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A General Conformable Diffusion Equation

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Abstract. We here derive a new mathematical model for a transport system in a diffusion process with involving general conformable fractional derivative. The model is called general conformable diffusion equation derived from continuous time random walk process considering stochastic aspects. The model is characterized by a function called fractional conformable function which has an intimate relationship with the waiting time of a cell to move in the diffusion process. Solutions to the general conformable diffusion equation are also studied. The application in modeling the transport system of chloride ions diffusing in non-saturated concretes is given to show that the general conformable diffusion equation can be used to model the diffusion process better by choosing a suitable fractional conformable function.

1. Introduction

Diffusion is a phenomenon related to the movement of molecules, particles, or cells from areas of higher concentration to areas of lower concentration. The diffusion process is widely applied in engineering, medicine, physics, biology, chemistry, economics or finance, and many other fields. In these fields, the diffusion process is applied to model phenomena that occur randomly and continuously under certain conditions. In physics, the diffusion process is used to model the movement of physical entities such as electric charge, mass, momentum, energy, or entropy in a physical system. In cell biology, the diffusion process is used to model the transport system or movement of needed substances such as amino acids in cells. The diffusion process is also involved in metabolism and respiration processes [1]. In finance, the diffusion process is used to model the movement of financial instrument values such as the movement of the exchange rate of US Dollar against the Japanese Yen [2]. In medicine, the diffusion process is used to model the

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transport system of drug molecules through cell membranes [3]. In addition, the diffusion process can also be applied to model the invasion and growth of cancer cells [4].

Mathematical modeling for the diffusion process has been carried out by several researchers. One of them is Othmer et al. [5] modeling the diffusion process via continuous time random walk process by the equation

$$\frac{\partial}{\partial t}p(x,t) = D\Delta p(x,t) \tag{1.1}$$

where $p(x,t)$ represents the probability density function of a cell being at position $x \in \mathbb{R}^n$ at time $t > 0$, and D represents the diffusion coefficient. The cell movement in the diffusion process modeled by the diffusion equation (1.1) follows the pattern

$$\langle x^2(t)\rangle \sim t$$

where $\langle x^2(t)\rangle$ represents the mean square displacement of the cell movement at t. However, in reality, there are several diffusion processes that cannot be well modeled by this diffusion equation, such as contaminant migration in groundwater systems [6], dispersive ion transport in column experiments [7], water diffusion in sand [8], dispersion in heterogeneous aquifers [9], and contaminant transport in geological formations [10]. These processes follow the pattern

$$\langle x^2(t)\rangle \sim t^\alpha, \quad 0 < \alpha < 1.$$

meaning that cells in these diffusion processes move more slowly than cells in the diffusion process modeled by the equation (1.1). Therefore, the equation (1.1) is less well for modeling the cell movement in such diffusion processes. These diffusion processes are called subdiffusion or slow diffusion. Mathematical modeling for the subdiffusion process via continuous time random walk process has been carried out, including by Mainardi [11] and Metzler-Klafter [12], of the form

$$\frac{\partial^\alpha}{\partial t^\alpha}p(x,t) = K_\alpha \Delta p(x,t) \tag{1.2}$$

where $0 < \alpha < 1$, K_α is the subdiffusion coefficient, and d^α/dt^α is Caputo fractional derivative operator defined by

$$\frac{d^\alpha}{dt^\alpha}f(t) = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\frac{d}{d\tau}f(\tau)d\tau. \tag{1.3}$$

However, the use of Caputo fractional derivative operator d^α/dt^α in subdiffusion equation (1.2) has several drawbacks:

1. the definition of Caputo fractional derivative operator d^α/dt^α is quite complex due to its nonlocal property, involving an integral operator. Consequently, solutions to equations involving Caputo fractional derivative operator is not easy to solve analytically and numerically;
2. Caputo fractional derivative operator d^α/dt^α does not satisfy several properties of usual derivative operator d/dt, such as the derivative rule for multiplication or division of two functions, the chain rule, Mean Value Theorem, and Rolle's Theorem.

Due to the weaknesses of Caputo fractional derivative operator, Khalil et al. [13] introduced a local fractional derivative operator whose definition is simpler than the definition of Caputo fractional derivative operator. The local fractional derivative operator is called conformable fractional derivative. The conformable fractional operator T_t^α of a function $f : [0, \infty) \to \mathbb{R}$ of order $\alpha \in (0, 1)$ is defined by

$$T_t^\alpha f(t) = \lim_{\varepsilon \to 0^+} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, t > 0. \tag{1.4}$$

The conformable fractional derivative (1.4) is analogous to the definition of usual derivative operator. It is a local operator since its definition does not involve an integral operator as used in the definition of Caputo fractional derivative operator. The conformable fractional derivative operator T_t^α satisfies the properties of usual derivative operator d/dt, such as the derivative rule for multiplication or division of two functions, the chain rule, the Mean Value Theorem, and Rolle's Theorem. Due to its simpler definition, the solutions of equations involving the conformable fractional derivative operator is also easier to solve analytically and numerically. The conformable fractional derivative operator can also be used to model subdiffusion or slow diffusion process, as showed by Zhou et al. [14], Yang et al. [15], and Guswanto et al. [16], which is expressed by the equation

$$T_t^\alpha p(x, t) = D_\alpha \Delta p(x, t) \tag{1.5}$$

where $0 < \alpha < 1$ and D_α is the subdiffusion coefficient. In [17], it was shown that the conformable fractional derivative offers an efficient and accurate approach for describing nonlinear diffusion-reaction systems. Meanwhile, in [18], the conformable fractional derivative was able to be used to model diffusion of pollutant in the ground. Also, in [19], the conformable fractional derivative could be used successfully to estimate the fate of pollutants in heterogeneous media.

We here offer a new mathematical model for the diffusion process by using other derivative operator introduced by Zhao and Luo [20]. It is a generalization of the conformable fractional derivative T_t^α. Both in [20] extended the definition of the conformable fractional derivative operator T_t^α defined by (1.4) to a derivative operator which is called general conformable fractional derivative operator D_{ψ_α} defined by

$$D_{\psi_\alpha} f(t) = \lim_{\varepsilon \to 0^+} \frac{f(t + \varepsilon \psi_\alpha(t)) - f(t)}{\varepsilon}. \tag{1.6}$$

where $\psi_\alpha : [0, \infty) \to [0, \infty)$ is a continuous function such that

$$\psi_1(t) = 1$$
$$\psi_\alpha \neq \psi_\beta \text{ for } \alpha \neq \beta \text{ and } \alpha, \beta \in (0, 1]$$

or ψ_α is a constant function with $\psi_\alpha(t) = 1$. The function ψ_α is called the fractional conformable function. The derivative operator D_{ψ_α} also satisfies the properties of usual deivative d/dt as mentioned previously. If $\psi_\alpha(t) = 1$ then D_{ψ_α} is usual derivative operator d/dt. If $\psi_\alpha(t) = t^{1-\alpha}$ then D_{ψ_α} is the conformable fractional derivative operator T_t^α defined by equation (1.4). Thus, the usual and conformable fractional derivative operators are special cases of the general conformable fractional derivative operator D_{ψ_α} defined by (1.6). In this paper :

1. a mathematical model for the cell transport system in diffusion process is derived using the theory of continuous time random walk with assumptions: cell jump length is a constant, the probability density function for the cell to move in a certain direction is a constant, and the probability density function for the cell to move after a waiting time is non-Poissonian associated with a fractional conformable function;
2. the mathematical model for the cell transport system in the diffusion process is obtained by employing the general conformable Laplace transform, Fourier transform, the general conformable fractional derivative, and stochastic process theory;
3. the solution to the mathematical model for the cell transport system in the diffusion process is obtained by applying the general conformable Laplace transform and Fourier transform.

