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Fuzzy Bases and Fuzzy Subbases of the Fuzzy Topological Space on A Fuzzy Space (X, I)

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Abstract. The fuzzy topological space was introduced by Dip in 1999, depending on the notion of the fuzzy space (X, I). In this paper, we reformulate the definitions of fuzzy space (X, I), fuzzy point, interior, exterior, boundary, cluster and isolated points. Then we define the fuzzy bases and fuzzy subbases of the fuzzy topological space $\big((X, I), \mathcal{T}\big)$ with theorems and examples.

1. Introduction

The fuzzy set was introduced in 1965 by Zadeh [9], who defined it as a collection of objects characterized by a membership function that assigns each object a membership grade in the range $[0, 1]$. This definition expanded upon the traditional sets: when A represents a collection of objects in the usual sense, its membership function can assume only two values, 0 or 1, depending on whether the object is in A or not. The concept of "belonging" that is fundamental in ordinary sets does not hold the same significance in fuzzy sets. It is not significant to say that a point x "belongs" to a fuzzy set A unless in the trivial context of its membership function being positive. Less trivially, one may define two thresholds α and β such that $0 < \beta < \alpha < 1$, and stipulate that x is a member of A if its membership function $\mu_A(x)$ is at least α, x is not a member of A if $\mu_A(x)$ is at most β, or x has an uncertain status regarding A if $\beta \leq \mu_A(x) \leq \alpha$. This results in a three-valued logic (Kleene, 1952, [10]) that includes three truth values: $T(\mu_A(x) \geq \alpha)$, $F(\mu \leq \beta)$, and $U(\beta \leq \mu_A(x) \leq \alpha)$.

Chang's method in 1968 [3] addressed the deficiency of the fuzzy universal set by presenting fuzzy topology on a base set X, as a collection σ of fuzzy subsets of X that meets the criteria of formal topology, serving as a generalization of Zadeh's idea. Chang introduced the concept of

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fuzzy topology (C-fuzzy topology) on a set X, defining it as a collection σ of fuzzy sets in X such that $\emptyset, X \in \sigma$, if $A, B \in \sigma$ then $A \cap B \in \sigma$, and if $A_i \in \sigma$ for all $i \in I$, then $U_I A_i \in \sigma$. The collection σ refers to fuzzy topology for X, and the combination (X, σ) is known as a fuzzy topological space (FTS). Subsequently, several generalizations, redefinitions, and proposals regarding fuzzy topology were developed and examined by Wong (1973) [8], Lowen (1976) [7], and Hazra et al. (1992) [6]. Unlike Chang's method, Dib [5] proposed fuzzy topology on the fuzzy space (X, I), consisting of a set of fuzzy subspaces that satisfy the conventional axioms of topology. In 1994 [4], Dip initiated the creation of fuzzy sets within algebra to demonstrate his concept, and later in 1999 [5] extended its application to topology. His concept focused on replacing standard universal sets with fuzzy spaces to address the discrepancy between the definitions of a closed fuzzy subset and the closure of a fuzzy subset in fuzzy topological spaces. In 2024, Jawarneh et al. have filled the lack of the concepts in Dip's approach by adding some definitions and theorems on cofinite and cocountable fuzzy topological spaeces and some separation axioms, see [2]. In this research, we continue jawarneh's approch by reformulating some important definitions and introducing the fuzzy bases, finite product of fuzzy spaces, and fuzzy subbases of the fuzzy topological space on the fuzzy space (X, I). Also, we present examples, and theorems on its properties.

This paper is organized as follows: In the next section, we reformulate the fuzzy space (X, I) and the fuzzy point with examples, and we present some necessary definitions related to fuzzy topology. In sections 3, 4, and 5 we introduce the definitions of the fuzzy bases, finite product of fuzzy spaces, and fuzzy subbases of the fuzzy topological space on the fuzzy space (X, I) with theorems and explained examples. Finally, the results are summarized in section 6.

2. PRELIMINARIES

Let X represent an ordinary set and I denote a closed interval $[0, 1]$. In this paper, we use some definitions and theorems of the fuzzy space (X, I) that are essential for this study see [1,2,5]. Also, we reformulate some definitions of the fuzzy space (X, I) that are mentioned in ([1,2,5] as follow

Definition 2.1. *The fuzzy space* (X, I) *is the set of all ordered pairs* (x, I) *such that* $x \in X$, *and* $(x.I) = \{(x, r_x) : x \in X$ *for all possible values* $r_x \in I$ *where* r_x*contains at least one value of* $(0, 1]$
besides $0\}$. *The order pair* (x, I) *is called a fuzzy element of the fuzzy space* (X, I), *and the order pair* (x, r_x) *is called a fuzzy point of the fuzzy space* (X, I).

