Int. J. Anal. Appl. (2026), 24:198

International Journal of Analysis and Applications

On Graded Multiplication-Like Modules

Rajáa Al-Qderat[1], Amani Al-Qderat[2], Khaldoun Al-Zoubi[3,*]

[1]*Rajáa Al-Qderat, Department of Basic Sciences, Princess Sumaya University for Technology, Amman, Jordan*

[2]*Amani Al-Qderat, Khawarizmi University Technical College, 36 Mirza Wasfi Street, Amman, Jordan*

[3]*Khaldoun Al-Zoubi, Department of Mathematics and Statistics, Faculty of Science and Arts, Jordan University of Science and Technology, P.O. Box 3030, Irbid 22110, Jordan*

* *Corresponding author:* kfzoubi@just.edu.jo

Abstract. In this article, we introduce the notion of graded multiplication-like modules over commutative graded rings and we obtain some related results. Let G be a group with identity e. Let $W = \bigoplus_{g\in G} W_g$ be a G-graded commutative ring and $D = \bigoplus_{\alpha\in G} D_\alpha$ a graded W-module. We say that D is a graded multiplication-like if for each graded ideal K of W, there exists a graded submodule L of D with $K = (L :_W D)$.

1. Introduction

Throughout this article, we assume that W is a commutative G-graded ring with identity and D is a unitary graded W-Module. Let G be a group with identity e and W be a commutative ring with identity 1_W. Then W is a G-graded ring if there exist additive subgroups W_g of W such that $W = \bigoplus_{g\in G} W_g$ and $W_g W_h \subseteq W_{gh}$ for all $g, h \in G$. If $a \in W$, then a can be written uniquely as $\sum_{g\in G} a_g$, where a_g is the component of a in W_g. Moreover, $h(W) = \bigcup_{g\in G} W_g$. Let K be an ideal of W. Then P is said to be a graded ideal of W if $K = \bigoplus_{g\in G}(K \bigcap W_g)$. We mean by $K \leq_G^{id} W$ that K is a G-graded ideal of W. Also, we mean by $K <_G^{id} W$ that K is a proper G-graded ideal of W, see [13]. A Left W-module D is called *a graded W-module* if there exists a family of additive subgroups $\{D_\alpha\}_{\alpha\in G}$ of D such that $D = \bigoplus_{\alpha\in G} D_\alpha$ and $W_\alpha D_\beta \subseteq D_{\alpha\beta}$ for all $\alpha, \beta \in G$. Moreover, $h(D) = \cup_{\alpha\in G} D_\alpha$. Let $W = \bigoplus_{\alpha\in G} W_\alpha$ be a G-graded ring. A submodule V of D is called *a graded submodule of D* if $V = \bigoplus_{\alpha\in G}(V \cap D_\alpha) := \bigoplus_{\alpha\in G} V_\alpha$. We mean by $V \leq_G^{sub} D$ that V is a G-graded submodule of D. Also, we mean by $V <_G^{sub} D$ that V is a proper G-graded submodule of D.

Received: May 5, 2026.

2020 *Mathematics Subject Classification.* 13A02, 16W50.

Key words and phrases. graded multiplication-like modules; graded multiplication modules; graded comultiplication modules.

https://doi.org/10.28924/2291-8639-24-2026-198
ISSN: 2291-8639

Also, by $V <_G^{sub} D$, we mean that V is a proper G-graded submodule of D, see [13]. Let W be a G-graded ring, D a graded W-module and V a graded submodule of D. Then $(V :_W D)$ is defined as $(V :_W D) = \{r \in W | rD \subseteq V\}$. The annihilator of D is defined as $(0 :_W D)$ and is denoted by $Ann_W(D)$, see [5]. The basic properties of graded rings and graded modules can be found in [2,13,14].

Graded multiplication modules over commutative graded ring have been studied by many authors extensively, for example, see [9,10,17]. A graded W-module D is called a graded multiplication module (briefly, gr-m) module if for every $V \leq_G^{sub} D$ there exists $J \leq_G^{id} W$ such that $V = JD$, see [9]. As a dual concept of gr-multiplication modules, graded comultiplication modules were introduced and studied by many authors extensively, for example, see [1,3,4,8,11]. A graded W-module D is called graded comultiplication (briefly, gr-com) module if for every $V \leq_G^{sub} D$ there exists $J \leq_G^{id} W$ such that $V = (0 :_D J)$, where $(0 :_D J) = \{d \in D : dJ = 0\}$. Also it was shown that D is a gr-com module if and only if for each $V \leq_G^{sub} D$, $V = (0 :_D Ann_W(V))$, see [8].