So far, no one has derived the mathematical model of the cell transport system in diffusion process by employing the general conformable fractional derivative, especially via continuous time random walk process considering stochastic aspect. The mathematical model is called general conformable diffusion equation and serves as a kind of "template model" for various types of diffusion processes characterized by the probability density functions of the waiting time of the cell to move involving the fractional conformable functions ψ_α's. The fractional conformable function ψ_α used in the mathematical model is chosen such that it is suitable for the waiting time of the cell to move in the diffusion process. In other words, the general conformable diffusion equation is more flexible than the model (1.1) and (1.5) which are its special case. The general conformable diffusion equation is also simpler than the model (1.2) in studying its solution due to the use of the local operator D_{ψ_α}. The application in modeling of the transport system of chloride ions diffusing in non-saturated concretes is also given to show that the general conformable diffusion equation can be used to model a diffusion process better by choosing a suitable fractional conformable function.

This paper consists of four sections. The second section provides briefly some notions concerning the general conformable fractional derivative and general conformable Laplace transform. In the third section, the derivation of the general conformable diffusion equation, its solution, and applications in modeling of the transport system of chloride ions diffusing in non-saturated concretes are given. Conclusion is given in the last section.

2. Preliminaries

Given a continuous function $\psi_\alpha : (0,\infty) \to [0,\infty)$ with $0 < \alpha \leq 1$ satisfying the conditions

$$\psi_1(t) = 1 \tag{2.1}$$

and

$$\psi_\alpha \neq \psi_\beta \text{ for } \alpha \neq \beta \text{ and } \alpha, \beta \in (0,1]. \tag{2.2}$$

Definition 2.1. *[20] A Continuous function $\psi_\alpha : (0,\infty) \to [0,\infty)$ with $0 < \alpha \leq 1$ is called a fractional conformable function if it satisfies the conditions* (2.1)-(2.2) *or $\psi_\alpha(t) = 1$.*

Definition 2.2. *[20] Let $\psi_\alpha : (0,\infty) \to [0,\infty)$ with $0 < \alpha \leq 1$ be a fractional conformable function. The general conformable fractional derivative (GCFD) of a function $f : [0,\infty) \to \mathbb{R}$ is defined as*

$$D_{\psi_\alpha} f(t) = \lim_{h\to 0} \frac{f(t + h\psi_\alpha(t)) - f(t)}{h}, \quad t > 0. \tag{2.3}$$

If the limit exists then f is said to be ψ_α-differentiable at $t > 0$.

Theorem 2.1. *[20] If a function $f : [0,\infty) \to \mathbb{R}$ is ψ_α-differentiable at $t > 0$ then for $0 < \alpha \leq 1$,*

$$D_{\psi_\alpha} f(t) = \psi_\alpha(t) \frac{d}{dt} f(t), \quad t > 0.$$

We now define $\Psi_\alpha(t)$ for $t \geq 0$ by

$$\Psi_\alpha(t) = \int_0^t \frac{1}{\psi_\alpha(\tau)} d\tau.$$

Henceforth, we consider $\psi_\alpha(t)$ such that $\Psi_\alpha(t)$ exists for $t \geq 0$ and $\Psi_\alpha(0) = 0$. We next define general conformable fractional integral operator as stated in the following definition.

Definition 2.3. *[20] Given a function $f : (0,t] \to \mathbb{R}$. The general conformable fractional integral I_{ψ_α} of f is defined by*

$$I_{\psi_\alpha} f(t) = \int_0^t f(\tau) d\Psi_\alpha(\tau) = \int_0^t \Psi'_\alpha(\tau) f(\tau) d\tau = \int_0^t \frac{f(\tau)}{\psi_\alpha(\tau)} d\tau.$$

If the integral exists then f is said to be ψ_α-integrable.

We then define General Conformable Laplace Transform (GCLT) as written in the following definition.

Definition 2.4. *[21] Let $f : [0,\infty) \to \mathbb{R}$ be a piecewise continuous and of the exponential order such that $|f(t)| \leq Me^{C\Psi_\alpha(t)}$ for some constants $C, M > 0$. The general conformable Laplace transform $\mathcal{L}_{\psi_\alpha}$ of f is defined by*

$$\mathcal{L}_{\psi_\alpha}\{f(t)\}(s) = \tilde{f}_{\psi_\alpha}(s) = \int_0^\infty \Psi'_\alpha(t) e^{-s\Psi_\alpha(t)} f(t) dt.$$

We now define the inverse $\mathcal{L}^{-1}_{\psi_\alpha}$ of the general conformable Laplace transform $\mathcal{L}_{\psi_\alpha}$ as defined in the following definition.

Definition 2.5. *[21] Let $0 < \alpha < 1$ and $\tilde{f}_{\psi_\alpha}(s)$ be the general conformable Laplace transform of function $f : [0,\infty) \to \mathbb{R}$. The inverse $\mathcal{L}^{-1}_{\psi_\alpha}$ of the general conformable Laplace transform $\tilde{f}_{\psi_\alpha}(s)$ of f is defined by*

$$\mathcal{L}^{-1}_{\psi_\alpha}\{\tilde{f}_{\psi_\alpha}(s)\}(t) = f(t) = \frac{1}{2\pi i} \int_\gamma e^{s\Psi_\alpha(t)} \tilde{f}_{\psi_\alpha}(s) ds \tag{2.4}$$

where γ is the vertical line $Re(s) = c$ such that c is greater than all real part of singularities of the integrand in the integral (2.4).

If $\mathcal{L}$ is usual Laplace transform then it is not difficult to show that

$$\mathcal{L}_{\psi_\alpha}\{f(t)\}(s) = \mathcal{L}\{f(\Psi_\alpha^{-1}(t))\}(s). \tag{2.5}$$

Moreover, we have the following theorem giving the general conformable Laplace transform for the general conformable fractional derivative of a ψ_α-differentiable function.

Theorem 2.2. *[22] If $f : [0,\infty) \to \mathbb{R}$ is a ψ_α-differentiable function at $t > 0$ then*

$$\mathcal{L}_{\psi_\alpha}\{D_{\psi_\alpha} f(t)\}(s) = s\tilde{f}_{\psi_\alpha}(s) - f(0).$$

Next, it is not difficult to obtain the general conformable laplace transform of the general conformable fractional integral as given in the following theorem.

Theorem 2.3. *If $f : [0,\infty) \to \mathbb{R}$ is a ψ_α-integrable function on $(0,t]$ then*

$$\mathcal{L}_{\psi_\alpha}\{I_{\psi_\alpha} f(t)\}(s) = \frac{\tilde{f}_{\psi_\alpha}(s)}{s}.$$

We now define ψ_α-convolution of two functions as written in the following definition.

Definition 2.6. *[22] If f and g are piecewise continuous functions on $[0,\infty)$ then ψ_α-convolution $f *_{\psi_\alpha} g$ of f and g is given by*

$$(f *_{\psi_\alpha} g)(t) = \int_0^t f(t-\tau)g(\tau)d\Psi_\alpha(\tau) = \int_0^t f(\Psi_\alpha^{-1}(\Psi_\alpha(t) - \Psi_\alpha(\tau)))\Psi'_\alpha(\tau)g(\tau)d\tau.$$

The general conformable Laplace transform of ψ_α-convolution of two functions is given by the following theorem.