A subset U of the fuzzy space (X, I) is called fuzzy subspace as follow

Definition 2.2. *The fuzzy subspace* U *of the fuzzy space* (X, I) *is a collection of ordered pairs* (x, u_x), *where* $x \in U_0$ *(for a given subset* U_0 *of* X*) and* u_x *is a subset of* I, *which contains at least one element besides zero element. If* $x \notin U_0$, *then* $u_x = \{0\}$. *The fuzzy subspace* U *is denoted by* $U = \{(x, u_x) : x \in U_0\}$, *where* (x, u_x) *is the fuzzy point of fuzzy subspace* U. U_0 *is called the support of* U *and denoted by* $S(U) = U_0$.

U_0 can be represented as $U_0 = \{x \in X : u_x = \{0, u'_x\}, u'_x \subset (0,1]\}$, $S(U) = \emptyset$ (empty fuzzy set [9]) if $u_x = 0$ for all $x \in X$, but the empty fuzzy subspace of the fuzzy space (X, I) is denoted by $\emptyset^F$ see [2] in which $\emptyset^F = \{(x, \phi_x) : x \in X$, where $S(\emptyset^F) = \emptyset$ as its membership function ϕ_x is identically zero on $X\}$. The concept subset of fuzzy subspaces is defined as follows

Definition 2.3. *The fuzzy subspace* $V = \{(x, v_x) : x \in V_0\}$ *is contained in fuzzy subspace* $U = \{(x, u_x) : x \in U_0\}$ *and denoted by* $V \subset U$*, if* $V_0 \subset U_0$ *and* $v_x \subset u_x$*, for all* $x \in V_0$*.*

The empty fuzzy subspace $\emptyset^F$ is clearly contained in any fuzzy subspace. Intersection "$\cap$" and union "$\cup$" operations in the fuzzy space (X, I) are defined in the following definition.

Definition 2.4. *Let* $U = \{(x, u_x) : x \in U_0\}$ *and* $V = \{(x, v_x) : x \in V_0\}$ *be fuzzy subspaces of the fuzzy spaces* (X, I)*. The union* $U \cup V$ *and the intersection* $U \cap V$ *of fuzzy subspaces are defined by the relations*
$U \cup V = \{(x, u_x \cup v_x) : x \in U_0 \cup V_0\}$,
$U \cap V = \{(x, u_x \cap v_x) : x \in U_0 \cap V_0\}$.
The support of these fuzzy subspaces satisfies that
$S(U \cup V) = S(U) \cup S(V) = U_0 \cup V_0$,
$S(U \cap V) \subset S(U) \cap S(V) = U_0 \cap V_0$,
the inclusion relation will be an equality, if $u_x \cap v_x \neq \{0\}$*, for all* $x \in U_0 \cap V_0$*.*

The difference between fuzzy subspaces is a fuzzy subspace.

Definition 2.5. *The difference* $U - V$ *between the fuzzy subspace* U *and* V *is defined by*
$U - V = \{(x, h_x) : x \in U_0, h_x = (u_x - v_x) \cup \{0\}\}$

Notice that $U_0 - V_0 \subset S(U - V)$ and the equality holds if $u_x \subset v_x$, for all $x \in U_0 \cap V_0$. The fuzzy topology $\mathcal{T}$ on the fuzzy space (X, I) is defined as following

Definition 2.6. *The family* $\mathcal{T} = \{U_\alpha : \alpha \in \Delta\}$ *of fuzzy subspaces of the fuzzy space* (X, I) *is called a fuzzy topology on the fuzzy space* (X, I) *if* $\emptyset^F, (X, I) \in \mathcal{T}$*,* $U_\alpha \cap V_\beta \in \mathcal{T}$ *for any* $\alpha, \beta \in \Delta$*, and* $\bigcup U_\alpha \in \mathcal{T}, \alpha \in \Delta$*. The ordered pair* $\big((X, I), \mathcal{T}\big)$ *is called a fuzzy topological space. The elements of* $\mathcal{T}$ *are open fuzzy subspaces. The fuzzy subspace* A *is closed if* $A^C = (X, I) - A$ *is open fuzzy subspace.*

The collection of closed fuzzy subspaces is closed under finite unions and arbitrary intersections. Moreover, $\emptyset^F$ and (X, I) are closed fuzzy subspaces. The trivial fuzzy topology contains only two elements (X, I) and $\emptyset^F$. Another example, the discrete fuzzy topology contains all the fuzzy subspaces of (X, I). Also we reformulate the interior, exterior, boundary, cluster, and isolated fuzzy points of the fuzzy subspace of the fuzzy topological space $\big((X, I), \mathcal{T}\big)$ as follow