In this paper, a graded multiplication-like module is introduced, along with some basic results and characteristics.

2. Multiplication-Like Modules

Definition 2.1. *A graded W-module D is called graded multiplication-like (briefly, gr-m-l) W-module, if for any $K \leq_G^{id} W$, there exists $L \leq_G^{sub} D$ with $K = (L :_W D)$.*

Lemma 2.1. *Let W be a G-graded ring and D be a graded W-module. Then D is gr-m-l W-module if and only if for every $K \leq_G^{id} W$, $K = (KD :_W D)$.*

Proof. $\Rightarrow$ Assume that D is a gr-m-l W-module. Hence, there exists $L \leq_G^{sub} D$ with $K = (L :_W D)$. It follows that $KD \subseteq L$. So $K \subseteq (KD :_W D) \subseteq (L :_W D) = K$. Thus $K = (KD :_W D)$ as desired.

$\Leftarrow$ clear □

Theorem 2.1. *Let W be a G-graded ring and D be a graded W-module. Then D is gr-m-l W-module if and only if for each $K \leq_G^{id} W$, there exist $L_i \leq_G^{sub} D$ $(i \in J)$, such that $K = \sum_{i \in J}(L_i :_W D) = (\sum_{i \in J} L_i :_W D)$.*

Proof. Suppose that D is a gr-m-l and $K \leq_G^{id} W$. By Lemma 2.1, $K = (KD :_W D)$. Thus, $K = \sum_{k_i \in K} Wk_i$ and for each $k_i \in K$, $Wk_i = (k_iD :_W D)$. Thus, $K = \sum_{k_i \in K} Wk_i = \sum_{k_i \in K}(k_iD :_W D) = (\sum_{k_i \in K} k_iD :_W D)$. □

Theorem 2.2. *Let W be a G-graded ring and D be a graded W-module. Then the following statements are equivalent.*

(i) *D is gr-m-l W-module.*

(ii) *For each $K \leq_G^{id} W$ and every $L \leq_G^{sub} D$ with $K \subset (L :_W D)$, there exists $U \leq_G^{sub} D$ such that $U \subset L$ and $K = (U :_W D)$.*

(iii) *For each $K \leq_G^{id} W$ and every $L \leq_G^{sub} D$ with $K \subset (L :_W D)$, there exists $U \leq_G^{sub} D$ such that $U \subset L$ and $K \subset (U :_W D)$.*

Proof. $(i) \Rightarrow (ii)$ Assume that $K \subset (L :_W D)$. Then by Lemma 2.1, $K = (KD :_W D)$. Put $U = KD \cap L$. Since $K = (KD :_W D) \subset (L :_W D)$, $U \subset L$. So we get $(U :_W D) = (KD \cap L :_W D) = (KD :_W D) \cap (L :_W D) = (KD :_W D) = K$.

$(ii) \Rightarrow (iii)$ It is clear.

$(iii) \Rightarrow (i)$ Assume that $K \leq_G^{id} W$ and $V = \{U : U \leq_G^{sub} D$ and $K \subset (U :_W D)\}$. Clearly V is a non-empty set, so by Zorn's Lemma, V has a minimal member like I and so $K \subset (I :_W D)$. Suppose that $K \neq (I :_W D)$. Then there exists $N \leq_G^{sub} D$ with $N \subset I$ and $K \subset (N :_W D)$ by part (iii). This is a contradiction by the choice of I. Hence we have $K = (I :_W D)$. This implies that D is gr-m-l W-module. □

a graded ring W is said to be a graded comultiplication (briefly, gr-com) ring if, as a graded W-module, W is a gr-com W-module, see [8]. A graded W-module D is said to be graded faithful (briefly, gr-faithful) if $sD = 0$ implies $s = 0$ for $s \in h(W)$, [5].

