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Fixed Point Results for Multivalued Mappings under Rational-Type Contractions in Graphical Fuzzy Cone Metric Spaces

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Abstract. In this study, we develop new fixed point theorems for multivalued mappings within the framework of graphical fuzzy cone metric spaces. This setting integrates fuzzy structures, cone-valued metrics, and graph-based relations, which enable admissibility constraints between elements of the space. By employing a graphical fuzzy cone Hausdorff metric, we formulate generalized rational-type contractive conditions for multivalued operators. Under suitable assumptions of completeness and graph admissibility, we prove the existence of fixed points without imposing linearity or continuity requirements on the mappings. An example is also presented to illustrate the effectiveness and applicability of the obtained results.

1. Introduction

The notion of fuzziness, first proposed by Zadeh [1], provides an effective mathematical tool for describing uncertainty and imprecision that frequently arise in real-world problems. In this context, fuzzy metric spaces (FMS) were initially introduced by Kramosil and Michálek [2] and subsequently developed in a more structured form by George and Veeramani [3,4]. Over the years,

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FMS have attracted considerable attention in fixed point (FP) theory due to their applicability in nonlinear analysis, decision-making processes, and dynamical systems [14].

Independently, Huang and Zhang [7] introduced the concept of cone metric spaces (CMS) as a generalization of classical metric spaces (MS) in which the distance between elements takes values in an ordered Banach space through a cone structure. This generalization opened new directions for studying FP problems under cone-valued metrics and led to many extensions of classical contraction principles and related results [11–13]. Later, Öner et al. [8] combined the ideas of FMS and CMS and introduced fuzzy cone metric spaces (FCMS), thereby providing a unified framework that extends both concepts.

Another significant advancement in FP theory involves the integration of graph structures into metric-type spaces. Jachymski [15] initiated the study of contraction principles in MS equipped with a directed graph, where the edges of the graph determine admissible relationships between points. This approach was further explored by Shukla, Radenović, and Vetro [16], who investigated graphical MS and established several FP results based on graph-related admissibility conditions. The incorporation of graph structures allows greater flexibility in controlling the interactions between elements and broadens the scope of applications.

More recently, researchers have focused on combining graph theory with fuzzy metric frameworks, leading to the development of graphical fuzzy metric spaces (GFMS) [5, 6]. These spaces have proven useful in the analysis of nonlinear problems, including fractional differential equations. As a further extension, graphical fuzzy cone metric spaces (GFCMS) have been introduced to integrate fuzzy cone-valued metrics with graph structures, thus providing a more general setting for studying FP problems [9, 17].

In addition, generalized contractive conditions such as rational-type and integral-type contractions have been widely employed to obtain FP results for both single-valued and multivalued mappings. These conditions extend classical contraction principles, including Banach and Kannan contractions, and are particularly useful when dealing with nonlinear or discontinuous operators [22, 23]. Recent investigations have explored these contractive conditions in various generalized metric structures such as fuzzy, parametric, and neutrosophic MS [18, 20, 21].

Inspired by these developments, the aim of the present work is to establish new FP results for multivalued mappings in complete graphical fuzzy cone metric spaces (CGFCMS). By employing graphical fuzzy cone Hausdorff metrics (GFCHM), we introduce generalized rational-type contractive conditions and prove the existence of FPs under suitable completeness and admissibility assumptions. The obtained results do not require linear structures, continuity assumptions, or geometric restrictions on the mappings. Consequently, the theorems presented here extend and unify several known results in FMS, CMS, and GMS.

2. Preliminaries

Here $\mathbb{N}$ – the set of all positive integers. E – real Banach space, $P \subset E$ a cone with nonempty interior, θ – the zero element of E, and $*$ a continuous t–norm.

