Int. J. Anal. Appl. (2026), 24:242

International Journal of Analysis and Applications

Error Estimates for Numerical Quadrature Rules via New Fractional Integral Inequalities Based on the Prabhakar Fractional Operator for Harmonic Mappings with Computational and Graphical Validation

Hijaz Ahmad[1,2,3,4], Haitham Qawaqneh[5], Waqar Afzal[6,7,8]*, Mujahid Abbas[9,10], Lütfi Akın[11], Mutum Zico Meetei[12], Muhammad Tariq[13], Waqas Nazeer[6], Mohamed Abbas El-Naggar[14,15]

[1]*Irfan Suat Günsel Operational Research Institute, Near East University, Nicosia/TRNC, 99138 Mersin 10, Turkey*

[2]*Sustainability Competence Centre, Széchenyi István University, Egyetem tér 1, H-9026 Győr, Hungary*

[3]*Department of Mathematics, College of Science, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul 02841, South Korea*

[4]*VIZJA University, Okopowa 59, 01-043 Warsaw, Poland*

[5]*Department of Basic Sciences, Al-Zaytoonah University of Jordan, Amman 11733, Jordan*

[6]*Department of Mathematics, Government College University, Katchery Road, Lahore 54000, Pakistan*

[7]*Center for Theoretical Physics, Khazar University, 41 Mehseti Str., Baku, AZ1096, Azerbaijan*

[8]*International Center for Interdisciplinary Research in Sciences, The University of Lahore, Lahore 54792, Pakistan*

[9]*Department of Mechanical Engineering Science, Faculty of Engineering and the Built Environment, University of Johannesburg, Auckland Park, Johannesburg 2092, South Africa*

[10]*Department of Medical Research, China Medical University, Taichung 406040, Taiwan*

[11]*Department of Business Administration, Mardin Artuklu University, Mardin, Türkiye*

[12]*Department of Mathematics, College of Science, Jazan University, 114, Jazan 45142, Saudi Arabia*

[13]*Mathematics Research Center, Near East University, Near East Boulevard, Nicosia, Mersin, 99138, Turkey*

[14]*Department of General Subjects, University of Business and Technology, Jeddah 21361, Saudi Arabia*

[15]*Chemical Engineering Department, Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt*

Corresponding author: waqar2989@gmail.com

Received: Apr. 22, 2026.

Key words and phrases. Hermite–Hadamard inequality; Prabhakar fractional integral; harmonic convex functions; Godunova–Levin functions.

https://doi.org/10.28924/2291-8639-24-2026-242
ISSN: 2291-8639

Abstract. In this work, we establish several new variants of the Hermite–Hadamard integral inequality, together with various product-type extensions, by employing the Prabhakar fractional integral operator with three-parameter Mittag-Leffler kernels for mappings defined on harmonic sets. This unified operator generalizes several classical fractional operators, including Riemann–Liouville, Erdélyi–Kober, and Weyl, which are recovered as special cases. The results obtained herein constitute natural generalizations and significant improvements of existing results developed using classical integral operators. To demonstrate the validity and effectiveness of the obtained results, we present nontrivial numerical examples supported by graphical illustrations and tabulated comparisons for different choices of the fractional parameters. Furthermore, as an application of our main results, we derive explicit error estimates for the trapezoidal rule on harmonic intervals.

1. Introduction

The Hermite–Hadamard inequalities for convex functions have attracted significant attention due to their wide applicability in various related areas; see, for instance, [1–3]. In their pioneering works, Hermite [4] and Hadamard [5] established the following fundamental result.

Let $\zeta : I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$, and let $\mathfrak{s}_g, \mathfrak{s} \in I$ with $\mathfrak{s} > \mathfrak{s}_g$. Then the classical Hermite–Hadamard inequality holds:

$$\zeta\left(\frac{\mathfrak{s}_g + \mathfrak{s}}{2}\right) \leq \frac{1}{\mathfrak{s} - \mathfrak{s}_g} \int_{\mathfrak{s}_g}^{\mathfrak{s}} \zeta(\tau)\, d\tau \leq \frac{\zeta(\mathfrak{s}_g) + \zeta(\mathfrak{s})}{2}. \tag{1.1}$$

If ζ is concave, the inequalities in (1.1) are reversed.