Theorem 2.3. *If W is a gr-com ring and D is a gr-faithful W-module, then D is a gr-m-l W-module.*

Proof. Let $K <_G^{id} W$ and $sD \subseteq KD$, for $s \in h(W)$. Thus $sAnn_W(K)D = 0$. We have $s \in K$, because D is gr-faithful and W is a gr-com ring. Hence D is a gr-m-l W-module. □

Since W is a gr-com ring if and only if $(K :_W T) = (Ann_W(T) :_W Ann_W(K))$, for any two graded ideals K and T of W. So by Theorem 2.3, we have:

Corollary 2.1. *If W is a G-graded ring such that for each $K \leq_G^{id} W$ and $T \leq_G^{id} W$, $(K :_W T) = (Ann_W(T) :_W Ann_W(K))$, then every gr-faithful W-module is gr-m-l module.*

Clearly, if D' is a gr-m-l W-module and $\alpha : D \to D'$ is a graded W-epimorphism, then D is a gr-m-l W-module. Also, Let D be a graded W-module and L be a graded submodule of D. If $\frac{D}{L}$ is a gr-m-l W-module, then we can conclude that D is a gr-m-l W-module.

Theorem 2.4. *Let W be a G-graded ring and D be a gr-m-l W-module.*

(i) *If for $L \leq_G^{sub} D$, $L \subseteq KD$ for each non-zero $K \leq_G^{id} W$ and $\frac{D}{L}$ is gr-faithful, then $\frac{D}{L}$ is a gr-m-l W-module.*
(ii) *If D' is a gr-faithful W-module, $\Omega : D \to D'$ is a graded epimorphism and for every non-zero $K \leq_G^{id} W$, $ker(\Omega) \subseteq KD$, then D' is a gr-m-l W-module.*

Proof. (i) Since $K(\frac{D}{L}) = \frac{KD}{L}$, so $\frac{D}{L}$ is a gr-m-l W-module.

(ii) It is clear by part (i). □

A non-zero graded submodule V of D is called graded second (gr-second) if $wV = V$ or $wV = 0$ for every $w \in h(W)$, see [7].

Corollary 2.2. *If L is a gr-faithful gr-second submodule of a gr-m-l W-module D, then for any non-zero $K \leq_G^{id} W$, there is $U \leq_G^{sub} \frac{D}{L}$ such that $K = (U :_W \frac{D}{L})$.*

Proof. Assume that L is gr-second and gr-faithful, then $KL = L$, for each non-zero $K \leq_G^{id} W$. This implies that $L \subseteq KD$. The proof is completely by Theorem 2.4(i). □

Theorem 2.5. *If D is a gr-m-l W-module and $K \leq_G^{id} W$, then $\frac{D}{KD}$ is a gr-m-l $\frac{W}{K}$-module.*

Proof. It is enough to show that $(\frac{T}{K}(\frac{D}{KD}) :_{\frac{W}{K}} \frac{D}{KD}) \subseteq \frac{T}{K}$ for each $T \leq_G^{id} W$ containing K. Suppose that D is gr-m-l, so we have $T = (TD :_W D)$. If $(r+K) \in (\frac{T}{K}(\frac{D}{KD}) :_{\frac{W}{K}} \frac{D}{KD})$, so for every $m \in h(D)$, $(r+K)(m+KD) \in \frac{T}{K}(\frac{D}{KD}) = \frac{TD}{KD}$. Thus $rm \in TD$. Hence $r \in T$. It follows that $r + K \in \frac{T}{K}$. □

Corollary 2.3. *If D is a gr-m-l W-module, then for every $K \leq_G^{id} W$ with $K \subseteq Ann_W(D)$, D is a gr-m-l $\frac{W}{K}$-module.*

Proof. It is clear, by Theorem 2.5. □

Recall that a graded W-module D is said to be graded finitely generated (briefly, gr-f-gen) if there exist $x_{g_1}, x_{g_2}, ..., x_{g_n} \in h(D)$ such that $D = Wx_{g_1} + ... + Wx_{g_n}$, see [13].

