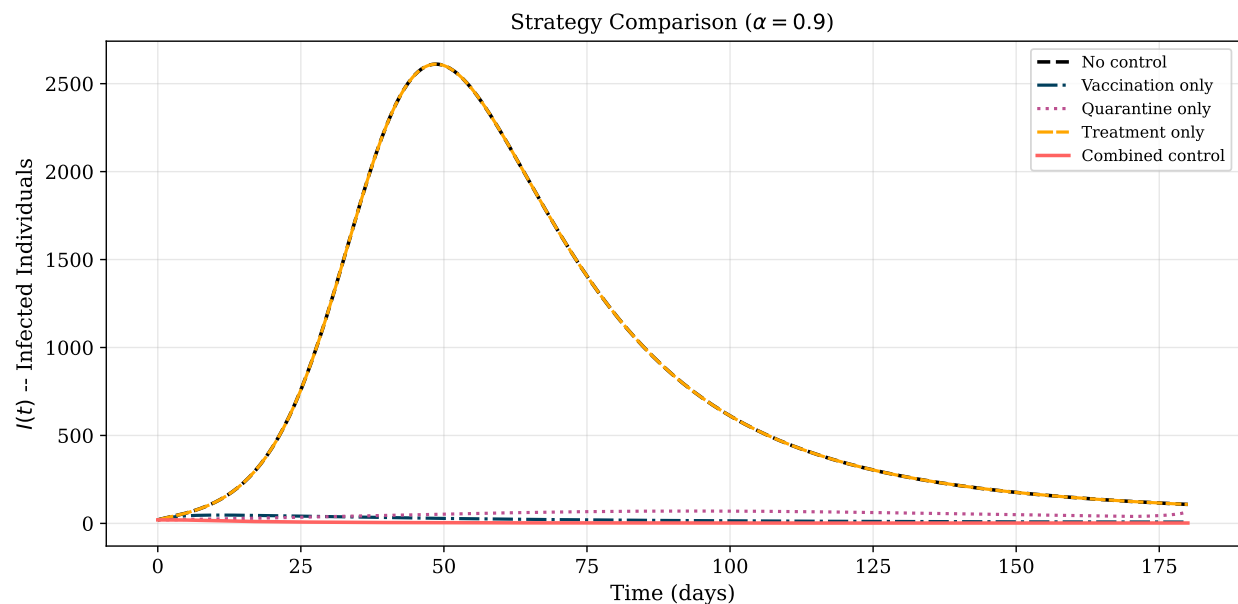


FIGURE 4. Individually optimized strategies (vaccination only, quarantine only, treatment only) against the combined optimal strategy for $\alpha = 0.9$. Treatment alone is indistinguishable from no control; quarantine alone rebounds as it is relaxed; the combined strategy gives the lowest infection curve.

cumulative infection burden $\int_0^T I dt$ by 155% ($9.9 \times 10^4 \rightarrow 2.5 \times 10^5$ person-days). The epidemic is therefore flatter but far more persistent. Both effects are predicted analytically in Sections 6.4 and 2: the early growth rate $r_{\text{eff}} = \zeta_+^{1/\alpha}$ falls from 0.166 to 0.077 day⁻¹ (doubling time 4.2 $\rightarrow$ 9.0 days, compared with peak delays by a factor 2.3), and the relaxation after the peak follows the

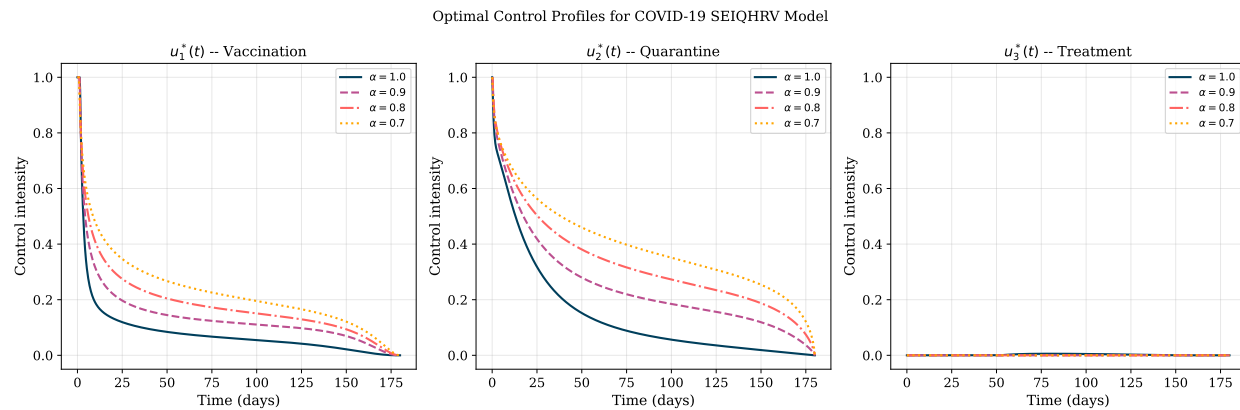


FIGURE 5. Optimal control profiles $u_1^*(t)$, $u_2^*(t)$, $u_3^*(t)$ for all fractional orders. Vaccination and quarantine start at the upper bound, decay rapidly and vanish at $t = T$ (Proposition 8.1); smaller α requires higher sustained levels. The treatment control is numerically negligible (maximum ≈ 0.006).