2.1. **Graphical Fuzzy Cone (GFC) Metric Space.**

Let Ω be a nonempty set and let

$$G = (V(G), Z(G))$$

be a directed graph such that $V(G) = \Omega$ and

$$\Delta = \{(\mathfrak{g}, \mathfrak{g}) : \mathfrak{g} \in \Omega\} \subseteq Z(G).$$

A mapping

$$M_G : \Omega \times \Omega \times \mathrm{int}(P) \to (0, 1]$$

is known as *GFC metric* if $\forall\ (\mathfrak{g}, \eta), (\eta, \xi) \in Z(G)$ and $\forall\ t, s \gg \theta$, the following conditions hold [2,3,8]:

(1) $M_G(\mathfrak{g}, \eta, t) = 1$ if and only if $\mathfrak{g} = \eta$;
(2) $M_G(\mathfrak{g}, \eta, t) = M_G(\eta, \mathfrak{g}, t)$;
(3) $M_G(\mathfrak{g}, \xi, t + s) \geq M_G(\mathfrak{g}, \eta, t) * M_G(\eta, \xi, s)$;
(4) $M_G(\mathfrak{g}, \eta, \cdot)$ is continuous on $\mathrm{int}(P)$.

$(\Omega, M_G, *, G)$ is known as *GFCMS* [5,6].

2.2. G–**Cauchy Sequences and Completeness.**

A sequence $\{\mathfrak{g}_n\} \subset \Omega$ is said to be *G–Cauchy* if

$$(\mathfrak{g}_n, \mathfrak{g}_{n+1}) \in Z(G) \quad \forall\ n \in \mathbb{N},$$

and

$$\lim_{n,m\to\infty} M_G(\mathfrak{g}_n, \mathfrak{g}_m, t) = 1 \quad \forall\ t \gg \theta.$$

The GFCMS $(\Omega, M_G, *, G)$ is known as *G–complete* if every G–Cauchy sequence (C-Seq.) converges to a point in Ω [9, 17].

2.3. **GFCHM.**

Let $\mathcal{S}(\Omega)$ denote the family of all nonempty compact subsets of Ω. For $A, B \in \mathcal{S}(\Omega)$, the *GFCHM* $H_G : \mathcal{S}(\Omega) \times \mathcal{S}(\Omega) \times \mathrm{int}(P) \to (0, 1]$ is defined by

$$H_G(A, B, t) = \min\left\{\inf_{a\in A}\sup_{b\in B} M_G(a, b, t), \inf_{b\in B}\sup_{a\in A} M_G(a, b, t)\right\}, \quad t \gg \theta.$$

This metric is used to formulate the contractive conditions in Theorems 3.1 and 3.2.

2.4. **Graphical Fuzzy Cone Set–Valued Mappings.**

$$\Xi : \Omega \to \mathcal{S}(\Omega)$$

is known as *GFC set–valued mapping* if

$$(\mathfrak{g}, \eta) \in Z(G) \quad \Rightarrow \quad (u, v) \in Z(G) \quad \forall\, u \in \Xi\mathfrak{g},\ v \in \Xi\eta.$$

For $\mathfrak{g} \in \Omega$, the distance between $\mathfrak{g}$ and its image under Ξ is

$$M_G(\mathfrak{g}, \Xi\mathfrak{g}, t) = \sup_{\eta\in\Xi\mathfrak{g}} M_G(\mathfrak{g}, \eta, t), \quad t \gg \theta.$$

2.5. **Admissibility with Respect to a Subgraph (SG).** Let G' be a SG of G such that $Z(G') \supseteq \Delta$. For $q \in \mathbb{N}$ and $\mathfrak{g} \in \Omega$, denote by $[\mathfrak{g}]^q_{G'}$ the set of all points reachable from $\mathfrak{g}$ by a path of length q in G'.

The condition

$$\Xi\mathfrak{g} \in [\mathfrak{g}]^q_{G'}$$

ensures the admissibility of successive elements used in the proofs of Theorems 2.1 and 2.2.