Theorem 2.6. *Let D be a graded W-module and B be a multiplicative closed subset of W.*

(i) *If D is a gr-f-gen gr-m-l W-module, then D_B is a gr-m-l W_B-module.*
(ii) *If D_B is a gr-m-l W_B-module and for any $K \leq_G^{id} W$ and any $r \in h(W) - K$, $B \cap (K :_W r) = \emptyset$, then D is a gr-m-l W-module.*

Proof. (i) Assume that D is a gr-m-l W-module, then by Lemma 2.1 $K = (KD :_W D)$ for any $K \leq_G^{id} W$. Since D is gr-f-gen, so we get $K_B = (K_B D_B :_{W_B} D_B)$.

(ii) If $K \leq_G^{id} W$ and $r \in (KD :_W D) \cap h(W)$, we get $\frac{r}{1} D_B \subseteq K_B D_B$. Since D_B is a gr-m-l W_B-module, we have $\frac{r}{1} \in K_B$. This implies that there exists $u \in B$ such that $ur \in K$. If $r \notin K$, hence $u \in B \cap (K :_W r)$ which is a contradiction so $r \in K$. □

A $K <_G^{id} W$ is called graded maximal (gr-maximal) ideal of W if B is a graded ideal of W such that $K \subseteq B \subseteq W$, then $B = K$ or $B = W$, see [13].

Lemma 2.2. *Let W be a graded ring and D be a graded W-module such that $K \neq (KD :_W D)$, for some $K \leq_G^{id} W$. Then there exists $I \leq_G^{id} W$ and $r \notin I$ such that $K \subseteq I$ and $(I :_W r)$ is graded maximal ideal of ring W.*

Proof. Assume that $s \in (KD :_W D)$ and $s \notin K$ for some $s \in h(W)$. Let B denote the collection of graded ideals J of W such that $K \subseteq J$ and $s \notin J$. B clearly is non-empty, so by Zorńs Lemma, B has a maximal member like I. Hence $K \subseteq I$ and $s \notin I$. Let $r \in h(W)$ such that $r \notin (I :_W s)$. Then I is a proper subset of $I + Wrs$ and hence $I + Wrs \notin B$. It implies that $s \in I + Wrs$. Thus, there exists $b \in h(W)$ and $u \in I \cap h(W)$ such that $s = u + brs$ and so $(1 - br)s \in I$. So we have $(I :_W s)$ is a graded maximal ideal of W. □

Theorem 2.7. *Let W be a G-graded ring and D be a graded W-module. Then the following are equivalent*

(i) *D is a gr-m-l W-module.*
(ii) *$K = (KD :_W D)$, for any $K \leq_G^{id} W$.*
(iii) *If $K \leq_G^{id} W$ and $T \leq_G^{id} W$ with $KD \subseteq TD$, then $K \subseteq T$.*
(iv) *If $K \leq_G^{id} W$ and $s \in h(W)$ with $sD \subseteq KD$, then $s \in K$.*
(v) *If $K \leq_G^{id} W$ and $s \in h(W)$ with $sD \subseteq KD$, then $(K :_W s)$ is not a graded maximal ideal.*

Proof. $(i) \Leftrightarrow (ii)$ By Lemma 2.1.

$(ii) \Rightarrow (iii)$ Suppose that $KD \subseteq TD$. Then $(KD :_W D) \subseteq (TD :_W D)$. By (ii), $K \subseteq T$.

$(iii) \Rightarrow (ii)$ Since $KD = (KD :_W D)D$. So by (iii), we have $K = (KD :_W D)$.

$(iii) \Rightarrow (iv)$ It is clear.

$(iv) \Rightarrow (v)$ Assume taht $sD \subseteq KD$. So by (iv), $s \in K$, and hence $(K :_W s) = W$. Thus, $(K :_W s)$ is not a graded maximal.

$(v) \Rightarrow (iv)$ Suppose that $sD \subseteq KD$ and $s \notin K$. So by Lemma 2.2, there exists $I \leq_G^{id} W$ with $K \subseteq I$, $s \notin I$ and $(I :_W s)$ is graded maximal ideal, which is a contradiction. □

3. Properties of Graded Multiplication-Like Modules

A graded W-module D is said to be a graded-r-multiplication (briefly, gr-r-mult) module, when $KD \neq D$ for every $K <_G^{id} W$. A $V <_G^{sub} W$ is called a graded prime (gr-prime) submodule if whenever $w \in h(W)$ and $d \in h(D)$ such that $wd \in V$, then either $w \in (V :_W D) = \{w \in W : wD \subseteq V\}$ or $d \in V$, see [5].

