

## Better Refinements of Fractional Inequalities through Harmonically Fuzzy Convexity and its Validation

Muhammad Haseeb<sup>1</sup>, Rana Safdar Ali<sup>2</sup>, Jamal Salah<sup>3,\*</sup>, Aseel Smerat<sup>4,5</sup>,  
Muhammad Imran Hafeez<sup>6</sup>

<sup>1</sup>Faculty of Sciences, Superior University Lahore, Sargodha Campus, Punjab, Pakistan

<sup>2</sup>Department of Mathematics and Statistics, The University of Lahore, Punjab, Pakistan

<sup>3</sup>College of Applied and Health Sciences, A'Sharqiyah University, P.O. Box 42, Post Code 400, Ibra,  
Sultanate of Oman

<sup>4</sup>Hourani Center for Applied Scientific Research, Al-Ahliyya Amman University, Amman 19328, Jordan

<sup>5</sup>Department of Biosciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical  
Sciences, Chennai, 602105, India

<sup>6</sup>Department of Computer Science, The University of Lahore, Sargodha Campus, Pakistan

\*Corresponding author: [damous73@yahoo.com](mailto:damous73@yahoo.com)

**Abstract.** There is a very close relation between convexity and generalized fractional operators (GFO) to improving the fractional inequalities. In this article, we describe the new form of fractional operators, named generalized proportional fuzzy fractional operators (GPFFO), and also discuss the refinements of well-known inequalities. We derive a modified version of Hermite-Hadamard-type inequalities and their refinements by utilizing the GPFFO with harmonically fuzzy convexity. Moreover, we discuss the numerical computations of our main results that show the existence of analytical and numerical behaviors. This type of validation represents the strength of our results. These new results and their numerical validations have a great contribution in the field of computational of inequalities.

### 1. INTRODUCTION

Convexity has provided a strong analytical mechanism, optimization and inequality theory. The concept of convex functions leads to many well known inequalities. Out of which Hermite-Hadamard ( $H - H$ ) type inequalities have a central position [10, 17]. Where Hermite-Hadamard

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inequality can be defined as if  $F : \mathbb{I} \subset \mathbb{R} \rightarrow \mathbb{R}$  a convex function which is defined on the interval  $\mathbb{I}$  of real numbers and  $\zeta_1, \zeta_2 \in \mathbb{I}$  with  $\zeta_1 < \zeta_2$ . The following inequality

$$F\left(\frac{\zeta_1 + \zeta_2}{2}\right) \leq \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} F(x) dx \leq \frac{F(\zeta_1) + F(\zeta_2)}{2},$$

holds. See [1, 6] for detail about different forms of Hermite-Hadamard inequality. This inequality provides brief estimates of integral mean through point wise values and has been generalized in numerous ways [10, 18]. Along with these generalizations, harmonically convexity has been developed into an effective forms of representation. Instead of using algorithmic means, the convexity condition is framed in terms of harmonic means, which makes it naturally suited to reciprocal-type relationships in both theory and application [1, 19, 20].

Along with these developments and generalizations, fractional calculus has significantly deepened the analysis of inequalities, including  $(H - H)$  type inequalities. Fractional integrals, such as Riemann-Liouville and generalized proportional fractional operator (*GPFO*), allow for a more refined interpolation between local and global behavior and can produce refined versions of classical inequalities [3, 12, 21]. In particular, generalized proportional fractional integrals (*GPFIs*) have unified and extended various fractional operators and have been effectively used to derive Hermite-Hadamard  $(H - H)$  inequalities under new structural assumption [19, 22].

At the same time, mathematical models for uncertainty and impression increasingly rely on fuzzy interval-valued functions (*FIVs*). This assign a fuzzy interval to each argument in the domain. This model obtains both interval-type uncertainty and fuzzy and fuzzy "degree of membership" structure. Which makes it appropriate for applications in decision making, systems with imprecision and uncertainty quantification [4, 23, 24]. In *FIV* model, one typically define lower and upper endpoints functions and then develop lower and upper fuzzy integrals by integrating these endpoints functions level-wise, restoring a fuzzy interval in the result [25, 26].

Integrating harmonic convexity with fuzziness establishes a structural approach. The idea of fuzzy harmonically convex functions deals with both the reciprocal structure of harmonic convexity and the interval uncertainty of fuzzy values. For such functions, previous work has established Hermite-Hadamard  $(H - H)$  and Fejèr type inequalities via fuzzy Riemann-Liouville integrals, using a fuzzy order relation on intervals [6, 9, 27]. These studies show that the fuzziness of the image can be addressed in a refined manner in inequality theory, preserving inclusion relations and integral estimates. Also, existing literature has not fully addressed the integration of fuzzy interval-valued functions under harmonic convexity with generalized proportional fuzzy fractional integrals (*GPFFI*).

Previous fundamental results to harmonically convex fractional inequalities include İşcan's work on  $H - H$  inequalities for harmonic convexity [1] and the generalization to fractional integrals by other authors [28]. Moreover, more recent advances in fractional convexity have expanded the operator toolbox. For instance, harmonic convexity has been combined with generalized fractional integrals in continuous difference models [26], and other works have explored  $H - H$  inequalities

using  $\Psi$ -conformable fractional operators [19] or extended Mittag–Leffler kernels [29]. There are also fractional  $H - H$  inequalities developed for exponentially convex functions [30], general pre-invex functions under fractional integrals [4], and weighted fractional operators applied to geometrically convex functions [9]. These advances underline the deep interrelation of convexity, fuzziness, and fractional calculus.

In this paper, We attempt to align these perspectives by introducing harmonically fuzzy convex IVFs and defining corresponding upper and lower GPFFIs. We develop some new Hermite–Hadamard type inequalities in this work, thus extending both the classical and fuzzy fractional literature. Our results generalize the work of Sahoo et.al [3] in fuzzy context and they also refine earlier harmonically fuzzy convex inequalities [6,9].

This paper is organized in a sequence such that in section three, we discuss preliminaries to understand the results derived in this work, and also describe the generalized proportional fuzzy fractional operators (GPFFOs). In sections four and fifth, we establish Hermite-Hadamard inequalities and their further refinements by implementation of GPFFO with harmonically convexity. Finally, in section six, we have discussed conclusions and future directions.

## 2. PRELIMINARIES

Before presenting the main results, we briefly recall the key ideas needed in this work. Since our inequalities combine harmonic convexity with fuzzy interval-valued functions, we first outline the basic definitions related to both concepts. These will help set the stage for the results that follow.

**Definition 2.1.** [2] A function  $F : [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$  is called convex function, if

$$F[p\zeta_1 + (1 - p)\zeta_2] \leq pF(\zeta_1) + (1 - p)F(\zeta_2).$$

**Definition 2.2.** [1–3] Let a real interval  $\mathbf{I} \subset \mathbb{R} \setminus \{0\}$ , then a function  $F : [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$  is known as harmonically convex function, if

$$F\left(\frac{xy}{px + (1 - p)y}\right) \leq pF((1 - p)y) + (1 - p)F(x)$$

**Definition 2.3.** [6, 15] A function  $F : [\zeta_1, \zeta_2] \subset \mathbb{R} \rightarrow \mathbb{K}$  is referred to as an FIV-function. For every  $\ell \in [0, 1]$ , the  $\ell$ -level set generate a family of interval-valued functions

$$F_\ell(w) = [F_*(w, \ell), F^*(w, \ell)], \forall w \in [\zeta_1, \zeta_2],$$

where, for each  $\ell \in [0, 1]$ , the midpoint function  $F_*(\cdot, \ell)$  and  $F^*(\cdot, \ell) : [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$  are termed as lower and upper function of  $F$ , respectively.

**Definition 2.4.** [14] Assume that  $\Pi_{bc}$  be the class of intervals and  $H \in \Pi_{bc}$ . Then

$$\widetilde{F} = [F_*, F^*] = \{\alpha \in \mathbb{R} \mid F_* \leq \alpha \leq F^*\}, (F_*, F^* \in \mathbb{R}).$$

If  $F_* = F^*$ , then we say that  $F$  is degenerate.

**Definition 2.5.** [13] Let  $\ell \in [0, 1]$ . A fuzzy interval-valued (FIV) mapping  $F : \Pi \subseteq \mathbb{R} \rightarrow \Pi_{bc}$  is defined by

$$F_\ell(x) = [F_*(x, \ell), F^*(x, \ell)], \quad \forall x \in \Pi,$$

where the left- and right-endpoint mappings  $F_*(x, \ell)$  and  $F^*(x, \ell) : \Pi \rightarrow \mathbb{R}$  are referred to as the lower and upper functions of  $F$ , respectively.

If, for  $\ell \in [0, 1]$ , both  $F_*(x, \ell)$  and  $F^*(x, \ell)$  are continuous at a point  $x \in \Pi$ , then the FIV-function  $F : \Pi \subseteq \mathbb{R} \rightarrow \Pi_{bc}$  is continuous at  $x \in \Pi$ .

