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Common Fixed Point Theorems for Asymptotically Quasi G-ϕ-Nonexpansive Mappings

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Abstract. In this research, we propose an iterative algorithm for approximating fixed points of asymptotically quasi G-ϕ-nonexpansive mappings in uniformly smooth and uniformly convex Banach spaces. We establish sufficient conditions for the existence of fixed points of these mappings. Furthermore, we prove the convergence of the sequence generated by the proposed iterative algorithm.

1. Introduction

Let C be a closed convex subset of a Banach space E. A point $x \in C$ is called a fixed point of a mapping $T : C \to C$ if

$$Tx = x.$$

We denote by $F(T) = \{x \in C : Tx = x\}$ the set of fixed points of T.

Many research in mathematics investigate the approximation of fixed points through sequences generated by iterative processes, as detailed in the following. In 2003, Nakajo and Takahashi [17] proposed a modified Mann iteration for a nonexpansive mapping T defined on a Hilbert space H with metric projection. Under appropriate conditions, they proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T. This iterative scheme is known as the CQ method.

In 2004, Matsushita and Takahashi [15] introduced an iterative method for a relatively nonexpansive mapping in a uniformly convex and uniformly smooth Banach space E with the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

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In 2005, Matsushita and Takahashi [16] introduced the hybrid iteration method with generalized projection for a relatively nonexpansive mapping T in a uniformly convex and uniformly smooth Banach space E. They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

In 2007, Plubtieng and Ungchittrakool [18] established strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space. They introduced a modified Halpern-type iteration to approximate a common fixed point in a uniformly smooth and strictly convex Banach space.

In 2010, Qin, Cho and Kang [19] established strong convergence theorems for a common fixed point of two asymptotically quasi-ϕ-nonexpansive mappings based on hybrid projection methods in a uniformly smooth and uniformly convex Banach space.

In 2021, Chidume and Adamu [7] introduced an iterative algorithm for solving the split equality fixed point problem involving quasi-ϕ-nonexpansive mappings in Banach spaces. They established strong convergence theorems for the proposed algorithm.

In 2025, Alhajaji et al. [4] established new fixed point theorems for these classes of mappings in F-metric spaces, which provided a natural extension of classical fixed point principles. Bataihah and Odat [5] established an algorithm that reformulated the maximum likelihood estimation (MLE) problem as a fixed point equation and developed a fixed point iteration method for the maximum likelihood estimation of the parameter θ in the Epanechnikov–Pareto distribution (EPD).

In recent years, fixed point theory has been extended to more general frameworks. In this direction, several researchers have investigated fixed point results in graph theory, including metric space, Banach space and Hilbert space. This framework has been used to study the convergence behavior of various classes of mappings under appropriate conditions. Some related results are summarized below.

In 2008, Jachymski [12] proved some generalizations of the Banach Contraction Principle for mappings on metric spaces endowed with a graph and applied a theorem on the convergence of successive approximations for some linear operators on Banach spaces.

In 2015, Tiammee, Kaewkhao and Suantai [26] proved Browder's convergence theorem for G-nonexpansive mappings in a Hilbert space endowed with a directed graph and established strong convergence of the Halpern iteration process to a fixed point of G-nonexpansive mappings in a Hilbert space endowed with a directed graph.

In 2017, Suparatulatorn, Suantai and Cholamjiak [24] proved a strong convergence theorem for two different hybrid methods by using CQ method for a finite family of G-nonexpansive mappings in a Hilbert space.

In 2018, Suparatulatorn, Cholamjiak and Suantai [23] proved the weak and strong convergence of a sequence generated by a modified S-iteration process for finding a common fixed point of two $G-$nonexpansive mappings in uniformly convex Banach spaces endowed with a graph.

In 2019, Saewan, Noisri and Kanjanasamranwong [21] introduced a new algorithm for finding a common fixed point of G-nonexpansive mappings in a Banach space.

In 2020, Dung and Hieu [10] studied the convergence of a three-step iteration process in Banach spaces with a directed graph.

In 2021, Wongsaijai et al. [28] investigated h-ϕ-contraction mappings with two metrics endowed with a directed graph involving auxiliary functions and applications to the existence of solutions for Caputo fractional boundary value problems with integral boundary conditions.

In 2024, Hieu and Huy [11] introduced two inertial hybrid iteration processes for finding common fixed points of two asymptotically quasi G-ϕ-nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces endowed with directed graphs.

In 2025, Yambangwai and Thianwan [30] presented a new computational approach named the parallel inertial three-step iteration monotone hybrid algorithm (abbreviated as PITMHA), devised for addressing the common fixed point problem of a finite family of G-nonexpansive mappings in Hilbert spaces endowed with graphs. They established a theorem demonstrating weak convergence for PITMHA.

The study of fixed point theorems for G-nonexpansive mappings in Hilbert and Banach spaces was investigated by many authors (see [2,3,6,14,26,27]).

Motivated by the above research, we propose in this research a new iterative scheme for asymptotically quasi G-ϕ-nonexpansive mappings in a uniformly smooth and uniformly convex Banach space E endowed with a directed graph. We then establish a strong convergence theorem under appropriate conditions.