Theorem 3.1. *Let D be a gr-m-l W-module. Then*

(i) *D is a gr-faithful W-module.*
(ii) *D is a gr-r-mult W-module.*
(iii) *The set of all graded prime submodules of a graded W-module D is non-empty ($Spec_W(D) \neq \emptyset$).*
(iv) *For any $K <_G^{id} W$, $Ann_W(K) = Ann_W(KD)$.*
(v) *$G\text{-}Z(W) = \{a \in h(W)$: there exists $\{0\} \neq L \leq_G^{sub} D$ such that $(L :_W D) \neq 0$ and $a(L :_W D) \neq 0\}$.*

Proof. (i) By Lemma 2.1, $0 = (0D :_W D) = Ann_W(D)$.

(ii) Let $K <_G^{id} W$ with $KD = D$, then by Lemma 2.1, $K = (KD :_W D) = W$. Which is a contradiction and so the proof is completed.

(iii) Assume that $x \in Max(h(W))$. Hence $x = (xD :_W D)$ by part (ii). This implies that xD is a graded prime submodule of D.

(iv) Assume that $s \in Ann_W(KD) \cap h(W)$, then $sKD = 0$. By part (i), we have $Ks = 0$. This prove that $Ann_W(KD) \subseteq Ann_W(K)$.

(v) Assume that $x \in G\text{-}Z(W)$. It follows that there exists $0 \neq y \in h(W)$ such that $xy = 0$. Since D is gr-m-l W-module, $x(yD :_W D) = 0$. The converse is clear. □

Theorem 3.2. *If D is a graded W-module, then D is a vector space if and only if D is gr-m-l and gr-second W-module.*

Proof. It is sufficient to prove that W is a graded field. Let $s \in h(W)$ such that s is non-unit, then $sD \neq D$ because D is gr-m-l W-module. Since D is gr-second and gr-faithful, $s = 0$. This implies that the set of non-units is zero graded ideal. Hence W is a graded field and D is a graded vector space. □

Theorem 3.3. *If D is a gr-r-mult W-module such that every proper graded submodule is gr-m-l W-module, then D is a gr-m-l. W-module.*

Proof. Assume that $K \leq_G^{id} W$. Then by Lemma 2.1, $K = (K^2D :_W KD)$. Let $sD \subseteq KD$. This implies that $KsD \subseteq K^2D$. Thus $s \in K$. It follows that, D is a gr-m-l W-module. □

Recall that a graded W-module D is said to be graded Noetherian (briefly, gr-Noeth) if D satisfies the ascending chain condition for graded submodules, see [12, 13]. A graded ring in which every nonzero homogeneous element is invertible is called a graded field (gr-field), see [13].

Theorem 3.4. *If W is a gr-Noeth domain which is not a gr-field and D is a gr-m-l W-module, then every non-zero graded maximal submodule of D, is gr-r-mult W-module.*

Proof. Assume that L is not a gr-r-mult where L is a non-zero graded maximal submodule of D. So there exists $K \leq_G^{id} W$ such that $KL = L$. As L is a graded maximal submodule and D is gr-m-l, we have $L = KD$ and $K = K^2 = (L :_W D)$. Thus there exists $a \in K$ such that $(1-a)K = 0$. It follows that $K = W$ or $K = 0$ since W is graded domain, which is a contradiction. □

Theorem 3.5. *If W is a local gr-Noeth ring that is not gr-field and D is a gr-m-l W-module, then every non-zero graded maximal submodule of D is gr-r-mult W-module.*

Proof. Assume that L is a non-zero graded maximal submodule of D and L is not a gr-r-mult W-module. So there exists $K \leq_G^{id} W$ such that $KL = L$. As L is a graded maximal submodule and D is gr-m-l W-module we get $L = KD$ and $K = K^2 = (L :_W D)$, by graded version of Nakayama's Lemma, $K = 0$, which is a contradiction. Thus L is a gr-r-mult W-module. □