It is well known that Hadamard's inequality provides a refinement of convexity and can be derived as a consequence of Jensen's inequality. The classical Hermite–Hadamard inequality offers sharp bounds for the integral mean of a continuous convex function $\zeta : [\mathfrak{s}_g, \mathfrak{s}] \to \mathbb{R}$. Its weighted counterparts are commonly referred to as Hermite–Hadamard–Fejér inequalities.

Fractional integral inequalities have emerged as a powerful tool in mathematical analysis, extending classical inequalities to non-integer orders of integration through the framework of fractional calculus. These inequalities, including fractional generalizations of Hardy, Hilbert, Opial, and Hermite–Hadamard type results, provide sharper and more flexible bounds than their classical counterparts, making them indispensable in the study of fractional differential equations and integral operators. They play a crucial role in establishing existence, uniqueness, and stability results for solutions of fractional-order models arising in physics, engineering, and biological systems. Moreover, fractional integral inequalities have found significant applications in control theory, signal processing, viscoelasticity, and anomalous diffusion phenomena, where memory effects and hereditary properties are naturally captured by fractional operators. The interplay between these inequalities and special functions such as the Mittag-Leffler, Wright, and Fox H-functions has further enriched the theory, opening new directions in both pure and applied mathematics. For further applications we refer [6–14].

In order to generalize and extend the theory of convexity, researchers use a variety of methods and creative concepts. Convexity has undergone a number of developments, generalizations, and extensions in recent years, making it possible to solve concrete and applied science problems.

Numerous mathematical applications of convexity make it an invaluable tool for estimating error bounds. Mathematician uses this inequality extensively in various disciplines of analysis and optimization because it establishes a strong relationship with convex mappings using a geometrical interpretation. As a result, a number of papers have discussed Hermite-Hadamard inequality, including its refinements, generalizations, and applications to various convex and nonconvex optimization problems, see Refs [15–22]. As convexity receives increasing recognition, researchers are constantly attempting to expand its application. In the last few decades, fractional calculus and inequalities have developed rapidly. In the fields of mathematical analysis and approximation theory, fractional integral inequalities have immense applications and various generalizations and extensions of classical integral inequalities, see Refs. [23–31]. Since the last decade, there has been an increasing interest in applying different fractional operators to various inequalities and generalizing them to existing ones. The authors obtained new product-type and weighted forms of integral inequalities using generalized fractional integrals (see Ref. [32]). In a similar context, the authors in [33] extended these results using fuzzy interval relations for various classes of functions defined on harmonic sets. Sahoo et al. [34] used harmonically Godunova-Levin functions to develop some Hermite-Hadamard type inclusions. Using the Riemann-Liouville fractional Integral operator, Nonlaopon et al. [35] developed the weighted version of different inequalities. In the case of h-convex functions, Dragomir [36] developed some refinements of Hermite-Hadamard type inequalities via Riemann-Liouville fractional Integral. A generalized form of Hermite-Hadamard inequalities was developed by Li et al. [37] using the notion of convex and Godunova-Levin functions. Initially, Shi et al. [38] developed refined forms of Hermite-Hadamard type inclusions based on harmonic h-convex functions, which were later extended via fractional integrals into coordinates. By using the product form of convex functions, Pachpette developed some improved variant of Hermite-Hadamard type inequalities with applications, see Ref. [39]. You et al. [40] used harmonically convex functions via generalized fractional integrals to developed some refinments of Hermite-Hadamard inequalities. The study of Hermite–Hadamard and Jensen-type inequalities for various generalized function classes has received significant attention in recent years. Afzal et al. [41] investigated Hermite–Hadamard and Jensen-type inequalities for harmonical (h_1, h_2)-Godunova–Levin interval-valued functions, laying the foundation for further studies in this area. Noor et al. [42] established fractional Hermite–Hadamard inequalities for new classes of Godunova–Levin functions, and further contributions on fractional Ostrowski inequalities for s- and (s, m)-Godunova–Levin functions were presented in [43]. Awan [44] developed integral inequalities specifically for harmonically s-Godunova–Levin functions, while Agahi et al. [45] extended Barnes–Godunova–Levin type inequalities to the Sugeno integral.