Definition 2.7. *Let* $\big((X, I), \mathcal{T}\big)$ *be a fuzzy topological space on the fuzzy space* (X, I)*, and let* A *be a fuzzy subspace of the fuzzy space* (X, I) *then*

(1) *A fuzzy point* $(x, r_x) \in A$ *(*$x \in A_0$*) is an interior fuzzy point of the fuzzy subspace* A *if there exists an open fuzzy subspace* $U \in \mathcal{T}$ *such that* $x \in U \subset A$*. The set of interior fuzzy points of the fuzzy subspace* A *is denoted by* A^0 *(*$Int(A)$*) which is the union of all its interior fuzzy points.*

(2) *A fuzzy point* $(x, r_x) \in (X, I)$ *is an exterior fuzzy point of the fuzzy subspace* A *if there exists an open fuzzy subspace* $U \in \mathcal{T}$ *containing* x*, such that* $U \cap A = \emptyset^F$*. The set of exterior fuzzy points of the fuzzy subspace* A *is denoted by* $Ext(A)$ *which is the union of all its exterior fuzzy points.*

(3) *A fuzzy point* $(x, r_x) \in (X, I)$ *is a fuzzy boundary point of the fuzzy subspace* A *if every open fuzzy subspace* $U \in \mathcal{T}$ *containing* x*, we have* $U \cap A \neq \emptyset^F$ *and* $U \cap \Big((X, I) - A\Big) \neq \emptyset^F$*. The set of boundary fuzzy points of the fuzzy subspace* A *is denoted by* $Bd(A)$ *which is the union of all its boundary fuzzy points.*

The definition tells us directly that $Int(A) \subset A$, $Ext(A) \subset (X, I) - A$, and the boundary fuzzy points of A may lie in either A or $(X, I) - A$. It also tells us that the three sets are pairwise disjoint. The closure, limit fuzzy point, and isolated fuzzy point in the fuzzy topological space $((X, I), \mathcal{T})$ are defined in the following definitions

Definition 2.8. *Let* $\Big((X, I), \mathcal{T}\Big)$ *be a fuzzy topological space on the fuzzy space* (X, I)*, and let* A *be a fuzzy subspace of the fuzzy space* (X, I) *then the closure of the fuzzy subspace* A *is the intersection of all closed fuzzy subspaces in* (X, I) *which contain* A*. It is denoted by* $\overline{A}$*.*

$\overline{A}$ is always a closed fuzzy subspace containing A.

Definition 2.9. *Let* $\Big((X, I), \mathcal{T}\Big)$ *be a fuzzy topological space on the fuzzy space* (X, I)*, and let* A *be a fuzzy subspace of the fuzzy space* (X, I) *then*

(1) *A fuzzy point* $(x, r_x) \in (X, I)$ *is a cluster point of* A *if every open fuzzy subspace containing* (x, r_x) *contains at least one point of* A *different from* (x, r_x)*.*

(2) *A point* $(x, r_x) \in A$ *is an isolated fuzzy point of* A *if there exists open fuzzy subspace* U *containing* (x, r_x) *such that* $U \cap A = (x, r_x)$

3. Fuzzy Bases

For a fuzzy topological space $((X, I), \mathcal{T})$ of the fuzzy space (X, I), it is often useful to describe $\mathcal{T}$ by defining a collection $\mathcal{B}$ of fuzzy subspaces of (X, I) which, in turn, yield the nonempty elements of $\mathcal{T}$ by forming all possible unions of members of $\mathcal{B}$. In so doing, we are able to investigate certain problems in the reduced collection $\mathcal{B}$ with out dealing directly with the fuzzy topological space, as well as to give some useful classifications of topologies. On the other hand, we see what restrictions must be placed on a collection of fuzzy subspaces of a fuzzy space (X, I) in order that the set of all possible unions of these fuzzy subspaces, together with $\emptyset^F$, form a fuzzy topology for (X, I).

Definition 3.1. *Let* $((X, I), \mathcal{T})$ *be a fuzzy topological space. A fuzzy base or basis for* $\mathcal{T}$ *is a collection* $\mathcal{B}$ *of fuzzy subspaces* (X, I) *such that*

(1) *Each fuzzy member of $\mathcal{B}$ is also a fuzzy member of $\mathcal{T}$.*

(2) *If U non-empty fuzzy subspace in $\mathcal{T}$, then U is the union of fuzzy subspaces of $\mathcal{B}$.*

Any fuzzy topology is a fuzzy base for itself, which means that any fuzzy topology always has at least one fuzzy base.