Lemma 3.1. *Let D be a gr-m W-module. Then D is a gr-m-l if and only if D is gr-f-gen and gr-faithful W-module.*

Proof. Assume that D is a gr-m-l W-module. Then by Theorem 3.1, D is gr-faithful and for every $K <_G^{id} W$, $KD \neq D$. This implies that D is gr-f-gen W-module. Conversely, suppose that $K <_G^{id} W$. Then $KD = (KD :_W D)D$. Since D is gr-m W-module, gr-faithful and gr-f-gen W-module, $K = (KD :_W D)$. Therefore, D is gr-m-l W-module. □

Theorem 3.6. *Let D be a gr-faithful, gr-m W-module. Then D is a gr-r-mult if and only if D is a gr-m-l W-module.*

Proof. Asuume that D is a gr-m-l W-module. Then by Theorem 3.1, D is gr-r-mult W-module. Conversely, suppose that $K \leq_G^{id} W$. Note that $KD = (KD :_W D)D$. Then D is gr-f-gen W-module, since D is gr-faithful, gr-m and gr-r-mult W-module. Now by Lemma 3.1, D is gr-m-l W-module. □

A graded W-module D is said to be graded strong comultiplication (strong gr-com), if for every graded submodule L of D there is exactly one graded ideal K of W with $L = (0 :_D K)$.

Theorem 3.7. *Let D be a gr-com and gr-m-l W-module. Then D is a strong gr-com W-module.*

Proof. Assume that $L \leq_G^{sub} D$, $K \leq_G^{id} W$ and $T \leq_G^{id} W$ with $L = (0 :_D K)$ and $L = (0 :_D T)$, then by [8, Theorem 3.7(c)], $KD = TD$. Now by Theorem 2.7, $K = T$. □

Theorem 3.8. . *If D is a gr-com and gr-m-l W-module, then for every $L \leq_G^{sub} D$, there exists $K \leq_G^{id} W$ such that $(L :_W D) = Ann_W(K)$.*

Proof. Assume that $L \leq_G^{sub} D$. Suppose that D is a gr-com W-module, this implies that there exists $K \leq_G^{id} W$ with $L = (0 :_D K)$, and so we get $(L :_W D) = ((0 :_D K) :_W D) = Ann_W(KD) = Ann_W(K)$. □

Theorem 3.9. *If D is a gr-m-l W-module, then for any $K \leq_G^{id} W$ and $T \leq_G^{id} W$, we have*

(i) $(KTD :_W D) = (KD :_W D)(TD :_W D)$.
(ii) $(KD + TD :_W D) = (KD :_W D) + (TD :_W D)$.
(iii) $((K \cap T)D :_W D) = (KD :_W D) \backslash (TD :_W D)$.

Proof. It is clear by Lemma 2.1. □

Theorem 3.10. *If D is a gr-Noeth gr-m-l W-module, then W is gr-Noeth.*

Proof. Assume that $K_1 \subseteq K_2 \subseteq K_3$... is an ascending chain of graded ideals of W. Then $K_1D \subseteq K_2D \subseteq K_3D$... is an ascending chain of graded submodules of D. It follows that there exists a positive integer m such that $K_mD = K_{m+1}D = ...$ Since D is gr-m-l W-module, so we have $K_m = K_{m+1} = ...$ □

Let $K \leq_G^{id} W$. The graded radical of K, denoted by $Gr(K)$, is the set of all $w = \sum_{h \in G} w_h \in w$ such that for each $h \in G$ there exists $t_h > 0$ with $w_h^{t_h} \in K$, see [15]. A proper graded ideal K of W is called a graded primary ideal, if whenever $a, b \in h(W)$ with $ab \in K$, then $a \in K$ or $b \in Gr(K)$, see [15].