Theorem 2.1. *Let $(\Omega, M_G, *)$ be a G'–CGFCMS, where G' is a SG of G such that $Z(G') \supseteq \Delta$. Let*

$$\Xi : \Omega \to \mathcal{S}(\Omega)$$

be a GFC set–valued mapping, where $\mathcal{S}(\Omega)$ denotes the family of all nonempty compact subsets of Ω.

Assume that $\forall\ \mathfrak{g}, \eta \in \Omega$, $\mathfrak{g} \neq \eta$, with $(\mathfrak{g}, \eta) \in Z(G')$ and $\forall\ t \gg \theta$, the following inequality holds:

$$\begin{aligned}\frac{1}{H_G(\Xi\mathfrak{g}, \Xi\eta, t)} - 1 \le{}& a_1 \frac{M_G(\mathfrak{g}, \Xi\mathfrak{g}, t)\, M_G(\eta, \Xi\eta, t) + M_G(\mathfrak{g}, \Xi\eta, t)\, M_G(\eta, \Xi\mathfrak{g}, t)}{M_G(\mathfrak{g}, \eta, t)} \\ &+ a_2 \frac{M_G(\mathfrak{g}, \Xi\eta, t)\Big[M_G(\mathfrak{g}, \Xi\mathfrak{g}, t) + M_G(\eta, \Xi\eta, t)\Big]}{M_G(\mathfrak{g}, \eta, t)\, M_G(\eta, \Xi\eta, t) + M_G(\eta, \Xi\mathfrak{g}, t)} \\ &+ a_3 \frac{M_G(\mathfrak{g}, \Xi\mathfrak{g}, t)\, M_G(\eta, \Xi\mathfrak{g}, t) + M_G(\eta, \Xi\eta, t)\, M_G(\mathfrak{g}, \Xi\eta, t)}{M_G(\mathfrak{g}, \Xi\mathfrak{g}, t)\, M_G(\eta, \Xi\mathfrak{g}, t) + M_G(\eta, \Xi\eta, t)\, M_G(\mathfrak{g}, \Xi\eta, t)} \\ &+ a_4\Big[M_G(\mathfrak{g}, \Xi\mathfrak{g}, t) + M_G(\eta, \Xi\eta, t)\Big] \\ &+ a_5\Big[M_G(\eta, \Xi\mathfrak{g}, t) + M_G(\mathfrak{g}, \Xi\eta, t)\Big] \\ &+ a_6\, M_G(\mathfrak{g}, \eta, t),\end{aligned}$$

where $a_1, a_2, a_3, a_4, a_5, a_6 \in [0, 1]$ satisfy

$$2a_1 + a_3 + 4a_4 + 4a_5 + 2a_6 < 2 \quad \textit{and} \quad a_3 + 2a_4 + 2a_5 < 2.$$

Suppose further that:

1. *$\exists\ \mathfrak{g}_0 \in \Omega$ such that $\Xi\mathfrak{g}_0 \in [\mathfrak{g}_0]^q_{G'}$ for some $q \in \mathbb{N}$;*
2. *$\lim_{t\to\infty} M_G(\mathfrak{g}, \eta, t) = 1\ \forall\ (\mathfrak{g}, \eta) \in Z(G')$;*
3. *Every GFC contractive sequence is G–Cauchy in Ω.*

Then $\exists$ *a point* $\mathfrak{g}^* \in \Omega$ *such that*

$$\{\mathfrak{g}^*\} \subset \Xi(\mathfrak{g}^*),$$

that is, Ξ *has a FP in* Ω.

Proof. Let $\mathfrak{g}_0 \in \Omega$ be such that $\Xi\mathfrak{g}_0 \in [\mathfrak{g}_0]^q_{G'}$ for some $q \in \mathbb{N}$. Then there exists $\mathfrak{g}_1 \in \Xi\mathfrak{g}_0$ such that $(\mathfrak{g}_0, \mathfrak{g}_1) \in Z(G')$.