Example 3.1. *The collection $\mathcal{B}$ of all open intervals in $\mathbb{R}^1$ with membership I serves as a base for the standard topology on $\mathbb{R}^1$.*

Example 3.2. *Take $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset^F, (X, I), U_1, U_2, U_3\}$, where*
$U_1 = \{(a, I), (b, \{0, 0.5\})\}$
$U_2 = \{(b, \{0, 0.3\}), (c, I)\}$
$U_3 = \{(a, I), (b, \{0, 0.3, 0.5\}), (c, I)\}$
In addition to $\mathcal{T}$ itself, the only fuzzy bases for $\mathcal{T}$ are
$\mathcal{B}_1 = \{\emptyset^F, (X, I), U_1, U_2\}$,
$\mathcal{B}_2 = \{(X, I), U_1, U_2\}$.

The following theorem has many helpful applications in this section.

Theorem 3.1. *The fuzzy subspace U is open in the fuzzy topological space $((X, I), \mathcal{T})$ iff for each $(x, \mu_x) \in U$ there exists an open fuzzy subspace $V_{(x,\mu_x)}$ containing (x, μ_x) such that $V_{(x,\mu_x)} \subset U$.*

Proof. Suppose first that U is open fuzzy subspace in (X, I). Then for each $(x, \mu_x) \in U$, $V_{(x,\mu_x)} = U$ is an open fuzzy subspace containing (x, μ_x) such that $V_{(x,\mu_x)} \subset U$.

Conversely, suppose that for each $(x, \mu_x) \in U$ there exists an open set $V_{(x,\mu_x)}$ containing (x, μ_x) such that $V_{(x,\mu_x)} \subset U$. It follows that

$$\bigcup_{(x,\mu_x)\in U} V_{(x,\mu_x)} \subset U.$$

On the other hand, if $(x, \mu_x) \in U$, then (x, μ_x) belongs to one of the open sets $V_{(x,\mu_x)}$ for which $V_{(x,\mu_x)} \subset U$. Consequently,

$$(x, \mu_x) \in \bigcup_{(x,\mu_x)\in U} V_{(x,\mu_x)},$$

and we have shown

$$U = \bigcup_{(x,\mu_x)\in U} V_{(x,\mu_x)}.$$

Now each $V_{(x,\mu_x)}$ is open so that

$$\bigcup_{(x,\mu_x)\in U} V_{(x,\mu_x)}$$

is open, which shows that

$$U = \bigcup_{(x,\mu_x)\in U} V_{(x,\mu_x)}$$

is open. □

The following theorem shows how a fuzzy base for a fuzzy topology may be used in a rather natural way to determine whether or not a set is open.

Theorem 3.2. *Let $((X,I),\mathcal{T})$ be a fuzzy topological space and $\mathcal{B}$ a fuzzy base for $\mathcal{T}$. Then for a non-empty $U \subset (X,I)$ is open fuzzy subspace iff for each $(x,\mu_x) \in U$ there is a fuzzy element $B \in \mathcal{B}$ such that $(x,\mu_x) \in B \subset U$.*

Proof. Let $U \subset (X,I)$ be open fuzzy subspace and $(x,\mu_x) \in U$. Then by Definition 5.1 we have

$$U = \bigcup_{\alpha\in\Delta} B_\alpha$$

where $B_\alpha \in \mathcal{B}$ and Δ is some indexing set. Therefore, $(x,\mu_x) \in B_\alpha \subset U$ for at least one $\alpha \in \Delta$.
For the converse, suppose U be a fuzzy subspace of (X,I), such that for each $(x,\mu_x) \in U$ there is a $B \in \mathcal{B}$ such that $(x,\mu_x) \in B \subset U$. Since each $B \in \mathcal{B}$ is open fuzzy subspace, Theorem 3.1 gives U open. □

Theorem 3.3. *For a fuzzy space $((X,I),\mathcal{T})$, a collection $\mathcal{B} \subset \mathcal{T}$ is a fuzzy base for $\mathcal{T}$ iff for each non-empty fuzzy subspace $U \in \mathcal{T}$ and each $(x,\mu_x) \in U$ there is a $B \in \mathcal{B}$ such that $(x,\mu_x) \in B \subset U$.*

Proof. If $\mathcal{B}$ is a fuzzy basis, then by definition, any open set U can be written as

$$U = \bigcup_{\alpha\in\Delta} B_\alpha$$

where $B_\alpha \in \mathcal{B}$. If $(x,\mu_x) \in U$, then $(x,\mu_x) \in B_\alpha$ for some α. Thus, there exist

$$B_\alpha \in \mathcal{B} \quad \text{such that } (x,\mu_x) \in B_\alpha \subseteq U.$$