**Proposition 2.1.** [10] For  $\zeta_1, \zeta_2 \in F$ ,  $c \in \mathbb{R}$  and  $\ell \in [0, 1]$ , we have

$$\begin{aligned} [\zeta_1 \bar{+} \zeta_2]^\ell &= [\zeta_1]^\ell + [\zeta_2]^\ell, \\ [\zeta_1 \bar{\times} \zeta_2]^\ell &= [\zeta_1]^\ell \times [\zeta_2]^\ell, \\ [c \cdot \zeta_1]^\ell &= c[\zeta_1]^\ell, \\ [c \bar{+} \zeta_1]^\ell &= c + [\zeta_1]^\ell. \end{aligned}$$

**Definition 2.6.** [4, 5] A fuzzy interval valued function  $\tilde{F} : \mathbf{I} \rightarrow \mathbf{R}$  is called convex if for every  $\ell \in [0, 1]$ , the  $\ell$ -level set  $\tilde{F} = [\tilde{F}_*(x, \ell), \tilde{F}^*(x, \ell)]$  satisfies:

$$\tilde{F}[p\zeta_1 + (1-p)\zeta_2] \leq p\tilde{F}(\zeta_2) \bar{+} (1-p)\tilde{F}(\zeta_1).$$

**Definition 2.7.** [9] The harmonically convex fuzzy interval valued function for a mapping  $\tilde{F} : [\zeta_1, \zeta_2] \rightarrow \mathbf{R}$  is defined as follows:

$$\tilde{F}\left(\frac{xy}{px + (1-p)y}\right) \leq p\tilde{F}(y) \bar{+} (1-p)\tilde{F}(x) \quad (2.1)$$

**Definition 2.8.** [6, 9] The Hermite-Hadamard inequality for fuzzy interval valued function is defined as:

$$\tilde{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \leq \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \tilde{F}(x) dx \leq \frac{\tilde{F}(\zeta_1) \bar{+} \tilde{F}(\zeta_2)}{2}. \quad (2.2)$$

**Definition 2.9.** For [8]  $\Re(b) > \Re(a) > 0$ , the Euler integral representation is:

$${}_1F_1(a; b; z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt. \quad (2.3)$$

**Definition 2.10.** [3, 31] Let  $F \in \mathcal{L}[\zeta_1, \zeta_2]$  and  $\gamma \in [0, 1]$  then the generalized proportional fractional integrals (GPF)  $\mathfrak{I}_{\zeta_1^+}^\alpha$  and  $\mathfrak{I}_{\zeta_2^-}^\alpha$  of order  $\alpha > 0$  are defined as:

$$\mathfrak{I}_{\zeta_1^+}^{\alpha, \gamma} F(x) := \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_{\zeta_1}^x e^{[\frac{\gamma-1}{\gamma}(x-u)]} (x-u)^{\alpha-1} F(u) du, \quad 0 \leq \zeta_1 < x < \zeta_2,$$

$$\mathfrak{I}_{\zeta_2^-}^{\alpha, \gamma} F(x) := \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_x^{\zeta_2} e^{[\frac{\gamma-1}{\gamma}(u-x)]} (u-x)^{\alpha-1} F(u) du, \quad 0 \leq \zeta_1 < x < \zeta_2.$$

**Definition 2.11.** The left GPFFO for a function  $F$  is defined as follows:

$$\begin{aligned} [\mathfrak{S}_{\zeta_1^+}^{\alpha,\gamma} F(\mathbf{x})]^\ell &= \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_{\zeta_1}^{\mathbf{x}} e^{[\frac{\gamma-1}{\gamma}(\mathbf{x}-\mathbf{u})]} (\mathbf{x}-\mathbf{u})^{\alpha-1} F_\ell(\mathbf{u}) d\mathbf{u} \\ &= \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_{\zeta_1}^{\mathbf{x}} e^{[\frac{\gamma-1}{\gamma}(\mathbf{x}-\mathbf{u})]} (\mathbf{x}-\mathbf{u})^{\alpha-1} [F_*(\mathbf{u}, \ell), F^*(\mathbf{u}, \ell)] d\mathbf{u} \quad 0 \leq \zeta_1 < \mathbf{x} < \zeta_2. \end{aligned}$$

Where

$$\mathfrak{S}_{\zeta_1^+}^{\alpha,\gamma} F_*(\mathbf{x}, \ell) = \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_{\zeta_1}^{\mathbf{x}} e^{[\frac{\gamma-1}{\gamma}(\mathbf{x}-\mathbf{u})]} (\mathbf{x}-\mathbf{u})^{\alpha-1} [F_*(\mathbf{u}, \ell)] d\mathbf{u}, \quad 0 \leq \zeta_1 < \mathbf{x} < \zeta_2,$$

and

$$\mathfrak{S}_{\zeta_1^+}^{\alpha,\gamma} F^*(\mathbf{x}, \ell) = \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_{\zeta_1}^{\mathbf{x}} e^{[\frac{\gamma-1}{\gamma}(\mathbf{x}-\mathbf{u})]} (\mathbf{x}-\mathbf{u})^{\alpha-1} [F^*(\mathbf{u}, \ell)] d\mathbf{u}, \quad 0 \leq \zeta_1 < \mathbf{x} < \zeta_2.$$

Similarly, we can define right GPFFO for a function.

**Remark 2.1.**

If  $u \neq 0$ , then we take function  $\psi(u) = \frac{1}{u}$ .

### 3. ANALYSIS OF HERMITE-HADAMARD INEQUALITIES BY GPFFO AND HARMONICALLY CONVEXITY

In this section, we will discuss the refinement of the Hermite-Hadamard-type inequality by implementation of the GPFFO with harmonically fuzzy convex function.

**Theorem 3.1.** Let  $\tilde{F} : [\zeta_1, \zeta_2] \rightarrow \mathbf{I}(\mathbf{R})$ , be a positive fuzzy interval valued function with  $0 < \zeta_1 < \zeta_2$  and  $\tilde{F} \in \mathcal{L}[\zeta_1, \zeta_2]$ . If  $F$  is a harmonically convex function, then we have:

$$\begin{aligned} \tilde{F}\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}\right) &\leq \frac{\gamma^\alpha \Gamma(\alpha + 1)}{2 \cdot {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1\zeta_2}\right)\right)} \left(\frac{\zeta_1\zeta_2}{\zeta_2 - \zeta_1}\right)^\alpha \\ \left[\left(\mathfrak{S}_{\frac{1}{\zeta_1}^-}^{\alpha,\gamma}(\tilde{F} \circ \psi)\right)\left(\frac{1}{\zeta_2}\right) \tilde{+} \left(\mathfrak{S}_{\frac{1}{\zeta_2}^+}^{\alpha,\gamma}(\tilde{F} \circ \psi)\right)\left(\frac{1}{\zeta_1}\right)\right] &\leq \frac{\tilde{F}(\zeta_1) \tilde{+} \tilde{F}(\zeta_2)}{2}. \end{aligned} \tag{3.1}$$

*Proof.* Let  $x, y \in [\zeta_1, \zeta_2]$ , if  $F$  is a harmonically convex function, then

$$\tilde{F}\left(\frac{2xy}{x+y}\right) \leq \frac{\tilde{F}(x) \tilde{+} \tilde{F}(y)}{2}, \tag{3.2}$$

for every  $\ell \in [\zeta_1, \zeta_2]$ , we can define lower and upper FIV function as

$$F^*\left(\frac{2xy}{x+y}, \ell\right) \leq \frac{F^*(x, \ell) \tilde{+} F^*(y, \ell)}{2}, \tag{3.3}$$

Similarly

$$F_*\left(\frac{2xy}{x+y}, \ell\right) \leq \frac{F_*(x, \ell) \tilde{+} F_*(y, \ell)}{2}, \tag{3.4}$$

Now solve for upper FIV function and put  $x = \frac{\zeta_1 \zeta_2}{k\zeta_2 + (1-k)\zeta_1}$  and  $y = \frac{\zeta_1 \zeta_2}{k\zeta_1 + (1-k)\zeta_2}$

$$\implies \frac{2xy}{x+y} = \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2},$$

put values of  $x$  and  $y$  in (3.3),

$$2F^*\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \leq F^*\left(\frac{\zeta_1 \zeta_2}{k\zeta_2 + (1-k)\zeta_1}, \ell\right) + F^*\left(\frac{\zeta_1 \zeta_2}{k\zeta_1 + (1-k)\zeta_2}, \ell\right). \quad (3.5)$$

Now multiply both sides by  $\frac{1}{\gamma^\alpha \Gamma(\alpha)} e^{\frac{\gamma-1}{\gamma} k \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)} k^{\alpha-1}$ , and performing integration with respect to  $k$  over the interval  $[0, 1]$ , we get:

$$\begin{aligned} & 2F^*\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}\right) \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_0^1 e^{\frac{\gamma-1}{\gamma} k \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)} k^{\alpha-1} dk \\ & \leq \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_0^1 e^{\frac{\gamma-1}{\gamma} k \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)} k^{\alpha-1} F^*\left(\frac{\zeta_1 \zeta_2}{k\zeta_2 + (1-k)\zeta_1}, \ell\right) dk \\ & + \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_0^1 e^{\frac{\gamma-1}{\gamma} k \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)} k^{\alpha-1} F^*\left(\frac{\zeta_1 \zeta_2}{k\zeta_1 + (1-k)\zeta_2}, \ell\right) dk \\ & = \mathfrak{J}_g + \mathfrak{J}. \end{aligned} \quad (3.6)$$

Now apply change of variable technique, we obtain

$$\begin{aligned} \mathfrak{J}_g &= \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_{\frac{1}{\zeta_2}}^{\frac{1}{\zeta_1}} e^{\frac{\gamma-1}{\gamma} \frac{\zeta_2 \zeta_1}{\zeta_2 - \zeta_1} \left(u - \frac{1}{\zeta_2}\right) \frac{\zeta_2 - \zeta_1}{\zeta_2 \zeta_1}} \left[ \frac{\zeta_1 \zeta_2}{\zeta_2 - \zeta_1} \left(u - \frac{1}{\zeta_2}\right) \right]^{\alpha-1} F^*\left(\frac{1}{u}, \ell\right) du \frac{\zeta_2 \zeta_1}{\zeta_2 - \zeta_1} \\ &= \left(\frac{\zeta_2 \zeta_1}{\zeta_2 - \zeta_1}\right)^\alpha \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_{\frac{1}{\zeta_2}}^{\frac{1}{\zeta_1}} e^{\frac{\gamma-1}{\gamma} \left(u - \frac{1}{\zeta_2}\right)} \left(u - \frac{1}{\zeta_2}\right)^{\alpha-1} F^*\left(\frac{1}{u}, \ell\right) du \\ &= \left(\frac{\zeta_2 \zeta_1}{\zeta_2 - \zeta_1}\right)^\alpha \left(\mathfrak{J}_{\frac{1}{\zeta_1}-}^{\alpha, \gamma} (F^* \circ \psi)\right)\left(\frac{1}{\zeta_2}\right). \end{aligned}$$