Theorem 3.11. *If D is a gr-faithful W-module over a gr-Noeth ring W such that for any graded primary ideal c of W, $c = (cD :_W D)$, then D is gr-m-l W-module.*

Proof. Assume that $K \leq_G^{id} W$ and let $K = \cap_{i=1}^n c_i$ be a gr-reduced primary decomposition of K in W, such that c_i are graded primary. This implies that $K(KD :_W D) = ((\cap_{i=1}^n c_i)D :_W D) \subseteq \cap_{i=1}^n (c_iD :_W D) = \cap_{i=1}^n c_i = K$. □

Theorem 3.12. *If D is a gr-m-l W-module and every graded submodule of D has a graded reduced primary decomposition, then any graded ideal of W has a graded reduced primary decomposition.*

Proof. Let $K \leq_G^{id} W$, then $K = (KD :_W D)$ as D is gr-m-l W-module. By hypothesis, $KD = \cap_{i=1}^n c_i$, when c_i are graded P_i-primary. So $K = (KD :_W D) = (\cap_{i=1}^n c_i :_W D) = \cap_{i=1}^n (c_i :_W D)$. It implies that K has graded reduced primary decomposition in W. □

Recall that the graded submodule L is called a pure graded submodule, if $KL = L \cap KD$ for every graded ideal K of W, see [5].

Theorem 3.13. *Let L be a pure graded submodule of a graded W-module D. If L is a gr-m-l W-module, then D is a gr-m-l W-module.*

Proof. Assume that $K \leq_G^{id} W$. This implies that $K = (KL :_W L)$. Suppose that $xD \subseteq KD$. So we have $xL \subseteq KL$ because L is a graded pure. Thus $x \in K$. Therefore, D is gr-m-l W-module. □

Recall that a graded ring W is graded discrete valuation ring *gr-DVR* if and only if it is graded valuation and gr-Noeth ring. If W is a *gr-DVR* then every non-zero $K \leq_G^{id} W$ is uniquely of the type $K = c^m$ for some $m \in \mathbb{N}$, where c is the unique graded maximal ideal W, see [16].

Theorem 3.14. *If D is a gr-faithul gr-f-gen W-module over a gr-DVR W, then D is a gr-m-l W-module.*

Proof. Assume that $K \leq_G^{id} W$ and c be the unique graded maximal ideal. It follows that there exists $m \in N$ such that $K = c^m$. This implies that $c^m(c^mD :_W D) \subseteq c^{m-1}$. Thus $c^m = (c^mD :_W D)$ or $(c^mD :_W D) = c^{m-1}$. If $(c^mD :_W D) = c^{m-1}$, then $c^mD = c^{m-1}D$. So by graded version of Nakayama's Lemma, $m = 0$ which is a contradiction. Hence $(c^mD :_W D) = mc$. □

Let $V \leq_G^{sub} D$. The graded radical of V, denoted by $\mathrm{Gr}_D(V)$, is defined to be the intersection of all gr-prime submodules of D containing V. If V is not contained in any gr-prime submodule of D, then $\mathrm{Gr}_D(V) = D$, see [6]. A $V <_G^{sub} D$ is called a graded primary-like (gr-primary-like) submodule if whenever $w \in h(W)$ and $d \in h(D)$ with $wd \in V$, then either $w \in (V :_W D)$ or $d \in \mathrm{Gr}_D(V)$, see [4].

Theorem 3.15. *.If $L \leq_G^{sub} D$ is a gr-m-l such that for any $K \leq_G^{id} W$, KL is a gr-primary-like submodule and $rad_W(KL) \subset L$, then D is a gr-m-l W-module.*

Proof. Let $K \leq_G^{id} W$. Thus $K = (KL :_W L)$, becuse L is a gr-m-l W-module. So $KD \subseteq KL$. It follows to prove that $K \subseteq (KL :_W D)$. Let $s \in K \cap h(W)$. Since $Gr_W(KL) \subset L$, so we can find an element $n \in L \cap h(D) - Gr_W(KL)$. Then $sn \in KL$. Therefore, $s \in (KL :_W D)$, as KL is a graded primary-like. Hence D is gr-m-l W-module. □

Theorem 3.16. *If D is a distributive gr-m-l W-module and for every two $L_1 \leq_G^{sub} D$, $L_2 \leq_G^{sub} D$, $(L_1 :_W D) + (L_2 :_W D) = (L_1 + L_2 :_W D)$. Then W is a graded distributive ring.*