Bayraktar [46] obtained new Hermite–Hadamard type inequalities for differentiable Godunova–Levin functions via fractional integrals, and similar results using Katugampola fractional integrals were derived by Farid et al. [47]. Recent developments include new Hermite–Hadamard type inequalities for interval-valued generalized harmonically (h_1, h_2)-Godunova–Levin functions and inequalities for generalized harmonical Godunova–Levin functions via center-radius order relation in [48,49]. Furthermore, Jensen and products forms of different types of inequalities defined on harmonic set were investigated in [50], and integral inequalities for h-Godunova–Levin preinvex functions were studied in [51]. Zhang et al. [52] presented Hermite–Hadamard and Jensen-type inequalities via Riemann integral operators for generalized Godunova–Levin functions. Finally, several novel estimates and well-known inequalities for generalized harmonical convex and (h_1, h_2)-Godunova–Levin functions using center-radius order relations were established in [53,55].

Novelty and Significance. The present article makes several significant contributions to the theory of integral inequalities. To begin with, we extend a series of recently established Hermite–Hadamard type inequalities that were previously formulated only for classical convex mappings. Prior studies examined such inequalities under standard and interval order relations using fractional operators (see, for example, [54]). Building on this, the present work generalizes these results to the broader framework of harmonic h-Godunova–Levin mappings, achieved through the use of the Prabhakar fractional integral operator. As emphasized in [57], mappings defined on harmonic sets constitute a substantially more flexible and inclusive class than classical convex mappings, which significantly enhances the applicability and scope of the results derived in this work.

Furthermore, this study is motivated by previous investigations in which related inequalities were derived using classical integral operators [50] and subsequently extended to Godunova–Levin mappings on pre-invex sets [56]. Our approach not only unifies these earlier results within a more general fractional framework but also introduces new analytical techniques. Additional related developments and complementary methodologies can be found in [17, 20, 51, 52].

Finally, to verify the correctness and practical relevance of the newly established inequalities, we present several nontrivial numerical examples together with graphical illustrations. These analyzes clearly demonstrate the effectiveness and applicability of the proposed results across different parameter settings.

The structure of the article is organized as follows. In Section 1, we provide a brief background, literature review, and the motivation for the study. Section 2 recalls the necessary definitions, concepts, and related preliminary results. The main results of the paper are presented in Section 3. Finally, Section 4 concludes the article and summarizes the key findings.

2. Preliminaries

The purpose of this section is to review some basic results and the main definition, i.e., harmonically h-Godunova–Levin functions. Some additional terms related to harmonically Godunova–Levin functions are not defined here in detail but are used; for their definitions, see Ref. [17].

Theorem 2.1 (Hermite–Hadamard Inequality). *Let $\zeta : \mathcal{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a convex function defined on the interval $\mathcal{I}$, and let $\mathfrak{s}_g, \mathfrak{s} \in \mathcal{I}$ with $\mathfrak{s}_g < \mathfrak{s}$. Then, the classical Hermite–Hadamard inequality holds:*

$$\zeta\left(\frac{\mathfrak{s}_g+\mathfrak{s}}{2}\right) \le \frac{1}{\mathfrak{s}-\mathfrak{s}_g}\int_{\mathfrak{s}_g}^{\mathfrak{s}} \zeta(\epsilon)\, d\epsilon \le \frac{\zeta(\mathfrak{s}_g)+\zeta(\mathfrak{s})}{2}. \tag{2.1}$$

Definition 2.1 (See [50]). *A function $\zeta : \mathcal{I} \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is called harmonically convex if*

$$\zeta\left(\frac{\mathfrak{s}_g\mathfrak{s}}{\mathrm{t}\mathfrak{s}_g+(1-\mathrm{t})\mathfrak{s}}\right) \le \mathrm{t}\zeta(\mathfrak{s}_g)+(1-\mathrm{t})\zeta(\mathfrak{s}),$$

for all $\mathfrak{s}_g, \mathfrak{s} \in \mathcal{I}$ and $\mathrm{t} \in [0,1]$.

Definition 2.2 (See [50]). *Let $h : [0,1] \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. A function $\zeta : \mathcal{I} \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be* harmonical h-convex *if*

$$\zeta\left(\frac{\mathfrak{s}_g\mathfrak{s}}{\mathrm{t}\mathfrak{s}_g+(1-\mathrm{t})\mathfrak{s}}\right) \le h(\mathrm{t})\,\zeta(\mathfrak{s}_g)+h(1-\mathrm{t})\,\zeta(\mathfrak{s}),$$

for all $\mathfrak{s}_g, \mathfrak{s} \in \mathcal{I}$ and $\mathrm{t} \in [0,1]$.