Theorem 2.4. *[22] If f and g are piecewise continuous functions on $[0,\infty)$ then the general conformable Laplace transform of ψ_α-convolution $f *_{\psi_\alpha} g$ of f and g is*

$$\mathcal{L}_{\psi_\alpha}\{(f *_{\psi_\alpha} g)(t)\}(s) = \tilde{f}_{\psi_\alpha}(s)\tilde{g}_{\psi_\alpha}(s).$$

The solution to an initial value problem of an ordinary differential equation with the general conformable fractional derivative is given by the following theorem.

Theorem 2.5. *[22] Let $\lambda \in \mathbb{R}$. The solution to the initial value problem*

$$D_{\psi_\alpha} y(t) = \lambda y(t), \quad y(0) = y_0, \quad t > 0$$

is

$$y = y_0 e^{\lambda \Psi_\alpha(t)}.$$

The definition of Fourier transform and its inverse are given below.

Definition 2.7. *Given a function g on $\mathbb{R}$. The Fourier transform $\mathcal{F}$ of g is defined by*

$$\mathcal{F}\{g(x)\}(k) = \hat{g}(k) = \int_{\mathbb{R}^n} e^{ik\cdot x} g(x)dx$$

and the Fourier transform inverse $\mathcal{F}^{-1}$ of $\tilde{g}(k)$ is defined by

$$\mathcal{F}^{-1}\{\hat{g}(k)\}(x) = g(x) = \frac{1}{2\pi}\int_{\mathbb{R}^n} e^{ik\cdot x}\hat{g}(k)dk$$

3. Main Results

3.1. **A General Conformable Diffusion Equation.** In this section, we derive a diffusion equation involving the general conformable fractional derivative via a continuous time random walk process. The equation is called general conformable diffusion equation. A cell moving in the diffusion process is assumed to do a sequence of jumps in $\mathbb{R}^n$ with a constant jump length Δx. We now denote the probability density function of the cell to jump after a waiting time $t > 0$ by $w(t)$ which is characterized by the fractional conformable function $\psi_\alpha(t)$. The function $w(t)$ satisfies the condition

$$\int_0^\infty w(t)d\Psi_\alpha(t) = 1.$$

If $\psi_\alpha(t) = 1$ then $\Psi_\alpha(t) = t$ and $w(t)$ is Poissonian waiting time probability density function. If $\psi_\alpha(t) = t^{1-\alpha}$ for $0 < \alpha < 1$ then $\Psi_\alpha(t) = t^\alpha/\alpha$ and $w(t)$ is non-Poissonian waiting time probability density function as considered in [16]. We next denote the probability density function of the cell to jump from position $x \in \mathbb{R}^n$ in a direction $\omega \in S^{n-1} = \{\omega \in \mathbb{R}^n : |\omega| = 1\}$ by $j(x;\omega)$. The function $j(x;\omega)$ satisfies the condition

$$\int_{S^{n-1}} j(x;\omega)d\omega = 1$$

with $d\omega$ is an isotropic probability measure on S^{n-1} satisfying

$$\int_{S^{n-1}} \omega_i d\omega = 0 \text{ and } \int_{S^{n-1}} \omega_i\omega_j d\omega = \frac{\delta_{ij}}{n}|S^{n-1}|$$

where

$$|S^{n-1}| = \int_{S^{n-1}} d\omega = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

Γ is Gamma function, and δ_{ij} is the Kronecker delta function i.e.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Next, let $P_k(x,t)$ be the conditional probability of the cell to reach position x at time t after k jumps and $P(x,t)$ be the probability of the cell to reach position x at time t. By modifying Othmer et al's way in [5] and Definition (2.6), we have

$$\begin{aligned} P_k(x,t) &= \int_0^t \int_{S^{n-1}} w(t-\tau)j(x-\omega\Delta x;\omega)P_{k-1}(x-\omega\Delta x,\tau)d\omega d\Psi_\alpha(\tau) \\ &= \int_0^t \int_{S^{n-1}} \Psi'_\alpha(\tau)w(\Psi_\alpha^{-1}(\Psi_\alpha(t)-\Psi_\alpha(\tau)))j(x-\omega\Delta x;\omega)P_{k-1}(x-\omega\Delta x,\tau)d\omega d\tau. \end{aligned} \tag{3.1}$$

Then

$$\begin{aligned} P(x,t) &= \sum_{k=0}^\infty P_k(x,t) \qquad (3.2) \\ &= P_0(x,t) + \int_0^t \int_{S^{n-1}} \Psi'_\alpha(\tau)w(\Psi_\alpha^{-1}(\Psi_\alpha(t)-\Psi_\alpha(\tau)))j(x-\omega\Delta x;\omega)P(x-\omega\Delta x,\tau)d\omega d\tau \end{aligned}$$

$$= \delta(x)\delta(t) + \int_0^t \int_{S^{n-1}} \Psi'_\alpha(\tau) w(\Psi_\alpha^{-1}(\Psi_\alpha(t) - \Psi_\alpha(\tau))) j(x - \omega\Delta x; \omega) P(x - \omega\Delta x, \tau) d\omega d\tau$$

where δ is Dirac delta function. Next, Let $p(x,t)$ be the probability of the cell to be at position x and time t with the initial position $x = 0$ and $t = 0$. Then

$$p(x,t) = \int_0^t W(t,\tau;x) P(x,\tau) d\Psi_\alpha(\tau) \tag{3.3}$$

where $W(t,\tau;x)$ stands for the probability of the cell to reach x at $\tau < t$ and not to jump during the time interval $t - \tau$. We next assume that

$$W(t,\tau;x) = W(t-\tau)$$

where

$$W(t) = \int_t^\infty w(\tau) d\Psi_\alpha(\tau) = 1 - \int_0^t w(\tau) d\Psi_\alpha(\tau) = 1 - \int_0^t \Psi'_\alpha(\tau) w(\tau) d\tau \tag{3.4}$$

which is a survival function. By the equations (3.2), (3.3), and (3.4), we obtain

$$\begin{aligned}
p(x,t) &= \delta(x) \int_0^t W(t-\tau)\delta(\tau) d\Psi_\alpha(\tau) \\
&\quad + \int_0^t \int_{S^{n-1}} \left(\int_r^t W(t-\tau) w(t-r) d\Psi_\alpha(\tau) \right) j(x - \omega\Delta x; \omega) P(x - \omega\Delta x, r) d\omega d\Psi_\alpha(r) \\
&= \delta(x) \int_0^t W(t-\tau)\delta(\tau) d\Psi_\alpha(\tau) \\
&\quad + \int_0^t \int_{S^{n-1}} \left(\int_r^t W(t-r) w(t-\tau) d\Psi_\alpha(\tau) \right) j(x - \omega\Delta x; \omega) P(x - \omega\Delta x, r) d\omega d\Psi_\alpha(r) \\
&= \delta(x) \int_0^t \Psi'_\alpha(\tau) W(\Psi_\alpha^{-1}(\Psi_\alpha(t) - \Psi_\alpha(\tau)))\delta(\tau) d\tau \\
&\quad + \int_0^t \int_{S^{n-1}} \Psi'_\alpha(\tau) w(\Psi_\alpha^{-1}(\Psi_\alpha(t) - \Psi_\alpha(\tau))) j(x - \omega\Delta x; \omega) p(x - \omega\Delta x, \tau) d\omega d\tau.
\end{aligned} \tag{3.5}$$