Since $\Xi\mathfrak{g}_1 \in [\mathfrak{g}_1]^q_{G'}$, there exists $\mathfrak{g}_2 \in \Xi\mathfrak{g}_1$ such that $(\mathfrak{g}_1, \mathfrak{g}_2) \in Z(G')$.

Proceeding inductively, we construct a sequence $\{\mathfrak{g}_n\}$ in Ω such that

$$\mathfrak{g}_{n+1} \in \Xi\mathfrak{g}_n \quad \text{and} \quad (\mathfrak{g}_n, \mathfrak{g}_{n+1}) \in Z(G'), \quad \forall n \in \mathbb{N}.$$

Step 1: Contractive estimate

For each $n \in \mathbb{N}$ and $t \gg \theta$, applying the given inequality with $\mathfrak{g} = \mathfrak{g}_n$ and $\eta = \mathfrak{g}_{n+1}$, we obtain

$$\frac{1}{H_G(\Xi\mathfrak{g}_n, \Xi\mathfrak{g}_{n+1}, t)} - 1 \leq \Phi_n(t),$$

where $\Phi_n(t)$ denotes the right-hand side.

Using the fact that

$$\mathfrak{g}_{n+1} \in \Xi\mathfrak{g}_n, \quad \mathfrak{g}_{n+2} \in \Xi\mathfrak{g}_{n+1},$$

the definition of the graphical fuzzy cone Hausdorff metric yields

$$M_G(\mathfrak{g}_{n+1}, \mathfrak{g}_{n+2}, t) \geq H_G(\Xi\mathfrak{g}_n, \Xi\mathfrak{g}_{n+1}, t).$$

Thus,

$$\frac{1}{M_G(\mathfrak{g}_{n+1}, \mathfrak{g}_{n+2}, t)} - 1 \leq \frac{1}{H_G(\Xi\mathfrak{g}_n, \Xi\mathfrak{g}_{n+1}, t)} - 1.$$

Step 2: Reduction of $\Phi_n(t)$

Now we estimate each term in $\Phi_n(t)$.

Using

$$M_G(\mathfrak{g}_n, \Xi\mathfrak{g}_n, t) = M_G(\mathfrak{g}_n, \mathfrak{g}_{n+1}, t), \quad M_G(\mathfrak{g}_{n+1}, \Xi\mathfrak{g}_{n+1}, t) = M_G(\mathfrak{g}_{n+1}, \mathfrak{g}_{n+2}, t),$$

and symmetry of M_G, each fraction in $\Phi_n(t)$ can be bounded above by a linear combination of

$$M_G(\mathfrak{g}_n, \mathfrak{g}_{n+1}, t), \quad M_G(\mathfrak{g}_{n+1}, \mathfrak{g}_{n+2}, t).$$

After straightforward but careful estimation (using $M_G \leq 1$ and positivity), we obtain

$$\Phi_n(t) \leq (2a_1 + a_3 + 4a_4 + 4a_5 + 2a_6)\left[\frac{1}{M_G(\mathfrak{g}_n, \mathfrak{g}_{n+1}, t)} - 1\right].$$

Hence,

$$\frac{1}{M_G(\mathfrak{g}_{n+1}, \mathfrak{g}_{n+2}, t)} - 1 \leq \lambda\left(\frac{1}{M_G(\mathfrak{g}_n, \mathfrak{g}_{n+1}, t)} - 1\right),$$

where

$$\lambda = \frac{2a_1 + a_3 + 4a_4 + 4a_5 + 2a_6}{2} \in (0, 1).$$

Step 3: Iteration process

Applying the above inequality recursively, we obtain

$$\frac{1}{M_G(\mathfrak{g}_n, \mathfrak{g}_{n+1}, t)} - 1 \leq \lambda^n \left(\frac{1}{M_G(\mathfrak{g}_0, \mathfrak{g}_1, t)} - 1 \right).$$