Similarly

$$\begin{aligned} \mathfrak{J} &= \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_{\frac{1}{\zeta_2}}^{\frac{1}{\zeta_1}} e^{\frac{\gamma-1}{\gamma} \frac{\zeta_1 \zeta_2}{\zeta_2 - \zeta_1} \left(\frac{1}{\zeta_1} - u\right) \frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}} \left[ \frac{\zeta_1 \zeta_2}{\zeta_2 - \zeta_1} \left(\frac{1}{\zeta_1} - u\right) \right]^{\alpha-1} F^*\left(\frac{1}{u}, \ell\right) du \frac{\zeta_1 \zeta_2}{\zeta_2 - \zeta_1} \\ &= \left(\frac{\zeta_2 \zeta_1}{\zeta_2 - \zeta_1}\right)^\alpha \left(\mathfrak{J}_{\frac{1}{\zeta_2}+}^{\alpha, \gamma} (F^* \circ \psi)\right)\left(\frac{1}{\zeta_1}\right). \end{aligned}$$

Also we have

$$\frac{2}{\gamma^\alpha \Gamma(\alpha)} \int_0^1 e^{\frac{\gamma-1}{\gamma} k \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)} k^{\alpha-1} dk = \frac{2}{\gamma^\alpha \Gamma(\alpha + 1)} {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right). \quad (3.7)$$

Now by substituting the value of  $\mathfrak{J}_g$ ,  $\mathfrak{J}$  and (3.7) into (3.6) we get,

$$F^*\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \leq \frac{\gamma^\alpha \Gamma(\alpha + 1)}{2 \cdot {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \left(\frac{\zeta_1 \zeta_2}{\zeta_2 - \zeta_1}\right)^\alpha$$

$$\left[ \left( \mathfrak{S}_{\frac{1}{\zeta_1}-}^{\alpha, \gamma} (F^* \circ \psi) \right) \left( \frac{1}{\zeta_2} \right) + \left( \mathfrak{S}_{\frac{1}{\zeta_2}+}^{\alpha, \gamma} (F^* \circ \psi) \right) \left( \frac{1}{\zeta_1} \right) \right]. \tag{3.8}$$

Similarly for lower FIV function we will have

$$F_* \left( \frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}, \ell \right) \leq \frac{\gamma^\alpha \Gamma(\alpha + 1)}{2 \cdot {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2} \right) \right)} \left( \frac{\zeta_1\zeta_2}{\zeta_2 - \zeta_1} \right)^\alpha$$

$$\left[ \left( \mathfrak{S}_{\frac{1}{\zeta_1}-}^{\alpha, \gamma} (F_* \circ \psi) \right) \left( \frac{1}{\zeta_2} \right) + \left( \mathfrak{S}_{\frac{1}{\zeta_2}+}^{\alpha, \gamma} (F_* \circ \psi) \right) \left( \frac{1}{\zeta_1} \right) \right]. \tag{3.9}$$

Now by combining (3.8) and (3.9) we obtain

$$\widetilde{F} \left( \frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2} \right) \leq \frac{\gamma^\alpha \Gamma(\alpha + 1)}{2 \cdot {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2} \right) \right)} \left( \frac{\zeta_1\zeta_2}{\zeta_2 - \zeta_1} \right)^\alpha$$

$$\left[ \left( \mathfrak{S}_{\frac{1}{\zeta_1}-}^{\alpha, \gamma} (\widetilde{F} \circ \psi) \right) \left( \frac{1}{\zeta_2} \right) + \left( \mathfrak{S}_{\frac{1}{\zeta_2}+}^{\alpha, \gamma} (\widetilde{F} \circ \psi) \right) \left( \frac{1}{\zeta_1} \right) \right]. \tag{3.10}$$

This Completes the first part of our theorem.

To establish the second part of the inequality , we utilize the definition of harmonically convex function for lower FIV function, which is given as follows:

$$F^* \left( \frac{\zeta_1\zeta_2}{k\zeta_2 + (1-k)\zeta_1}, \ell \right) + F^* \left( \frac{\zeta_1\zeta_2}{k\zeta_1 + (1-k)\zeta_2}, \ell \right) \leq F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell). \tag{3.11}$$

By multiplying both sides of inequality by  $\frac{1}{\gamma^\alpha \Gamma(\alpha)} e^{\frac{\gamma-1}{\gamma} k \left( \frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2} \right)} k^{\alpha-1}$  , and integrating with respect to  $k$  over the interval  $[0, 1]$  , the following results is obtained:

$$\left( \frac{\zeta_1\zeta_2}{\zeta_2 - \zeta_1} \right)^\alpha \left[ \left( \mathfrak{S}_{\frac{1}{\zeta_1}-}^{\alpha, \gamma} (F^* \circ \psi) \right) \left( \frac{1}{\zeta_2} \right) + \left( \mathfrak{S}_{\frac{1}{\zeta_2}+}^{\alpha, \gamma} (F^* \circ \psi) \right) \left( \frac{1}{\zeta_1} \right) \right]$$

$$\leq \frac{F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell)}{\gamma^\alpha \Gamma(\alpha + 1)} {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2} \right) \right). \tag{3.12}$$

Similarly for upper FIV function

$$\left( \frac{\zeta_1\zeta_2}{\zeta_2 - \zeta_1} \right)^\alpha \left[ \left( \mathfrak{S}_{\frac{1}{\zeta_1}-}^{\alpha, \gamma} (F_* \circ \psi) \right) \left( \frac{1}{\zeta_2} \right) + \left( \mathfrak{S}_{\frac{1}{\zeta_2}+}^{\alpha, \gamma} (F_* \circ \psi) \right) \left( \frac{1}{\zeta_1} \right) \right]$$

$$\leq \frac{F_*(\zeta_1, \ell) + F_*(\zeta_2, \ell)}{\gamma^\alpha \Gamma(\alpha + 1)} {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2} \right) \right). \tag{3.13}$$

After combining (3.12) and (3.13) and after some rearrangements we obtain:

$$\frac{\gamma^\alpha \Gamma(\alpha + 1)}{2 \cdot {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2} \right) \right)} \left( \frac{\zeta_1\zeta_2}{\zeta_2 - \zeta_1} \right)^\alpha \left[ \left( \mathfrak{S}_{\frac{1}{\zeta_1}-}^{\alpha, \gamma} (\widetilde{F} \circ \psi) \right) \left( \frac{1}{\zeta_2} \right) + \left( \mathfrak{S}_{\frac{1}{\zeta_2}+}^{\alpha, \gamma} (\widetilde{F} \circ \psi) \right) \left( \frac{1}{\zeta_1} \right) \right]$$

$$\leq \frac{\widetilde{F}(\zeta_1) + \widetilde{F}(\zeta_2)}{2}. \tag{3.14}$$

After combining equation (3.10) and (3.14) we get our desired results of equation (3.1). □

**Example 3.1.** We consider a fuzzy interval valued function  $\tilde{F}(x) = \left[ \frac{1}{x^2}, \frac{3}{x^2} \right]$ ,  $x \in [1, 2]$  and solve it to validate the theorem 3.1. We will consider  $\zeta_1 = 1$  and  $\zeta_2 = 2$ ,  $\gamma = \frac{1}{2}$  and we manipulate the graph for different values of  $\alpha$ . For the left and right term of inequality we will get a fix value, i.e.

$$F_* \left( \frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2} \right) = 0.5625,$$

and

$$\frac{F_*(\zeta_1) + F_*(\zeta_2)}{2} = 0.6250.$$

For different values of  $\alpha$ , we get different values of middle term as table below is given,

$\alpha$	Middle Term
0.1	0.61486
0.2	0.60702
0.3	0.60093
0.4	0.59618
0.5	0.59248
0.6	0.58960
0.7	0.58736
0.8	0.58565
0.9	0.58434
1.0	0.58337

FIGURE 1. table

Values of middle term for different values of  $\alpha$   
for  $F_*$

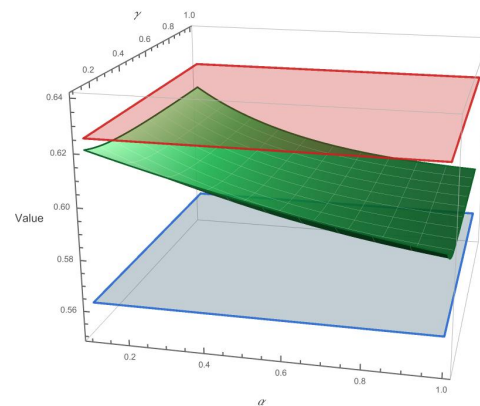


FIGURE 2. figure

Graph for lower bound of function

Similarly,

$$F^* \left( \frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2} \right) = 1.6875,$$

and

$$\frac{F^*(\zeta_1) + F^*(\zeta_2)}{2} = 1.8750.$$

For different values of  $\alpha$ , we will have the following values:

$\alpha$	Middle Term
0.1	1.84458
0.2	1.82107
0.3	1.80280
0.4	1.78856
0.5	1.77745
0.6	1.76880
0.7	1.76210
0.8	1.75695
0.9	1.75304
1.0	1.75013

FIGURE 3. table

Values of middle term for different values of  $\alpha$  for  $F^*$

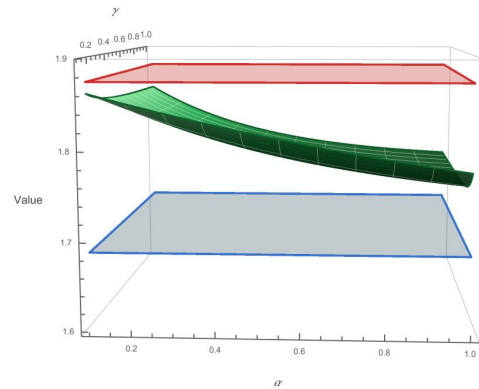


FIGURE 4. figure

Graph for upper bound of function

**Theorem 3.2.** Let  $\tilde{F} : [\zeta_1, \zeta_2] \rightarrow \mathbf{I}(\mathbf{R})$ , be a positive fuzzy interval-valued function with  $0 < \zeta_1 < \zeta_2$  and  $\tilde{F} \in \mathcal{L}[\zeta_1, \zeta_2]$ . If  $\tilde{F}$  is harmonically convex then following inequality holds:

$$\tilde{F}\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}\right) \leq \frac{\gamma^\alpha \Gamma(\alpha + 1)}{2 \cdot {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{2\zeta_1\zeta_2}\right)\right)} \left(\frac{2\zeta_1\zeta_2}{\zeta_2 - \zeta_1}\right)^\alpha$$

$$\left[ \mathfrak{J}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^-}^{\alpha, \gamma} (\tilde{F} \circ \psi)\left(\frac{1}{\zeta_2}\right) + \mathfrak{J}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^+}^{\alpha, \gamma} (\tilde{F} \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] \leq \frac{\tilde{F}(\zeta_1) + \tilde{F}(\zeta_2)}{2} \tag{3.15}$$

*Proof.* Let  $x, y \in [\zeta_1, \zeta_2]$ . If  $\tilde{F}$  be a positive fuzzy integral valued and harmonically convex function, then

$$\tilde{F}\left(\frac{2xy}{x+y}\right) \leq \frac{\tilde{F}(x) + \tilde{F}(y)}{2}, \tag{3.16}$$

for every  $\ell \in [\zeta_1, \zeta_2]$ , we can write lower and upper fuzzy integral valued function as follows:

$$F_*\left(\frac{2xy}{x+y}, \ell\right) \leq \frac{F_*(x, \ell) + F_*(y, \ell)}{2}, \tag{3.17}$$

and

$$F^*\left(\frac{2xy}{x+y}, \ell\right) \leq \frac{F^*(x, \ell) + F^*(y, \ell)}{2}. \tag{3.18}$$

Now, first we will solve for upper fuzzy interval valued function and choosing  $x = \frac{\zeta_1\zeta_2}{\frac{k}{2}\zeta_2 + \frac{2-k}{2}\zeta_1}$  and  $y = \frac{\zeta_1\zeta_2}{\frac{k}{2}\zeta_1 + \frac{2-k}{2}\zeta_2}$ , we have

$$2F^*\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \leq F^*\left(\frac{\zeta_1\zeta_2}{\frac{k}{2}\zeta_2 + \frac{2-k}{2}\zeta_1}\right) + F^*\left(\frac{\zeta_1\zeta_2}{\frac{k}{2}\zeta_1 + \frac{2-k}{2}\zeta_2}\right). \tag{3.19}$$

Multiply both sides by  $\frac{1}{\gamma^\alpha \Gamma(\alpha)} e^{\frac{\gamma-1}{\gamma} k \left( \frac{\zeta_2 - \zeta_1}{2\zeta_1 \zeta_2} \right)} k^{\alpha-1}$  and applying integration with respect to  $k$  over the interval  $[0, 1]$ , we get:

$$\begin{aligned}
 & 2F^* \left( \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell \right) \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_0^1 e^{\frac{\gamma-1}{\gamma} k \left( \frac{\zeta_2 - \zeta_1}{2\zeta_1 \zeta_2} \right)} k^{\alpha-1} dk \\
 & \leq \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_0^1 e^{\frac{\gamma-1}{\gamma} k \left( \frac{\zeta_2 - \zeta_1}{2\zeta_1 \zeta_2} \right)} k^{\alpha-1} F^* \left( \frac{2\zeta_1 \zeta_2}{k\zeta_2 + (2-k)\zeta_1}, \ell \right) dk \\
 & \quad + \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_0^1 e^{\frac{\gamma-1}{\gamma} k \left( \frac{\zeta_2 - \zeta_1}{2\zeta_1 \zeta_2} \right)} k^{\alpha-1} F^* \left( \frac{2\zeta_1 \zeta_2}{k\zeta_1 + (2-k)\zeta_2}, \ell \right) dk. \\
 & = \mathfrak{I}_g^* + \mathfrak{I}^*. \tag{3.20}
 \end{aligned}$$

Applying change of variable technique, we get the following result:

$$\begin{aligned}
 \mathfrak{I}_g^* &= \left( \frac{2\zeta_1 \zeta_2}{\zeta_2 - \zeta_1} \right)^\alpha \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_{\frac{1}{\zeta_2}}^{\frac{\zeta_1 + \zeta_2}{2\zeta_1 \zeta_2}} e^{\frac{\gamma-1}{\gamma} \left( u - \frac{1}{\zeta_2} \right)} \left( u - \frac{1}{\zeta_2} \right)^{\alpha-1} F^* \left( \frac{1}{u}, \ell \right) du \\
 &= \left( \frac{2\zeta_1 \zeta_2}{\zeta_2 - \zeta_1} \right)^\alpha \left( \mathfrak{J}_{\left( \frac{\zeta_1 + \zeta_2}{2\zeta_1 \zeta_2} \right)^-}^{\alpha, \gamma} \left( F^* \circ \psi \right) \right) \left( \frac{1}{\zeta_2} \right).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \mathfrak{I}_2^* &= \left( \frac{2\zeta_1 \zeta_2}{\zeta_2 - \zeta_1} \right)^\alpha \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_{\frac{\zeta_1 + \zeta_2}{2\zeta_1 \zeta_2}}^{\frac{1}{\zeta_1}} e^{\frac{\gamma-1}{\gamma} \left( \frac{1}{\zeta_1} - u \right)} \left( \frac{1}{\zeta_1} - u \right)^{\alpha-1} F^* \left( \frac{1}{u}, \ell \right) du \\
 &= \left( \frac{2\zeta_1 \zeta_2}{\zeta_2 - \zeta_1} \right)^\alpha \left( \mathfrak{J}_{\left( \frac{\zeta_1 + \zeta_2}{2\zeta_1 \zeta_2} \right)^+}^{\alpha, \gamma} \left( F^* \circ \psi \right) \right) \left( \frac{1}{\zeta_1} \right)
 \end{aligned}$$

Also we have,

$$\begin{aligned}
 & F^* \left( \frac{2\zeta_1 \zeta_2}{\zeta_2 - \zeta_1}, \ell \right) \frac{2}{\gamma^\alpha \Gamma(\alpha)} \int_0^1 e^{\frac{\gamma-1}{\gamma} k \left( \frac{\zeta_2 - \zeta_1}{2\zeta_1 \zeta_2} \right)} k^{\alpha-1} dk \\
 & = F^* \left( \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell \right) \frac{2}{\gamma^\alpha \Gamma(\alpha + 1)} {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{2\zeta_1 \zeta_2} \right) \right). \tag{3.21}
 \end{aligned}$$

Put the value of  $\mathfrak{I}_g^*$ ,  $\mathfrak{I}_2^*$  and (3.21) in (3.20), we get:

$$\begin{aligned}
 F^* \left( \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell \right) &\leq \frac{\gamma^\alpha \Gamma(\alpha + 1)}{2 \cdot {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{2\zeta_1 \zeta_2} \right) \right)} \left( \frac{2\zeta_1 \zeta_2}{\zeta_2 - \zeta_1} \right)^\alpha \\
 &\quad \left[ \mathfrak{J}_{\left( \frac{\zeta_1 + \zeta_2}{2\zeta_1 \zeta_2} \right)^-}^{\alpha, \gamma} \left( F^* \circ \psi \right) \left( \frac{1}{\zeta_2} \right) + \mathfrak{J}_{\left( \frac{\zeta_1 + \zeta_2}{2\zeta_1 \zeta_2} \right)^+}^{\alpha, \gamma} \left( F^* \circ \psi \right) \left( \frac{1}{\zeta_1} \right) \right]. \tag{3.22}
 \end{aligned}$$

Similarly for lower fuzzy integral valued function we can have,

$$F_* \left( \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell \right) \leq \frac{\gamma^\alpha \Gamma(\alpha + 1)}{2 \cdot {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{2\zeta_1 \zeta_2} \right) \right)} \left( \frac{2\zeta_1 \zeta_2}{\zeta_2 - \zeta_1} \right)^\alpha$$

$$\left[ \mathfrak{S}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^-}^{\alpha,\gamma} \left(F_* \circ \psi\right)\left(\frac{1}{\zeta_2}\right) + \mathfrak{S}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^+}^{\alpha,\gamma} \left(F_* \circ \psi\right)\left(\frac{1}{\zeta_1}\right) \right]. \tag{3.23}$$

After combining and simplifying (3.22) and (3.23) we get following:

$$\begin{aligned} \widetilde{F}\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}, \ell\right) &\leq \frac{\gamma^\alpha \Gamma(\alpha + 1)}{2 \cdot {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{2\zeta_1\zeta_2}\right)\right)} \left(\frac{2\zeta_1\zeta_2}{\zeta_2 - \zeta_1}\right)^\alpha \\ &\left[ \mathfrak{S}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^-}^{\alpha,\gamma} \left(\widetilde{F} \circ \psi\right)\left(\frac{1}{\zeta_2}\right) + \mathfrak{S}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^+}^{\alpha,\gamma} \left(\widetilde{F} \circ \psi\right)\left(\frac{1}{\zeta_1}\right) \right]. \end{aligned} \tag{3.24}$$

This completes the proof of first part of theorem.