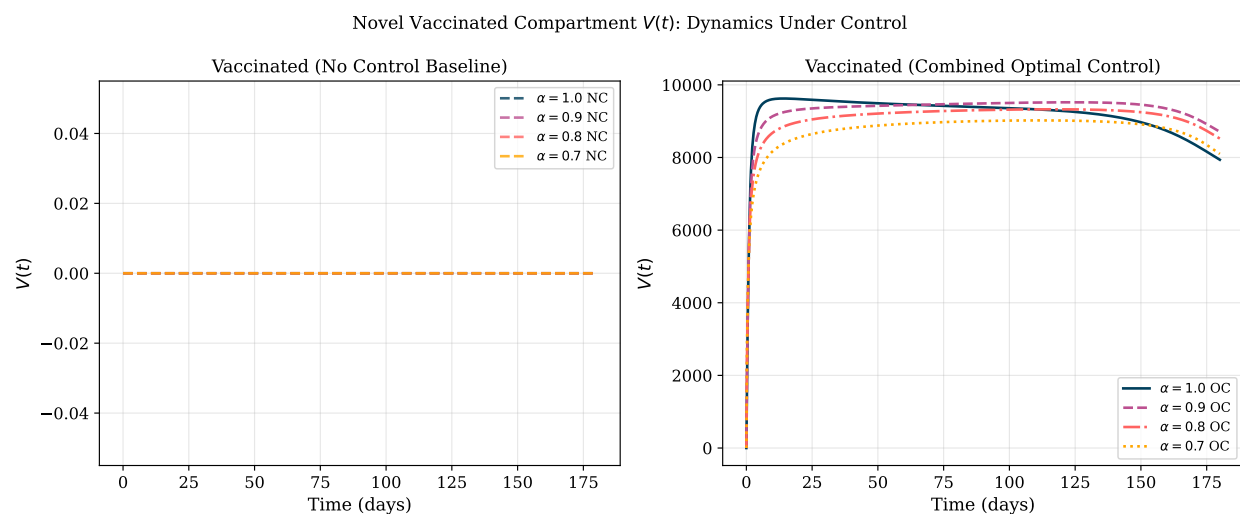


FIGURE 6. Vaccinated population $V(t)$ without control (left) and under combined optimal control (right) for different fractional orders. Under control, V saturates near 9×10^3 within weeks and then declines through waning once u_1^* has relaxed.

algebraic law $t^{-\alpha}$ rather than an exponential (Table 1). Biologically, this corresponds to prolonged viral shedding, delayed seroconversion and accumulated immunological memory, none of which are captured by integer-order models.

11.2. Impact of Individual Interventions. For $\alpha = 0.9$ (Tables 9–10, Fig. 4), the optimal vaccination-only strategy reduces the infection peak by 98.2% ($2611 \rightarrow 47.4$) and the cumulative burden by 97.5%, but the peak is reached only at day 10.75 because vaccine coverage must accumulate and waning (governed by ω) partly offsets it. The optimal quarantine-only strategy acts faster but less durably: it lowers the peak by 97.3% (70.3), yet this peak occurs late (day 93)

TABLE 9. Effect of α on the uncontrolled (NC) and combined-control (OC) epidemic: peak infected, time of peak, infected at T , cumulative infection burden $\int_0^T I dt$ (person-days), and optimal cost J^* .

α	No control				Combined control		
	peak I	t_{peak}	$I(T)$	$\int I$	peak I	$\int I$	J^*
1.0	2850	39.5	0.2	99494	21.3	370	1133
0.9	2611	48.5	107.9	149736	20.6	764	2018
0.8	2397	63.8	551.1	214325	20.4	1238	3127
0.7	2206	92.0	1352.7	253639	20.5	1851	4431

TABLE 10. Objective functional J (7.1) for each strategy (each optimized within its class).