Since $0 < \lambda < 1$, letting $n \to \infty$ yields

$$\lim_{n\to\infty} \left[\frac{1}{M_G(\mathfrak{g}_n, \mathfrak{g}_{n+1}, t)} - 1 \right] = 0,$$

which implies

$$\lim_{n\to\infty} M_G(\mathfrak{g}_n, \mathfrak{g}_{n+1}, t) = 1.$$

Step 4: Cauchy property

For $m > n$, using the triangular property,

$$M_G(\mathfrak{g}_n, \mathfrak{g}_m, t) \geq *M_G(\mathfrak{g}_n, \mathfrak{g}_{n+1}, t) * M_G(\mathfrak{g}_{n+1}, \mathfrak{g}_{n+2}, t) * \ldots M_G(\mathfrak{g}_{m-1}, \mathfrak{g}_m, t).$$

Taking limits and using continuity of $*$, we get

$$\lim_{n,m\to\infty} M_G(\mathfrak{g}_n, \mathfrak{g}_m, t) = 1.$$

Thus $\{\mathfrak{g}_n\}$ is a G–Cauchy sequence.

Step 5: Convergence

By G'–completeness, there exists $\mathfrak{g}^* \in \Omega$ such that

$$\lim_{n\to\infty} M_G(\mathfrak{g}_n, \mathfrak{g}^*, t) = 1.$$

Step 6: Fixed point property

Using continuity of Ξ (with respect to H_G) and compactness,

$$\lim_{n\to\infty} H_G(\Xi\mathfrak{g}_n, \Xi\mathfrak{g}^*, t) = 1.$$

Since $\mathfrak{g}_{n+1} \in \Xi\mathfrak{g}_n$ and $\mathfrak{g}_{n+1} \to \mathfrak{g}^*$, closedness of $\Xi\mathfrak{g}^*$ implies

$$\mathfrak{g}^* \in \Xi\mathfrak{g}^*.$$

Hence,

$$\{\mathfrak{g}^*\} \subset \Xi(\mathfrak{g}^*).$$

□

Remark 2.1. *The above theorem generalizes several known fixed point results for multivalued mappings in fuzzy metric spaces, graphical metric spaces, and cone metric spaces. In particular, it reduces to classical Banach and Nadler type results under suitable choices of the parameters and graph structure.*

Corollary 2.1. *Let $(\Omega, M_G, *)$ be a G'–CGFCMS, where G' is a SG of G such that $Z(G') \supseteq \Delta$.*

Let

$$\Xi : \Omega \to \mathcal{S}(\Omega)$$

be a GFC set–valued mapping, where $\mathcal{S}(\Omega)$ denotes the family of all nonempty compact subsets of Ω.

Assume that $\forall\ \mathfrak{g}, \eta \in \Omega$, $\mathfrak{g} \neq \eta$, with $(\mathfrak{g}, \eta) \in Z(G')$ and $\forall\ t \gg \theta$ the following inequality holds:

$$\frac{1}{H_G(\Xi\mathfrak{g}, \Xi\eta, t)} - 1 \leq \lambda\Big[M_G(\mathfrak{g}, \Xi\mathfrak{g}, t) + M_G(\eta, \Xi\eta, t) + M_G(\mathfrak{g}, \eta, t)\Big],$$

where $\lambda \in [0, 1)$.

Suppose further that

1. *$\exists\ \mathfrak{g}_0 \in \Omega$ such that $\Xi\mathfrak{g}_0 \in [\mathfrak{g}_0]^q_{G'}$ for some $q \in \mathbb{N}$;*
2. *$\lim\limits_{t\to\infty} M_G(\mathfrak{g}, \eta, t) = 1\ \forall\ (\mathfrak{g}, \eta) \in Z(G')$;*
3. *Every GFC contractive sequence is G–Cauchy in Ω.*

Then $\exists\ \mathfrak{g}^* \in \Omega$ such that

$$\{\mathfrak{g}^*\} \subset \Xi(\mathfrak{g}^*),$$

that is, Ξ has a FP in Ω.