By applying the general conformable Laplace transform to the equation (3.5) and Theorem (2.3)-(2.4), we get

$$\tilde{p}_{\psi_\alpha}(x,s) = \delta(x) \frac{1 - \tilde{w}_{\psi_\alpha}(s)}{s} + \tilde{w}_{\psi_\alpha}(s) \int_{S^{n-1}} j(x - \omega\Delta x; \omega) \tilde{p}_{\psi_\alpha}(x - \omega\Delta x, s) d\omega. \tag{3.6}$$

We then divide and subtract both sides of the equation (3.6) by $1 - \tilde{w}_{\psi_\alpha}(s)$ and $\tilde{p}_{\psi_\alpha}(x,s)\tilde{w}_{\psi_\alpha}(s)(1 - \tilde{w}_{\psi_\alpha}(s))^{-1}$ with $\tilde{w}_{\psi_\alpha}(s) \neq 1$, respectively, implying

$$s\tilde{p}_{\psi_\alpha}(x,s) - \delta(x) = s\tilde{H}_{\psi_\alpha}(s) \left(\int_{S^{n-1}} j(x - \omega\Delta x; \omega) \tilde{p}_{\psi_\alpha}(x - \omega\Delta x, s) d\omega - \tilde{p}_{\psi_\alpha}(x,s) \right) \tag{3.7}$$

with

$$\tilde{H}_{\psi_\alpha}(s) = \frac{\tilde{w}_{\psi_\alpha}(s)}{1 - \tilde{w}_{\psi_\alpha}(s)} = \frac{1 - s\tilde{W}_{\psi_\alpha}(s)}{s\tilde{W}_{\psi_\alpha}(s)}. \tag{3.8}$$

We here consider the waiting time density probability density function $w(t)$ with

$$\tilde{w}_{\psi_\alpha}(s) = 1 - C_{\psi_\alpha} s + (C_{\psi_\alpha} s)^2 - \cdots$$

for some constant $C_{\psi_\alpha} > 0$. Such a $w(t)$ has the survival function $W(t)$ of the form

$$W(t) = e^{-\frac{\Psi_\alpha(t)}{C_{\psi_\alpha}}}. \tag{3.9}$$

Then

$$\tilde{H}_{\psi_\alpha}(s) = \frac{1 - C_{\psi_\alpha}s + (C_{\psi_\alpha}s)^2 - \cdots}{C_{\psi_\alpha}s - (C_{\psi_\alpha}s)^2 + \cdots} = \frac{(C_{\psi_\alpha}s)^{-1} - 1 + C_{\psi_\alpha}s - \cdots}{1 - C_{\psi_\alpha}s + \cdots}. \tag{3.10}$$

If $\psi_\alpha(t) = 1$ then $\Psi_\alpha(t) = t$ and $W(t)$ is a Poissonian survival function. If $\psi_\alpha(t) = t^{1-\alpha}$ for $0 < \alpha < 1$ then $\psi_\alpha(t) = t^\alpha/\alpha$ and $W(t)$ is a non-Poissonian survival function as considered in [16]. If $j(x;\omega)$ is a constant then

$$j(x;\omega) = \frac{1}{|S^{n-1}|} = \frac{\Gamma(n/2)}{2\pi^{n/2}}.$$

Consequently, together with the equation (3.10), the equation (3.7) is reduced to

$$s\tilde{p}_{\psi_\alpha}(x,s) - \delta(x) = \frac{s\tilde{H}_{\psi_\alpha}(s)}{|S^{n-1}|}\int_{S^{n-1}}\Big(\tilde{p}_{\psi_\alpha}(x - \omega\Delta x, s) - \tilde{p}_{\psi_\alpha}(x,s)\Big)d\omega. \tag{3.11}$$

We now consider the following lemma which is useful in the derivation of the general conformable diffusion equation.

Lemma 3.1. *[23] If $f(x)$ is a continuously twice differentiable function with respect to x then*

$$\int_{S^{n-1}}[f(x+\omega\Delta x) - f(x)]d\omega = \frac{1}{2n}|S^{n-1}|(\Delta x)^2\Delta f(x) + o((\Delta x)^2)$$

as $\Delta x \to 0$.

Then, by Lemma (3.1), the equation (3.11) is reduced to

$$s\tilde{p}_{\psi_\alpha}(x,s) - \delta(x) = s\tilde{H}_{\psi_\alpha}(s)\frac{(\Delta x)^2}{2n}\Delta\tilde{p}_{\psi_\alpha}(x,s) + o((\Delta x)^2) \tag{3.12}$$

as $\Delta x \to 0$. If $\Delta x \to 0$, $C_{\psi_\alpha} \to 0$, and $(\Delta x)^2/(2nC_{\psi_\alpha})$ is kept finite then the equation (3.12) becomes

$$s\tilde{p}_{\psi_\alpha}(x,s) - \delta(x) = \frac{(\Delta x)^2}{2nC_{\psi_\alpha}}\Delta\tilde{p}_{\psi_\alpha}(x,s). \tag{3.13}$$

By applying the inverse of general conformable Laplace transform to the equation (3.13) and employing Theorem (2.2), we obtain

$$D_{\psi_\alpha}p(x,t) = K_{\psi_\alpha}\Delta p(x,t) \tag{3.14}$$

with

$$K_{\psi_\alpha} = \frac{(\Delta x)^2}{2nC_{\psi_\alpha}}. \tag{3.15}$$

We call the equation (3.14) the general conformable diffusion equation and K_{ψ_α} ψ_α-diffusion constant. If $\psi_\alpha(t) = 1$, the equation (3.14) is usual diffusion equation (1.1). If $\psi_\alpha(t) = t^{1-\alpha}$, the equation (3.14) is the conformable diffusion equation (1.5) as derived in [16]

3.2. **A Fundamental Solution to General Conformable Diffusion Equation.** We here find a solution to Cauchy problem of the general conformable diffusion equation (3.14)

$$\begin{aligned} D_{\psi_\alpha} p(x,t) &= K_{\psi_\alpha} \Delta p(x,t) \\ p(x,0) &= p_0(x). \end{aligned} \tag{3.16}$$

If we apply Fourier transform to the problem (3.16) then we have

$$\begin{aligned} D_{\psi_\alpha} \hat{p}(k,t) &= -K_{\psi_\alpha} |k|^2 \hat{p}(k,t) \\ \hat{p}(k,0) &= \hat{p}_0(k). \end{aligned} \tag{3.17}$$

By using Theorem (2.5), the solution to the initial value problem (3.17) is

$$\hat{p}(k,t) = \hat{p}_0(k) e^{-K_{\psi_\alpha} |k|^2 \Psi_\alpha(t)}. \tag{3.18}$$

By the inverse of Fourier transform, we obtain the solution to Cauchy problem (3.16) as follows

$$p(x,t) = \int_{\mathbb{R}^n} p_0(x-y) \left(\frac{1}{4\pi K_{\psi_\alpha} \Psi_\alpha(t)} \right)^{n/2} e^{-\frac{1}{4K_{\psi_\alpha} \Psi_\alpha(t)} |y|^2} dy. \tag{3.19}$$

If $p_0(x) = \delta(x)$, we have the fundamental solution to the Cauchy problem (3.16), that is

$$p(x,t) = \left(\frac{1}{4\pi K_{\psi_\alpha} \Psi_\alpha(t)} \right)^{n/2} e^{-\frac{1}{4K_{\psi_\alpha} \Psi_\alpha(t)} |x|^2}. \tag{3.20}$$

