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Geometric Properties of Starlike and Spiral-like Functions Defined by a Complex Fractional Operator

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Abstract. This paper considers subclasses of the analytic function known as starlike and spiral-like functions of real order in the open unit disk. We derive necessary and sufficient coefficient conditions, distortion bounds, and radii of starlikeness and spiral-likeness, and show how the parameters influence the geometric domains in the unit disk. The class is generated via a complex fractional operator, which extends earlier results by allowing a rotation parameter to take complex values. This operator unifies several classical differential and integral operators, including the Sălăgean and Libera operators, with the former generalized from integer to real order. The operator provides a useful approach for generating subclasses of starlike and spiral-like functions, describing both rotational and radial behaviour. We also study geometric properties of these functions, including conjugate symmetry, and examine the effect of the operator on their mapping behaviour.

1. Introduction

The open unit disk: $\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ serves as a fundamental domain in geometric function theory due to its invariance under rotations, reflections, and other conformal transformations. The unit disk remains invariant under the transformation $z \mapsto e^{i\theta} z$ as well as under inversion through the origin given by $z \mapsto -z$ highlighting its intrinsic rotational structure. The classical subclasses, such as starlike functions of order α and spiral-like functions with argument λ were unified and extended through Ma–Minda [1], where geometric properties are characterized by subordinations of the form $zf'(z)/f(z) \prec \phi(z)$, $\Re e(\phi(z)) > 0$. Bazilevič functions induce rotational

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mappings in the image domain, leading to spiral-like deformation effects [2], and Robertson-type functions [3,4] show spiral-like behavior in their derivatives.

In recent years, significant advancements in geometric function theory have emerged through new methods. The integration of quantum calculus has led to new subclasses of analytic functions [5], while fractional q-operators have extended classical results [6]. The simultaneous, interest in symmetry properties [6] and spiral-like functions [7, 8] has enriched our understanding of geometric mappings. Here in the sequel we introduce a complex rotational operator that combines fractional calculus with controlled complex rotations, addressing the need for operators capable of generating diverse geometric patterns while maintaining analyticity. Earlier studies have used Caputo's definition [9] of fractional derivatives to generalize classical starlike and convex functions [10, 11], and Sekine and Owa [12] employed integral means to extend subclasses of analytic functions.

Based on these geometric structures, the present article investigates starlike and spiral-like functions via a complex rotational operator, which unifies and extends several known subclasses, including Sălăgean-type, Ma–Minda, and Robertson-type classes [1, 3, 13], and recovers various classical operator-defined subclasses as special cases. This operator-based formulation enables systematic analysis of geometric properties such as angular rotation, inclusion relationships, and distortion bounds within symmetric regions of the unit disk. Although previous studies have explored starlike and spiral-like functions using real rotation parameters or specific integral operators [10,12], however a gap remains for the unified approach that allows to take complex values while simultaneously generalizing multiple classical operators. This leads us to introduce the Complex Fractional Operator known to be CFO.

Let $\mathcal{A}$ denote the class of normalized analytic functions in the open unit disk $\mathcal{U}$. A function $f \in \mathcal{A}$ has the series representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (a_k \geq 0). \tag{1.1}$$

The analytic function $f(z)$ defined by equation (1.1) is said to be starlike of order λ $(0 \leq \lambda < 1)$, with $f(0) = 0$, $f'(0) = 1$, if it satisfies the condition

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) > \lambda$$

Geometrically, a domain S is said to be starlike with respect to a point $w_o \in S$ if the line segment joining $w_o = 0$ to every other point $w \in S$ lies entirely inside the unit disk $\mathcal{U}$. This property ensures that all radial segments originating from the origin remain within the image domain under the mapping f, thereby preserving starlikeness.

Let $\mathcal{T}$ denote the subclass of normalized functions with negative coefficients belonging to $\mathcal{A}$. The function $g(z)$ is analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and has the representation

$$g(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad (a_k \geq 0) \tag{1.2}$$

To facilitate the development of our main results, we first recall several classical definitions and known results related to starlike functions with negative coefficients, together with their fundamental coefficient characterizations.

2. Preliminary Results and Definitions

The necessary and sufficient condition for a function $g(z) \in \mathcal{T}$ to be starlike of order λ $(0 \leq \lambda < 1)$ is due to Silverman [14].

$$\Re e\left(\frac{zg'(z)}{g(z)}\right) > \lambda \iff \sum_{k=2}^{\infty}(k-\lambda)a_k \leq 1-\lambda. \tag{2.1}$$

As a direct consequence of (2.1), the coefficients satisfy the following bounds.

Corollary 2.1. *If $g \in T^*(\lambda)$ then $|a_k| \leq \dfrac{1-\lambda}{k-\lambda}$, with equality only for functions of the form $g_k(z) = z - \dfrac{1-\lambda}{k-\lambda}z^k$ for all $k \geq 2$.*

An extended version of this class was studied by Sekine and Owa [12], where they considered the class $\mathcal{A}(n,\theta)$, with the constraints

$$\theta \in \mathbb{R},\ n \in \mathcal{N} = \{1, 2, 3, \cdots\}, a_k \geq 0.$$

This is a subclass of $\mathcal{A}$ consisting of all functions $g(z)$ of the form:

$$g_n(z) = z - \sum_{k=2}^{\infty} e^{in\theta} a_k z^k. \tag{2.2}$$

By using a similar structure, Rehman et al. [15] established that a function belonging to $\mathcal{A}(n,\phi)$ is contained within $\mathcal{H}_\lambda^*(n,b,\phi)$ subject to the conditions stated in the following lemma.