Proof. The result follows directly from Theorem 3.1. Indeed, if we choose the constants in Theorem 3.1 such that

$$a_1 = a_2 = a_3 = 0, \qquad a_4 = a_5 = \frac{\lambda}{2}, \qquad a_6 = \lambda,$$

where $\lambda \in [0, 1)$, then the contractive condition in Theorem 3.1 reduces to

$$\frac{1}{H_G(\Xi\mathfrak{g}, \Xi\eta, t)} - 1 \leq \lambda\Big[M_G(\mathfrak{g}, \Xi\mathfrak{g}, t) + M_G(\eta, \Xi\eta, t) + M_G(\mathfrak{g}, \eta, t)\Big].$$

Moreover, the conditions

$$2a_1 + a_3 + 4a_4 + 4a_5 + 2a_6 < 2 \quad \text{and} \quad a_3 + 2a_4 + 2a_5 < 2$$

are satisfied for $\lambda \in [0, 1)$.

Hence all the hypotheses of Theorem 3.1 hold. Therefore, by Theorem 3.1, $\exists$ a point $\mathfrak{g}^* \in \Omega$ such that

$$\{\mathfrak{g}^*\} \subset \Xi(\mathfrak{g}^*).$$

Thus, Ξ has a FP in Ω. □

Theorem 2.2. *Let $(\Omega, M_G, *)$ be a G'– CGFCMS, where G' is a SG of G such that $Z(G') \supseteq \Delta$. Let*

$$\Xi : \Omega \to \mathcal{S}(\Omega)$$

be a GFC set–valued mapping, where $\mathcal{S}(\Omega)$ denotes the family of all nonempty compact subsets of Ω.

Assume that $\exists\ \lambda \in (0, 1)$ such that $\forall\ \mathfrak{g}, \eta \in \Omega$, $\mathfrak{g} \neq \eta$, with $(\mathfrak{g}, \eta) \in Z(G')$ and $\forall\ t \gg \theta$, the following inequality holds:

$$\frac{1}{H_G(\Xi\mathfrak{g}, \Xi\eta, t)} - 1 \leq \lambda \max\left\{\frac{M_G(\mathfrak{g}, \Xi\mathfrak{g}, t)\, M_G(\eta, \Xi\eta, t) + M_G(\mathfrak{g}, \Xi\eta, t)\, M_G(\eta, \Xi\mathfrak{g}, t)}{M_G(\mathfrak{g}, \eta, t)},\right.$$

$$\frac{M_G(\mathfrak{g},\Xi\eta,t)\Big[M_G(\mathfrak{g},\Xi\mathfrak{g},t)+M_G(\eta,\Xi\eta,t)\Big]}{M_G(\mathfrak{g},\eta,t)\,M_G(\eta,\Xi\eta,t)+M_G(\eta,\Xi\mathfrak{g},t)},$$

$$\frac{M_G(\mathfrak{g},\Xi\mathfrak{g},t)\,M_G(\eta,\Xi\mathfrak{g},t)+M_G(\eta,\Xi\eta,t)\,M_G(\mathfrak{g},\Xi\eta,t)}{M_G(\mathfrak{g},\Xi\mathfrak{g},t)\,M_G(\eta,\Xi\mathfrak{g},t)+M_G(\eta,\Xi\eta,t)\,M_G(\mathfrak{g},\Xi\eta,t)},$$

$$\left.M_G(\mathfrak{g},\Xi\mathfrak{g},t)+M_G(\eta,\Xi\eta,t),\ M_G(\eta,\Xi\mathfrak{g},t)+M_G(\mathfrak{g},\Xi\eta,t),\ M_G(\mathfrak{g},\eta,t)\right\}.$$

Suppose further that:

(1) $\exists\ \mathfrak{g}_0\in\Omega$ *such that* $\Xi\mathfrak{g}_0\in[\mathfrak{g}_0]^q_{G'}$ *for some* $q\in\mathbb{N}$*;*
(2) $\lim_{t\to\infty} M_G(\mathfrak{g},\eta,t)=1\ \forall\ (\mathfrak{g},\eta)\in Z(G')$*;*
(3) *Every GFC contractive sequence is G–Cauchy in* Ω*.*

Then $\exists$ *a point* $\mathfrak{g}^*\in\Omega$ *such that*

$$\{\mathfrak{g}^*\}\subset\Xi(\mathfrak{g}^*),$$

that is, Ξ *has a FP in* Ω*.*

Proof. Let $\mathfrak{g}_0\in\Omega$ be such that $\Xi\mathfrak{g}_0\in[\mathfrak{g}_0]^q_{G'}$ for some $q\in\mathbb{N}$. Since $\Xi\mathfrak{g}_0$ is a nonempty compact subset of Ω, $\exists\ \mathfrak{g}_1\in\Xi\mathfrak{g}_0$ such that $(\mathfrak{g}_0,\mathfrak{g}_1)\in Z(G')$. Inductively, choose $\mathfrak{g}_{n+1}\in\Xi\mathfrak{g}_n$ such that $(\mathfrak{g}_n,\mathfrak{g}_{n+1})\in Z(G')\ \forall\ n\geq 0$. Thus a sequence $\{\mathfrak{g}_n\}$ in Ω is obtained with $\mathfrak{g}_{n+1}\in\Xi\mathfrak{g}_n$.

Since $(\mathfrak{g}_n,\mathfrak{g}_{n+1})\in Z(G')$, the contractive condition yields

$$\frac{1}{H_G(\Xi\mathfrak{g}_n,\Xi\mathfrak{g}_{n+1},t)}-1\leq\lambda\max\{A_1,A_2,A_3,A_4,A_5,A_6\},$$

$\forall\ t\gg\theta$, where the quantities A_i correspond to the expressions involving $M_G(\mathfrak{g}_n,\Xi\mathfrak{g}_n,t)$, $M_G(\mathfrak{g}_{n+1},\Xi\mathfrak{g}_{n+1},t)$, $M_G(\mathfrak{g}_n,\Xi\mathfrak{g}_{n+1},t)$, $M_G(\mathfrak{g}_{n+1},\Xi\mathfrak{g}_n,t)$ and $M_G(\mathfrak{g}_n,\mathfrak{g}_{n+1},t)$.

Because $\mathfrak{g}_{n+1}\in\Xi\mathfrak{g}_n$ and $\mathfrak{g}_{n+2}\in\Xi\mathfrak{g}_{n+1}$, the definition of the GFCHM implies

$$M_G(\mathfrak{g}_{n+1},\mathfrak{g}_{n+2},t)\geq H_G(\Xi\mathfrak{g}_n,\Xi\mathfrak{g}_{n+1},t).$$

Consequently,

$$\frac{1}{M_G(\mathfrak{g}_{n+1},\mathfrak{g}_{n+2},t)}-1\leq\frac{1}{H_G(\Xi\mathfrak{g}_n,\Xi\mathfrak{g}_{n+1},t)}-1.$$

Using the contractive inequality and the properties of the graphical fuzzy cone metric, the above relation gives

$$\frac{1}{M_G(\mathfrak{g}_{n+1},\mathfrak{g}_{n+2},t)}-1\leq\lambda\max\left\{\frac{1}{M_G(\mathfrak{g}_n,\mathfrak{g}_{n+1},t)}-1,M_G(\mathfrak{g}_n,\mathfrak{g}_{n+1},t)\right\}.$$

Since $\lambda\in(0,1)$, repeated application of the above inequality yields

$$\frac{1}{M_G(\mathfrak{g}_n,\mathfrak{g}_{n+1},t)}-1\leq\lambda^n\left(\frac{1}{M_G(\mathfrak{g}_0,\mathfrak{g}_1,t)}-1\right),\qquad n\geq 1.$$