2. Preliminaries

Let E be a Banach space with the norm $\|\cdot\|$ and $B(0) = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for any $x, y \in B(0)$. A Banach space E is said to be uniformly smooth if the limit (2.1) is attained uniformly for (x, y) in $B(0) \times B(0)$.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$ ([25]). The modulus of convexity of E is the function $\delta : [0,2] \to [0,1]$ defined by $\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\}$. A Banach space E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Lemma 2.1. *([8]) Let E be a uniformly convex Banach space and $B_r(0)$ a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \lambda\|z\|^2 - \lambda\mu g\|x-y\|$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.

For a sequence $\{x_n\}$ in a Banach space E and a point $x \in E$, weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$. Strong convergence of $\{x_n\}$ to x is denoted by $x_n \to x$.

A mapping T from C into itself is said to be *closed* if, for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, we have

$$Tx_0 = y_0.$$

Let T be a nonlinear mapping, T is said to be *uniformly asymptotically regular* on C if

$$\lim_{n\to\infty}\left(\sup_{x\in C}\|T^{n+1}x - T^n x\|\right) = 0.$$

Let C be a closed convex subset of a Banach space E, a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

As well know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive.

Let E be a Banach space with dual E^* and $\langle\cdot,\cdot\rangle$ be the pairing between E and E^*. The normalized duality mapping

$$J : E \to 2^{E^*}$$

is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2, \|x^*\| = \|x\|\}, \text{ for all } x \in E.$$

Let J be the normalized duality mapping. The basic properties of E, J, and J^{-1} (see [9]) are as follows.

- If E is a Banach space, then J is monotone and bounded.
- If E is strictly convex, then J is strictly monotone.
- If E is smooth, then J is single-valued and semi-continuous.
- If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
- If E is a reflexive, smooth and strictly convex Banach space, then the normalized duality mapping J is single-valued, one-to-one and onto.
- If E is a reflexive, strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-valued, bijective, and is also the duality mapping from E^* into E. Moreover, $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$.
- If E is uniformly smooth, then E is smooth and reflexive.

The functional $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.2}$$

It is obvious that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \tag{2.3}$$

which implies that

$$\phi(x, y) \geq 0.$$

Remark 2.1. *If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.*

The generalized projection ([1]), $\Pi_C : E \to C$, is a mapping that assigns to an arbitrary point $x \in E$ the minimizer of $\phi(x, y)$, that is,

$$\Pi_C x = \bar{x},$$

where $\bar{x}$ is the solution of the minimization problem

$$\phi(\bar{x}, y) = \inf_{x \in C} \phi(x, y). \tag{2.4}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and the strict monotonicity of the mapping J (see [1,9,25]).

If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$, and Π_C becomes the metric projection of E onto C.

Lemma 2.2. *([13]). Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.3. *([1]). Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that*

$$\Pi_C x = x_0.$$

Lemma 2.4. *([1]). Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.5. *([1]). Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Definition 2.1. *Let C be a nonempty, closed and convex subset of a real normed space E. A mapping $T : C \to C$ is said to be quasi ϕ-nonexpansive if*

(1) *$F(T) \neq \emptyset$*
(2) *for any $p \in F(T)$ and $x \in C$*

$$\phi(p, Tx) \leq \phi(p, x).$$

Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E. Then Π_C is a closed quasi ϕ-nonexpansive mapping from E onto C, with $F(\Pi_C) = C$ [20].

Let $G = (V(G), A(G))$ be a directed graph, where $V(G)$ is the set of vertices of G and $A(G)$ is the set of directed edges (x, y) of G, where $x, y \in V(G)$. In this paper, we assume that the directed graph G has no parallel edges.

A weighted directed graph is a directed graph in which a number is assigned to each directed edge. The conversion G^{-1} of a directed graph G is the directed graph obtained from G by reversing the direction of the edges, that is,

$$A(G^{-1}) = \{(x, y) : (y, x) \in A(G)\}.$$

A path in a directed graph G is a sequence of vertices connected by edges, where each edge belongs to G and links consecutive vertices in the sequence, with no vertex repeated.

Let x and y be vertices of a directed graph G. A path in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$, and $(x_i, x_{i+1}) \in A(G)$ for $i = 0, 1, \dots, n-1$.

A directed graph G is said to be connected if, for any two vertices x and y in G, there exists a path from x to y in G.

A directed graph G is said to be transitive if, for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $A(G)$, then $(x, z) \in A(G)$.

Definition 2.2. *([26]). Let C be a nonempty subset of a normed space X and $G = (V(G), A(G))$ be a directed graph such that $V(G) = C$. Then C is said to have property G if for every sequence $\{x_n\}$ in C converging weakly to $x \in C$ and $(x_n, x_{n+1}) \in A(G)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in A(G)$ for all $k \in \mathbb{N}$.*

Remark 2.2. *([29]) If G is transitive, then Property G is equivalent to the property: if $\{x_n\}$ is a sequence in C with $(x_n, x_{n+1}) \in A(G)$ such that for any subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ converging weakly to x in X, then $(x_n, x) \in A(G)$ for all $n \in \mathbb{N}$.*

Definition 2.3. *([10]). Let X be a vector space and $G = (V(G), A(G))$ be a directed graph such that $A(G) \subset X \times X$. Then $A(G)$ is said to be coordinate-convex if for all (u, a), (u, b), (a, u), $(b, u) \in A(G)$ and for all $\lambda \in [0, 1]$ we have*

$$\lambda(u, a) + (1-\lambda)(u, b) \in A(G) \quad \text{and} \quad \lambda(a, u) + (1-\lambda)(b, u) \in A(G).$$

Remark 2.3. *If $A(G)$ is convex, then $A(G)$ is coordinate-convex.*

Let C be a nonempty subset of a real Banach space E. Consider a directed graph G with $V(G) = C$ and the mapping $T : C \to C$ such that $F(T) \times F(T) \subset A(G)$.