To find the second part of inequality, we apply the definition of upper fuzzy interval valued function on harmonically fuzzy convex function, then we will have:

$$F^*\left(\frac{\zeta_1\zeta_2}{\frac{k}{2}\zeta_2 + \frac{2-k}{2}\zeta_1}, \ell\right) \leq \frac{2-k}{k} F^*(\zeta_2, \ell) + \frac{k}{2} F^*(\zeta_1, \ell), \tag{3.25}$$

and

$$F^*\left(\frac{\zeta_1\zeta_2}{\frac{k}{2}\zeta_1 + \frac{2-k}{2}\zeta_2}, \ell\right) \leq \frac{2-k}{k} F^*(\zeta_1, \ell) + \frac{k}{2} F^*(\zeta_2, \ell), \tag{3.26}$$

adding both equations, we get:

$$F^*\left(\frac{2\zeta_1\zeta_2}{k\zeta_2 + (2-k)\zeta_1}, \ell\right) + F^*\left(\frac{2\zeta_1\zeta_2}{k\zeta_1 + (2-k)\zeta_2}, \ell\right) \leq F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell). \tag{3.27}$$

Multiply both sides by  $\frac{1}{\gamma^\alpha \Gamma(\alpha)} e^{\frac{\gamma-1}{\gamma} k \left(\frac{\zeta_2-\zeta_1}{2\zeta_1\zeta_2}\right)} k^{\alpha-1}$  and applying integration with respect to  $k$  over the interval  $[0, 1]$ , we get:

$$\begin{aligned} &\frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_0^1 F^*\left(\frac{2\zeta_1\zeta_2}{k\zeta_2 + (2-k)\zeta_1}\right) e^{\frac{\gamma-1}{\gamma} k \left(\frac{\zeta_2-\zeta_1}{2\zeta_1\zeta_2}\right)} k^{\alpha-1} dk \\ &+ \frac{1}{\gamma^\alpha \Gamma(\alpha)} \int_0^1 F^*\left(\frac{2\zeta_1\zeta_2}{k\zeta_1 + (2-k)\zeta_2}\right) e^{\frac{\gamma-1}{\gamma} k \left(\frac{\zeta_2-\zeta_1}{2\zeta_1\zeta_2}\right)} k^{\alpha-1} dk \\ &\leq \left(F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell)\right) \int_0^1 \frac{1}{\gamma^\alpha \Gamma(\alpha)} e^{\frac{\gamma-1}{\gamma} k \left(\frac{\zeta_2-\zeta_1}{2\zeta_1\zeta_2}\right)} k^{\alpha-1} dk, \end{aligned} \tag{3.28}$$

which gives us:

$$\begin{aligned} &\left(\frac{2\zeta_1\zeta_2}{\zeta_2 - \zeta_1}\right)^\alpha \left[ \mathfrak{S}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^-}^{\alpha,\gamma} \left(F^* \circ \psi\right)\left(\frac{1}{\zeta_2}\right) + \mathfrak{S}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^+}^{\alpha,\gamma} \left(F^* \circ \psi\right)\left(\frac{1}{\zeta_1}\right) \right] \\ &\leq \frac{F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell)}{\gamma^\alpha \Gamma(\alpha + 1)} {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{2\zeta_1\zeta_2}\right)\right) \end{aligned} \tag{3.29}$$

Similarly for upper fuzzy interval valued function we will have:

$$\begin{aligned} & \left(\frac{2\zeta_1\zeta_2}{\zeta_2 - \zeta_1}\right)^\alpha \left[ \mathfrak{J}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^-}^{\alpha,\gamma} (F_* \circ \psi) \left(\frac{1}{\zeta_2}\right) + \left( \mathfrak{J}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^+}^{\alpha,\gamma} (F_* \circ \psi) \right) \left(\frac{1}{\zeta_1}\right) \right] \\ & \leq \frac{F_*(\zeta_1, \ell) + F_*(\zeta_2, \ell)}{\gamma^\alpha \Gamma(\alpha + 1)} {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma - 1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{2\zeta_1\zeta_2} \right) \right) \end{aligned} \quad (3.30)$$

After combining (3.29) and (3.30) we will have:

$$\begin{aligned} & \left(\frac{2\zeta_1\zeta_2}{\zeta_2 - \zeta_1}\right)^\alpha \left[ \mathfrak{J}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^-}^{\alpha,\gamma} (\widetilde{F} \circ \psi) \left(\frac{1}{\zeta_2}\right) + \left( \mathfrak{J}_{\left(\frac{\zeta_1+\zeta_2}{2\zeta_1\zeta_2}\right)^+}^{\alpha,\gamma} (\widetilde{F} \circ \psi) \right) \left(\frac{1}{\zeta_1}\right) \right] \\ & \leq \frac{\widetilde{F}(\zeta_1, \ell) + \widetilde{F}(\zeta_2, \ell)}{\gamma^\alpha \Gamma(\alpha + 1)} {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma - 1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{2\zeta_1\zeta_2} \right) \right) \end{aligned} \quad (3.31)$$

By combining (3.24) and (3.31) and applying few rearrangements, we will get our desired results.  $\square$

#### 4. REFINEMENTS OF HERMITE-HADAMARD INEQUALITY WITH GPFFO

In this section, further refinements of Hermite-Hadamard type inequalities are obtained by applying GPFFO with harmonically fuzzy convexity.

**Theorem 4.1.** Let  $\widetilde{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{I}(\mathcal{R})$  be a twice differentiable, fuzzy interval-valued function with  $0 < \zeta_1 < \zeta_2$  and  $\widetilde{F} \in \mathcal{L}[\zeta_1, \zeta_2]$ . If  $\widetilde{F}''$  is bounded on  $[\zeta_1, \zeta_2]$  and  $\widetilde{F}(\mathbf{x}) = \left(\frac{\zeta_1\zeta_2}{\mathbf{x}}\right)$ , then we have the following inequalities for GPFFIs:

$$\begin{aligned} & \frac{m\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma - 1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2} \right) \right)} \int_{\zeta_1}^{\frac{\delta_1 + \zeta_2}{2}} \left( \frac{\zeta_2 - \zeta_1}{2} - \mathbf{x} \right)^2 \\ & \left[ e^{\frac{\gamma - 1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha - 1} + e^{\frac{\gamma - 1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha - 1} \right] d\mathbf{x} \\ & \leq \frac{\gamma^\alpha \Gamma(\alpha + 1) (\zeta_1 \zeta_2)^\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma - 1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2} \right) \right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}^-}^{\alpha,\gamma} (\widetilde{F} \circ \psi) \left(\frac{1}{\zeta_2}\right) + \mathfrak{J}_{\frac{1}{\zeta_2}^+}^{\alpha,\gamma} (\widetilde{F} \circ \psi) \left(\frac{1}{\zeta_1}\right) \right] - \widetilde{F} \left( \frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2} \right) \\ & \leq \frac{M\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma - 1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2} \right) \right)} \\ & \int_{\zeta_1}^{\frac{\zeta_1 + \zeta_2}{2}} \left( \frac{\zeta_1 + \zeta_2}{2} + \mathbf{x} \right)^2 \left[ e^{\frac{\gamma - 1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha - 1} + e^{\frac{\gamma - 1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha - 1} \right] d\mathbf{x}. \end{aligned}$$

*Proof.* By applying change of variable technique, we have

$$\frac{\gamma^\alpha \Gamma(\alpha + 1) (\zeta_1 \zeta_2)^\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma - 1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2} \right) \right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}^-}^{\alpha,\gamma} (F_* \circ \psi) \left(\frac{1}{\zeta_2}\right) + \mathfrak{J}_{\frac{1}{\zeta_2}^+}^{\alpha,\gamma} (F_* \circ \psi) \left(\frac{1}{\zeta_1}\right) \right]$$

$$\begin{aligned}
 &= \frac{\gamma^\alpha \Gamma(\alpha + 1)(\zeta_1 \zeta_2)^\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \\
 &\quad \int_{\frac{1}{\zeta_2}}^{\frac{1}{\zeta_1}} \frac{1}{\gamma^\alpha \Gamma(\alpha)} \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \frac{1}{\zeta_2})^{\alpha-1}} \left(\mathbf{x} - \frac{1}{\zeta_2}\right)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\frac{1}{\zeta_2} - \mathbf{x})} \left(\frac{1}{\zeta_1} - \mathbf{x}\right)^{\alpha-1} \right] F^*\left(\frac{1}{\mathbf{x}}, \ell\right) d\mathbf{x} \\
 &= \frac{\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \\
 &\quad \int_{\zeta_1}^{\zeta_2} \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{u} - \zeta_1)} (\mathbf{u} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{u})} (\zeta_2 - \mathbf{u})^{\alpha-1} \right] F^*\left(\frac{\zeta_1 \zeta_2}{\mathbf{u}}, \ell\right) d\mathbf{u} \\
 &= \frac{\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \\
 &\quad \int_{\zeta_1}^{\zeta_2} \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{u} - \zeta_1)} (\mathbf{u} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{u})} (\zeta_2 - \mathbf{u})^{\alpha-1} \right] F^*\left(\frac{\zeta_1 \zeta_2}{\zeta_1 + \zeta_2 - \mathbf{u}}, \ell\right) d\mathbf{u},
 \end{aligned}$$

which now gives,

$$\begin{aligned}
 &\frac{\gamma^\alpha \Gamma(\alpha + 1)(\zeta_1 \zeta_2)^\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}^-}^{\alpha, \gamma} (F^* \circ \psi)\left(\frac{1}{\zeta_2}\right) + \mathfrak{J}_{\frac{1}{\zeta_2}^+}^{\alpha, \gamma} (F^* \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] \\
 &\quad - F^*\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \\
 &= \frac{\alpha}{4(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \int_{\zeta_1}^{\zeta_2} \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha-1} \right] \\
 &\quad \times \left[ F^*\left(\frac{\zeta_1 \zeta_2}{\mathbf{x}}, \ell\right) + F^*\left(\frac{\zeta_1 \zeta_2}{\zeta_1 + \zeta_2 - \mathbf{x}}, \ell\right) - 2F^*\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \right] d\mathbf{x}. \tag{4.1}
 \end{aligned}$$