Lemma 2.1. *Let $g(z) \in \mathcal{A}(n,\phi)$ be of the form (2.2). Then $g(z)$ belongs to the class $\mathcal{H}_\lambda^*(n,b,\phi)$ if and only if,*

$$\sum_{k=2}^{\infty}(k-\lambda)a_k \leq 1-\lambda,\ (0 \leq \lambda < 1).$$

The necessary and sufficient condition for an analytic function to be $\gamma-$spiral-like is due to Silverman et al. [16]. In particular, a function $g \in \mathcal{T}$ is $\gamma-$spiral-like of order 0 if and only if

$$\Re e\left(e^{i\gamma}\frac{zg'(z)}{g(z)}\right) > 0, \quad |\gamma| < \frac{\pi}{2} \tag{2.3}$$

As seen from the article by [17], that the family of $T_\gamma(\lambda)$ of the functions $\gamma-$spiral-like of order λ is given by:

$$\Re e\left(e^{i\gamma}\frac{zg'(z)}{g(z)}\right) > \lambda cos\gamma$$

equivalently becomes;

$$\sum_{k=2}^{\infty}[k-\lambda)\,cos\gamma - sin\gamma]a_k \leq (1-\lambda)cos\gamma, \quad |\gamma| < \frac{\pi}{2}.$$

From the above inequality, the sharp coefficient bound for the function g is:

$$a_k \leq \frac{(1-\lambda)cos\gamma}{(k-\lambda)\,cos\gamma - sin\gamma}$$

Equality is obtained for the function of type:

$$g_k(z) = z - \frac{(1-\lambda)cos\gamma}{(k-\lambda)\,cos\gamma - sin\gamma} z^k.$$

This result is the exact analogue of Silverman's starlike condition given in (2.1) together with the sharp coefficient bounds stated in Corollary 2.1 in the special case $\lambda = 0$. In particular, when $\lambda = 0$, condition (2.3) reduces to Silverman's starlike criterion, and the classical class of starlike functions is recovered from the class of γ-spiral-like functions by taking $\gamma = 0$.

The Alexander operator serves as a base for creating more complex integral and differential operators, such as the Hohlov operator or various q-calculus extensions [18–20].

Definition 2.1. *[21] For a function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$, the Alexander transform is defined by:*

$$\mathbf{A}(f) = \int_0^z \frac{f(t)}{t} dt = z + \sum_{k=2}^{\infty} \left(\frac{1}{k}\right) a_k z^k, \tag{2.4}$$

Definition 2.2. *The generalized Bernardi integral operator J_c [22] for a function f(z) belonging to the class $\mathcal{A}$, is defined as under:*

$$J_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt = z + \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right) a_k\, z^k, \quad (c \geq 0), \tag{2.5}$$

and for $c = 1$, we obtain

$$J_1(f) = \frac{2}{z} \int_0^z f(t) dt = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right) a_k\, z^k. \tag{2.6}$$

The operator J_c, when $c = \{1, 2, 3, \cdots\}$ was introduced by Bernardi [22]. In particular, the operator $J_1(f)$, was studied earlier by Libera [4] and Livingston [23].

Based on the foregoing preliminaries, we proceed to introduce the complex fractional operator. For particular parameter values, the proposed operator reduces to the Bernardi and Alexander operators, while providing a broader generalization.

3. The Complex Fractional Operator (CFO)

The study of starlike functions is a geometric consequence of analytic functions, linking the geometric properties of image domains with analytic behavior and serving as a bridge between pure mathematical theory and practical applications in science and engineering, such as modified sigmoid functions mapped to starlike classes [24,25], spiral flows in engineering applications [6,26] and in airfoil theory and designing [27,28]. In addition, we consider a complex rotation parameter $\phi \in \mathbb{C}$, rather than the real rotation parameter $\theta \in \mathbb{R}$ studied by Sekine and Owa [12], thereby allowing both angular and fractional radial effects. Motivated by these applications and the properties of the new parameter, we introduce a coefficient multiplier operator $\mathcal{T}_n^m(\phi, \eta_k)$, which gives rise to new subclasses of starlike and spiral-like functions.

Definition 3.1 (CFO)**.** *Let $\phi \in \mathbb{C}$, $m > 0$ with $m \neq 1$, $n \in \mathcal{N}_\circ = \{0, 1, 2, \cdots\}$ and $\{\eta_k\}_{k\geq 2}$, be a non-negative real sequence. Define the operator $\mathcal{R}_n^m(\phi, \eta_k)$ acting on $\mathcal{T}$ by*

$$\begin{aligned}\mathcal{R}_n^m(\phi, \eta_k)[g](z) &= z - \sum_{k=2}^{\infty} \left(\eta_k + 1\right)^{\frac{in\phi}{ln(m)}} a_k z^k \\ &= z - \sum_{k=2}^{\infty} e^{in\phi\, log_m(\eta_k+1)} a_k z^k \qquad (3.1)\end{aligned}$$

where the principal branch of the logarithm is assumed.

3.1. **The Operator Characterization.**

The operator $\mathcal{R}_n^m(\phi, \eta_k)$ defined by (3.1) depends on several parameters that allow flexible control over the induced geometric behaviour. The complex parameter ϕ governs rotational and spiral effects, while the parameter m specifies the base of the logarithmic deformation. The sequence $\{\eta_k\}_{k\geq 2}$ determines the coefficient-wise fractional structure, with $\{a_k\}$ denoting the standard Taylor coefficients. The coefficient multiplier $exp\{i\, n\phi\, log_m(\eta_k + 1)\}$, introduces complex rotation directly at the coefficient level, enabling the operator to generate a wide range of geometric effects. In particular, suitable choices of the sequence $\{\eta_k\}$ yield several well-known operators as special cases, as illustrated below.

