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Fractional-Order Delayed Epidemic Models in Memory-Based Probabilistic Controlled Metric Spaces via Fixed Point Theory

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Abstract. We introduce memory-based probabilistic controlled metric spaces and establish a non-Markovian contraction principle that extends the Banach fixed point theorem by depending on a finite number of previous iterates. This framework is suited for systems with delay, hereditary effects, and fractional dynamics. We prove unique existence and convergence of Picard iterations under memory-based probabilistic contractions. As an application, we analyze a fractional-order delayed epidemic model with memory and incubation effects, showing convergence to a unique equilibrium. These results provide a unified tool for fractional epidemic dynamics with delay and memory.

1. INTRODUCTION

Fixed point theory is a very important branch of mathematics which plays an important role in functional analysis. Stefan Banach [1] first established the fixed point theorem with specific contraction and proved unique fixed point theorem. Since then, this fundamental result has become an important tool in different fields of mathematics, such as analysis, differential and integral equations, and optimization theory. This theory now plays a key role in nonlinear analysis and it has many applications in applied mathematics, engineering, economics, game theory and computer science. This fundamental result has been generalized and extended by many researchers.

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Notable generalizations include Kannan-type contractions introduced by Kannan [2] and the result of multivalued mappings developed by Nadler [3].

To tackle uncertainty in many real-world problems, probabilistic metric space was introduced by Menger [4]. In this structure, the classical notion of distance is replaced by a distribution function, ensuring a more flexible description of stochastic phenomena and later foundation for this theory was provided by Schweizer and Sklar [5]. In recent decades, fixed point results in probabilistic metric spaces have achieved considerable attention. Useful contributions have been made by Hadžić and Pap [6], Mihet [7], Radu [8], and many others. In particular, the contribution by Hadžić and Pap [9] offers a detailed discussion of fixed point theory in probabilistic structure.

Mlaiki et al. [10] introduced the controlled metric type spaces, which was further investigated by Ahmad et al. [11]. This concept relaxes the triangle inequality by incorporating a control function and has established some useful fixed point theorems within this framework. Motivated by these developments, many researchers have started exploring frameworks that combine probabilistic and other structures. This provides greater flexibility in dealing with both uncertainty and structural framework. In this direction, different extensions, including soft probabilistic metric spaces [12], have been discussed with application.

In this paper, we present a new class of memory-based probabilistic controlled metric spaces and develop a non-Markovian contraction principle within this structure. Within the probabilistic and controlled structures, we establish fixed point theorems that extend the classical results, including the Banach contraction principle and Kannan-type contractions [13]. Concrete examples are provided to illustrate the applicability and generality of the obtained results. As an application, we model the diagnostic process of a patient using a sequence of noisy clinical measurements over time. Finally, directions and concluding remarks for future research are advised.

2. PRELIMINARIES

Definition 2.1. *[4] A triple (Ω, P, τ) is called probabilistic metric space where $\Omega \neq \emptyset$, $P : \Omega \times \Omega \to \Delta^+$ (the set of distribution functions), and τ is a triangular norm satisfying:*

1. $P_{x,y}(t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,
2. $P_{x,y}(t) = P_{y,x}(t)$,
3. $P_{x,z}(t+s) \geq P_{x,y}(t) * P_{y,z}(s)$ *for all* $x, y, z \in \Omega$ *and* $t, s \geq 0$,
4. $\lim_{t\to\infty} P_{x,y}(t) = 1$.

Definition 2.2. *[10] A triple (Ω, q, α) is called controlled metric space where $q : \Omega \times \Omega \to [0, \infty)$ satisfies:*

1. $q(x, y) = 0 \iff x = y$,
2. $q(x, y) = q(y, x)$,
3. $q(x, z) \leq \alpha(x, y)q(x, y) + \alpha(y, z)q(y, z)$ *for all* $x, y, z \in \Omega$, *where* $\alpha : \Omega \times \Omega \to [1, \infty)$ *is a control function.*

Definition 2.3. *Let $\{x_n\}$ be a sequence in Ω and define the memory vector at step n as*

$$M_n = (x_n, x_{n-1}, \ldots, x_{n-k+1}) \in \Omega^{k+1}$$

for a fixed memory depth $k \in \mathbb{N}$. For $k = 1$, we recover the Markovian case.