Since

$$\begin{aligned}
 &\left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha-1} \right] \\
 &\quad \left[ F^*\left(\frac{\zeta_1 \zeta_2}{\mathbf{x}}, \ell\right) + F^*\left(\frac{\zeta_1 \zeta_2}{\zeta_1 + \zeta_2 - \mathbf{x}}, \ell\right) - 2F^*\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \right]
 \end{aligned}$$

is symmetric about  $\frac{\zeta_1 + \zeta_2}{2}$  and we get,

$$\begin{aligned}
 &\frac{\alpha}{4(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \int_{\zeta_1}^{\zeta_2} \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha-1} \right] \\
 &\quad \times \left[ F^*\left(\frac{\zeta_1 \zeta_2}{\mathbf{x}}, \ell\right) + F^*\left(\frac{\zeta_1 \zeta_2}{\zeta_1 + \zeta_2 - \mathbf{x}}, \ell\right) - 2F^*\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \right] d\mathbf{x} \\
 &= \frac{\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \int_{\zeta_1}^{\frac{\zeta_1 + \zeta_2}{2}} \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha-1} \right]
 \end{aligned}$$

$$\left[ F^* \left( \frac{\zeta_1 \zeta_2}{\mathbf{x}}, \ell \right) + F^* \left( \frac{\zeta_1 \zeta_2}{\zeta_1 + \zeta_2 - \mathbf{x}}, \ell \right) - 2F^* \left( \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell \right) \right] d\mathbf{x}. \quad (4.2)$$

Which gives,

$$\begin{aligned} & \frac{\gamma^\alpha \Gamma(\alpha + 1) (\zeta_1 \zeta_2)^\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2} \right) \right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}}^{\alpha, \gamma} (F^* \circ \psi) \left( \frac{1}{\zeta_2} \right) + \mathfrak{J}_{\frac{1}{\zeta_2}}^{\alpha, \gamma} (F^* \circ \psi) \left( \frac{1}{\zeta_1} \right) \right] \\ & - F^* \left( \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell \right) \\ & = \frac{\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1 \left( \alpha, \alpha + 1, \frac{\gamma-1}{\gamma} \left( \frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2} \right) \right)} \int_{\zeta_1}^{\frac{\zeta_1 + \zeta_2}{2}} \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha-1} \right] \\ & \times \left[ F^* \left( \frac{\zeta_1 \zeta_2}{\mathbf{x}}, \ell \right) + F^* \left( \frac{\zeta_1 \zeta_2}{\zeta_1 + \zeta_2 - \mathbf{x}}, \ell \right) - 2F^* \left( \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell \right) \right] d\mathbf{x}. \quad (4.3) \end{aligned}$$

Since

$$F^* \left( \frac{\zeta_1 \zeta_2}{\zeta_1 + \zeta_2 - \mathbf{x}}, \ell \right) - F^* \left( \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell \right) = F^*(\zeta_1 + \zeta_2 - \mathbf{x}, \ell) - F^* \left( \frac{\zeta_1 + \zeta_2}{2}, \ell \right) = \int_{\frac{\zeta_1 + \zeta_2}{2}}^{\zeta_1 + \zeta_2 - \mathbf{x}} F'^*(\mathbf{k}, \ell) d\mathbf{k}$$

and

$$F^* \left( \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell \right) - F^* \left( \frac{\zeta_1 \zeta_2}{\mathbf{x}}, \ell \right) = F^* \left( \frac{\zeta_1 + \zeta_2}{2}, \ell \right) - F^*(\mathbf{x}, \ell) = \int_{\mathbf{x}}^{\frac{\zeta_1 + \zeta_2}{2}} F'^*(\mathbf{k}, \ell) d\mathbf{k},$$

then, we get

$$\begin{aligned} & F^* \left( \frac{\zeta_1 \zeta_2}{\mathbf{x}}, \ell \right) + F^* \left( \frac{\zeta_1 \zeta_2}{\zeta_1 + \zeta_2 - \mathbf{x}}, \ell \right) - 2F^* \left( \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell \right) = \int_{\frac{\zeta_1 + \zeta_2}{2}}^{\zeta_1 + \zeta_2 - \mathbf{x}} F'^*(\mathbf{k}, \ell) d\mathbf{k} - \int_{\mathbf{x}}^{\frac{\zeta_1 + \zeta_2}{2}} F'^*(\mathbf{k}, \ell) d\mathbf{k} \\ & = \int_{\mathbf{x}}^{\frac{\zeta_1 + \zeta_2}{2}} F'^*(\zeta_1 + \zeta_2 - \mathbf{k}, \ell) d\mathbf{k} - \int_{\mathbf{x}}^{\frac{\zeta_1 + \zeta_2}{2}} F'^*(\mathbf{k}, \ell) d\mathbf{k} = \int_{\mathbf{x}}^{\frac{\zeta_1 + \zeta_2}{2}} \left[ F'^*(\zeta_1 + \zeta_2 - \mathbf{k}, \ell) - F'^*(\mathbf{k}, \ell) \right] d\mathbf{k}. \quad (4.4) \end{aligned}$$

Since

$$F'^*(\zeta_1 + \zeta_2 - \mathbf{k}, \ell) - F'^*(\mathbf{k}, \ell) = \int_{\mathbf{k}}^{\zeta_1 + \zeta_2 - \mathbf{k}} F''^*(\mathbf{y}) d\mathbf{y},$$

for  $\mathbf{k} \in \left[ \zeta_1, \frac{\zeta_1 + \zeta_2}{2} \right]$ , one has

$$m(\zeta_1 + \zeta_2 - 2\mathbf{k}) \leq F'^*(\zeta_1 + \zeta_2 - \mathbf{k}, \ell) - F'^*(\mathbf{k}, \ell) \leq M(\zeta_1 + \zeta_2 - 2\mathbf{k}).$$

Hence

$$\begin{aligned} \int_{\mathbf{x}}^{\frac{\zeta_1 + \zeta_2}{2}} m(\zeta_1 + \zeta_2 - 2\mathbf{k}) d\mathbf{k} & \leq F^* \left( \frac{\zeta_1 \zeta_2}{\mathbf{x}}, \ell \right) + F^* \left( \frac{\zeta_1 \zeta_2}{\zeta_1 + \zeta_2 - \mathbf{x}}, \ell \right) - 2F^* \left( \frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell \right) \\ & \leq \int_{\mathbf{x}}^{\frac{\zeta_1 + \zeta_2}{2}} M(\zeta_1 + \zeta_2 - 2\mathbf{k}) d\mathbf{k}. \quad (4.5) \end{aligned}$$

So,

$$m\left(\frac{\zeta_1 + \zeta_2}{2} - \mathbf{x}\right)^2 \leq F^*\left(\frac{\zeta_1 \zeta_2}{\mathbf{x}}, \ell\right) + F^*\left(\frac{\zeta_1 \zeta_2}{\zeta_1 + \zeta_2 - \mathbf{x}}, \ell\right) - 2F^*\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \leq M\left(\frac{\zeta_1 + \zeta_2}{2} - \mathbf{x}\right)^2,$$

i.e

$$\begin{aligned} & \frac{m\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \\ & \int_{\zeta_1}^{\frac{\delta_1 + \zeta_2}{2}} \left(\frac{\zeta_2 - \zeta_1}{2} - \mathbf{x}\right)^2 \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha-1} \right] d\mathbf{x} \\ & \leq \frac{\gamma^\alpha \Gamma(\alpha + 1) (\zeta_1 \zeta_2)^\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}^-}^{\alpha, \gamma}(F^* \circ \psi)\left(\frac{1}{\zeta_2}\right) + \mathfrak{J}_{\frac{1}{\zeta_2}^+}^{\alpha, \gamma}(F^* \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] \\ & \quad - F^*\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \\ & \leq \frac{M\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \\ & \int_{\zeta_1}^{\frac{\zeta_1 + \zeta_2}{2}} \left(\frac{\zeta_1 + \zeta_2}{2} + \mathbf{x}\right)^2 \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha-1} \right] d\mathbf{x}. \end{aligned} \tag{4.6}$$