Linear family. For $\eta_k = ck - n$, $0 < c \leq 1, n \in \mathcal{N}_\circ$, the operator reduces to a Sălăgean-type differential operator of complex order β [13], given by

$$\mathcal{R}_\beta[g](z) = z - \sum_{k=2}^{\infty} k^\beta a_k z^k = 2z - D^\beta(f)(z), \quad \beta = in\phi. \qquad (3.2)$$

Reciprocal family. For $\eta_k = \frac{c}{(k-1)^p}$, $c > 0$, $p > 0$ multiplier approaches unity as $k \to \infty$, and the operator converges to the identity. In the particular case $\eta_k = 1/k - 1$ with $m = e$, the operator coincides with a Libera-type integral operator of complex order β [4]:

$$\mathcal{R}_\beta[g](z) = z - \sum_{k=2}^{\infty} \left(\frac{k}{k-1}\right)^{-in\phi} a_k z^k = D^\beta(f)(z) - f(z) + z, \quad \beta = -i\, n\phi. \qquad (3.3)$$

Exponential family. For $\eta_k = b^{1-k} - 1$, $b > 0$. When $b \neq 1$, the coefficients exhibit unbounded rotational behaviour, whereas for $b = 1$ the operator reduces to a constant-phase transformation. In this case the operator produces a geometric progression in the phase factors.

$$\mathcal{R}_n^m[g](z) = z - \sum_{k=2}^{\infty} \rho^{1-k} a_k z^k = z - \rho\Big(g(z/\rho) - z/\rho\Big), \quad \rho = e^{i\,n\phi\,\log_m b} \tag{3.4}$$

Trigonometric family. For $\eta_k = \sin^2(\omega k + \theta)$ for $\omega, \theta \in \mathbb{R}$, the multiplier induces periodic oscillations, producing alternating rotational effects governed by the parameters ω and θ. Finally generates periodic oscillations between 0 and $log_m 2$, in the coefficient rotations.

$$\mathcal{R}_n^m[g](z) = z - \sum_{k=2}^{\infty} e^{i\,n\phi log_m\left(1+\sin^2(\omega k+\theta)\right)} a_k z^k. \tag{3.5}$$

Next, we consider three possibilities for the parameter ϕ. The choices of m allow us to explore the effect of the logarithmic scaling factor in (3.1) on the geometric properties of the image domains.

Purely real: When $\phi = \theta : \theta \in \mathbb{R}$, which induces purely rotational deformation of the coefficients without affecting their magnitudes.

(1) For $\mathcal{R}_1^m(1,1)[g](z) = e^{i\theta_m} g(z) + z(1 - e^{i\theta_m})$, for $\theta_m = log_m 2$.
(2) For $\mathcal{R}_1^e(1,1)[g](z) = z + 2^i(g(z) - z)$.
(3) For $\mathcal{R}_n^m(\phi, m-1)[g](z) = g_n(z)$, returns to equation (2.2).

Purely imaginary: When $\phi = \pm i\beta : \beta \in \mathbb{R}$, which introduces fractional radial scaling without any rotational effect. The complex-rotation fractional operator $\mathcal{R}_n^m(\phi, \eta_k)$ reduces to several known operators for specific choices of parameters:

(1) For $\mathcal{R}_0^m(0,0)[g](z) = g(z)$, Identity operator (1.2).
(2) For $\mathcal{R}_1^e(-i, k-1)[g](z) = zg'(z)$, classical differential operator [3].
(3) For $\mathcal{R}_n^e(-i, k-1)[g](z) = 2z - D^n(f)(z)$, Sălăgean operator [13].
(4) For $\mathcal{R}_n^e(-i, (k-1)\lambda)[g](z) = D_\lambda^n(g)(z)$, the Al–Oboudi operator [29].
(5) For $\mathcal{R}_n^e(-i\beta, k-1)[g](z) = 2z - D^{n\beta}(f)(z)$, Sălăgean operator of order β [13].
(6) For $\mathcal{R}_1^e(i, 1/k-1)[g](z) = \mathbf{A}(f)(z) - f(z) + z$, where $\mathbf{A}$ is the Alexander operator [21].
(7) For $\mathcal{R}_n^e(i\beta, 1/k-1)[g](z) = \mathbf{A}^{n\beta}(f)(z) - f(z) + z$, where $\mathbf{A}^{n\beta}$ denotes the fractional Alexander operator of order $n\beta$. The integral form of this operator is intuited from [30, 31] and it takes the following form.

$$\mathcal{R}_n(\beta)[g](z) = z - \frac{1}{\Gamma(n\beta)} \int_0^z \left(1 - \frac{t}{z}\right)^{n\beta-1} \left(\frac{f(t)}{t} - 1\right) dt, \quad z \in \mathcal{U}. \tag{3.6}$$

Note that for $n\beta = 0$, (3.6) returns to Identity and for $n\beta = 1$, it returns to (6). Moreover when $n\beta > 0$, it becomes the fractional Alexander–Bernardi integral operator and for $n\beta < 0$, this becomes the Hadamard-type fractional differential operator see [32].

General complex: When $\phi = \theta + i\beta$, with $\theta, \beta \in \mathbb{R}$, the operator combines both angular rotation and fractional radial scaling, thereby generating new subclasses of spiral-like and starlike functions.