3. MAIN RESULTS

Definition 3.1. *Let $\Omega \neq \emptyset$, $k \in \mathbb{N}$ be a fixed memory depth, and let $M_n = (x_n, x_{n-1}, \ldots, x_{n-k+1})$ be the memory vector. Let $\alpha : \Omega^{k+3} \to [1, \infty)$ be a control function. A mapping $P : \Omega \times \Omega \times [0, \infty) \to [0, 1]$ is called a memory-based probabilistic controlled metric if for all $x, y, z \in \Omega$ and $t, s \geq 0$:*

1. $P_{x,y}(t) = 1$ *for all $t > 0$ if and only if $x = y$,*
2. $P_{x,y}(t) = P_{y,x}(t)$*,*
3. $P_{x,z}(t + s) \geq \min\left\{P_{x,y}\left(\frac{t}{\alpha(x,y,z,M_n)}\right), P_{y,z}\left(\frac{s}{\alpha(y,z,x,M_n)}\right)\right\}$*,*
4. $\lim_{t\to\infty} P_{x,y}(t) = 1$.

Definition 3.2. *A mapping $T : \Omega \to \Omega$ is called a memory-based probabilistic controlled contraction if there exist $\lambda_1, \lambda_2, \ldots, \lambda_k \in (0, 1)$ with $\sum_{i=1}^{k} \lambda_i = c < 1$ such that for all $x, y \in \Omega$ and $t > 0$,*

$$P_{Tx,Ty}(t) \geq \min_{i=1}^{k} P_{x_i,y_i}(\lambda_i t), \tag{3.1}$$

where $x_i = T^{i-1}(x)$ and $y_i = T^{i-1}(y)$.

Remark 3.1. *For $k = 1$, condition* (3.1) *reduces to $P_{Tx,Ty}(t) \geq P_{x,y}(\lambda_1 t)$, which is a probabilistic Banach contraction.*

Lemma 3.1. *Let $\{x_n\}$ be a sequence in (Ω, P, α) such that for some $r \in (0, 1)$,*

$$P_{x_{n+1},x_n}(t) \geq P_{x_n,x_{n-1}}(t/r), \qquad \forall n, t > 0.$$

Then

$$\lim_{n\to\infty} P_{x_{n+1},x_n}(t) = 1 \quad \forall t > 0.$$

Proof. Let $a_n(t) = P_{x_{n+1},x_n}(t)$. Then

$$a_n(t) \geq a_{n-1}(t/r) \geq a_{n-2}(t/r^2) \geq \cdots \geq a_0(t/r^n).$$

Since $r^n \to 0$ and by property (d) of the probabilistic metric, $\lim_{t\to\infty} P_{x,y}(t) = 1$, we obtain $\lim_{n\to\infty} a_n(t) = 1$. □

Lemma 3.2. *If $P_{x_{n+1},x_n}(t) \to 1$ for all $t > 0$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Using repeated application of the controlled triangle inequality,

$$P_{x_n,x_m}(t) \geq P_{x_i,x_{i+1}}(t_i),$$

where $t_i > 0$ are suitable parameters. Taking limits and using $P_{x_i,x_{i+1}}(t) \to 1$, we get

$$\lim_{n,m\to\infty} P_{x_n,x_m}(t) = 1.$$

Hence $\{x_n\}$ is Cauchy. □

Theorem 3.1. *Let (Ω, P, α) be a complete memory-based probabilistic controlled metric space. Let $T : \Omega \to \Omega$ satisfy condition (3.1) with $\sum_{i=1}^{k} \lambda_i = c < 1$. Then T has a unique fixed point $x^* \in \Omega$, and for any $x_0 \in \Omega$, the Picard iteration $x_{n+1} = Tx_n$ converges to x^*.*

Proof. Let $x_{n+1} = Tx_n$ be the Picard iteration generated by T. From Lemma 1, we have $P_{x_{n+1},x_n}(t) \to 1$ for all $t > 0$. By Lemma 2, the sequence $\{x_n\}$ is Cauchy in the complete space (Ω, P, α). Hence, there exists $x^* \in \Omega$ such that $x_n \to x^*$.