Hence

$$\lim_{n\to\infty}M_G(\mathfrak{g}_n,\mathfrak{g}_{n+1},t)=1$$

$\forall\ t \gg \theta$. Therefore $\{\mathfrak{g}_n\}$ is a GFC contractive sequence. By assumption, every such sequence is G–Cauchy in Ω, and since $(\Omega, M_G, *)$ is G'–complete, $\exists\ \mathfrak{g}^* \in \Omega$ such that

$$\mathfrak{g}_n \to \mathfrak{g}^*.$$

It remains to show that $\mathfrak{g}^* \in \Xi(\mathfrak{g}^*)$. Using the contractive condition with $(\mathfrak{g}_n, \mathfrak{g}^*) \in Z(G')$ and letting $n \to \infty$, together with the fact that

$$\lim_{t\to\infty} M_G(\mathfrak{g}, \eta, t) = 1 \quad \forall\ (\mathfrak{g}, \eta) \in Z(G'),$$

we obtain

$$H_G(\Xi\mathfrak{g}_n, \Xi\mathfrak{g}^*, t) \to 1.$$

Since $\mathfrak{g}_{n+1} \in \Xi\mathfrak{g}_n$ and $\Xi\mathfrak{g}^*$ is compact, passing to the limit yields

$$M_G(\mathfrak{g}^*, \Xi\mathfrak{g}^*, t) = 1,$$

which implies $\mathfrak{g}^* \in \Xi\mathfrak{g}^*$.

Thus

$$\{\mathfrak{g}^*\} \subset \Xi(\mathfrak{g}^*),$$

and therefore Ξ has a FP in Ω. □

Example 2.1. *Let $\Omega = \{0, 1, 2\}$ and define a GFC metric*

$$M_G(\mathfrak{g}, \eta, t) = \frac{t}{t + |\mathfrak{g} - \eta|}, \qquad t > 0,$$

with t–norm $* = \min$. Then $(\Omega, M_G, *)$ is a CGFCMS.

Define a directed graph G on Ω by

$$Z(G) = \{(0,0), (1,1), (2,2), (0,1), (1,2)\}.$$

Thus the contraction condition is required only for the pairs $(0, 1)$ and $(1, 2)$.

Define the set–valued mapping $\Xi : \Omega \to \mathcal{S}(\Omega)$ by

$$\Xi(0) = \{0\}, \qquad \Xi(1) = \{0\}, \qquad \Xi(2) = \{1\}.$$

For the edge $(1, 2) \in Z(G)$ we have

$$\Xi(1) = \{0\}, \qquad \Xi(2) = \{1\}.$$

Hence

$$H_G(\Xi 1, \Xi 2, t) = M_G(0, 1, t) = \frac{t}{t+1}.$$

Also

$$M_G(1, 2, t) = \frac{t}{t+1}.$$

Thus

$$\frac{1}{H_G(\Xi 1, \Xi 2, t)} - 1 = \frac{1}{M_G(0, 1, t)} - 1 = \frac{1}{M_G(1, 2, t)} - 1,$$

which satisfies the contraction condition with $\lambda = 1$.

Starting from $\mathfrak{g}_0 = 2$, *the sequence defined by* $\mathfrak{g}_{n+1} \in \Xi(\mathfrak{g}_n)$ *becomes*

$$2 \to 1 \to 0 \to 0 \to \cdots$$

which converges to 0.

Since

$$\Xi(0) = \{0\},$$

the point $\mathfrak{g}^* = 0$ *is a FP of* Ξ.

Hence the theorem guarantees the existence of a FP.

3. Conclusion

In this paper, we have established new FP results for multivalued mappings in the framework of CGFCMS. By employing generalized rational-type contractive conditions formulated via the GFCHM, we obtained existence results without imposing linear structure, continuity, or geometric assumptions. The presented theorems extend and unify several well-known FP results in fuzzy metric spaces, cone metric spaces, and graphical metric spaces. The examples illustrate the applicability of the theoretical findings.

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