(1) For $\mathcal{R}_n^e(\theta + i\beta, 1/k - 1)[g](z) = z - D^{\beta}\{\mathbf{A}^{n\beta}(f) - f(z)\}, \quad \beta = in\theta$, we obtain the series representation

$$\mathcal{R}_n(\theta, \beta)[g](z) = z - \sum_{k=2}^{\infty} \left(\frac{k-1}{k}\right)^{n(\beta - i\theta)} a_k z^k. \tag{3.7}$$

(2) In particular, for $n = 1$ and $(\theta + i\beta) = (1 + i)$, equation (3.7) reduces to

$$\mathcal{R}[g](z) = z - \sum_{k=2}^{\infty} \left(\frac{k-1}{k}\right)^{1-i} a_k z^k. \tag{3.8}$$

Remark 3.1. *By applying Corollary 2.1 with* $a_2 = \frac{1}{2}$ *and* $k = 2$ *to equation* (3.8)*, we obtain the explicit function*

$$\mathcal{R}(z) = z - 4^{-1+i} z^2,$$

which is starlike, as illustrated in Figure 1.

By employing the Möbius transformation method, it follows directly that the function $\mathcal{R}(z)$ is starlike, as the computation yields

$$\Re e \left\{ \frac{z\mathcal{R}'[g](z)}{\mathcal{R}(z)} \right\} > 0, \quad z \in \mathcal{U}$$

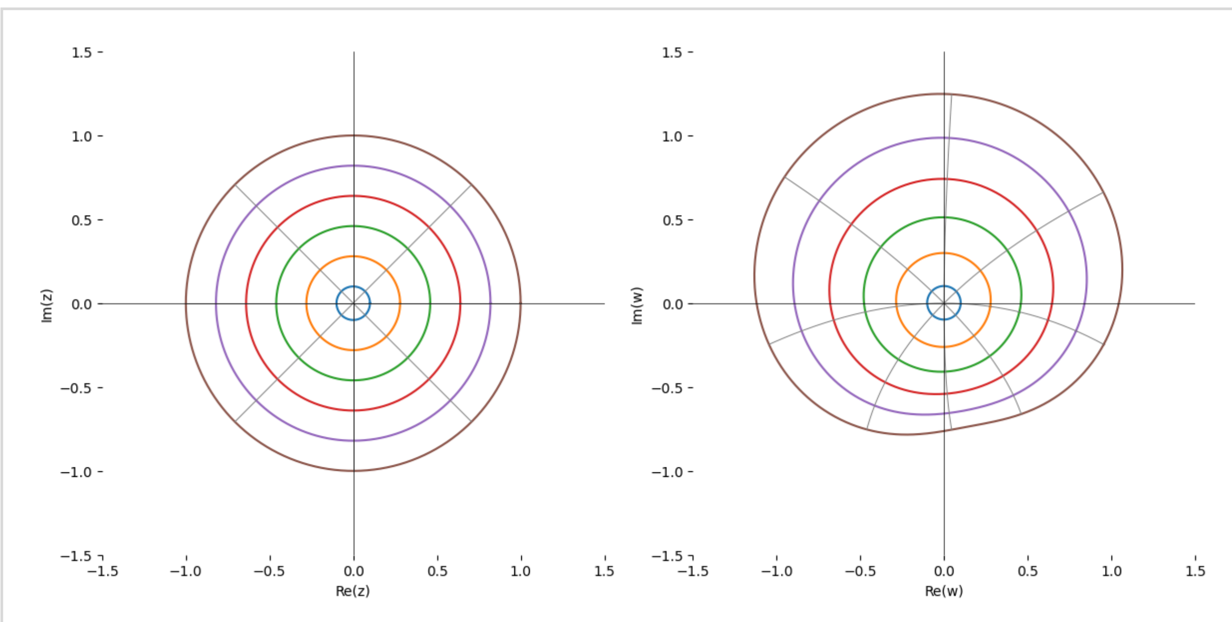
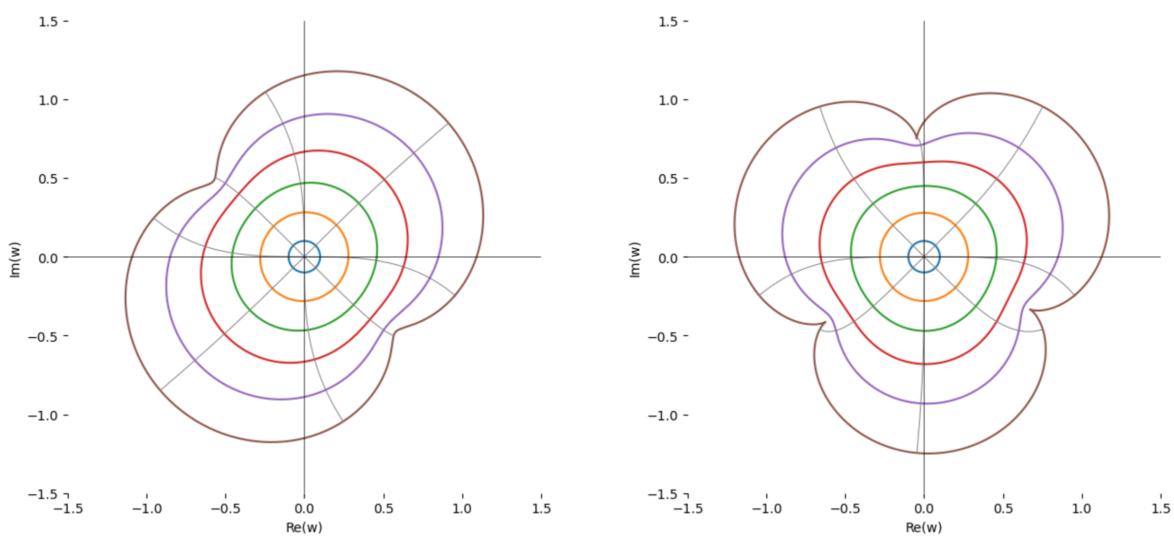


FIGURE 1. Starlike mapping of Unit disk under $\mathcal{R}(z) = z - 4^{-1+i} z^2$

Note that the function is analytic and hence smooth (differentiable). Moreover that the above complex quadratic function exhibits a conjugate symmetry. This function carries additional interesting symmetrical properties when expressed in the more general form.

$$\mathcal{R}_n(z) = z - 4^{-1+i} z^n. \tag{3.9}$$

For integer values $n \in \{3, 4\}$, the function exhibits rotational or conjugate symmetry about the origin and preserve starlikeness as evident from the Figure 2.