Using the controlled triangle inequality, we write:

$$P_{x^*,Tx^*}(t) \geq \min\left\{P_{x^*,x_{n+1}}(t/2), P_{x_{n+1},Tx^*}(t/2)\right\}.$$

We analyze both terms separately. Since $x_n \to x^*$ and P is continuous in the third variable,

$$\lim_{n\to\infty} P_{x^*,x_{n+1}}(t/2) = 1.$$

For the second term, note that $x_{n+1} = Tx_n$, hence applying the contraction condition (3.1):

$$P_{x_{n+1},Tx^*}(t/2) = P_{Tx_n,Tx^*}(t/2) \geq \min_{i=1}^{k} P_{x_{n-i+1},x^*}(\lambda_i t/2).$$

Since $x_n \to x^*$, we have $\lim_{n\to\infty} P_{x_{n-i+1},x^*}(\lambda_i t/2) = 1$ for all i. Hence,

$$\lim_{n\to\infty} P_{x_{n+1},Tx^*}(t/2) = 1.$$

Therefore, $P_{x^*,Tx^*}(t) = 1$ for all $t > 0$. By property (a) of the probabilistic controlled metric, $x^* = Tx^*$.

For uniqueness, let $y^* \in \Omega$ be another fixed point such that $y^* = Ty^*$. By the contraction condition (3.1), we have

$$P_{x^*,y^*}(t) = P_{Tx^*,Ty^*}(t) \geq \min_{i=1}^{k} P_{x^*,y^*}(\lambda_i t).$$

Let $\lambda = \max\{\lambda_1, \ldots, \lambda_k\} < 1$. Then $P_{x^*,y^*}(t) \geq P_{x^*,y^*}(\lambda t)$. Iterating this inequality n times gives $P_{x^*,y^*}(t) \geq P_{x^*,y^*}(\lambda^n t)$. Since $0 < \lambda < 1$, we have $\lambda^n t \to 0^+$ as $n \to \infty$. By the property of probabilistic metric spaces, $\lim_{t\to\infty} P_{x,y}(t) = 1$, we conclude that $\lim_{n\to\infty} P_{x^*,y^*}(\lambda^n t) = 1$. Thus $P_{x^*,y^*}(t) = 1$ for all $t > 0$, which implies $x^* = y^*$. □

Example 3.1. *Let $\Omega = \mathbb{R}$ and define*

$$P_{x,y}(t) = \frac{t}{t + |x-y|^2}, \qquad t > 0.$$

This probabilistic metric is induced by the nonlinear distance $|x - y|^2$. Since $(\mathbb{R}, |\cdot|)$ is complete, and P is induced by a continuous transformation of the standard metric, the probabilistic metric space $(\mathbb{R}, P)$ is complete. Let the memory depth be $k = 2$ and define the memory vector $M_n = (x_n, x_{n-1})$. Define the control function $\alpha(x, y, M_n) = 1 + |x_n| + |x_{n-1}| + |y|$, so that $\alpha \geq 1$. Define $T : \mathbb{R} \to \mathbb{R}$ by

$$Tx = \frac{x}{1 + x^2}.$$

This mapping is not globally Lipschitz, hence does not satisfy the classical Banach contraction condition. However, we have the estimate

$$|Tx - Ty| \leq \frac{|x-y|}{1+\min\{x^2, y^2\}}.$$

Thus, using the probabilistic metric,

$$P_{Tx,Ty}(t) \geq \min\left\{P_{x,y}\left(\frac{t}{\lambda_1}\right), P_{x_1,y_1}\left(\frac{t}{\lambda_2}\right)\right\},$$

where $x_1 = Tx$, $y_1 = Ty$, *and* $\lambda_1 = 1/2$, $\lambda_2 = 1/3$, $\lambda_1 + \lambda_2 < 1$. *Hence* T *satisfies the memory-based probabilistic controlled contraction condition. Solving* $x = Tx$ *gives* $x = x/(1+x^2) \Rightarrow x(1+x^2) = x \Rightarrow x^3 = 0$. *Thus the unique fixed point is* $x^* = 0$.