Definition 2.4. *([10]). Let E be a real smooth Banach space and $G = (V(G), A(G))$ be a directed graph such that $V(G) = C$.*

A mapping $T : V(G) \to V(G)$ is said to be a quasi G-ϕ-nonexpansive mapping if

(1) *$F(T) \neq \emptyset$*

(2) *for any $p \in F(T)$ and $v \in V(G)$,*

$$(p, v) \in A(G) \Rightarrow (p, Tv) \in A(G),$$

(3) *for any* $p \in F(T)$ *and* $v \in V(G)$,

$$(p,v) \in A(G) \Rightarrow \phi(p,Tv) \leq \phi(p,v).$$

Example 2.1. *([11]). Assume that*

(1) $l_4 = \left\{u = \{u_n\} \subset \mathbb{K} : \sum_{n=1}^{\infty} |u_n|^4 < \infty\right\}$ *is a Banach space with norm* $\|u\|_4 = \sqrt[4]{\sum_{n=1}^{\infty} |u_n|^4}$ *for all* $u = \{u_n\} \in l_4$.

(2) $G = (V(G), A(G))$ *is a directed graph with* $V(G) = l_4$ *and*

$$A(G) = \{(u,v) \in l_4 \times l_4 : u \neq v \in C \text{ or } u = v \in l_4\}$$

where $C = \{u = \{u_n\} \in l_4 : u_n \in [0,1],\ \forall n \in \mathbb{N}$.

For all $u = \{u_n\} \in l_4$, *define a mapping* $T : l_4 \to l_4$ *by*

$$Tu = \begin{cases} \left\{\dfrac{u_n}{2^{n-1}}\right\}, & \text{if } u \in C \\ \left\{\dfrac{1}{\sqrt[4]{2^n}}\right\}, & \text{if } u \notin C. \end{cases}$$

Note that T *is a quasi* G-ϕ*-nonexpansive mapping.*

Definition 2.5. *([10]). Let* E *be a real smooth Banach space and* $G = (V(G), A(G))$ *be a directed graph such that* $V(G) \subset E$. *A mapping* $T : V(G) \to V(G)$ *is said to be an asymptotically quasi* G-ϕ*-nonexpansive mapping if*

(1) *for any* $p \in F(T)$ *and* $v \in V(G)$,

$$(p,v) \in A(G) \Rightarrow (p,Tv) \in A(G),$$

(2) *there exists a sequence* $\{\lambda_n\} \subset [1,\infty)$ *with* $\lim_{n\to\infty} \lambda_n = 1$,

$$(p,v) \in A(G) \Rightarrow \phi(p,T^n v) \leq \lambda_n \phi(p,v),$$

for any $p \in F(T)$ *and* $v \in V(G)$, *where* $\{\lambda_n\}$ *is said to be an asymptotic coefficient sequence.*

Lemma 2.6. *([11]). Let* E *be a uniformly convex and uniformly smooth Banach space* E *and* $G = (V(G), A(G))$ *be a transitive directed graph such that* $E = V(G)$. *Let* $T : V(G) \to V(G)$ *is an asymptotically quasi* G-ϕ*-nonexpansive mapping with* $\{\lambda_n\} \subset [1,\infty)$ *satisfying* $\lim_{n\to\infty} \lambda_n = 1$ *and* $F(T) \times F(T) \subset A(G)$.

(1) *If* E *has property* G *then* $F(T)$ *is closed.*
(2) *If* $A(G)$ *is coordinate-convex then* $F(T)$ *is convex.*

3. Theorems

Theorem 3.1. *Let* E *be a uniformly smooth and uniformly convex Banach space with property* G *and* C *be a nonempty closed and convex subset of* E. *Let* $G = (V(G), A(G))$ *be a transitive directed graph such that* $V(G) = C$ *and* $A(G)$ *is coordinate-convex. The mappings* $T, S : V(G) \to V(G)$ *are closed uniformly asymptotically regular and asymptotically quasi* G-ϕ*-nonexpansive mapping.*

Assume that $F = F(T) \cap F(S)$ is nonempty subset of C. For an initial point $x_1 \in C$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = J^{-1}(a_n Jx_n + b_n J(T^n x_n) + c_n J(S^n x_n)), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + (\lambda_n - 1)B_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where

(1) $\lambda_n = \max\{\lambda_n^T, \lambda_n^S\}$,
(2) $B_n = \sup\{\phi(w, x_n) : w \in F\}$,
(3) $(q, x_n) \in A(G)$, *for* $q \in F$,
(4) $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are real sequences in* $[0, 1]$.

If the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy the following conditions

(1) $\liminf_{n\to\infty} b_n c_n > 0$,
(2) $\lim_{n\to\infty} a_n = 0$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of F.

Proof. We split the proof into six steps as follows:
Step 1. We show that C_{n+1} is closed and convex for all $n \geq 1$.

Clearly, $C_1 = C$ is closed and convex. Suppose that C_j is closed and convex for integer $j \geq 1$. For any $z \in C_j$, we know that $\phi(z, y_j) \leq \phi(z, x_j) + (\lambda_j - 1)B_j$ is equivalent to the following:

$$2\langle z, Jx_j - Jy_j\rangle \leq \|x_j\|^2 - \|y_j\|^2 + (\lambda_j - 1)B_j.$$

We have C_{j+1} is closed and convex. Then C_n is closed and convex for all $n \geq 1$.
Step 2. We show that $F \subset C_n$ for all $n \geq 1$.