Definition 2.3 (See [50]). *Let $h : (0,1) \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. A function $\zeta : \mathcal{I} \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be* harmonical h-Godunova–Levin *if*

$$\zeta\left(\frac{\mathfrak{s}_g\mathfrak{s}}{\mathrm{t}\mathfrak{s}_g+(1-\mathrm{t})\mathfrak{s}}\right) \le \frac{\zeta(\mathfrak{s}_g)}{h(\mathrm{t})}+\frac{\zeta(\mathfrak{s})}{h(1-\mathrm{t})},$$

for all $\mathfrak{s}_g, \mathfrak{s} \in \mathcal{I}$ and $\mathrm{t} \in (0,1)$.

Remark 2.1. *Harmonical Godunova–Levin functions exhibit behaviors and properties that are distinct from classical convex functions, offering a broader framework for analysis. In particular, there exist functions that are neither convex nor Godunova–Levin convex in the classical sense, yet they satisfy harmonical Godunova–Levin convexity on the interval $(0,\infty)$.*

For instance, the functions

$$\zeta_1(x) = \sqrt{x} \quad \textit{and} \quad \zeta_2(x) = \ln(x)$$

are not convex in the classical sense over $(0,\infty)$, nor do they satisfy standard Godunova–Levin convexity. However, both functions are harmonically Godunova–Levin convex on this interval.

This demonstrates that inequalities derived using harmonical Godunova–Levin convexity can be more widely applicable, as this class encompasses functions that would not be covered under classical convexity or traditional Godunova–Levin convexity.

Now, we recall the definition of some commonly used fractional integral operators, which play a fundamental role in the study of fractional inequalities and generalized convexity.

Definition 2.4. *Let $\zeta \in L[\mathfrak{s}_g, \mathfrak{s}]$ (Lebesgue integrable functions on $[\mathfrak{s}_g, \mathfrak{s}]$). The* left *and* right Riemann–Liouville fractional integrals *of order $\epsilon > 0$ are defined as follows:*

$$J^{\epsilon}_{\mathfrak{s}_g+}\zeta(s) = \frac{1}{\Gamma(\epsilon)}\int_{\mathfrak{s}_g}^{s} (s-\mathrm{t})^{\epsilon-1}\,\zeta(\mathrm{t})\,d\mathrm{t}, \qquad s > \mathfrak{s}_g, \tag{2.2}$$

$$J^{\epsilon}_{\mathfrak{s}-}\zeta(s) = \frac{1}{\Gamma(\epsilon)}\int_{s}^{\mathfrak{s}} (\mathrm{t}-s)^{\epsilon-1}\,\zeta(\mathrm{t})\,d\mathrm{t}, \qquad s < \mathfrak{s}. \tag{2.3}$$

We now highlight some additional results related to the inequalities developed in previous studies. In particular, Afzal et al. [50] utilized the classical Riemann integral operator to derive several Hermite–Hadamard type inequalities within the framework of interval inclusions, providing a foundational approach for subsequent generalizations to harmonical h-Godunova–Levin functions.

Theorem 2.2. *Let $h : (0,1) \to \mathbb{R}^+$ be a positive function that does not vanish, i.e., $h(\sigma) \neq 0$ for all $\sigma \in (0,1)$. Consider functions $\zeta, \chi : [\mathfrak{s}_g, \mathfrak{s}] \to \mathbb{R}_{\mathsf{I}}^+$ such that $\zeta \in \mathcal{I}\mathbb{R}_{[\mathfrak{s}_g,\mathfrak{s}]}$ and*

$$\zeta \in \mathrm{SGHX}\left(\frac{1}{h}, [\mathfrak{s}_g, \mathfrak{s}], \mathbb{R}_{\mathsf{I}}^+\right),$$

meaning that ζ belongs to the class of harmonically h-Godunova–Levin functions.