Similarly for upper fuzzy integral valued function, we will have:

$$\begin{aligned} & \frac{m\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \\ & \int_{\zeta_1}^{\frac{\delta_1 + \zeta_2}{2}} \left(\frac{\zeta_2 - \zeta_1}{2} - \mathbf{x}\right)^2 \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha-1} \right] d\mathbf{x} \\ & \leq \frac{\gamma^\alpha \Gamma(\alpha + 1) (\zeta_1 \zeta_2)^\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}^-}^{\alpha, \gamma}(F_* \circ \psi)\left(\frac{1}{\zeta_2}\right) + \mathfrak{J}_{\frac{1}{\zeta_2}^+}^{\alpha, \gamma}(F_* \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] \\ & \quad - F_*\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \\ & \leq \frac{M\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \\ & \int_{\zeta_1}^{\frac{\zeta_1 + \zeta_2}{2}} \left(\frac{\zeta_1 + \zeta_2}{2} + \mathbf{x}\right)^2 \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha-1} \right] d\mathbf{x}. \end{aligned} \tag{4.7}$$

Combining and rearranging (4.6) and (4.7), we have:

$$\frac{m\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}\right)\right)} \int_{\zeta_1}^{\frac{\delta_1 + \zeta_2}{2}} \left(\frac{\zeta_2 - \zeta_1}{2} - \mathbf{x}\right)^2$$

$$\begin{aligned}
& \left[ e^{\frac{\gamma-1}{\gamma}(x-\zeta_1)}(x-\zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2-x)}(\zeta_2-x)^{\alpha-1} \right] dx \\
& \ll \frac{\gamma^\alpha \Gamma(\alpha+1)(\zeta_1 \zeta_2)^\alpha}{2(\zeta_2-\zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha+1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1 \zeta_2}\right)\right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}}^{\alpha, \gamma}(\bar{F} \circ \psi)\left(\frac{1}{\zeta_2}\right) \tilde{+} \mathfrak{J}_{\frac{1}{\zeta_2}}^{\alpha, \gamma}(\bar{F} \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] - \bar{F}\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}\right) \\
& \ll \frac{M\alpha}{2(\zeta_2-\zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha+1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1 \zeta_2}\right)\right)} \\
& \int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} \left( \frac{\zeta_1 + \zeta_2}{2} + x \right) \left[ e^{\frac{\gamma-1}{\gamma}(x-\zeta_1)}(x-\zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2-x)}(\zeta_2-x)^{\alpha-1} \right] dx.
\end{aligned}$$

□

**Theorem 4.2.** Let  $\bar{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{I}(\mathcal{R})$  be a twice differentiable, fuzzy interval-valued function with  $0 < \zeta_1 < \zeta_2$  and  $\bar{F} \in \mathcal{L}[\zeta_1, \zeta_2]$ . If  $\bar{F}''$  is bounded on  $[\zeta_1, \zeta_2]$  and  $\bar{F}(x) = \left(\frac{\zeta_1 \zeta_2}{x}\right)$ , then we ave the following inequalities for F-GPFIs:

$$\begin{aligned}
& \frac{-M\alpha}{2(\zeta_2-\zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha+1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1 \zeta_2}\right)\right)} \int_{\zeta_1}^{\frac{\delta_1+\zeta_2}{2}} \left( \frac{\zeta_2-\zeta_1}{2} - x \right)^2 \\
& \left[ e^{\frac{\gamma-1}{\gamma}(x-\zeta_1)}(x-\zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2-x)}(\zeta_2-x)^{\alpha-1} \right] dx \\
& \ll \frac{\gamma^\alpha \Gamma(\alpha+1)(\zeta_1 \zeta_2)^\alpha}{2(\zeta_2-\zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha+1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1 \zeta_2}\right)\right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}}^{\alpha, \gamma}(\bar{F} \circ \psi)\left(\frac{1}{\zeta_2}\right) \tilde{+} \mathfrak{J}_{\frac{1}{\zeta_2}}^{\alpha, \gamma}(\bar{F} \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] - \bar{F}\left(\frac{2\zeta_1 \zeta_2}{\zeta_1 + \zeta_2}\right) \\
& \ll \frac{-m\alpha}{2(\zeta_2-\zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha+1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1 \zeta_2}\right)\right)} \\
& \int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} \left( \frac{\zeta_1 + \zeta_2}{2} + x \right) \left[ e^{\frac{\gamma-1}{\gamma}(x-\zeta_1)}(x-\zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2-x)}(\zeta_2-x)^{\alpha-1} \right] dx. \tag{4.8}
\end{aligned}$$

*Proof.* Using the same notations and assumptions as theorem (4.1) , and equation (4.3) and foe upper fuzzy integral valued function, one gets:

$$\begin{aligned}
& \frac{\gamma^\alpha \Gamma(\alpha+1)(\zeta_1 \zeta_2)^\alpha}{2(\zeta_2-\zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha+1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1 \zeta_2}\right)\right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}}^{\alpha, \gamma}(F^* \circ \psi)\left(\frac{1}{\zeta_2}\right) + \mathfrak{J}_{\frac{1}{\zeta_2}}^{\alpha, \gamma}(F^* \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] \\
& - \frac{F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell)}{2} \\
& = \frac{\alpha}{4(\zeta_2-\zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha+1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1 \zeta_2}\right)\right)} \int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} \left[ e^{\frac{\gamma-1}{\gamma}(x-\zeta_1)}(x-\zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2-x)}(\zeta_2-x)^{\alpha-1} \right] \\
& \times \left[ F^*\left(\frac{\zeta_1 \zeta_2}{x}, \ell\right) + F^*\left(\frac{\zeta_1 \zeta_2}{\zeta_1 + \zeta_2 - x}, \ell\right) - [F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell)] \right] dx. \tag{4.9}
\end{aligned}$$

since

$\left[ e^{\frac{\gamma-1}{\gamma}(x-\zeta_1)}(x-\zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2-x)}(\zeta_2-x)^{\alpha-1} \right] \left[ F^*\left(\frac{\zeta_1\zeta_2}{x}, \ell\right) + F^*\left(\frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2-x}, \ell\right) - [F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell)] \right]$   
 is symmetric about  $\frac{\zeta_1+\zeta_2}{2}$ , we get

$$\begin{aligned} & \frac{\gamma^\alpha \Gamma(\alpha+1)(\zeta_1\zeta_2)^\alpha}{2(\zeta_2-\zeta_1)^\alpha F_1\left(\alpha, \alpha+1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1\zeta_2}\right)\right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}^-}^{\alpha, \gamma}(F^* \circ \psi)\left(\frac{1}{\zeta_2}\right) + \mathfrak{J}_{\frac{1}{\zeta_2}^+}^{\alpha, \gamma}(F^* \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] \\ & - \frac{F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell)}{2} \\ = & \frac{\alpha}{2(\zeta_2-\zeta_1)^\alpha F_1\left(\alpha, \alpha+1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1\zeta_2}\right)\right)} \int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} \left[ e^{\frac{\gamma-1}{\gamma}(x-\zeta_1)}(x-\zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2-x)}(\zeta_2-x)^{\alpha-1} \right] \\ & \times \left[ F^*\left(\frac{\zeta_1\zeta_2}{x}, \ell\right) + F^*\left(\frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2-x}, \ell\right) - [F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell)] \right] dx. \end{aligned} \tag{4.10}$$

Since

$$F^*(\zeta_1, \ell) - F^*\left(\frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2-x}, \ell\right) = F^*(\zeta_2, \ell) - F^*(\zeta_1+\zeta_2-x, \ell) = \int_{\zeta_1+\zeta_2-x}^{\zeta_2} F'(\mathbf{k}, \ell) d\mathbf{k}$$

and

$$F^*\left(\frac{\zeta_1\zeta_2}{x}, \ell\right) - F^*(\zeta_2, \ell) = F^*(x, \ell) - F^*(\zeta_1, \ell) = \int_{\zeta_1}^x F'(\mathbf{k}, \ell) d\mathbf{k}$$

then

$$\begin{aligned} & F^*\left(\frac{\zeta_1\zeta_2}{x}, \ell\right) + F^*\left(\frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2-x}, \ell\right) - [F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell)] = \int_{\zeta_1}^x F'(\mathbf{k}, \ell) d\mathbf{k} - \int_{\zeta_1+\zeta_2-x}^{\zeta_2} F'(\mathbf{k}, \ell) d\mathbf{k} \\ = & - \int_{\zeta_1}^x [F'(\zeta_1+\zeta_2-\mathbf{k}) - F'(\mathbf{k}, \ell)] d\mathbf{k}. \end{aligned} \tag{4.11}$$

We again have

$$[F'(\zeta_1+\zeta_2-\mathbf{k}) - F'(\mathbf{k}, \ell)] = \int_{\mathbf{k}}^{\zeta_1+\zeta_2-\mathbf{k}} F''(\mathbf{z}, \ell) d\mathbf{z}.$$

Now for  $\mathbf{k} \in \left[ \zeta_1, \frac{\zeta_1+\zeta_2}{2} \right]$ , we have:

$$m(\zeta_1+\zeta_2-2\mathbf{k}) \leq F'(\zeta_1+\zeta_2-\mathbf{k}, \ell) - F'(\mathbf{k}, \ell) \leq M(\zeta_1+\zeta_2-2\mathbf{k}).$$

Hence

$$\begin{aligned} - \int_x^{\frac{\zeta_1+\zeta_2}{2}} M(\zeta_1+\zeta_2-2\mathbf{k}) d\mathbf{k} & \leq F^*\left(\frac{\zeta_1\zeta_2}{x}, \ell\right) + F^*\left(\frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2-x}, \ell\right) - [F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell)] \\ & \leq - \int_x^{\frac{\zeta_1+\zeta_2}{2}} m(\zeta_1+\zeta_2-2\mathbf{k}) d\mathbf{k}, \end{aligned}$$

i.e

$$-M(x-\zeta_1)(\zeta_2-x) \leq F^*\left(\frac{\zeta_1\zeta_2}{x}, \ell\right) + F^*\left(\frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2-x}, \ell\right) - [F^*(\zeta_1, \ell) + F^*(\zeta_2, \ell)]$$

$$-m(\mathbf{x} - \zeta_1)(\zeta_2 - \mathbf{x}). \quad (4.12)$$

Similarly for lower fuzzy integral valued function

$$-M(\mathbf{x} - \zeta_1)(\zeta_2 - \mathbf{x}) \leq F_*\left(\frac{\zeta_1\zeta_2}{\mathbf{x}}\right) + F_*\left(\frac{\zeta_1\zeta_2}{\zeta_1 + \zeta_2 - \mathbf{x}}\right) - \left[F_*(\zeta_1, \ell) + F_*(\zeta_2, \ell)\right] - m(\mathbf{x} - \zeta_1)(\zeta_2 - \mathbf{x}). \quad (4.13)$$

After combining and simplifying (4.12) and (4.13) we get our desired result, which is (4.8).  $\square$

Next, we will prove the results of theorem (4.1) and (4.2) without harmonically fuzzy convex property.