(A) $\mathcal{R}_3(z) = z - 4^{-1+i} z^3$ (*Starlike*) (B) $\mathcal{R}_4(z) = z - 4^{-1+i} z^4$ (*Weak Starlike*)

Figure 2. Starlike Functions yielded by (3.9)

3.2. **A Subclass defined by $\mathcal{R}_n^m(\phi, \eta_k)$ Operator.** Utilizing the operator $\mathcal{R}_n^m(\phi, \eta_k)$ introduced in Definition 3.1, we now define a subclass of analytic functions as follows.

Definition 3.2. *Let $\phi \in \mathbb{C}$, $m > 0$ with $m \neq 1$, $n \in \mathcal{N}_\circ$, and let $\{\eta_k\}_{k \geq 2}$ be a non-negative real sequence. A function $g(z) \in \mathcal{A}$ is said to be in the class $\mathcal{A}_\lambda(m, n, \phi)$ if and only if*

$$\Re e \left\{ \frac{\mathcal{R}_n^m(\phi, \eta_k)[g](z)}{g(z)} \right\} > \lambda, \quad 0 \leq \lambda < 1, \, z \in \mathcal{U}. \tag{3.10}$$

Observed that the inequality in (3.10) generalizes the classical class of starlike functions of order λ, since

$$\mathcal{A}_\lambda(e, 1, -i) = S^*(\lambda),$$

where

$$S^*(\lambda) = \{ f \in \mathcal{A} : \Re e(z f'(z)/f(z)) > \lambda \}.$$

Furthermore, if $T^*(\lambda) = T \cap S^*(\lambda)$, it represents the class of starlike functions of order λ with negative coefficients [14].

Next, we define a subclass $\mathcal{T}_\lambda^*(m, n, \phi)$ consisting of functions with negative coefficients.

Definition 3.3. *Let $\mathcal{T}$ denote the collection of functions in the form (1.2). Then the subclass of functions with negative coefficients is defined by*

$$\mathcal{T}_\lambda^*(m, n, \phi) = \mathcal{A}_\lambda(m, n, \phi) \cap \mathcal{T}.$$

In particular, for $m = e$, $n = 1$, $\phi = -i$, and $\eta_k = k - 1$, the inequality in (3.10) reduces to

$$\mathcal{T}^*_\lambda(e, 1, -i) = S^*(\lambda) \cap \mathcal{T} = T^*(\lambda),$$

which corresponds to the classical class studied by Silverman [14].

Remark 3.2. *The complex-rotation fractional operator $\mathcal{R}^m_n(\phi, \eta_k)$ reduces to several known subclasses for specific parameter choices:*

(1) $\mathcal{R}^m_n(\theta, m-1)[g](z) = g_n(z)$ *with* $\theta \in \mathbb{R}$, $m > 0$, $m \neq 1$, *considered by Sekine and Owa [12].*
(2) $\mathcal{R}^b_n(\phi, k-2)[g](z) = H^*_\beta(n, b, \phi)$ *with* $\phi = \phi + 2n\pi, n \in \mathbb{N}$, *considered by Rehman et al. [15].*
(3) $\mathcal{R}^m_n(0, \eta_k)[g](z) = g(z)$, *with* $\Re e(\phi(z)) > 0$ *via subordination, corresponding to starlike subclasses studied by Ma-Minda et al. [1].*

4. Main Results

Theorem 4.1. *Let the function $g(z)$ defined by (1.2), then $g(z) \in \mathcal{A}_\lambda(m, n, \phi)$ if and only if*

$$\sum_{k=2}^{\infty} (|\Psi(k)| - \lambda) a_k \leqq 1 - \lambda. \tag{4.1}$$

where from Definition 3.1

$$|\Psi(k)| = \left(\eta_k + 1\right)^{-\frac{n\Im(\phi)}{ln(m)}} > 0 \tag{4.2}$$

Proof. Consider (4.1) holds and let $|z| = 1$. Then we get

$$\begin{aligned}\left|\frac{\mathcal{R}^m_n(\phi, \eta_k)[g](z)}{g(z)} - 1\right| &\leqslant \frac{\sum_{k=2}^{\infty} (|\Psi(k)| - \lambda) a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k |z|^{k-1}} \\ &= \frac{\sum_{k=2}^{\infty} (|\Psi(k)| - \lambda) a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k |z|^{k-1}} \\ &\leqslant 1 - \lambda \end{aligned} \tag{4.3}$$

which implies (3.10). Thus $g(z) \in \mathcal{T}^*_\lambda(m, n, \phi)$.