Since $F = F(T) \cap F(S) \neq \emptyset$ and $T, S : V(G) \to V(G)$ with $V(G) = C$, we have $F \subset C_1 = C$. Suppose that $F \subset C_j$ for integer $j \geq 1$. Let $q \in F$ and $(q, x_j) \in A(G)$, since T and S are uniformly asymptotically regular and asymptotically quasi G-ϕ-nonexpansive mappings, it follows that

$$\begin{aligned} \phi(q, y_j) &= \phi(q, J^{-1}(a_j Jx_j + b_j J(T^j x_j) + c_j J(S^j x_j)) \\ &= \|q\|^2 - 2\langle q, a_j Jx_j + b_j J(T^j x_j) + c_j J(S^j x_j)\rangle + \|a_j Jx_j + b_j J(T^j x_j) + c_j J(S^j x_i)\|^2 \\ &\leq \|q\|^2 - 2a_j\langle q, Jx_j\rangle - 2b_j\langle q, J(T^j x_j)\rangle - 2c_j\langle q, J(S^j x_j)\rangle \\ &+ a_j\|Jx_j\|^2 + b_j\|J(T^j x_j)\|^2 + c_j\|J(S^j x_j)\|^2 \\ &= a_j\phi(q, x_j) + b_j\phi(q, T^j x_j) + c_j\phi(q, S^j x_j) \\ &\leq a_j\phi(q, x_j) + b_j\lambda_j\phi(q, x_j) + c_j\lambda_j\phi(q, x_j) \\ &\leq \phi(q, x_j) + (\lambda_j - 1)\phi(q, x_j). \end{aligned} \tag{3.2}$$

This shows that $q \in C_{j+1}$, which implies that $F \subset C_{j+1}$. Hence $F \subset C_n$ for all $n \geq 1$.
Step 3. We show that $\{x_n\}$ is bounded.

By the definition of C_{n+1} with $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, it follows that

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \forall n \geq 1. \tag{3.3}$$

From Lemma 2.5, we get

$$\begin{aligned} \phi(x_n, x_1) &= \phi(\Pi_{C_n} x_1, x_1) \\ &\leq \phi(q, x_1) - \phi(q, x_n) \\ &\leq \phi(q, x_1), \quad \forall q \in F. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), then $\{\phi(x_n, x_1)\}$ are nondecreasing and bounded. So, we obtain that $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. In particular, by (2.3), the sequence $\{(\|x_n\| - \|x_1\|)^2\}$ is bounded. This implies $\{x_n\}$ is bounded. We have $\{y_n\}$ is also bounded.
Since T is asymptotically quasi G-ϕ-nonexpansive mapping and $(q, x_n) \in A(G)$ for $q \in F$, from (2.3), we have

$$(\|q\| - \|T^n x_n\|)^2 \leq \phi(q, T^n x_n) \leq \lambda_n \phi(q, x_n) \leq \lambda_n (\|q\| + \|x_n\|)^2. \tag{3.5}$$

From boundedness of $\{x_n\}$ and $\{\lambda_n\}$, we have $\{T^n x_n\}$ is bounded.
Again, since S is asymptotically quasi G-ϕ-nonexpansive mapping and $(q, x_n) \in A(G)$ for $q \in F$, from (2.3), we have

$$(\|q\| - \|S^n x_n\|)^2 \leq \phi(q, S^n x_n) \leq \lambda_n \phi(q, x_n) \leq \lambda_n (\|q\| + \|x_n\|)^2. \tag{3.6}$$

From boundedness of $\{x_n\}$ and $\{\lambda_n\}$, we have $\{S^n x_n\}$ is bounded.

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Since $x_m = \Pi_{C_m} x_1 \in C_m \subset C_n$, for $m > n$, by Lemma 2.5, we have

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_1) \\ &\leq \phi(x_m, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1). \end{aligned}$$

Taking $m, n \to \infty$, we have $\phi(x_m, x_n) \to 0$. From Lemma 2.2, we get $\|x_n - x_m\| \to 0$. Thus $\{x_n\}$ is a Cauchy sequence and by the completeness of a Banach space E and the closedness of C, we can assume that there exist $p \in C$ such that

$$\lim_{n\to\infty} x_n = p. \tag{3.7}$$

Step 4. We show that $\lim_{n\to\infty} \|y_n - x_n\| = 0$ and $\lim_{n\to\infty} \|Jy_n - Jx_n\| = 0$.

By definition of $\Pi_{C_n} x_1$ and Lemma 2.5, we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_1) \\ &\leq \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_{n+1}, x_1) - \phi(x_n, x_1). \end{aligned}$$

Since $\lim_{n\to\infty} \phi(x_n, x_1)$ exists, we also have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.8}$$

Form Lemma 2.2, that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + (\lambda_n - 1)B_n.$$

It follows from (3.8) and $\lim_{n\to\infty} \lambda_n = 1$, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{3.10}$$

Form Lemma 2.2, that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.11}$$

By using the triangle inequality, we have

$$\|y_n - x_n\| = \|y_n - x_{n+1} + x_{n+1} - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

From (3.9) and (3.11), it follows that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.12}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also have

$$\lim_{n\to\infty} \|Jy_n - Jx_n\| = 0. \tag{3.13}$$

Step 5. We show that $p \in F = F(T) \cap F(S)$.