For the converse, if for every $(x,\mu_x) \in U$ there exists $B \in \mathcal{B}$ with $(x,\mu_x) \in B \subseteq U$, then

$$U = \bigcup_{(x,\mu_x)\in U} B_{(x,\mu_x)}.$$

Since U is the union of sets from $\mathcal{B}$, we get $\mathcal{B}$ is a fuzzy basis for the fuzzy topology. □

Theorem 3.4. *Let $\mathcal{B}$ be a collection of fuzzy subspaces of (X,I). Then there exists a fuzzy topology on (X,I) for which $\mathcal{B}$ is a fuzzy base iff*

a) *For each $(x,\mu_x) \in (X,I)$ there is at least one fuzzy subspace $U \in \mathcal{B}$ such that $(x,\mu_x) \in U$, and*
b) *if $U \in \mathcal{B}$, $V \in \mathcal{B}$, and $(x,\mu_x) \in U \cap V$, then there exists a $W \in \mathcal{B}$ such that $(x,\mu_x) \in W \subset U \cap V$.*

Proof. Let $\mathcal{B}$ be a fuzzy base for a fuzzy topology $\mathcal{T}$. Since each fuzzy members of $\mathcal{B}$ is also fuzzy members of $\mathcal{T}$, conditions (a) and (b) follow readily.
Conversely, suppose $\mathcal{B}$ is a collection of fuzzy subspaces of (X,I) for which (a) and (b) hold. Define $\mathcal{T}(\mathcal{B})$ as a set consisting of all possible unions of fuzzy members of $\mathcal{B}$ together with $\emptyset^F$. We establish that $\mathcal{T}(\mathcal{B})$ is a fuzzy topology for (X,I) having $\mathcal{B}$ as a fuzzy base. By (a) there is a fuzzy subspace $U_{(x,\mu_x)} \in \mathcal{B}$ containing (x,μ_x) for each $(x,\mu_x) \in (X,I)$ and, therefore, the definition of $\mathcal{T}(\mathcal{B})$ gives

$$\bigcup_{(x,\mu_x)\in(X,I)} U_{(x,\mu_x)} = (X,I) \in \mathcal{T}(B)$$

Since each fuzzy element of $\mathcal{T}(\mathcal{B})$ is the union of fuzzy members of $\mathcal{B}$, an arbitrary union of fuzzy elements in $\mathcal{T}(\mathcal{B})$ is again a union of members of $\mathcal{B}$ and, thus, belongs to $\mathcal{T}(\mathcal{B})$. Now we need only show that if U and V belong to $\mathcal{T}(\mathcal{B})$, then $U \cap V$ belongs to $\mathcal{T}(\mathcal{B})$. To this end, let $U = \bigcup\limits_{\alpha\in\Delta} B_\alpha$ and $V = \bigcup\limits_{\beta\in\Omega} C_\beta$ belong to $\mathcal{T}(\mathcal{B})$. Then $U \cap V = \bigcup\limits_{\alpha\in\Delta} B_\alpha \cap \bigcup\limits_{\beta\in\Omega} C_\beta = \bigcup\limits_{\substack{\alpha\in\Delta\\ \beta\in\Omega}} B_\alpha \cap C_\beta$. By (b), it is easy to see that each $B_\alpha \cap C_\beta$ is the union of members of $\mathcal{B}$ or is empty. Therefore, $U \cap V$ is a fuzzy member of $\mathcal{T}(\mathcal{B})$, and this completes the proof that $\mathcal{T}(\mathcal{B})$ is a fuzzy topology for (X, I). From the construction of $\mathcal{T}(\mathcal{B})$, $\mathcal{B}$ is a fuzzy base for $\mathcal{T}(\mathcal{B})$. □

Definition 3.2. *Two collections $\mathcal{B}_1$ and $\mathcal{B}_2$ of fuzzy subspace (X, I) are equivalent fuzzy bases iff there exists a fuzzy topology $\mathcal{T}$ for (X, I) such that $\mathcal{B}_1$ and $\mathcal{B}_2$ are both fuzzy bases for $\mathcal{T}$.*

Our last theorem give useful information about equivalent bases

Theorem 3.5. *Let $((X, I), \mathcal{T})$ be a fuzzy topological space, $\mathcal{B}_1$ a fuzzy base for $\mathcal{T}$ and $\mathcal{B}_2$ a collection of fuzzy subspaces of (X, I). Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent fuzzy bases iff*

a) *For each $U_1 \in \mathcal{B}_1$ and $(x, \mu_x) \in U_1$ there is a $U_2 \in \mathcal{B}_2$ such that $(x, \mu_x) \in U_2 \subset U_1$.*
b) *For each $U_2 \in \mathcal{B}_2$ and $(x, \mu_x) \in U_2$, there is a $U_1 \in \mathcal{B}_1$ such that $(x, \mu_x) \in U_1 \subset U_2$.*

Proof. By left hand side, consider $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent base.