Conversely suppose that $g(z) \in \mathcal{T}^*_\lambda(m, n, \phi)$, then

$$\Re e\left\{\frac{\mathcal{R}^m_n(\phi, \eta_k)[g](z)}{g(z)}\right\} = \Re e\left\{\frac{1 - \sum_{k=2}^{\infty} (|\Psi(k)| - 1) a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} a_k z^{k-1}}\right\} > \lambda. \tag{4.4}$$

Since the condition (3.10) must hold for all $z \in \mathcal{U}$, and the class $\mathcal{T}$ has real coefficients a_k, we can consider z to be real and let $z \to 1^-$, to obtain the necessary coefficient inequality.

$$1 - \sum_{k=2}^{\infty} (|\Psi(k)| - 1) a_k \geq \lambda \left(1 - \sum_{k=2}^{\infty} a_k \right) \tag{4.5}$$

which implies (3.10). The result is sharp with the extremal function f defined by

$$g(z) = z - \frac{1-\lambda}{|\Psi(k)| - \lambda} z^k \quad , (k \geq 2). \tag{4.6}$$

This completes the proof asserted in (4.1). □

Corollary 4.1. *Let the function defined by (1.2) be in the class $\mathcal{T}^*_\lambda(m, n, \phi)$, then*

$$a_k \leq \frac{1-\lambda}{|\Psi(k)| - \lambda} \quad , (k \geq 2). \tag{4.7}$$

The inequality in (4.7) is attained for the function $g(z)$ given by equation (4.6).

Theorem 4.2. *A function f defined by (1.2) belongs to class $g(z) \in \mathcal{T}^*_\lambda(m, n, \phi)$, then*

$$|g(z)| \geq |z| - \frac{1-\lambda}{|\Psi(2)| - \lambda} |z^2| \quad \textit{and} \quad |g(z)| \leq |z| + \frac{1-\lambda}{|\Psi(2)| - \lambda} |z|^2 \tag{4.8}$$

where $|\Psi(2)| = \left(\eta_k + 1\right)^{-\frac{n\mathfrak{I}(\phi)}{ln(m)}}$.

Proof. Let the function $f \in \mathcal{T}^*_\lambda(m, n, \phi)$, then by virtue of Theorem 4.1, we have

$$(|\Psi(2)| - \lambda) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} (|\Psi(k)| - \lambda) a_k \leq 1 - \lambda$$

From above we can write

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\lambda}{|\Psi(2)| - \lambda} = \frac{1-\lambda}{|\Psi(2)| - \lambda}$$

Now taking into account the function $g(z)$, we have

$$g(z) \geq |z| - |z|^2 \sum_{k=2}^{\infty} a_k \geq |z| - \frac{1-\lambda}{|\Psi(2)| - \lambda} |z|^2$$

and

$$g(z) \leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \leq |z| + \frac{1-\lambda}{|\Psi(2)| - \lambda} |z|^2$$

which proves both the lower and upper distortion bounds asserted in (4.8). □

5. The Complex Fractional Operator Radius Problem

One of the fundamental questions in geometric function theory concerns the radius of a geometric property. For a given class of function, what is the largest disk centered at the origin in which all functions in that class possess a specific geometric property? Hence we need to find the largest disk on the origin of starlikness and Spiral-likeness.

5.1. **Radius of Starlikeness.**

Theorem 5.1 (Radius of Starlikeness). *Let $g(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathcal{T}_{\lambda}^{*}(m, n, \phi)$. Then $g(z)$ is starlike of order λ in the disk $|z| < R_S$, where:*

$$R_S = \inf_{k\geq 2}\left\{\frac{|\Psi(k)| - \lambda}{k-\lambda}\right\}^{\frac{1}{k-1}} \tag{31}$$

where $|\Psi(k)| = \left(\eta_k + 1\right)^{-\frac{n\Im(\phi)}{ln(m)}}$.

Moreover that, the radius here is sharp, with extremal functions given by:

$$g_k(z) = z - \frac{1-\lambda}{|\Psi(k)| - \lambda} z^k, \quad k \geq 2 \tag{32}$$

Proof. For starlikeness of order λ, we require:

$$\Re e\left\{\frac{zg'(z)}{g(z)}\right\} > \lambda \quad \text{for} \quad |z| < R$$

This is equivalent to:

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < 1 - \lambda \quad \text{for} \quad |z| < R$$

Using the coefficient bounds from Theorem 4.1, we have:

$$\left|\frac{zg'(z)}{g(z)} - 1\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k|z|^{k-1}}$$

Since $a_k \leq \frac{1-\lambda}{|\Psi(k)|-\lambda}$ where $|\Psi(k)| = [1 + (k-1)\alpha]^{n\alpha+\eta}$, we obtain:

$$\left|\frac{zg'(z)}{g(z)} - 1\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)\frac{1-\lambda}{|\Psi(k)|-\lambda}|z|^{k-1}}{1 - \sum_{k=2}^{\infty}\frac{1-\lambda}{|\Psi(k)|-\lambda}|z|^{k-1}}$$

The condition $\left|\frac{zg'(z)}{g(z)} - 1\right| < 1 - \lambda$ holds if:

$$\sum_{k=2}^{\infty}\frac{k-\lambda}{|\Psi(k)|-\lambda}|z|^{k-1} \leq 1$$

The largest R satisfying this for all k is given by (31). Sharpness follows by considering the extremal functions $g_k(z)$. □

5.2. **Radius of Spiral-likeness.**

Theorem 5.2 (Radius of γ-Spiral-likeness). *Let $g(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathcal{T}_\lambda^*(m,n,\phi)$ with $\phi \in \mathbb{R} \setminus \{0\}$. Then $g(z)$ is γ-spiral-like (i.e., $\Re e\left\{e^{i\gamma}\frac{zg'(z)}{g(z)}\right\} > 0$) in the disk $|z| < R_{SP}$, where:*

$$R_{SP} = \inf_{k\geq 2}\left\{\frac{2\cos\gamma \cdot [(\eta_k+1)^{\frac{in\phi}{ln(m)}} - \lambda]}{(k-\lambda) + \sqrt{(k-\lambda)^2 - 4\cos^2\gamma(\eta_k+1)^{\frac{in\phi}{ln(m)}} - \lambda)^2}}\right\}^{\frac{1}{k-1}} \tag{5.1}$$

for $0 < \gamma < \pi/2$.