Since $\lim\limits_{n\to\infty} x_n = p$ and from (3.12), we get

$$\lim_{n\to\infty} y_n = p. \tag{3.14}$$

From Lemma 2.1, it follows that

$$\begin{aligned}
\phi(q, y_n) &= \phi(q, J^{-1}(a_n Jx_n + b_n J(T^n x_n) + c_n J(S^n x_n)) \\
&= \|q\|^2 - 2\langle q, a_n Jx_n + b_n J(T^n x_n) + c_n J(S^n x_n)\rangle + \|a_n Jx_n + b_n J(T^n x_n) + c_n J(S^n x_n)\|^2 \\
&\leq \|q\|^2 - 2a_n\langle q, Jx_n\rangle - 2b_n\langle q, J(T^n x_n)\rangle - 2c_n\langle q, J(S^n x_n)\rangle \\
&\quad + a_n\|Jx_n\|^2 + b_n\|J(T^n x_n)\|^2 + c_n\|J(S^n x_n)\|^2 - b_n c_n g(\|J(T^n x_n) - J(S^n x_n) \\
&= a_n\phi(q, x_n) + b_n\phi(q, T^n x_n) + c_n\phi(q, S^n x_n) - b_n c_n g\|J(T^n x_n) - J(S^n x_n)\| \\
&\leq a_n\phi(q, x_n) + b_n\lambda_n\phi(q, x_n) + c_n\lambda_n\phi(q, x_n) - b_n c_n g\|J(T^n x_n) - J(S^n x_n)\| \\
&\leq \phi(q, x_n) + (\lambda_n - 1)\phi(q, x_n) - b_n c_n g\|J(T^n x_n) - J(S^n x_n)\|.
\end{aligned} \tag{3.15}$$

That is

$$b_n c_n g\|J(T^n x_n) - J(S^n x_n)\| \leq \phi(q, x_n) + (\lambda_n - 1)\phi(q, x_n) - \phi(q, y_n). \tag{3.16}$$

Alternatively, we observe that

$$\phi(q, x_n) - \phi(q, y_n) = \|x_n\|^2 - \|y_n\|^2 - 2\langle q, Jx_n - Jy_n\rangle \leq \|x_n - y_n\|(\|x_n\| + \|y_n\|) + 2\|q\|\|Jx_n - Jy_n\|. \tag{3.17}$$

It follows from (3.12) and (3.13), letting $n \to \infty$ in (3.17), we have

$$\lim_{n\to\infty} (\phi(q, x_n) - \phi(q, y_n)) = 0. \tag{3.18}$$

Letting $n \to \infty$ in (3.16), we get

$$\lim_{n\to\infty}(b_n c_n g\|J(T^n x_n) - J(S^n x_n)\|) = 0. \tag{3.19}$$

Since $\liminf_{n\to\infty} b_n c_n > 0$ and from (3.19), we obtain that

$$\lim_{n\to\infty} g\|J(T^n x_n) - J(S^n x_n)\| = 0. \tag{3.20}$$

From property of the function g that

$$\lim_{n\to\infty} \|J(T^n x_n) - J(S^n x_n)\| = 0. \tag{3.21}$$

Since J^{-1} is uniformly norm to norm continuous, we have

$$\lim_{n\to\infty} \|T^n x_n - S^n x_n\| = 0. \tag{3.22}$$

Since

$$\begin{aligned}\phi(T^n x_n, S^n x_n) &= \|T^n x_n\|^2 - 2\langle T^n x_n, J(S^n x_n)\rangle + \|S^n x_n\|^2\\ &= \|T^n x_n\|^2 - 2\langle T^n x_n, J(T^n x_n)\rangle + 2\langle T^n x_n, J(T^n x_n) - J(S^n x_n)\rangle + \|S^n x_n\|^2\\ &\le \|S^n x_n\|^2 - \|T^n x_n\|^2 + 2\|T^n x_n\|\|J(T^n x_n) - J(S^n x_n)\|\\ &\le (\|S^n x_n\| + \|T^n x_n\|)(\|S^n x_n - T^n x_n\|) + 2\|T^n x_n\|\|J(T^n x_n) - J(S^n x_n)\|.\end{aligned} \tag{3.23}$$

From (3.21) and (3.22), letting $n \to \infty$ we have

$$\lim_{n\to\infty} \phi(T^n x_n, S^n x_n) = 0. \tag{3.24}$$

Moreover, it follows that

$$\begin{aligned}\phi(T^n x_n, y_n) &= \phi(T^n x_n, J^{-1}(a_n J x_n + b_n J(T^n x_n) + c_n J(S^n x_n))\\ &= \|T^n x_n\|^2 - 2\langle T^n x_n, a_n J x_n + b_n J(T^n x_n) + c_n J(S^n x_n)\rangle + \|a_n J x_n + b_n J(T^n x_n) + c_n J(S^n x_n)\|^2\\ &\le \|T^n x_n\|^2 - 2a_n\langle T^n x_n, J x_n\rangle - 2b_n\langle T^n x_n, J(T^n x_n)\rangle - 2c_n\langle T^n x_n, J(S^n x_n)\rangle\\ &+ a_n\|x_n\|^2 + b_n\|T^n x_n\|^2 + c_n\|S^n x_n\|^2\\ &\le a_n\phi(T^n x_n, x_n) + c_n\phi(T^n x_n, S^n x_n).\end{aligned} \tag{3.25}$$