α	No control	Vaccination only	Quarantine only	Treatment only	Combined
1.0	99494	2729	28132	99494	1133
0.9	149736	4386	32730	149736	2018
0.8	214325	6203	30088	213344	3127
0.7	253639	7905	23405	237684	4431

and the burden is only reduced by 93.7%, because Proposition 6.1 shows that quarantine alone can never bring the controlled reproduction number below one ($\mathcal{R}_c(0, 1, 0) = 1.22$), so the epidemic merely smolders at a low level. Treatment alone is never used by the optimizer ($u_3^* \equiv 0$) and yields J identical to no control. In contrast, vaccination alone can reach $\mathcal{R}_c < 1$ once coverage exceeds the herd-immunity threshold (6.12) (87.9%).

11.3. Superiority and Complementarity of Combined Control. The combined strategy gives the lowest objective value for every α (Table 10), as it must since it optimizes over a larger set. For $\alpha = 0.9$ it reduces the peak by 99.2% ($2611 \rightarrow 20.6$, essentially the initial level $I_0 = 20$, i.e. the outbreak is prevented) and the burden by 99.5% ($149,736 \rightarrow 764$). Its cost $J^* = 2018$ is 54% lower than the best single strategy (vaccination, 4386) and 94% lower than quarantine only (32,730); for $\alpha = 1.0$ and 0.7 the corresponding reductions relative to vaccination only are 58% and 44%. The two interventions are complementary rather than additive: quarantine acts within days and keeps I at its initial level while the vaccinated fraction builds up, whereas vaccination provides the durable protection that quarantine alone cannot (Proposition 6.1 and Table 4: $\mathcal{R}_c(1, 1, 1) = 0.17$).

11.4. Optimal Control Timing and Intensity. The optimal profiles (Fig. 5, Table 11) show a characteristic pattern: vaccination (u_1^*) and quarantine (u_2^*) start at the upper bound, decay within a few days (for $\alpha = 1$, $u_1^* \approx 0.09$ and $u_2^* \approx 0.17$ at day 45) and are then relaxed gradually, balancing marginal cost against marginal benefit. All controls vanish at $t = T$, with the power-law behavior $u_2^* \propto (T - t)^\alpha$ and $u_1^* = O((T - t)^{2\alpha})$ predicted by Proposition 8.1. Memory sustains the controls: at

TABLE 11. Selected state and control values under the combined optimal strategy ($T = 180$ days).

α	t	$I(t)$	$H(t)$	$Q(t)$	$V(t)$	$u_1^*(t)$	$u_2^*(t)$	$u_3^*(t)$
1.0	0	20.00	0.000	0.00	0.0	1.000	1.000	0.0000
1.0	30	2.67	0.000	13.74	9567.0	0.109	0.269	0.0000
1.0	60	0.55	0.001	1.76	9461.5	0.076	0.120	0.0036
1.0	90	0.21	0.002	0.24	9385.8	0.060	0.067	0.0054
1.0	120	0.13	0.001	0.05	9284.0	0.045	0.039	0.0032
1.0	150	0.11	0.000	0.02	8970.3	0.022	0.019	0.0002
1.0	180	0.16	0.000	0.00	7937.9	0.000	0.000	0.0000
0.9	0	20.00	0.000	0.00	0.0	1.000	1.000	0.0000
0.9	30	6.50	0.000	20.44	9373.5	0.181	0.379	0.0000
0.9	60	3.26	0.000	7.91	9441.9	0.135	0.251	0.0000
0.9	90	2.25	0.000	3.75	9489.4	0.116	0.197	0.0000
0.9	120	1.81	0.000	2.19	9519.5	0.101	0.162	0.0000
0.9	150	1.66	0.000	1.43	9454.6	0.069	0.119	0.0000
0.9	180	2.21	0.000	0.74	8709.0	0.000	0.000	0.0000
0.8	0	20.00	0.000	0.00	0.0	1.000	1.000	0.0000
0.8	30	10.03	0.000	24.56	9097.9	0.254	0.470	0.0000
0.8	60	6.43	0.000	15.08	9243.1	0.189	0.351	0.0000
0.8	90	4.95	0.000	9.83	9306.1	0.159	0.289	0.0000
0.8	120	4.22	0.000	6.95	9326.7	0.134	0.243	0.0000
0.8	150	3.98	0.000	5.14	9247.8	0.094	0.187	0.0000
0.8	180	5.23	0.000	3.18	8521.6	0.000	0.000	0.0000
0.7	0	20.00	0.000	0.00	0.0	1.000	1.000	0.0000
0.7	30	13.45	0.000	25.93	8717.8	0.324	0.537	0.0000
0.7	60	10.21	0.000	20.93	8925.9	0.248	0.432	0.0000
0.7	90	8.60	0.000	16.69	9004.6	0.207	0.369	0.0000
0.7	120	7.76	0.000	13.61	9018.1	0.172	0.317	0.0000
0.7	150	7.56	0.000	11.23	8913.8	0.121	0.254	0.0000
0.7	180	9.97	0.000	7.88	8098.3	0.000	0.000	0.0000