(a) let U_1 be a fuzzy subspace of (X, I), and $U_1 \in \mathcal{B}_1$ then $U_1 \in \mathcal{T}$ and for each (x, μ_x) such that $(x, \mu_x) \in U_1$ there exists $U_2 \in \mathcal{B}_2$ such that $(x, \mu_x) \in U_2 \subset U_1$, by Theorem 3.2.
(b) since $U_2 \in \mathcal{B}_2$, then $U_2 \in \mathcal{T}$ for each $(x, \mu_x) \in U_2$ there exists $U_1 \in \mathcal{B}_1$ such that $(x, \mu_x) \in U_1 \subset U_2$, by Theorem 3.2.

By right hand side, Assume conditions (a) and (b) hold. We must show that $\mathcal{T}_{B_1} = \mathcal{T}_{B_2}$.

$\mathcal{T}_{B_1} \subseteq \mathcal{T}_{B_2}$: Let $V \in \mathcal{T}_{B_1}$. Then

$$V = \bigcup_{i\in I} U_{1,i}, \quad \text{where } U_{1,i} \in B_1.$$

For every fuzzy point $(x, \mu_x) \in V$, there is some $U_{1,k}$ such that $(x, \mu_x) \in U_{1,k}$. By condition (a), there exists $U_{2,x} \in B_2$ such that $(x, \mu_x) \in U_{2,x} \subseteq U_{1,k} \subseteq V$. Since V is the union of such $U_{2,x}$ for all its fuzzy points, $V \in \mathcal{T}_{B_2}$.

$\mathcal{T}_{B_2} \subseteq \mathcal{T}_{B_1}$: Let $W \in \mathcal{T}_{B_2}$. By condition (b), for every fuzzy point $(x, \mu_x) \in W$, there exists $U_{1,x} \in B_1$ such that $(x, \mu_x) \in U_{1,x} \subseteq W$. Thus, W is the union of elements from B_1, meaning $W \in \mathcal{T}_{B_1}$. Since both inclusions hold, the topologies are identical, and the bases are equivalent. □

4. Finite Product of Fuzzy Topological Space

For a finite family of fuzzy topological spaces $\big((X_1, I), \mathcal{T}_1\big)$, $\big((X_2, I), \mathcal{T}_2\big)$, $\ldots$, $\big((X_n, I), \mathcal{T}_n\big)$, there is a rather natural way to topologize the product $(X_1, I) \times (X_2, I) \times \cdots \times (X_n, I)$ by essentially producting the given topologies. This new fuzzy topology is called the product fuzzy topology.

Example 4.1. *For* $X = Y = \mathbb{R}$*, then* $(\mathbb{R}, I) \times (\mathbb{R}, I) = (\mathbb{R} \times \mathbb{R}, I \times I) = (\mathbb{R}^2, I^2)$*,* $I^2 = [0,1] \times [0,1]$*.*

The fuzzy space $(X \times Y, I^2)$ is the set of all $\Big((x,y), I^2)\Big)$, where the fuzzy element $\Big((x,y), I^2)\Big) = \{\Big((x,y), (\{r_x\}, \{r_y\})\Big) : x \in X, y \in Y$ for all possible membership values $r_x, r_y \in I$ such that r_x and r_y contain at least one value of $(0,1]$ besides $0\}$.
$\Big((x,y), (\{0\} \times \{0\})\Big) = \Big((x,y), (\{0\})\Big) \in \emptyset^F$ because its membership is zero.