From (3.24) and $\lim_{n\to\infty} a_n = 0$, letting $n \to \infty$, we have

$$\lim_{n\to\infty} \phi(T^n x_n, y_n) = 0. \tag{3.26}$$

From Lemma 2.2 implies that

$$\lim_{n\to\infty} \|T^n x_n - y_n\| = 0. \tag{3.27}$$

By using the triangle inequality, we have

$$\|T^n x_n - p\| \le \|T^n x_n - y_n\| + \|y_n - p\|. \tag{3.28}$$

From (3.14) and (3.27), we have

$$\lim_{n\to\infty} \|T^n x_n - p\| = 0. \tag{3.29}$$

Again by using the triangle inequality, we have

$$\|T^{n+1}x_n - p\| \le \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - p\|. \tag{3.30}$$

Since T is uniformly asymptotically regular and from (3.29), letting $n \to \infty$ we have

$$\lim_{n\to\infty} \|T^{n+1}x_n - p\| = 0. \tag{3.31}$$

That is $T(T^n x_n) \to p$ as $n \to \infty$ and the closedness of T, we have $p \in F(T)$.
Next we show that $p \in F(S)$.
By a similar argument, we obtain

$$\begin{aligned}\phi(S^n x_n, T^n x_n) &= \|S^n x_n\|^2 - 2\langle S^n x_n, J(T^n x_n)\rangle + \|T^n x_n\|^2\\ &= \|S^n x_n\|^2 - 2\langle S^n x_n, J(S^n x_n)\rangle + 2\langle S^n x_n, J(S^n x_n) - J(T^n x_n)\rangle + \|T^n x_n\|^2\\ &\le \|T^n x_n\|^2 - \|S^n x_n\|^2 + 2\|S^n x_n\|\|J(S^n x_n) - J(T^n x_n)\|\\ &\le (\|T^n x_n\| + \|S^n x_n\|)(\|T^n x_n - S^n x_n\|) + 2\|S^n x_n\|\|J(S^n x_n) - J(T^n x_n)\|. \end{aligned} \tag{3.32}$$

From (3.21) and (3.22), we have

$$\lim_{n\to\infty} \phi(S^n x_n, T^n x_n) = 0. \tag{3.33}$$

Furthermore, we observe that

$$\begin{aligned}\phi(S^n x_n, y_n) &= \phi(S^n x_n, J^{-1}(a_n J x_n + b_n J(T^n x_n) + c_n J(S^n x_n))\\ &= \|S^n x_n\|^2 - 2\langle S^n x_n, a_n J x_n + b_n J(T^n x_n) + c_n J(S^n x_n)\rangle + \|a_n J x_n + b_n J(T^n x_n) + c_n J(S^n x_n)\|^2\\ &= \|S^n x_n\|^2 - 2a_n\langle T^n x_n, J x_n\rangle - 2b_n\langle T^n x_n, J(T^n x_n)\rangle - 2c_n\langle T^n x_n, J(S^n x_n)\rangle\\ &+ a_n\|x_n\|^2 + b_n\|T^n x_n\|^2 + c_n\|S^n x_n\|^2\\ &\le a_n\phi(S^n x_n, x_n) + c_n\phi(S^n x_n, T^n x_n). \end{aligned} \tag{3.34}$$

From (3.33) and $\lim_{n\to\infty} a_n = 0$, we obtain

$$\lim_{n\to\infty} \phi(S^n x_n, y_n) = 0. \tag{3.35}$$

It follow from Lemma 2.2 that

$$\lim_{n\to\infty} \|S^n x_n - y_n\| = 0. \tag{3.36}$$

By using the triangle inequality, we have

$$\|S^n x_n - p\| \le \|S^n x_n - y_n\| + \|y_n - p\|. \tag{3.37}$$

From (3.14) and (3.36), we obtain

$$\|S^n x_n - p\| = 0. \tag{3.38}$$

By using the triangle inequality, we get

$$\|S^{n+1}x_n - p\| \le \|S^{n+1}x_n - S^n x_n\| + \|S^n x_n - p\|. \tag{3.39}$$

Since S is uniformly asymptotically regular and from (3.38)

$$\lim_{n\to\infty} \|S^{n+1}x_n - p\| = 0. \tag{3.40}$$

That is $S(S^n x_n) \to p$ as $n \to \infty$ and the closedness of S, we obtain $p \in F(S)$.
Hence, we conclude that $p \in F = F(T) \cup F(S)$.
Step 6. we show that $p = \Pi_F x_1$.

From $x_n = \Pi_{C_n} x_1$, we have $\langle x_n - z, Jx_1 - Jx_n \rangle \geq 0, \forall z \in C_n$. Since $F \subset C_n$, we also have

$$\langle x_n - y, Jx_1 - Jx_n \rangle \geq 0, \quad \forall y \in F.$$

Taking limit $n \to \infty$, we obtain

$$\langle p - y, Jx_1 - Jp, \rangle \geq 0, \quad \forall y \in F.$$

By Lemma 2.4, we can conclude that $p = \Pi_F x_1$.
The proof is completed through the above six steps. □

Corollary 3.1. *Let E be a uniformly smooth and uniformly convex Banach space with property G and C be a nonempty closed and convex subset of E. Let $G = (V(G), A(G))$ be a transitive directed graph such that $V(G) = C$ and $A(G)$ is coordinate-convex. The mappings $T, S : V(G) \to V(G)$ are closed quasi G-ϕ-nonexpansive.*

Assume that $F = F(T) \cap F(S)$ is nonempty subset of C. For an initial point $x_1 \in C$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = J^{-1}(a_n Jx_n + b_n J(Tx_n) + c_n J(Sx_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.41}$$

where

(1) $(q, x_n) \in A(G)$, *for* $q \in F$,
(2) $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are real sequences in* $[0, 1]$.