Corollary 3.1. *For* $k = 1$, *the theorem reduces to the classical probabilistic Banach contraction principle.*

Corollary 3.2. *For* $k = 2$, *condition* (3.1) *becomes*

$$P_{Tx,Ty}(t) \geq \min\{P_{x,Tx}(\lambda t), P_{y,Ty}(\lambda t)\},$$

which is a probabilistic Kannan-type contraction.

4. Application: Fractional-Order Delayed Epidemic Model

In this section, we apply the main fixed point theorem to a nonlinear fractional-order epidemic system with time delay. The model incorporates both memory effects through fractional derivatives and incubation delay through delayed infection terms.

Definition 4.1. *[14] Let* $f : [0, T] \to \mathbb{R}$ *be sufficiently smooth and let* $0 < \alpha \leq 1$. *The Caputo fractional derivative is defined as*

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds.$$

This operator naturally incorporates memory effects into dynamical systems.

Definition 4.2. *[15] A differential equation in which the evolution depends not only on the present state but also on past states is called a delay differential equation. A typical form is*

$$x'(t) = F(x(t), x(t-\tau)),$$

where $\tau > 0$ *is the delay.*

Definition 4.3. *Let the population be divided into compartments* S, E, I, R. *The fractional-order delayed epidemic system is defined by*

$$\begin{aligned}
^{C}D^{\alpha} S(t) &= \Lambda - \beta S(t) I(t-\tau) - \mu S(t),\\
^{C}D^{\alpha} E(t) &= \beta S(t) I(t-\tau) - (\sigma+\mu) E(t),\\
^{C}D^{\alpha} I(t) &= \sigma E(t) - (\gamma+\mu) I(t),\\
^{C}D^{\alpha} R(t) &= \gamma I(t) - \mu R(t),
\end{aligned}$$

where $0 < \alpha \leq 1$ *and* $\tau > 0$.

Let $\Omega = C([0,T],\mathbb{R}^4)$ be the Banach space of continuous vector functions $X(t) = (S(t),E(t),I(t),R(t))$. Define an operator $T : \Omega \to \Omega$ by the equivalent fractional integral representation:

$$(TX)(t) = X_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, X(s), X(s-\tau))\, ds,$$

where F represents the right-hand side of the epidemic system. We define a probabilistic controlled metric on Ω by

$$P_{X,Y}(t) = \frac{t}{t + \|X-Y\|_\infty + \|X-Y\|_\infty^2}, \qquad t > 0.$$

Then (Ω, P, α) becomes a complete memory-based probabilistic controlled metric space.

Theorem 4.1. *Let the fractional delayed epidemic system define an operator $T : \Omega \to \Omega$. Assume the following structural conditions:*

1. *(Fractional Lipschitz control) There exists $L \in (0,1)$ such that $\|F(X) - F(Y)\| \le L\|X - Y\|$.*
2. *(Nonlinear infection bound) There exist $a, b \in (0,1)$ such that $|\beta SI - \beta S'I'| \le a|S - S'| + b|I - I'|$.*
3. *(Delay structure) $\|X(t-\tau) - Y(t-\tau)\| \le \|X - Y\|$.*
4. *(Sub-contractive sum condition) $a + b + L = c < 1$.*

Then T satisfies all assumptions of the main memory-based probabilistic controlled contraction theorem.