Proof. For γ-spiral-likeness, we require:

$$\left|\frac{zg'(z)}{g(z)} - e^{-i\gamma}\right| < \cos\gamma \quad \text{for} \quad |z| < R$$

Proceeding similarly to Theorem 5.1 and using the inequality:

$$\left|\frac{zg'(z)}{g(z)} - e^{-i\gamma}\right| \leq \frac{\sum_{k=2}^{\infty}|k - e^{-i\gamma}|a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k|z|^{k-1}}$$

we obtain the radius condition shown in (5.1) after some algebraic manipulation. □

6. Geometric Examples and Symmetry Properties

Geometric function theory suggests that the CFO defined by (3.1) has the potential to produce a starlike mappings when $\phi = 0$ (or when ϕ is purely imaginary) and spiral-like mappings when $\Re e(\phi) \neq 0$. By appropriately selecting the parameters and coefficient bounds, the operator can be constrained to exhibit starlike or spiral-like properties.

Example 6.1. *Let if we choose $m = e$, $n = 1$, $\phi = -i$ and $\eta_k = \frac{1}{2^k}$ (a sequence from exponential family section 3.1), the growth factor in the main function becomes: $\frac{1}{2^k}$, and hence the new function with coefficient of z^k we obtain is*

$$\mathcal{R}(z) = z - \sum_{k=2}^{\infty}\left(\frac{1}{2^k}\right)a_k z^k \tag{6.1}$$

To map the open unit disc under $\mathcal{R}(z)$, we consider few terms from the above equation (6.1).

$$\mathcal{R}(z) = z - \left(\frac{1}{4}\right)a_2z^2 - \left(\frac{1}{8}\right)a_3z^3 - \left(\frac{1}{16}\right)a_4z^4 - \cdots$$

For the above function to be a starlike the suitable choice for a_k is given by: $a_k = \frac{1}{k}$. Since the chosen coefficients satisfy the below condition.

$$\sum_{k=2}^{\infty} k\left(\frac{1}{2^k}\right)|a_k| \leq 1$$

thus it can be called a starlike function. The final closed form becomes.

$$\mathcal{R}(z) = z - \sum_{k=2}^{\infty}\left(\frac{1}{k.2^k}\right)z^k = \frac{3z}{2} + ln\left(1 - \frac{z}{2}\right). \tag{6.2}$$

The following Figure 3 demonstrates that the exact function with radical coefficients produces a well-behaved starlike mapping $|z| < 1$, with a slight distortion along x-axis from identity mapping. Note that by using the properties of complex conjugation for linear term ($\frac{3\bar{z}}{2} = \overline{\frac{3z}{2}}$) and for the

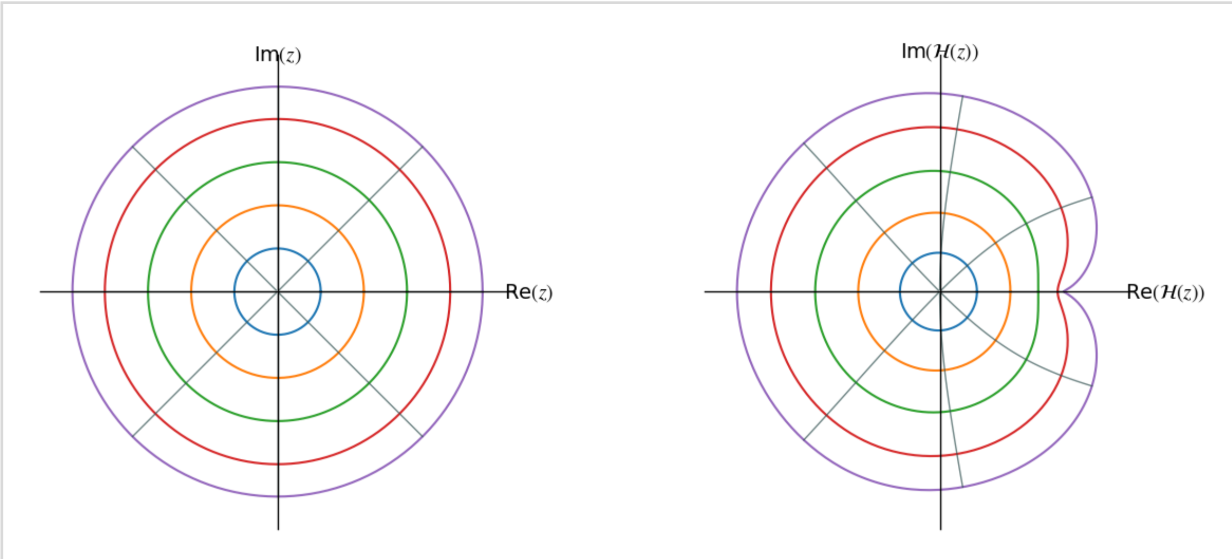


FIGURE 3. Starlike mapping of Unit disk under $\mathcal{R}(z)$

logarithmic term ($ln\,\bar{z} = \overline{ln\,z}$) the function $\mathcal{R}(z)$ in (6.2), satisfy $\mathcal{R}(\bar{z}) = \overline{\mathcal{R}(z)}$ and hence the complex mapping shows the conjugate symmetry.

In view of (2.3) to its further consequences for $\mathcal{R}(z)$ to be γ-spiral-like, the coefficients must satisfy:

$$\Re e\left(e^{i\gamma}\frac{z\mathcal{R}'[g](z)}{\mathcal{R}[g](z)}\right) > 0, \quad |z| < 1.$$

To obtain the rotating factor $e^{i\theta}$, in coefficients the parameters n, γ and $\Re e(\phi)$ must be non-zero.