Example 4.2. *Take* $X = \{a, b, c\}$*,* $\mathcal{T} = \{\emptyset^F, (X, I), U_1, U_2, U_3\}$*, where*
$U_1 = \{(a, I), (b, \{0, 0.5\})\}$
$U_2 = \{(b, \{0, 0.3\}), (c, I)\}$*,*
$U_3 = \{(a, I), (b, \{0, 0.3, 0.5\}), (c, I)\}$
then
$U_1 \times U_2 = \{\Big((a,b), (I \times \{0, 0.3\})\Big), \Big((a,c), (I \times I)\Big), \Big((b,b), (\{0, 0.5\} \times \{0, 0.3\})\Big), \Big((b,c), (\{0, 0.5\} \times I)\Big)\}$
Notice that $\Big((b,b), (\{0, 0.5\} \times \{0, 0.3\})\Big) = \Big((b,b), (\{0, 0.5\}, \{0, 0.3\})\Big) = \Big((b,b), (\{0, 0.3, 0.5)\})\Big)$ *because the first coordinate b identical to the second coordinate b, but if we have* $\Big((b,c), (\{0, 0.5\} \times \{0, 0.3\})\Big)$ *then* $\Big((b,c), (\{0, 0.5\} \times \{0, 0.3\})\Big) = \Big((b,c), (\{0, 0.5\}, \{0, 0.3\})\Big)$ *where the first coordinate b has the membership* $\{0, 0.5\}$*, and the second coordinate c has the membership* $\{0, 0.3\}$*.* $U_1 \times U_3 = \{\Big((a,a), (I \times I)\Big), \Big((a,b), (I \times \{0, 0.3, 0.5\})\Big), \Big((a,c), (I \times I)\Big), \Big((b,a), (\{0, 0.5\} \times I)\Big),$
$\Big((b,b), (\{0, 0.5\} \times \{0, 0.3, 0.5\})\Big), \Big((b,c), (\{0, 0.5\} \times I)\Big)\}$

Theorem 4.1. *Let* $\Big((X_1, I), \mathcal{T}_1\Big), \Big((X_2, I), \mathcal{T}_2\Big), \ldots, \Big((X_n, I), \mathcal{T}_n\Big)$ *be a finite collection of fuzzy topological spaces, and let* $\mathcal{B}$ *be the collection of all sets in* $(X_1, I) \times (X_2, I) \times \cdots \times (X_n, I)$ *of the form* $U_1 \times U_2 \times \cdots \times U_n$ *where* U_k *is an open fuzzy subspace set in* (X_k, I) *for* $k = 1, 2, \ldots, n$*. Then* $\mathcal{B}$ *is the base for a topology on* $(X_1, I) \times (X_2, I) \times \cdots \times (X_n, I)$*.*

Proof. We verify that conditions (a) and (b) of the theorem 3.4 hold for the collection $\mathcal{B}$

(a) Let $\Big((x_1, r_{x_1}), (x_2, r_{x_2}), \ldots, (x_n . r_{x_n})\Big) \in (X_1, I) \times (X_2, I) \times \cdots \times (X_n, I)$. Since each $(x_k, r_{x_k}) \in (X_k, I)$, and (X_k, I) is open in itself, we have

$$\Big((x_1, r_{x_1}), (x_2, r_{x_2}), \ldots, (x_n . r_{x_n})\Big) \in (X_1, I) \times (X_2, I) \times \cdots \times (X_n, I) \in \mathcal{B}.$$

(b) Let

$$U = U_1 \times U_2 \times \cdots \times U_n, \quad V = V_1 \times V_2 \times \cdots \times V_n$$

be elements of $\mathcal{B}$, and suppose

$$\Big((x_1, r_{x_1}), (x_2, r_{x_2}), \ldots, (x_n . r_{x_n})\Big) \in U \cap V.$$

Then

$$U \cap V = (U_1 \cap V_1) \times (U_2 \cap V_2) \times \cdots \times (U_n \cap V_n).$$

Let

$$W = (U_1 \cap V_1) \times (U_2 \cap V_2) \times \cdots \times (U_n \cap V_n).$$

Then $\Big((x_1, r_{x_1}), (x_2, r_{x_2}), \ldots, (x_n.r_{x_n})\Big) \in W \subseteq U \cap V$, and since each $U_k \cap V_k$ is open in (X_k, I), it follows that $W \in \mathcal{B}$. Thus condition (b) is satisfied. Hence, $\mathcal{B}$ is a base, and there exists a topology $\mathcal{T}$ on $(X_1, I) \times (X_2, I) \times \cdots \times (X_n, I)$ for which $\mathcal{B}$ is a base. □

Definition 4.1. *The topology $\mathcal{T}$ generated by the base $\mathcal{B}$ is called the **product topology** on $(X, I) = (X_1, I) \times (X_2, I) \times \cdots \times (X_n, I)$. The space $(X_1, I) \times (X_2, I) \times \cdots \times (X_n, I)$ with this topology is called the **product space**.*

5. Fuzzy Subbases

We define the idea of a fuzzy subbase for a fuzzy topology and then show that any set $\mathscr{S}$ of subspaces of (X, I) may be used to construct a fuzzy topology on (X, I) having $\mathscr{S}$ as a subbase.