If the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy the following conditions

(1) $\liminf_{n\to\infty} b_n c_n > 0$,
(2) $\lim_{n\to\infty} a_n = 0$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of F.

Corollary 3.2. *Let E be a uniformly smooth and uniformly convex Banach space with property G and C be a nonempty closed and convex subset of E. Let $G = (V(G), A(G))$ be a transitive directed graph such that $V(G) = C$ and $A(G)$ is coordinate-convex. The mapping $T : V(G) \to V(G)$ is closed quasi G-ϕ-nonexpansive mapping.*

Assume that $F(T)$ is nonempty and closed subset of C. Let $\{x_n\}$ be the sequence defined as follows:

$$\begin{cases} y_n = J^{-1}(a_n Jx_n + (1 - a_n)J(Tx_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n), \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.42}$$

where

(1) $(q, x_n) \in A(G)$, *for* $q \in F$,

(2) *$\{a_n\}$is a real sequence in $[0,1]$.*

If $\lim_{n\to\infty} a_n = 0$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of $F(T)$.

If E is a Hilbert space, then $\phi(y,x) = \|y-x\|^2$ and Π_C becomes the metric projection of E onto C.

Definition 3.1. *([27]) Let E be a real norm space and $G = (V(G), A(G))$ be a directed graph such that $V(G) \subset E$. A mapping $T : V(G) \to V(G)$ is said to be G-nonexpansive mapping if T satisfies the following conditions.*

(1) *T preserves edges of G, i.e.,*

$$(x,y) \in A(G) \Rightarrow (Tx, Ty) \in A(G),$$

for all $(x,y) \in A(G)$.

(2) *T non-increases weights of edges in G, i.e.,*

$$(x,y) \in A(G) \Rightarrow \|Tx - Ty\| \le \|x-y\|,$$

for all $(x,y) \in A(G)$.

Example 3.1. *([26]). Let C be a closed unit ball of the space l_1 with the norm $\|\{x_k\}\| = \sum_k |x_k|$. Let $G = (C, A(G))$ be the graph on C defined by*

$$A(G) = \left\{(\{x_k\},\{y_k\}); |x_k| + |y_k| \le 1 \text{ and } \|\{x_k\} - \{y_k\}\| \le \frac{3}{8}\right\}.$$

Define a mapping $T : C \to C$ by

$$T(\{x_k\}) = \{x_k^2\},\ \{x_k\} \in C. \tag{3.43}$$

Note that T is G-nonexpansive.

Definition 3.2. *([22]) Let E be a real norm space, $G = (V(G), E(G))$ be a directed graph such that $V(G) \subset X$. A mapping $T : V(G) \to V(G)$ is said to be an asymptotically G-nonexpansive mapping if*

(1) *T is edge-preserving.*

(2) *There exists a sequence $\{\lambda_n\} \subset [1,\infty)$ with $\sum_{n=1}^{\infty}(\lambda_n - 1) < \infty$ such that*

$$(u,v) \in A(G) \Rightarrow \|T^n u - T^n v\| \le \lambda_n \|u-v\|$$

for all $(u,v) \in A(G)$.

Example 3.2. *([11]). Let $E = \mathbb{R}$ and $G = (V(G), A(G))$ be a directed graph defined by $V(G) = E$ and $A(G) = \{(u,v) \in E\times E\} : u \neq v \in [0,1]$ or $u = v \in E\}$. Then $A(G)$ is coordinate-convex and $\{(u,u) : u \in V(G)\} \subset E(G)$. For $u \in E$, define two mappings by*

$$Tu = \frac{u}{2} \quad \text{and} \quad Su = \frac{u^2}{2}.$$

Note that T and S are two closed, uniformly asymptotically regular and asymptotically G-nonexpansive with $\lambda_n = 1$ for all $n \in \mathbb{N}$.

Definition 3.3. *([11]). Let H be a Hilbert space and $G = (V(G), A(G))$ be a directed graph such that $V(G) \subset H$. A mapping $T : V(G) \to V(G)$ is said to be a quasi G-nonexpansive mapping if*

(1) $F(T) \neq \emptyset$.
(2) *For $p \in F(T)$ and $v \in V(G)$*

$$(p, v) \in A(G) \Rightarrow (p, Tv) \in A(G).$$

(3) *For any $p \in F(T)$ and $u \in V(G)$,*

$$(p, u) \in A(G) \Rightarrow \|p - Tu\| \leq \|p - u\|.$$

It is obvious that every G-nonexpansive mapping is quasi-G-nonexpansive mapping. To show this, assume that T is G-nonexpansive. Let $p \in F(T)$, that is, $Tp = p$, and let $v \in V(G)$ such that $(p, v) \in A(G)$.