The mean square displacement $\langle x^2(t) \rangle$ of the cell movement in the continuous time random walks process is

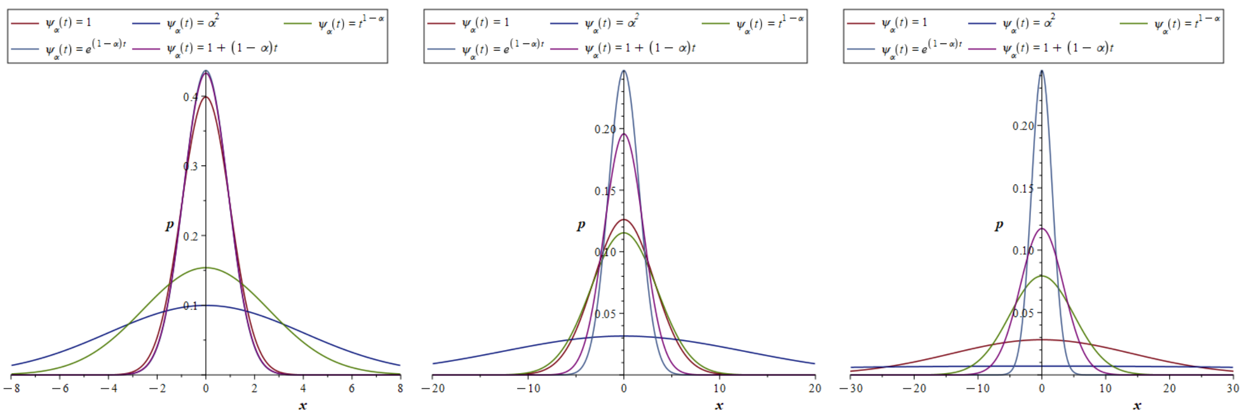
$$\langle x^2(t) \rangle = \int_{\mathbb{R}^n} |x|^2 \left(\frac{1}{4\pi K_{\psi_\alpha} \Psi_\alpha(t)} \right)^{n/2} e^{-\frac{1}{4K_{\psi_\alpha} \Psi_\alpha(t)} |x|^2} dx = 2n K_{\psi_\alpha} \Psi_\alpha(t). \tag{3.21}$$

Next, we here consider the following ψ_α's :

(1) $\psi_\alpha(t) = 1$;
(2) $\psi_\alpha(t) = \alpha^2$;
(3) $\psi_\alpha(t) = t^{1-\alpha}$;
(4) $\psi_\alpha(t) = e^{(1-\alpha)t}$;
(5) $\psi_\alpha(t) = 1 + (1-\alpha)t$.

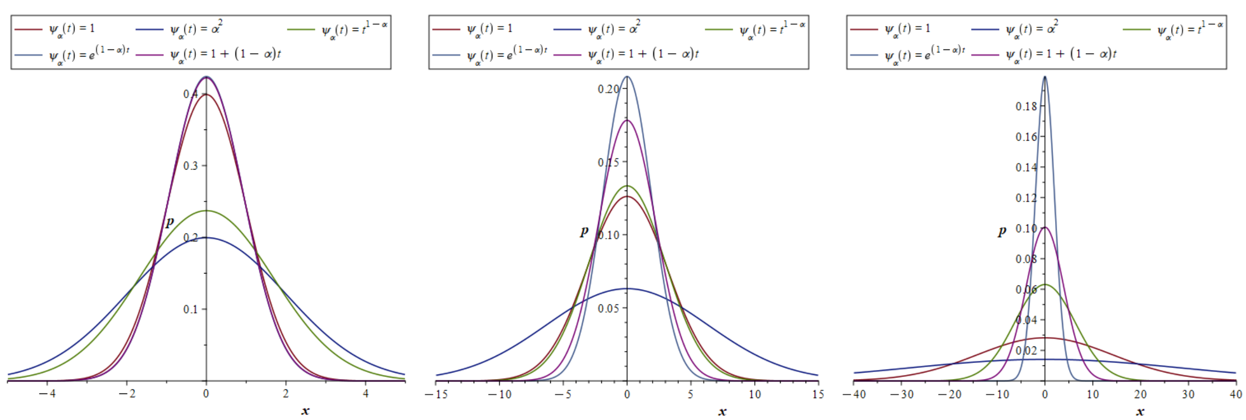
Based on Figure 1-3, it seems that the larger the order α, the closer the curve of $p(x,t)$ with $\psi_\alpha(t) = e^{(1-\alpha)t}$, $\psi_\alpha(t) = 1 + (1-\alpha)t$, $\psi_\alpha(t) = t^{1-\alpha}$, and $\psi_\alpha(t) = \alpha^2$ to the curve of $p(x,t)$ with $\psi_\alpha(t) = 1$ which is associated with usual diffusion. At x around 0, the value of $p(x,t)$ with $\psi_\alpha(t) = e^{(1-\alpha)t}$ is the greatest and the value of $p(x,t)$ with $\psi_\alpha(t) = \alpha^2$ is the least. At $x \to \pm\infty$, the value of $p(x,t)$ with the five $\psi_\alpha(t)$'s from the least one to the greatest one is $p(x,t)$ with $\psi_\alpha(t) = e^{(1-\alpha)t}$, $\psi_\alpha(t) = 1 + (1-\alpha)t$, $\psi_\alpha(t) = t^{1-\alpha}$, $\psi_\alpha(t) = 1$, and $\psi_\alpha(t) = \alpha^2$, respectively. The curve of $p(x,t)$ with $\psi_\alpha(t) = e^{(1-\alpha)t}$ decreases most rapidly at x around 0 and most slowly at $x \to \pm\infty$. The curve of $p(x,t)$ with $\psi_\alpha(t) = \alpha^2$ decreases most slowly at x around 0 and most rapidly at $x \to \pm\infty$. These facts mean that the cell movement in diffusion process characterized by $\psi_\alpha(t) = e^{(1-\alpha)t}$ is the slowest and the cell movement in diffusion process characterized by $\psi_\alpha(t) = \alpha^2$ is the fastest.

Figure 4 shows the pattern of the cell movements in the diffusion processes described by mean square displacements $\langle x^2(t)\rangle$'s for the five $\psi_\alpha(t)$'s. At t sufficiently large, the cell movements in diffusion processes with $\psi_\alpha(t) = e^{(1-\alpha)t}$, $\psi_\alpha(t) = 1 + (1-\alpha)t$, $\psi_\alpha(t) = t^{1-\alpha}$ are slow indicated by the positions of their curves of $\langle x^2(t)\rangle$'s which are under the position of the curve of $x^2(t)$ with $\psi_\alpha(t) = 1$ which is associated with usual diffusion. The diffusion processes characterized $\psi_\alpha(t) = e^{(1-\alpha)t}$ is the slowest since its curve of $\langle x^2(t)\rangle$ is at the bottom. The diffusion process characterized $\psi_\alpha(t) = \alpha^2$ is usual diffusion but it is faster than usual diffusion with $\psi_\alpha(t) = 1$ since the curve of $\langle x^2(t)\rangle$ with $\psi_\alpha(t) = \alpha^2$ is above the curve of $\langle x^2(t)\rangle$ with $\psi_\alpha(t) = 1$. It also indicates that the cell movement in diffusion process characterized by $\psi_\alpha(t) = \alpha^2$ is the fastest. Figure 4 also shows that the larger the order α the faster the cell movement in diffusion process. These facts are in accordance with what was explained in Figure 1-3.