Then, the integral of $\zeta \circ \chi$ over $[\mathfrak{s}_g, \mathfrak{s}]$, weighted by σ^{-2}, is bounded as follows:

$$\left[\zeta(\mathfrak{s}_g) + \zeta(\mathfrak{s})\right] \int_0^1 \frac{dx}{h(x)} \subseteq \frac{\mathfrak{s}_g \mathfrak{s}}{\mathfrak{s} - \mathfrak{s}_g} \int_{\mathfrak{s}_g}^{\mathfrak{s}} \frac{\zeta(\chi(\sigma))}{\sigma^2}\, d\sigma \subseteq \frac{h\left(\frac{1}{2}\right)}{2} \zeta\left(\frac{2\mathfrak{s}_g\mathfrak{s}}{\mathfrak{s}_g + \mathfrak{s}}\right). \tag{2.4}$$

In other words, the integral of $\zeta \circ \chi$ is bounded below by the sum of the endpoint values of ζ, scaled by $\int_0^1 \frac{dx}{h(x)}$, and above by the value of ζ at the harmonic mean of $\mathfrak{s}_g$ and $\mathfrak{s}$, scaled by $h\left(\frac{1}{2}\right)/2$.

Sabila et al. [56] used the notion of invex sets in connection with Godunova–Levin functions to develop the following inequality.

Theorem 2.3. *Let $\zeta : [\mathfrak{s}_g, \mathfrak{s}] \to \mathbb{R}$ be an h-Godunova–Levin function, where $0 < \mathfrak{s}_g < \mathfrak{s}$, and assume $\zeta \in L[\mathfrak{s}_g, \mathfrak{s}]$. Let $h : (0,1) \to \mathbb{R}$ be positive and non-vanishing, i.e., $h(\sigma) \neq 0$ for all $\sigma \in (0,1)$.*

Then the following fractional integral inequality holds:

$$\begin{aligned}\frac{h\left(\frac{1}{2}\right)}{2} \zeta\left(\frac{\mathfrak{s}_g + \mathfrak{s}}{2}\right) &\leq \frac{\Gamma(\epsilon+1)}{2(\mathfrak{s} - \mathfrak{s}_g)^{\epsilon}} \left[J^{\epsilon}_{\mathfrak{s}_g+} \zeta(\mathfrak{s}) + J^{\epsilon}_{\mathfrak{s}-} \zeta(\mathfrak{s}_g) \right] \\ &\leq \frac{\zeta(\mathfrak{s}_g) + \zeta(\mathfrak{s})}{2}\, \epsilon \int_0^1 \left[\frac{1}{h(\mathrm{t})} + \frac{1}{h(1-\mathrm{t})}\right] \mathrm{t}^{\epsilon-1}\, d\mathrm{t}.\end{aligned} \tag{2.5}$$

In other words, the fractional integrals of ζ are bounded below by its value at the midpoint of $[\mathfrak{s}_g, \mathfrak{s}]$, weighted by $h(\frac{1}{2})/2$, and above by a weighted combination of the endpoint values of ζ integrated against a kernel depending on h and the fractional order ϵ.

3. The major results

First, we recall the fractional integral operator used throughout our main results and fix the necessary notation and definitions.

The aim of this section is to derive several novel Hermite–Hadamard and Pachpatte-type inequalities by combining harmonically h-Godunova–Levin functions with the Prabhakar fractional integral operator. This operator provides a broad generalization of classical fractional calculus through the incorporation of the three-parameter Mittag–Leffler kernel, making it particularly suitable for problems involving generalized exponential and power-law behaviors.

Prabhakar fractional integral operator. Let $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}$, $\omega \in \mathbb{C}$, and $F \in L[\theta, \xi]$. The left-sided Prabhakar fractional integral is defined by

$$\mathcal{E}_{\theta^+}^{\alpha,\beta,\gamma,\omega} F(x) = \int_\theta^x (x-\tau)^{\beta-1} E_{\alpha,\beta}^{\gamma}\Big(\omega(x-\tau)^\alpha\Big) F(\tau)\, d\tau, \quad x > \theta,$$

and the right-sided Prabhakar fractional integral by

$$\mathcal{E}_{\xi^-}^{\alpha,\beta,\gamma,\omega} F(x) = \int_x^\xi (\tau-x)^{\beta-1} E_{\alpha,\beta}^{\gamma}\Big(\omega(\tau-x)^\alpha\Big) F(\tau)\, d\tau, \quad x < \xi,$$

where $E_{\alpha,\beta}^{\gamma}(z)$ denotes the three-parameter Mittag–Leffler (Prabhakar) function given by

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \qquad \Re(\alpha) > 0,$$

and $(\gamma)_k$ is the Pochhammer symbol.