Definition 5.1. *Let $\Big((X, I), \mathcal{T}\Big)$ be a fuzzy topological space. A subbase for $\mathcal{T}$ is a collection of subspaces $\mathscr{S}$ of (X, I) having the following properties:*

- (a) *Each element of $\mathscr{S}$ is also a member of $\mathcal{T}$.*
- (b) *The collection of all finite intersections of elements of $\mathscr{S}$, together with (X, I), forms a base for $\mathcal{T}$.*

Example 5.1. *For $\Big((\mathbb{R}, I), \mathcal{T}_{standard}\Big)$, the collection $\mathscr{S}$ of all open left rays $U^l = \{\Big((-\infty, l), u_x\Big) : u_x \subset I, x \in (-\infty, l), l \in \mathbb{R}\}$ together with all open right rays $U^r = \{\Big((r, \infty), u_x\Big) : u_x \subset I, x \in (r, \infty), r \in \mathbb{R}\}$ forms a subbase for the fuzzy standard topology.*

Theorem 5.1. *Let $\Big((X, I), \mathcal{T}\Big)$ be a fuzzy topology with a subbase $\mathscr{S} = \{S_\alpha : \alpha \in \Delta\}$, then the nonempty subspace U of (X, I) is open iff for each $(x, \mu_x) \in U$ there is a finite intersection $\cap_{k=1}^{n} S_{\alpha_k}$ such that $(x, \mu_x) \in \cap_{k=1}^{n} S_{\alpha_k} \subset U$.*

Proof. Let $U \subset (X, I)$ be open and $(x, \mu_x) \in U$. Then by Definition 5.1 we have

$$U = \bigcup_{\alpha \in \Delta} (\bigcap_{k=1}^{n} S_{\alpha_k})$$

for some indexing set Δ. Therefore $(x, \mu_x) \in \cap_{k=1}^{n} S_{\alpha_k} \subset U$ for some $n \in \mathbb{N}$, for some $\alpha \in \Delta$.
For the converse, suppose the subspace $U \subset (X, I)$ such that for each $(x, \mu_x) \in U$ there is a $\cap_{k=1}^{n} S_{\alpha_k}$ such that $(x, \mu_x) \in \cap_{k=1}^{n} S_{\alpha_k} \subset U$. Since $\cap_{k=1}^{n} S_{\alpha_k}$ is open, U is open. □

Theorem 5.2. *Let $X \neq \emptyset$ be a set, and let $\mathcal{S} = \{S_\alpha : \alpha \in \Delta\}$ be any collection of fuzzy subspaces of (X, I). Define $\mathcal{B}(\mathcal{S})$ to be the collection of all possible finite intersections of members of $\mathcal{S}$ together with (X, I). Then $\mathcal{B}(\mathcal{S})$ is a base for a topology on (X, I).*

Proof. We show conditions (a) and (b) of theorem 3.4 are satisfied by $\mathcal{B}(\mathcal{S})$. Since $(X, I) \in \mathcal{B}(\mathcal{S})$, condition (a) is clearly satisfied. To see condition (b), let $U = \bigcap_{k=1}^{n} S_{\alpha_k}$ and $V = \bigcap_{j=1}^{m} S_{\beta_j}$ be two of the finite intersections in $\mathcal{B}(\mathcal{S})$, and let $(x,\mu_x) \in U \cap V = \left(\bigcap_{k=1}^{n} S_{\alpha_k}\right) \cap \left(\bigcap_{j=1}^{m} S_{\beta_j}\right)$. Then (x,μ_x) belongs to each S_{α_k} and also to each S_{β_j} and, therefore, to the finite intersection of the sets

$$W = S_{\alpha_1} \cap S_{\alpha_2} \cap \cdots \cap S_{\alpha_n} \cap S_{\beta_1} \cap S_{\beta_2} \cap \cdots \cap S_{\beta_m}.$$

Consequently, we have demonstrated the existence of an element $W \in \mathcal{B}(\mathcal{S})$ such that $(x,\mu_x) \in W \subseteq U \cap V$, which shows that (b) holds. We have now shown that $\mathcal{B}(\mathcal{S})$ satisfies (a) and (b) of Theorem 3.4, and hence $\mathcal{B}(\mathcal{S})$ is a base for a topology on (X,I). □

6. CONCLUSION

In this paper, we review the concept of fuzzy topological spaces, which is based on the concept of a fuzzy space (X,I) and was first introduced by Dip in 1999. To improve clarity and coherence within the framework, we revise several basic terms, such as fuzzy space, fuzzy point, interior, exterior, boundary, cluster points, and isolated points. Additionally, we present the concepts of fuzzy bases and fuzzy subbases for a fuzzy topological space $\left((X,I),\mathcal{T}\right)$, supported by theorems and examples. These advances help us better understand fuzzy topology and provide a more organized foundation for future studies and applications in this area.

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