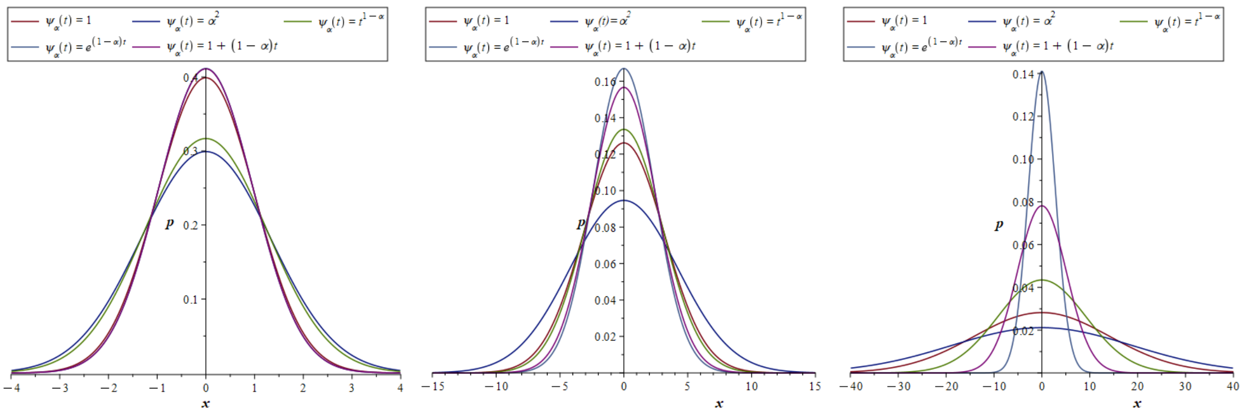
(A) Graph of $p(x,t)$ at $t = 0.5$ (B) Graph of $p(x,t)$ at $t = 5$ (C) Graph of $p(x,t)$ at $t = 100$

Figure 1. plots of $p(x,t)$'s with the specified $\psi_\alpha(t)$'s for $\alpha = 0.25$



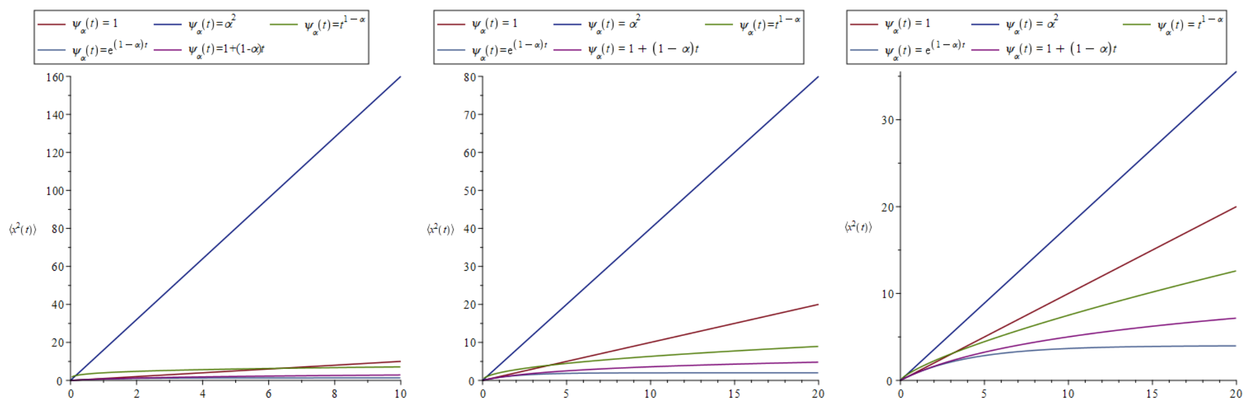
(A) Graph of $p(x,t)$ at $t = 0.5$ (B) Graph of $p(x,t)$ at $t = 5$ (C) Graph of $p(x,t)$ at $t = 100$

Figure 2. plots of $p(x,t)$'s with the specified $\psi_\alpha(t)$'s for $\alpha = 0.5$



(A) Graph of $p(x,t)$ at $t = 0.5$ (B) Graph of $p(x,t)$ at $t = 5$ (C) Graph of $p(x,t)$ at $t = 100$

FIGURE 3. plots of $p(x,t)$'s with the specified $\psi_\alpha(t)$'s for $\alpha = 0.75$



(A) Mean Square Displacement $\langle x^2(t)\rangle$ for $\alpha = 0.25$ (B) Mean Square Displacement $\langle x^2(t)\rangle$ for $\alpha = 0.5$ (C) Mean Square Displacement $\langle x^2(t)\rangle$ for $\alpha = 0.75$

FIGURE 4. Mean Square Displacements $\langle x^2(t)\rangle$'s with the specified $\psi_\alpha(t)$'s for $\alpha = 0.25, 0.5, 0.75$

3.3. **A Solution to General Conformable Diffusion Equation in** $0 < x < +\infty$**.** We here find a solution to the following the general conformable diffusion equation with some boundary conditions

$$\begin{aligned} &D_{\psi_\alpha} p(x,t) = K_{\psi_\alpha} \Delta p(x,t) \\ &\lim_{x\to 0^+} p(x,t) = p_s, \ \lim_{x\to+\infty} p(x,t) = 0. \end{aligned} \tag{3.22}$$

We use a dimensionless Boltzman transform $z = x/\sqrt{4K_{\psi_\alpha}\Psi_\alpha(t)}$. Then

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial z}\frac{\partial z}{\partial t} = -\frac{1}{2} x \frac{4K_{\psi_\alpha}\Psi'_\alpha(t)}{(4K_{\psi_\alpha}\Psi_\alpha(t))^{3/2}}\frac{\partial p}{\partial z}, \tag{3.23}$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial z}\frac{\partial z}{\partial x} = \frac{1}{\sqrt{4K_{\psi_\alpha}\Psi_\alpha(t)}}\frac{\partial p}{\partial z}, \tag{3.24}$$

and

$$\frac{\partial^2 p}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) = \frac{1}{4K_{\psi_\alpha}\Psi_\alpha(t)}\frac{\partial^2 p}{\partial z^2}. \tag{3.25}$$

Substituting the equation(3.23) and the equation (3.25) into the equation (3.22), we have

$$-\psi_\alpha(t)\frac{1}{2}x\frac{4K_{\psi_\alpha}\Psi'_\alpha(t)}{(4K_{\psi_\alpha}\Psi_\alpha(t))^{3/2}}\frac{\partial p}{\partial z} = K_{\psi_\alpha}\frac{1}{4K_{\psi_\alpha}\Psi_\alpha(t)}\frac{\partial^2 p}{\partial z^2} \tag{3.26}$$

or

$$-2\frac{x}{\sqrt{4K_{\psi_\alpha}\Psi_\alpha(t)}}\frac{\partial p}{\partial z} = \frac{\partial^2 p}{\partial z^2}. \tag{3.27}$$

Consequently, we have the ordinary differential equation

$$\frac{\partial^2 p}{\partial z^2} + 2z\frac{\partial p}{\partial z} = 0. \tag{3.28}$$

The general solution to the equation (3.28) is

$$p(z) = C_1\int_0^z e^{-y^2}dy + C_2. \tag{3.29}$$

Since

$$\lim_{x\to 0^+} p(x,t) = p_s, \ \lim_{x\to +\infty} p(x,t) = 0$$

then the particular solution to the equation (3.28) is

$$p(z) = p_s - \frac{2}{\sqrt{\pi}}p_s\int_0^z e^{-y^2}dy. \tag{3.30}$$

Thus, the solution to the problem (3.22) is

$$p(x,t) = p_s - \frac{2}{\sqrt{\pi}}p_s\int_0^{x/\sqrt{4K_{\psi_\alpha}\Psi_\alpha(t)}} e^{-y^2}dy. \tag{3.31}$$

We next model the transport process of chloride ions diffusing in three types of non-saturated concrete as studied in [24] by applying the general conformable diffusion equation (3.22) with the five specified conformable fractional functions ψ_α's mentioned in previous section. The three types of the concretes are the first one with only CEM I 52.2 R (CP), the second one with 10% of silica fume (CPHS), and the third one with 20% of blast furnace slag (CPEAH). An experimental program was designed in five phases representing the natural conditions of a full year to test the mechanical properties of the concretes. At the end of each phase, the experimental data for the chloride concentration profiles of each concrete were obtained. We here use the experimental data for the chloride concentration profiles obtained at the end of Phase 1, i.e. at age of 102 days. In this case, $p(x,t)$ represents the chloride concentration at depth x in mm and at age in day. The notation p_s means the chloride concentration at surface. The best model is obtained based on the value of root mean square error (RMSE).