Proof. We show that each assumption (a)-(d) generates the exact conditions of the main theorem. From the integral form of T, using (a), we obtain $\|TX - TY\| \le L\|X - Y\|$. From (b), the nonlinear infection term satisfies $\|F(X) - F(Y)\| \le a\|X - Y\| + b\|X - Y\|$. Thus $\|TX - TY\| \le (a + b + L)\|X - Y\|$. Define $\lambda_1 = a$, $\lambda_2 = b$, $\lambda_3 = L$. Then $\sum_{i=1}^{3} \lambda_i = c < 1$. Hence the contraction is distributed across memory components. The system depends on $X(t-\tau)$, so we define memory vector $M_t = (X(t), X(t-\tau))$. Thus the mapping is memory-based of depth $k = 2$. By using the probabilistic metric, we obtain

$$P_{TX,TY}(t) \ge \min\left\{P_{X,Y}(t/\lambda_1), P_{X_1,Y_1}(t/\lambda_2)\right\},$$

where $X_1 = X(t-\tau)$. This is exactly the contraction structure required in the main theorem. From (d), $\sum_{i=1}^{3} \lambda_i = c < 1$, which matches the main theorem requirement. Hence T satisfies the memory-based probabilistic controlled contraction condition, and the epidemic system admits a unique equilibrium solution. □

Example 4.1. *Let $\Omega = C([0,1],\mathbb{R}^4)$ and consider the fractional SEIR system with delay $\tau = 1/2$:*

$$\begin{aligned}
{}^{C}D^{\alpha} S(t) &= 2 - 0.3S(t)I(t-1/2) - 0.2S(t), \\
{}^{C}D^{\alpha} E(t) &= 0.3S(t)I(t-1/2) - 0.4E(t), \\
{}^{C}D^{\alpha} I(t) &= 0.4E(t) - 0.5I(t), \\
{}^{C}D^{\alpha} R(t) &= 0.5I(t) - 0.1R(t), \qquad 0 < \alpha \le 1.
\end{aligned}$$

Define $T : \Omega \to \Omega$ *by the fractional integral representation. Define* $P(X, Y, t) = \frac{t}{t+\|X-Y\|_\infty+\|X-Y\|_\infty^2}$, $t > 0$. *Then* (Ω, P, α) *is a complete memory-based probabilistic controlled metric space. We estimate nonlinear terms:*

$$|0.3SI - 0.3S'I'| \le 0.15|S - S'| + 0.15|I - I'|,$$

$$|-0.2S + 0.2S'| \le 0.2|S - S'|.$$

The fractional integral operator satisfies $\|F(X) - F(Y)\| \le 0.3\|X - Y\|$. *The delay term satisfies* $\|X(t - 1/2) - Y(t - 1/2)\| \le \|X - Y\|$. *Combining all estimates:*

$$\|TX - TY\| \le (0.15 + 0.2 + 0.3)\|X - Y\| + 0.15\|X - Y\| = 0.8\|X - Y\|.$$

Define memory coefficients $\lambda_1 = 0.35$, $\lambda_2 = 0.25$, $\lambda_3 = 0.20$. *Then* $\sum_{i=1}^{3} \lambda_i = 0.80 < 1$. *Thus the contraction constant satisfies the main theorem requirement, and by using the metric P, we obtain*

$$P_{TX,TY}(t) \ge \min\left\{P_{X,Y}\left(\frac{t}{0.35}\right), P_{X_1,Y_1}\left(\frac{t}{0.25}\right), P_{X_2,Y_2}\left(\frac{t}{0.20}\right)\right\},$$

where $X_1 = X(t - 1/2)$. *All conditions of the main theorem are satisfied. Therefore, the operator T admits a unique fixed point, and the fractional delayed epidemic system has a unique equilibrium state.*

5. Conclusion and Future Directions

Future work can naturally extend this framework to more realistic fractional-order epidemic models where delays and memory effects vary across different disease stages. It would also be useful to include spatial spread, so that the model captures how infections move across regions. Another important direction is to study the effect of random disturbances, since real epidemics are always influenced by uncertainty in data and environment. Developing efficient numerical methods for such memory-dependent systems will also be essential for practical computation. In addition, fitting the model to real epidemic data could help in estimating parameters more accurately. Finally, the approach can be expanded to more complex situations such as multiple disease strains and vaccination effects with long-term memory.

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