Proof. Assume that $A_1 \leq_G^{id} D$, $A_2 \leq_G^{id} W$ and $A_3 \leq_G^{id} W$. Then there exist $L_1 \leq_G^{sub} D$, $L_2 \leq_G^{sub} D$ and $L_3 \leq_G^{sub} D$ with $A_1 = (L_1 :_W D)$, $A_2 = (L_2 :_W D)$ and $A_3 = (L_3 :_W D)$, Because D is gr-m-l W-module. So we get $(A_1 + A_2) \cap A_3 = ((L_1 :_W D) + (L_2 :_W D)) \cap (L_3 :_W D) = (L_1 + L_2 :_W D) \cap (L_3 :_W D) = ((L_1 + L_2) \cap L_3 :_W D) = ((L_1 \cap L_3) + (L_2 \cap L_3) :_W D), = (L_1 \cap L_3 :_W D) + (L_2 \cap L_3 :_W D) = (L_1 :_W D) \cap (L_3 :_W D) + (L_2 :_W D) \cap (L_3 :_W D) = A_1 \cap A_3 + A_2 \cap A_3$: □

Acknowledgments: The authors wish to thank sincerely the referees for their valuable comments and suggestions.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R. Abu-Dawwas, M. Bataineh, A. Da'Keek, Graded Weak Comultiplication Modules, Hokkaido Math. J. 48 (2019), 253–261. https://doi.org/10.14492/hokmj/1562810507.

[2] K. Al-Zoubi, M. Al-Azaizeh, On Graded Weakly 2-Absorbing Primary Submodules, Vietnam J. Math. 47 (2018), 297–307. https://doi.org/10.1007/s10013-018-0321-z.

[3] K. Al-Zoubi, A. Al-Qderat, Some Properties of Graded Comultiplication Modules, Open Math. 15 (2017), 187–192. https://doi.org/10.1515/math-2017-0016.

[4] K. Al-Zoubi, M. Al-Dolat, On Graded Primary-Like Submodules of Graded Modules over Graded Commutative Rings, Proyecciones 40 (2021), 859–871. https://doi.org/10.22199/issn.0717-6279-3467.

[5] S.E. Atani, On Graded Prime Submodules, Chiang Mai J. Sci. 33 (2006), 3–7.

[6] S. Ebrahimi Atani, F. Farzalipour, On Graded Secondary Modules, Turk. J. Math. 31 (2007), 371–378.

[7] H. Ansari-Toroghy, F. Farshadifar, On Graded Second Modules, Tamkang J. Math. 43 (2012), 499–505. https://doi.org/10.5556/j.tkjm.43.2012.1319.

[8] H. Ansari-Toroghy, F. Farshadifar, Graded Comultiplication Modules, Chiang Mai J. Sci. 38 (2011), 311–320.

[9] J. Escoriza, B. Torrecillas, Multiplication Objects in Commutative Grothendieck Categories, Commun. Algebra 26 (1998), 1867–1883. https://doi.org/10.1080/00927879808826244.

[10] P. Ghiasvand, F. Farzalipour, On Graded Weak Multiplication Modules, Tamkang J. Math. 43 (2012), 171–177. https://doi.org/10.5556/j.tkjm.43.2012.712.

[11] M. Hamoda, K. Al-Zoubi, On Graded S-Comultiplication Modules, Bol. Soc. Paranaense Mat. 42 (2024), 1–11. https://doi.org/10.5269/bspm.63403.

[12] L. Lu, Gr-Injective Modules and Gr-Projective Modules over G-Graded Commutative Rings, J. Math. Sci. 247 (2020), 613–625. https://doi.org/10.1007/s10958-020-04823-z.

[13] C. Nastasescu, F. Van Oystaeyen, Graded Ring Theory, North Holand, Amsterdam, 1982.

[14] C. Nastasescu, F. Van Oystaeyen, Methods of Graded Rings, Springer, 2004. https://doi.org/10.1007/b94904.

[15] M. Refai, K. Al-Zoubi, On Graded Primary Ideals, Turk. J. Math. 28 (2004), 217–229.

[16] M. Van Den Berg, Graded Dedekind Rings, J. Pure Appl. Algebra 35 (1985), 105–115. https://doi.org/10.1016/0022-4049(85)90033-7.

[17] N. Zamani, Finitely Generated Graded Multiplication Modules, Glasg. Math. J. 53 (2011), 693–705. https://doi.org/10.1017/s0017089511000279.