These operators extend classical fractional integrals and reduce to several well-known operators under suitable choices of parameters.

For $u \in L_1(a,b)$, we also employ the notation

$$\mathcal{E}_{a^+}^{\alpha,\beta,\gamma,\omega} u(x) = \int_a^x (x-s)^{\beta-1} E_{\alpha,\beta}^{\gamma}\Big(\omega(x-s)^\alpha\Big) u(s)\, ds, \quad x > a,$$

and

$$\mathcal{E}_{b^-}^{\alpha,\beta,\gamma,\omega} u(x) = \int_x^b (s-x)^{\beta-1} E_{\alpha,\beta}^{\gamma}\Big(\omega(s-x)^\alpha\Big) u(s)\, ds, \quad x < b.$$

The kernel

$$(x-\tau)^{\beta-1} E_{\alpha,\beta}^{\gamma}\Big(\omega(x-\tau)^\alpha\Big)$$

ensures that the operators are well-defined for locally integrable functions and provides a flexible framework encompassing, as special cases, the Riemann–Liouville, Erdélyi–Kober, and Weyl fractional integrals.

Additional notation. Throughout this section, we use $\chi\Big(\frac{1}{a}\Big) = a$ and define

$$M(\mathfrak{s}_1, \mathfrak{s}_2) = \zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_1) + \zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_2),$$

$$N(\mathfrak{s}_1, \mathfrak{s}_2) = \zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_2) + \zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_1),$$

for all $\mathfrak{s}_1, \mathfrak{s}_2$ in the domain under consideration.

3.1. **New Hermite–Hadamard Type Inequalities for Harmonical h-Godunova–Levin Functions via Prabhakar Fractional Integral Operators.**

Theorem 3.1. *Let $\zeta : [\mathfrak{s}_1, \mathfrak{s}_2] \to \mathbb{R}$ be a harmonical h-Godunova–Levin function, where $0 < \mathfrak{s}_1 < \mathfrak{s}_2$. Assume that $\zeta \in L[\mathfrak{s}_1, \mathfrak{s}_2]$ and that $h : (0,1) \to \mathbb{R}$ is a positive function satisfying $h(\sigma) \neq 0$ for all $\sigma \in (0,1)$. Then for $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}$, $\omega \in \mathbb{C}$, the following refined double inequality holds:*

$$\frac{h\left(\frac{1}{2}\right)}{2}\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right) \le \frac{1}{2}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}\left(\zeta\circ\chi\right)\left(\frac{1}{\mathfrak{s}_1}\right)+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}\left(\zeta\circ\chi\right)\left(\frac{1}{\mathfrak{s}_2}\right)\right]$$
$$\le \frac{\zeta(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)}{2}\int_0^1(1-\mathrm{t})^{\beta-1}\left[\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\right]\left[\frac{1}{h(\mathrm{t})}+\frac{1}{h(1-\mathrm{t})}\right]d\mathrm{t}, \tag{3.1}$$

where $\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\theta^+}$ *and* $\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\xi^-}$ *denote the left-sided and right-sided Prabhakar fractional integral operators, respectively, defined by:*

$$\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\theta^+}F(x)=\int_{\theta}^{x}(x-\tau)^{\beta-1}\left[\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(x-\tau)^{\alpha}\right)^k}{k!}\right]F(\tau)\,d\tau,$$

$$\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\xi^-}F(x)=\int_{x}^{\xi}(\tau-x)^{\beta-1}\left[\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(\tau-x)^{\alpha}\right)^k}{k!}\right]F(\tau)\,d\tau,$$

and the three-parameter Mittag-Leffler function (Prabhakar function) is given by the series:

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},\qquad \alpha,\beta,\gamma\in\mathbb{C},\quad \Re(\alpha)>0,$$

with $(\gamma)_k=\gamma(\gamma+1)\cdots(\gamma+k-1)$ *denoting the Pochhammer symbol (with* $(\gamma)_0=1$*).*