Example 6.2. *Let us choose $m = e$, $n = 1$, $\theta = \frac{\pi}{4}$, $\beta = 0$, and $\eta_k = k - 3$ (a sequence from linear family section 3.1),. Then the growth factor in the main function becomes $e^{i(k-3)\frac{\pi}{4}}$, and hence the new function with coefficient of z^k is given by*

$$\mathcal{R}(z) = z - \sum_{k=2}^{\infty} e^{i(k-3)\frac{\pi}{4}} a_k z^k. \tag{6.3}$$

To map the open unit disc under $\mathcal{R}(z)$, we consider a few terms from (6.3). Thus,

$$\mathcal{R}(z) = z - e^{-i\frac{\pi}{4}} a_2 z^2 - e^{i0} a_3 z^3 - e^{i\frac{\pi}{4}} a_4 z^4 - \cdots .$$

For the above function to be spiral-like, a suitable choice of the coefficients is $a_k = \frac{1}{k \cdot 3^k}$. Since the chosen coefficients satisfy

$$\sum_{k=2}^{\infty} |a_k| = \sum_{k=2}^{\infty} \frac{1}{k \cdot 3^k} < \infty,$$

the function $\mathcal{R}(z)$ belongs to the class of spiral-like functions associated with the angle $\theta = \frac{\pi}{4}$. Hence, the function $\mathcal{R}(z)$ takes the form

$$\mathcal{R}(z) = z - \sum_{k=2}^{\infty} \frac{e^{i(k-3)\frac{\pi}{4}}}{k \cdot 3^k} z^k. \tag{6.4}$$

Using the logarithmic series expansion

$$\sum_{k=1}^{\infty} \frac{w^k}{k} = -ln(1-w), \qquad |w| < 1,$$

equation (6.4) can be written in the closed form

$$\mathcal{R}(z) = e^{-i\frac{3\pi}{4}} ln\left(1 - \frac{e^{i\frac{\pi}{4}}}{3} z\right) + \frac{4}{3} z, \qquad |z| < 1. \tag{6.5}$$

The following Figure 4 demonstrates that the exact function with radical coefficients produces a well-behaved Spiral-like mapping $|z| < 1$, with a slight deformation from identity mapping.

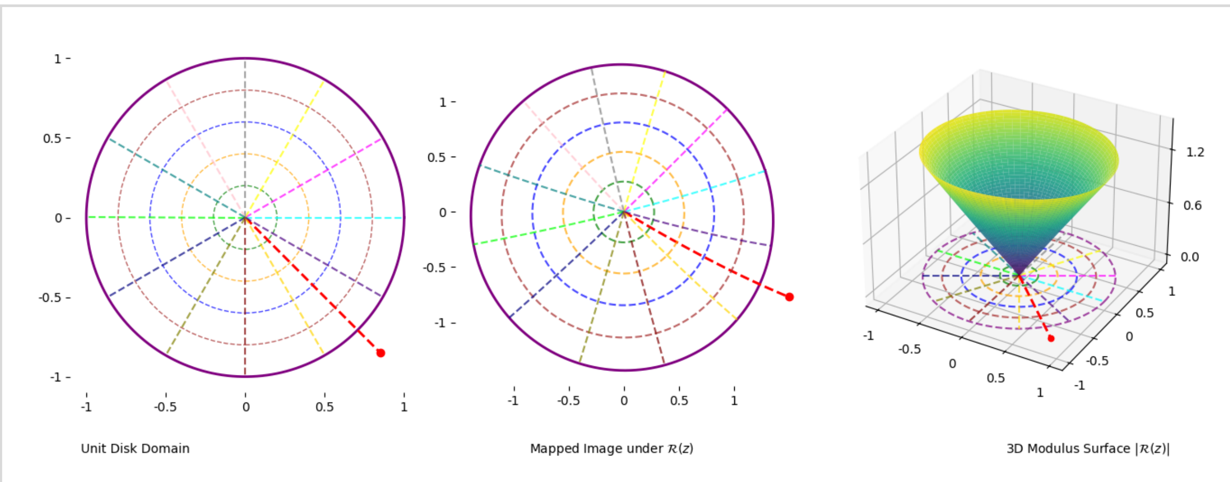


FIGURE 4. Spiral-like mapping of Unit disk under $\mathcal{R}(z)$

Note that the function $\mathcal{R}(z)$ defined by (6.5) satisfies the property of $\mathcal{R}(e^{i\frac{\pi}{4}} z) = e^{i\frac{\pi}{4}} \mathcal{R}(z) + C$ where C is a constant that does not affect the spiral motion, and hence the complex mapping shows the spiral symmetry. The branch cut occurs where the argument of the logarithm becomes real and negative, which happens along the ray from the $z_\circ$ in the direction of the branch point at $z = 3e^{-i\frac{\pi}{4}}$.

7. CONCLUSION

In this article, we have introduced the Complex Fractional Operator (CFO) and consequently developed two subclasses of analytic functions, $\mathcal{A}_\lambda(m, n, \phi)$ and $\mathcal{T}_\lambda^*(m, n, \phi)$, for generating starlike and spiral-like functions. By allowing the parameter ϕ to be complex, we have demonstrated that this operator generalizes both the Sălăgean differential operator and the Libera integral operator. We have shown that this approach naturally gives rise to both starlike and spiral-like functions, with the parameter ϕ directly influencing the spiral angle. Furthermore, we have established fundamental properties of the new classes, including coefficient bounds and distortion theorems. Finally, we have proved explicit radius formulas showing parameter dependence. Interested researchers and scholars may further study the convex properties of these classes or apply this operator to other subclasses of univalent functions.

Conflict of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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