TABLE 1. Parameters of the general conformable diffusion equation (3.22) for chloride transport in CP

$\psi_\alpha(t)$	K_{ψ_α}	α	p_s	RMSE
1	0.334	1	18.125	0.446846
α^2	0.14654	0.66633	18.125	0.444961
$t^{1-\alpha}$	2.55	0.28729	18.125	0.44495
$e^{(1-\alpha)t}$	5.625	0.83267	18.125	0.444917
$1+(1-\alpha)t$	6.04167	0.20892	18.125	0.444923

TABLE 2. Parameters of the general conformable diffusion equation (3.22) for chloride transport in CPHS

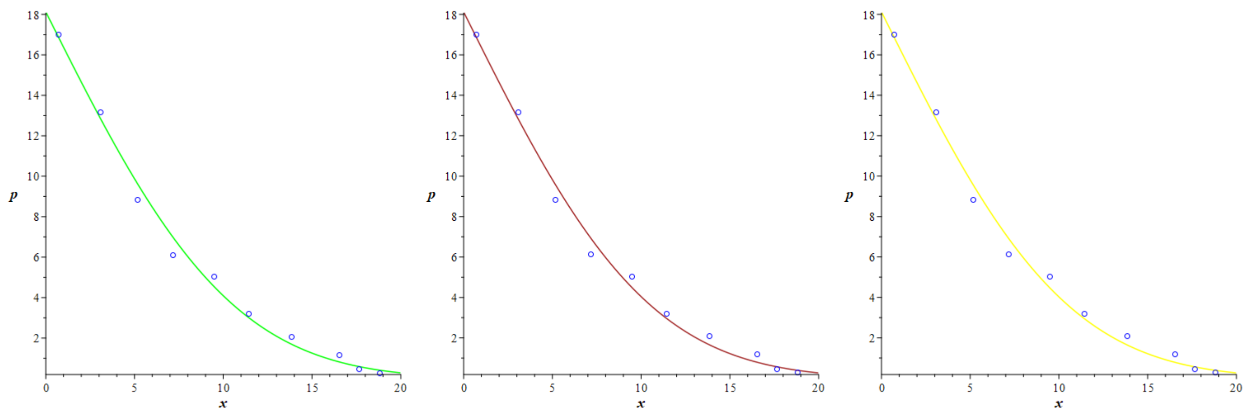
$\psi_\alpha(t)$	K_{ψ_α}	α	p_s	RMSE
1	0.08425	1	16.04167	0.314656
α^2	0.02988	0.5817	15.958833	0.305797
$t^{1-\alpha}$	0.25	0.69933	15.833	0.306979
$e^{(1-\alpha)t}$	7.1125	0.20892	15.91667	0.305771
$1+(1-\alpha)t$	1.75	0.12575	15.91667	0.305797

TABLE 3. Parameters of the general conformable diffusion equation (3.22) for chloride transport in CPEAH

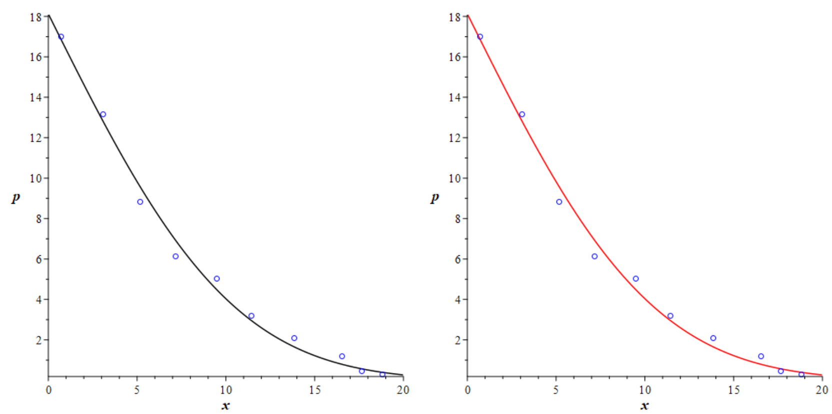
$\psi_\alpha(t)$	K_{ψ_α}	α	p_s	RMSE
1	0.25075	1	22.08333	0.643798
α^2	0.10496	0.62475	21.66667	0.619105
$t^{1-\alpha}$	0.85	0.66187	21.66667	0.619121
$e^{(1-\alpha)t}$	9.16667	0.66633	21.66667	0.619042
$1+(1-\alpha)t$	5.5	0.08417	21.75	0.61811

Tables 1-3 show the parameters of the models (3.22) describing the transport processes of chloride ions diffusing in the three types of non-saturated concrete studied in [24] with the five specified conformable fractional functions ψ_α's. Figure 5-7 show comparison between the models (3.22) with the five conformable fractional functions ψ_α's and the experimental data of the chloride transport in CP, CPHS, and CPEAH. Based on Table 1, the model (3.22) with $\psi_\alpha = e^{(1-\alpha)t}$ is the best model to describe the chloride transport in CP. From Table 2, the model (3.22) with $\psi_\alpha = e^{(1-\alpha)t}$ is the best model to describe the chloride transport in CPHS. According to Table 3, the model (3.22) with

$\psi_\alpha = 1 + (1-\alpha)t$ is the best model to describe the chloride transport in CPEAH. Generally, the models (3.22) with $\psi_\alpha(t) = \alpha^2$, $\psi_\alpha(t) = t^{1-\alpha}$, $\psi_\alpha(t) = e^{(1-\alpha)t}$, and $\psi_\alpha(t) = 1 + (1-\alpha)t$ agree better with the experimental data than the model (3.22) with $\psi_\alpha(t) = 1$ which is the usual diffusion model.