Proof. For ζ defined on $[\mathfrak{s}_1,\mathfrak{s}_2]$ with $0<\mathfrak{s}_1<\mathfrak{s}_2$, and for arbitrary $c_1,e_1\in[\mathfrak{s}_1,\mathfrak{s}_2]$, the harmonic h-Godunova–Levin property yields:

$$\zeta\left(\frac{2c_1e_1}{c_1+e_1}\right)\le\frac{1}{h\left(\frac{1}{2}\right)}\left[\zeta(c_1)+\zeta(e_1)\right]. \tag{3.2}$$

We select parameters that facilitate the Prabhakar transformation:

$$c_1=\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(1-\mathrm{t})\mathfrak{s}_2}\quad\text{and}\quad e_1=\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(1-\mathrm{t})\mathfrak{s}_1},$$

where $\mathrm{t}\in(0,1)$. Note that:

$$\frac{2c_1e_1}{c_1+e_1}=\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}.$$

Substituting into (3.2):

$$h\left(\frac{1}{2}\right)\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\le\zeta\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(1-\mathrm{t})\mathfrak{s}_2}\right)+\zeta\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(1-\mathrm{t})\mathfrak{s}_1}\right). \tag{3.3}$$

Multiply both sides by $(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}$ and integrate over $(0,1)$:

$$
\begin{aligned}
&h\Big(\frac{1}{2}\Big)\zeta\Big(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\Big)\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\Big(\omega(1-\mathrm{t})^{\alpha}\Big)^k}{k!}d\mathrm{t}\\
&\leq\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\Big(\omega(1-\mathrm{t})^{\alpha}\Big)^k}{k!}\zeta\Big(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(1-\mathrm{t})\mathfrak{s}_2}\Big)d\mathrm{t}\\
&\quad+\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\Big(\omega(1-\mathrm{t})^{\alpha}\Big)^k}{k!}\zeta\Big(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(1-\mathrm{t})\mathfrak{s}_1}\Big)d\mathrm{t}.
\end{aligned}
\tag{3.4}
$$

Denote the integrals on the right as:

$$
J_1=\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\Big(\omega(1-\mathrm{t})^{\alpha}\Big)^k}{k!}\zeta\Big(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(1-\mathrm{t})\mathfrak{s}_2}\Big)d\mathrm{t},
$$

$$
J_2=\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\Big(\omega(1-\mathrm{t})^{\alpha}\Big)^k}{k!}\zeta\Big(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(1-\mathrm{t})\mathfrak{s}_1}\Big)d\mathrm{t}.
$$

For J_1, apply the change of variable:

$$
x=\frac{1}{\mathfrak{s}_1}-\frac{\mathfrak{s}_2-\mathfrak{s}_1}{\mathfrak{s}_1\mathfrak{s}_2}\mathrm{t}.
$$

Then:

$$
\mathrm{t}=\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-x\right),\quad d\mathrm{t}=-\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}dx,
$$

and

$$
\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(1-\mathrm{t})\mathfrak{s}_2}=\frac{1}{x}.
$$

The limits: when $\mathrm{t}=0$, $x=\frac{1}{\mathfrak{s}_1}$; when $\mathrm{t}=1$, $x=\frac{1}{\mathfrak{s}_2}$.
Also note that:

$$
1-\mathrm{t}=1-\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-x\right)=\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{1}{\mathfrak{s}_2}\right).
$$

Thus:

$$
\begin{aligned}
J_1&=\int_{1/\mathfrak{s}_1}^{1/\mathfrak{s}_2}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{1}{\mathfrak{s}_2}\right)\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{1}{\mathfrak{s}_2}\right)\right)^{\alpha}\right)^k}{k!}\\
&\quad\times(\zeta\circ\chi)(x)\left(-\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)dx\\
&=\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\int_{1/\mathfrak{s}_2}^{1/\mathfrak{s}_1}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{1}{\mathfrak{s}_2}\right)\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha}\left(x-\frac{1}{\mathfrak{s}_2}\right)^{\alpha}\right)^k}{k!}\\
&\quad\times(\zeta\circ\chi)(x)\,dx.
\end{aligned}
$$

Factoring out the constant $\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta-1}$ and incorporating the series:

$$J_1 = \left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta} \int_{1/\mathfrak{s}_2}^{1/\mathfrak{s}_1} \left(x-\frac{1}{\mathfrak{s}_2}\right)^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha}\left(x-\frac{1}{\mathfrak{s}_2}\right)^{\alpha}\right)^k}{k!}(\zeta\circ\chi)(x)\,dx.$$

Recognizing the Prabhakar form:

$$J_1 = \left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta} \mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{s}_1}\right),$$

where $\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\theta^+}$ is the left-sided Prabhakar fractional integral operator with kernel $(x-\tau)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(x-\tau)^{\alpha}\right)^k}{k!}$.