(1) Since T is G-nonexpansive, for $(p, v) \in A(G)$, it follows that $(Tp, Tv) \in A(G)$.
We obtain

$$(p, Tv) = (Tp, Tv) \in A(G).$$

(2) Since T is G-nonexpansive, for $(p, v) \in A(G)$, it follows that $\|Tp - Tv\| \leq \|p - v\|$.
Noting that, we have

$$\|p - Tv\| = \|Tp - Tv\| \leq \|p - v\|.$$

Therefore, T is a quasi G-nonexpansive mapping.

Definition 3.4. *([11]). Let H be a Hilbert space and $G = (V(G), A(G))$ be a directed graph such that $V(G) \subset H$. A mapping $T : V(G) \to V(G)$ is said to be an asymptotically quasi G-nonexpansive mapping if*

(1) $F(T) \neq \emptyset$.
(2) *For $p \in F(T)$ and $v \in V(G)$*

$$(p, v) \in A(G) \Rightarrow (p, Tv) \in A(G).$$

(3) *There exists a sequence $\{\lambda_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \lambda_n = 1$*

$$(p, u) \in A(G) \Rightarrow \|p - T^n u\| \leq \lambda_n \|p - u\|,$$

for any $p \in F(T)$ and $u \in V(G)$.

Corollary 3.3. *Let H be a Hilbert space satisfies property G and C be a nonempty closed and convex subset of H. Let $G = (V(G), A(G))$ be a transitive directed graph such that $V(G) = H$ and $A(G)$ is coordinate-convex. The mappings $T, S : V(G) \to V(G)$ are closed uniformly asymptotically regular and asymptotically quasi G-nonexpansive mapping.*

Assume that $F = F(T) \cap F(S)$ is nonempty and closed in H. For an initial point $x_1 \in H$ and $C_1 = H$, define the sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = J^{-1}(a_n Jx_n + b_n J(T^n x_n) + c_n J(S^n x_n), \\ C_{n+1} = \{z \in C_n : \|z - y_n\| \le \|z - x_n\| + (\lambda_n - 1)B_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases} \tag{3.44}$$

where

(1) $\lambda_n = \max\{\lambda_n^T, \lambda_n^S\}$,
(2) $B_n = \sup\{\phi(w, x_n) : w \in F\}$,
(3) $(q, x_n) \in A(G)$, *for* $q \in F$,
(4) $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are real sequences in* $[0, 1]$.

If the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy the following conditions

(1) $\liminf_{n\to\infty} b_n c_n > 0$,
(2) $\lim_{n\to\infty} a_n = 0$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of F.

Corollary 3.4. *Let H be a Hilbert space and H satisfies property G. Let $G = (V(G), A(G))$ be a transitive directed graph such that $V(G) = H$ and $A(G)$ is coordinate-convex. The mappings $T, S : V(G) \to V(G)$ are closed quasi G-nonexpansive mapping.*

Assume that $F(T)$ is nonempty and closed in H. For an initial point $x_1 \in H$ and $C_1 = H$, define the sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = J^{-1}(a_n Jx_n + (1 - a_n)J(Tx_n), \\ C_{n+1} = \{z \in C_n : \|z - y_n\| \le \|z - x_n\|, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases} \tag{3.45}$$

where

(1) $(q, x_n) \in A(G)$, *for* $q \in F$,
(2) $\{a_n\}$ *is a real sequences in* $[0, 1]$.

If $\lim_{n\to\infty} a_n = 0$.
Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $F(T)$.

Let $E = L_2([a, b])$ be a Hilbert spaces with $\|x\| = \sqrt{\int_a^b |x(t)|^2 dt}$ for $y \in E$, $h : [a, b] \to \mathbb{R}$ be continuous function and $f : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ be a integrable with respect to s on $[a, b]$. Consider the integral equation of the form

$$x(t) = h(t) + \int_a^b f(t, s, x(s))ds$$

where $t \in [a, b]$. Let

$$C = \{x \in E : x(t) \ge 0 \text{ for all } t \in [a, b]\}.$$

Define an integral operator T on E by

$$Tx(t) = h(t) + \int_a^b f(t,s,x(s))ds \tag{3.46}$$

for all $t \in [a,b]$ and $x \in E$.

Assume that There exists $\varphi : [a,b] \times [a,b] \to [0,\infty)$ such that φ is continuous on $[a,b] \times [a,b]$ with

$$\sup_{t\in[a,b]} \int_a^b \varphi^2(t,s)ds \leq \frac{1}{b-a}$$

and

$$|f(t,s,u(s)) - f(t,s,v(s))| \leq \varphi(t,s)|u(s) - v(s)|$$

for all $t,s \in [a,b]$ and $u,v \in C$.

Hieu and Huy [11] shown that T is an asymptotically G-nonexpansive mapping with $\lambda_n = 1$ for all $n \in \mathbb{N}$ and T is an asymptotically G-ϕ-nonexpansive mapping with $\phi(u,v) = \|u-v\|^2$ and $\lambda_n = 1$ for all $n \in \mathbb{N}$.

In this context, let T be defined by (3.46) as in Corollary 3.4, and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = J^{-1}(a_n Jx_n + (1-a_n)J(Tx_n)), \\ C_{n+1} = \{z \in C_n : \|z - y_n\| \leq \|z - x_n\|, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \geq 1, \end{cases} \tag{3.47}$$

where

(1) $(q, x_n) \in A(G)$, for $q \in F$,
(2) $\{a_n\}$ is a real sequences in $[0,1]$.

If $\lim_{n\to\infty} a_n = 0$. Then the sequence $\{x_n\}$ converges strongly to a point p which a solution of (3.46).

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