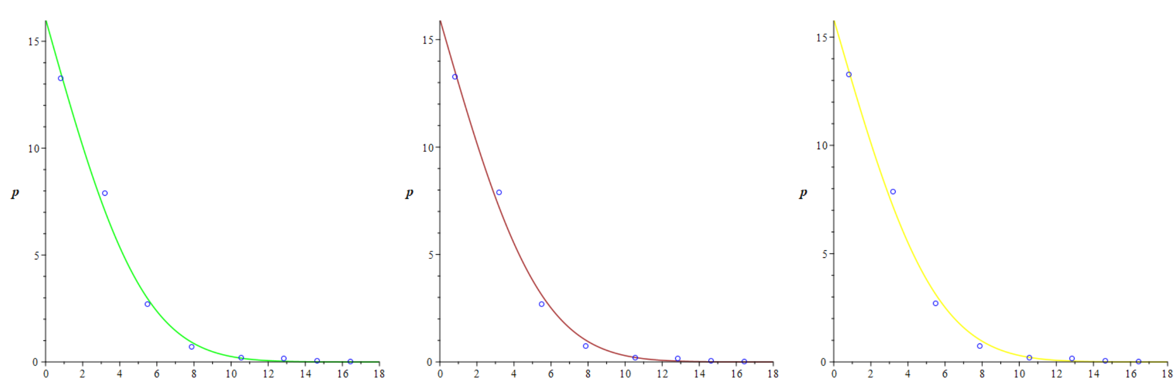


(A) $\psi_\alpha(t) = 1$ (B) $\psi_\alpha(t) = \alpha^2$ (C) $\psi_\alpha(t) = t^{1-\alpha}$

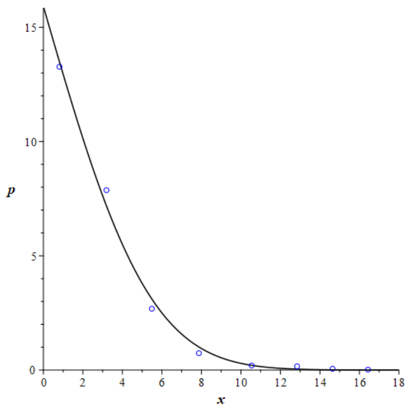


(D) $\psi_\alpha(t) = e^{(1-\alpha)t}$ (E) $\psi_\alpha(t) = 1 + (1-\alpha)t$

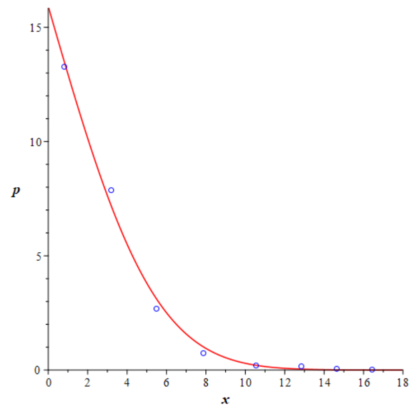
FIGURE 5. Experimental Data (CP) vs $p(x,t)$'s with the specified $\psi_\alpha(t)$'s



(A) $\psi_\alpha(t) = 1$ (B) $\psi_\alpha(t) = \alpha^2$ (C) $\psi_\alpha(t) = t^{1-\alpha}$

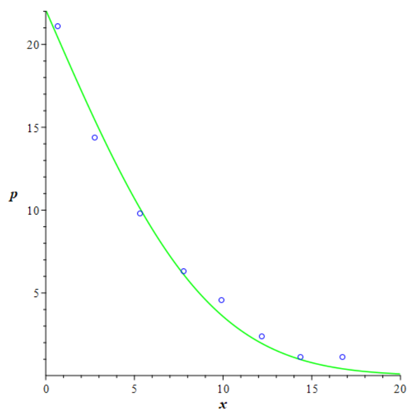


(D) $\psi_\alpha(t) = e^{(1-\alpha)t}$

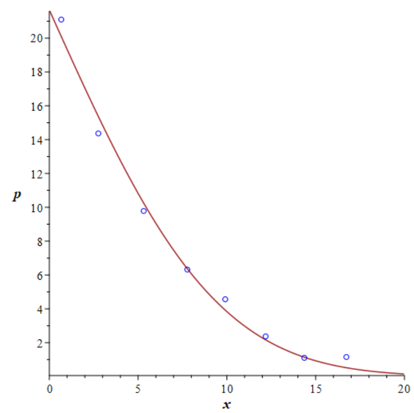


(E) $\psi_\alpha(t) = 1 + (1-\alpha)t$

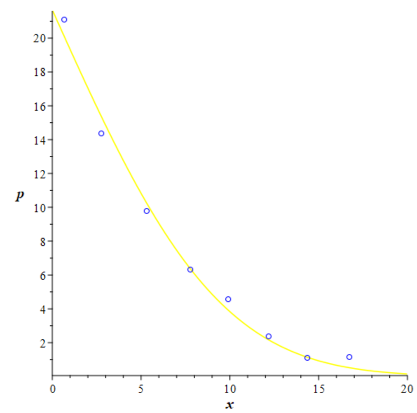
FIGURE 6. Experimental Data (CPHS) vs $p(x,t)$'s with the specified $\psi_\alpha(t)$'s



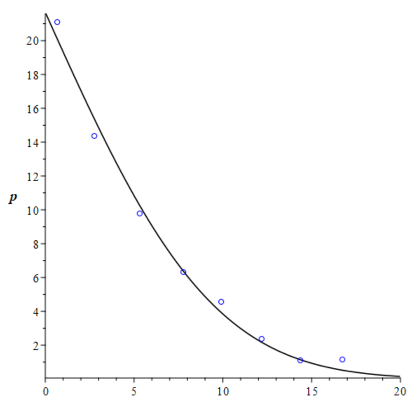
(A) $\psi_\alpha(t) = 1$



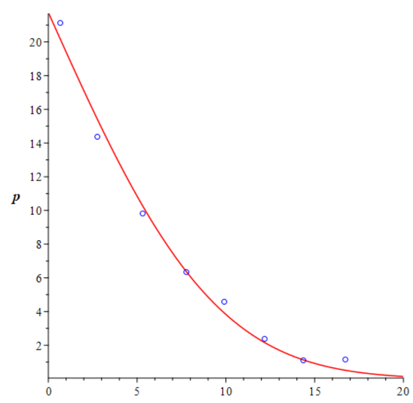
(B) $\psi_\alpha(t) = \alpha^2$



(C) $\psi_\alpha(t) = t^{1-\alpha}$



(D) $\psi_\alpha(t) = e^{(1-\alpha)t}$



(E) $\psi_\alpha(t) = 1 + (1-\alpha)t$

FIGURE 7. Experimental Data (CPEAH) vs $p(x,t)$'s with the specified $\psi_\alpha(t)$'s

4. Conclusion

We derived the general conformable diffusion equation that describes the cell transport system in (slow) diffusion processes. We also found a solution to the general conformable diffusion equation. For all conformable fractional functions ψ_α's, the higher the order α, the closer the curve of the fundamental solution $p(x,t)$ of the equation (3.16) to the curve of $p(x,t)$ for the usual diffusion which is associated with $\psi_\alpha(t) = 1$. The models (3.16) with some conformable fractional functions such as $\psi_\alpha(t) = t^{1-\alpha}$, $\psi_\alpha(t) = e^{(1-\alpha)t}$, and $\psi_\alpha(t) = 1 + (1-\alpha)t$ describe the transport system of cells in slow diffusion. The models (3.22) with $\psi_\alpha(t) = \alpha^2$, $\psi_\alpha(t) = t^{1-\alpha}$, $\psi_\alpha(t) = e^{(1-\alpha)t}$, and $\psi_\alpha(t) = 1 + (1-\alpha)t$ agree better with the experimental data than the model (3.22) with $\psi_\alpha(t) = 1$ which is the usual diffusion model. Other conformable fractional functions, not mentioned here, may give much better diffusion models for the experimental data.

For further study, motivated by the modeling of the transport process of chloride ions diffusing in three types of non-saturated concrete as discussed in section 3.3, it is very interesting to develop the diffusion model using the general conformable fractional derivative of the order α depending on t. It is also very interesting to develop the model for a nonconstant $j(x;\omega)$.

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