**Theorem 4.3.** Let  $\widetilde{F} : \rightarrow \mathbb{I}(\mathbb{R})$  be a positive and twice differentiable function with  $0 \leq \zeta_1 < \zeta_2$ , and  $\widetilde{F} \in \mathcal{L}[\zeta_1, \zeta_2]$ . If  $\widetilde{F}'(\zeta_1 + \zeta_2 - \mathbf{x}) \geq \widetilde{F}'(\mathbf{x})$ ,  $\forall \mathbf{x} \in \left[\zeta_1, \frac{\zeta_1 + \zeta_2}{2}\right]$ , where  $\widetilde{F}(\mathbf{x}) = \widetilde{F}\left(\frac{\zeta_1\zeta_2}{\mathbf{x}}\right)$ , then we have the following inequalities for GPFIs:

$$\begin{aligned} \widetilde{F}\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}, \ell\right) &\leq \frac{\gamma^\alpha \Gamma(\alpha + 1)}{2 \cdot {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{2\zeta_1\zeta_2}\right)\right)} \left(\frac{2\zeta_1\zeta_2}{\zeta_2 - \zeta_1}\right)^\alpha \\ \left[ \mathfrak{J}_{\left(\frac{\zeta_1 + \zeta_2}{2\zeta_1\zeta_2}\right)^-}^{\alpha, \gamma} (\widetilde{F} \circ \psi)\left(\frac{1}{\zeta_2}\right) \right] + \left[ \mathfrak{J}_{\left(\frac{\zeta_1 + \zeta_2}{2\zeta_1\zeta_2}\right)^+}^{\alpha, \gamma} (\widetilde{F} \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] &\leq \frac{\widetilde{F}(\zeta_1) + \widetilde{F}(\zeta_2)}{2}. \end{aligned} \quad (4.14)$$

*Proof.* From equation ((4.3)) and ((4.4)) we can get:

$$\begin{aligned} &\frac{\gamma^\alpha \Gamma(\alpha + 1)(\zeta_1\zeta_2)^\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2}\right)\right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}}^{\alpha, \gamma} (F^* \circ \psi)\left(\frac{1}{\zeta_2}\right) + \mathfrak{J}_{\frac{1}{\zeta_2}}^{\alpha, \gamma} (F^* \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] \\ &- F^*\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \\ &= \frac{\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2}\right)\right)} \int_{\zeta_1}^{\frac{\zeta_1 + \zeta_2}{2}} \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha-1} \right] \\ &\times \left[ F^*\left(\frac{\zeta_1\zeta_2}{\mathbf{x}}, \ell\right) + F^*\left(\frac{\zeta_1\zeta_2}{\zeta_1 + \zeta_2 - \mathbf{x}}, \ell\right) - 2F^*\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \right] d\mathbf{x}. \\ &= \frac{\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2}\right)\right)} \int_{\zeta_1}^{\frac{\zeta_1 + \zeta_2}{2}} \left[ e^{\frac{\gamma-1}{\gamma}(\mathbf{x} - \zeta_1)} (\mathbf{x} - \zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2 - \mathbf{x})} (\zeta_2 - \mathbf{x})^{\alpha-1} \right] \\ &\times \left[ \int_{\mathbf{x}}^{\frac{\zeta_1 + \zeta_2}{2}} \left[ F^{*'}(\zeta_1 + \zeta_2 - \mathbf{k}, \ell) - F^{*}'(\mathbf{k}, \ell) \right] d\mathbf{k} \right]. \end{aligned}$$

Similarly for lower fuzzy integral-valued function, we will have:

$$\frac{\gamma^\alpha \Gamma(\alpha + 1)(\zeta_1\zeta_2)^\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2}\right)\right)} \left[ \mathfrak{J}_{\frac{1}{\zeta_1}}^{\alpha, \gamma} (F_* \circ \psi)\left(\frac{1}{\zeta_2}\right) + \mathfrak{J}_{\frac{1}{\zeta_2}}^{\alpha, \gamma} (F_* \circ \psi)\left(\frac{1}{\zeta_1}\right) \right]$$

$$\begin{aligned}
 & -F_*\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}, \ell\right) \\
 = & \frac{\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1\zeta_2}\right)\right)} \int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} \left[ e^{\frac{\gamma-1}{\gamma}(x-\zeta_1)}(x-\zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2-x)}(\zeta_2-x)^{\alpha-1} \right] \\
 & \times \left[ \int_x^{\frac{\zeta_1+\zeta_2}{2}} \left[ F'_*(\zeta_1 + \zeta_2 - \mathbf{k}, \ell) - F'_*(\mathbf{k}, \ell) \right] d\mathbf{k} \right].
 \end{aligned}$$

Combining and simplifying the valued at upper and lower fuzzy integral valued function, we will have the following:

$$\begin{aligned}
 & \frac{\gamma^\alpha \Gamma(\alpha + 1)(\zeta_1\zeta_2)^\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1\zeta_2}\right)\right)} \left[ \mathfrak{S}_{\frac{1}{\zeta_1}}^{\alpha, \gamma}(\bar{F} \circ \psi)\left(\frac{1}{\zeta_2}\right) \bar{\mathfrak{S}}_{\frac{1}{\zeta_2}}^{\alpha, \gamma}(\bar{F} \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] - \bar{F}\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}\right) \\
 = & \frac{\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1\zeta_2}\right)\right)} \int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} \left[ e^{\frac{\gamma-1}{\gamma}(x-\zeta_1)}(x-\zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2-x)}(\zeta_2-x)^{\alpha-1} \right] \\
 & \times \left[ \int_x^{\frac{\zeta_1+\zeta_2}{2}} \left[ \bar{F}(\zeta_1 + \zeta_2 - \mathbf{k}) - \bar{F}(\mathbf{k}) \right] d\mathbf{k} \right] \geq 0.
 \end{aligned}$$

Similarly from equation (4.10) and (4.11), and after simplifying upper and lower fuzzy interval-valued function we will have the following:

$$\begin{aligned}
 & \frac{\gamma^\alpha \Gamma(\alpha + 1)(\zeta_1\zeta_2)^\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1\zeta_2}\right)\right)} \left[ \mathfrak{S}_{\frac{1}{\zeta_1}}^{\alpha, \gamma}(\bar{F} \circ \psi)\left(\frac{1}{\zeta_2}\right) \bar{\mathfrak{S}}_{\frac{1}{\zeta_2}}^{\alpha, \gamma}(\bar{F} \circ \psi)\left(\frac{1}{\zeta_1}\right) \right] \\
 & - \frac{\bar{F}(\zeta_1) \bar{F}(\zeta_2)}{2} \\
 = & \frac{\alpha}{2(\zeta_2 - \zeta_1)^\alpha {}_1F_1\left(\alpha, \alpha + 1, \frac{\gamma-1}{\gamma}\left(\frac{\zeta_2-\zeta_1}{\zeta_1\zeta_2}\right)\right)} \int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} \left( \left[ e^{\frac{\gamma-1}{\gamma}(x-\zeta_1)}(x-\zeta_1)^{\alpha-1} + e^{\frac{\gamma-1}{\gamma}(\zeta_2-x)}(\zeta_2-x)^{\alpha-1} \right] \right. \\
 & \left. \times \left[ \int_{\zeta_1}^x \left[ \bar{F}(\zeta_1 + \zeta_2 - \mathbf{k}) - \bar{F}(\mathbf{k}) \right] d\mathbf{k} \right] \right) dx \leq 0.
 \end{aligned}$$

□

### CONCLUSION

In this study, we described the newly generalized proportional fuzzy fractional operators (GPF-FOs) and modified the Hermite–Hadamard inequalities to fuzzy interval-valued functions (FIVFs) under the concepts of harmonically convex functions by implementation of newly developed GPFFO. After defining upper and lower fuzzy proportional fractional integrals, we established

further refinements of Hermite-Hadamard-type inequalities with harmonically convexity. Theoretical concepts will provide a strong foundation for further investigations in fuzzy fractional analysis, offering potential applications in uncertain modeling and systems with imprecise or interval-valued data. Moreover, we discussed the numerical computations of our main results, which show the validation of analytical and numerical behaviors. Future research may explore more general convexity types, alternative fractional operators, and higher-dimensional extensions. We concluded that such type of refinements could be possible for analytical and numerical behavior by implementation of GPFFO with harmonically fuzzy convexity.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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