day 45, $u_1^* = 0.28$ and $u_2^* = 0.48$ for $\alpha = 0.7$, and the optimal cost increases from $J^* = 1133$ at $\alpha = 1$ to 4431 at $\alpha = 0.7$, a factor 3.9. The treatment control u_3^* is numerically negligible (maximum 0.006 for $\alpha = 1$ and below 10^{-6} for $\alpha \leq 0.9$), as anticipated from the terminal analysis: with the burden weights $B = 2 > A = 1$ a hospitalized day costs more than an infected day, so diverting patients to H is not worthwhile unless $B \lesssim A$. This conclusion is therefore specific to the chosen weights; for $B = 0.5$ the optimizer uses u_3^* up to about 0.20.

11.5. Vaccinated Compartment Dynamics. Under combined control $V(t)$ rises to about 9×10^3 within the first month (9567, 9374, 9098, 8718 at day 30 for $\alpha = 1.0, \dots, 0.7$), then declines as waning (ωV) is no longer compensated once u_1^* has relaxed; $V(180)$ is 7938, 8709, 8522, 8098. For smaller α the initial build-up is slower, because the early dynamics are slower, so more of the vaccination effort is spread over time. These observations suggest that booster scheduling matters most when memory effects are pronounced. We stress that the very rapid early build-up reflects the permissive bound $u_1 \leq 1$ (vaccination of up to 85% of susceptibles per day); realistic capacity constraints would reduce the early benefit and should be examined in applications.

11.6. Cost Analysis, Limitations and Public Health Implications. The combined optimal strategy achieves the lowest objective value across all fractional orders, confirming that coordinated interventions are epidemiologically and economically superior within the model. The quadratic control penalty keeps the profiles bounded away from implausible persistence, but the horizon is finite: because the costates vanish at T , the controls are relaxed near the end (e.g. the quarantine-only run for $\alpha = 0.8$ shows I rising again at day 180), so results close to T should be interpreted with care. The parameters are illustrative rather than fitted, and the fractional order is not calibrated to data. Within these limits the results support integrated response plans that combine early isolation of cases with rapid vaccination.

12. CONCLUSION

This paper developed and analyzed a seven-compartment fractional-order SEIQHRV model for COVID-19 transmission, incorporating quarantine, hospitalization, and vaccination as distinct epidemiological states. The Caputo fractional derivative framework captured memory effects in immune and clinical progression. Three time-dependent controls, representing vaccination, quarantine-isolation, and clinical treatment, were formulated and their optimal profiles determined through the fractional Pontryagin Maximum Principle.

The mathematical analysis established well-posedness (with an explicit Lipschitz constant and a Mittag-Leffler continuous-dependence estimate), positivity and two-sided bounds of the state variables. The basic reproduction number $\mathcal{R}_0 = 3.954$ was derived analytically and shown to govern the local stability of the disease-free equilibrium for every $\alpha \in (0, 1]$; sensitivity indices identified the contact rate and the recovery rate as the dominant parameters; an explicit controlled reproduction number and herd-immunity threshold (87.9%) were derived; and a unique endemic equilibrium was characterized in closed form. The existence of optimal controls was proved in detail, the exact costate equations were derived, and their terminal asymptotics were shown to explain the shape of the optimal controls.

Numerical simulations, validated by convergence tests, an integer-order comparison and an adjoint gradient test, showed that the fractional order materially affects epidemic persistence (a 155% larger infection burden at $\alpha = 0.7$), that combined control is markedly superior to the best

individual strategy (a 44–58% lower cost), and that vaccination and quarantine are complementary, whereas the treatment control is inactive for the chosen cost weights.

Future work will extend this framework to age-stratified contact patterns, multiple SARS-CoV-2 variants with different transmissibility and vaccine escape, capacity-constrained controls, stochastic forcing, and spatial diffusion across networked regions. Data-driven calibration of the fractional order to empirical COVID-19 time series from multiple countries will also be pursued.

Data Availability Statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

Author Contributions: Amit Jain: Conceptualization, Methodology, Formal analysis, Investigation, and Writing – original draft.

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S. Donganont: Investigation and Funding acquisition.

Artificial Intelligence (AI) Declaration: The authors declare that no Artificial Intelligence (AI) tools were used in the preparation, writing, or analysis of this manuscript. All content, including the mathematical derivations, numerical computations, and interpretations, is entirely the work of the authors.

Funding: This research was supported by the University of Phayao and the Thailand Science Research and Innovation Fund (Fundamental Fund 2027).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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