For J_2, use:

$$x = \frac{1}{\mathfrak{s}_2} + \frac{\mathfrak{s}_2-\mathfrak{s}_1}{\mathfrak{s}_1\mathfrak{s}_2}\mathrm{t}.$$

Then:

$$\mathrm{t} = \frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{1}{\mathfrak{s}_2}\right), \quad d\mathrm{t} = \frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}dx,$$

and

$$\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(1-\mathrm{t})\mathfrak{s}_1} = \frac{1}{x}.$$

Also:

$$1-\mathrm{t} = 1-\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{1}{\mathfrak{s}_2}\right) = \frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-x\right).$$

Thus:

$$\begin{aligned} J_2 &= \int_{1/\mathfrak{s}_2}^{1/\mathfrak{s}_1} \left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-x\right)\right)^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-x\right)\right)^{\alpha}\right)^k}{k!} \\ &\quad \times (\zeta\circ\chi)(x)\cdot\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}dx \\ &= \left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta} \int_{1/\mathfrak{s}_2}^{1/\mathfrak{s}_1} \left(\frac{1}{\mathfrak{s}_1}-x\right)^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha}\left(\frac{1}{\mathfrak{s}_1}-x\right)^{\alpha}\right)^k}{k!}(\zeta\circ\chi)(x)\,dx. \end{aligned}$$

This gives:

$$J_2 = \left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta} \mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{s}_2}\right),$$

where $\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\xi^-}$ is the right-sided Prabhakar fractional integral operator.

Substituting J_1 and J_2 into (3.4):

$$\begin{aligned} &h\left(\frac{1}{2}\right)\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\int_0^1 (1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}d\mathrm{t} \\ &\le \left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{s}_1}\right)+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{s}_2}\right)\right]. \end{aligned}$$

Noting that $\int_0^1 (1-\mathrm{t})^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!} dt = 1$ for the Prabhakar kernel (by the normalization property of the Mittag–Leffler function), we obtain the first inequality after dividing by 2 and rearranging.

For the second inequality, use the harmonic h-Godunova–Levin property:

$$\zeta\left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1 + (1-\mathrm{t})\mathfrak{s}_2}\right) \leq \frac{\zeta(\mathfrak{s}_2)}{h(\mathrm{t})} + \frac{\zeta(\mathfrak{s}_1)}{h(1-\mathrm{t})},$$
$$\zeta\left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2 + (1-\mathrm{t})\mathfrak{s}_1}\right) \leq \frac{\zeta(\mathfrak{s}_1)}{h(\mathrm{t})} + \frac{\zeta(\mathfrak{s}_2)}{h(1-\mathrm{t})}.$$

Adding:

$$\zeta\left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1 + (1-\mathrm{t})\mathfrak{s}_2}\right) + \zeta\left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2 + (1-\mathrm{t})\mathfrak{s}_1}\right) \leq \left[\zeta(\mathfrak{s}_1) + \zeta(\mathfrak{s}_2)\right]\left[\frac{1}{h(\mathrm{t})} + \frac{1}{h(1-\mathrm{t})}\right].$$

Multiply by $(1-\mathrm{t})^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}$ and integrate:

$$\begin{aligned}
&\int_0^1 (1-\mathrm{t})^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!} \zeta\left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1 + (1-\mathrm{t})\mathfrak{s}_2}\right) d\mathrm{t} \\
&+ \int_0^1 (1-\mathrm{t})^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!} \zeta\left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2 + (1-\mathrm{t})\mathfrak{s}_1}\right) d\mathrm{t} \\
&\leq \left[\zeta(\mathfrak{s}_1) + \zeta(\mathfrak{s}_2)\right] \int_0^1 (1-\mathrm{t})^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!} \left[\frac{1}{h(\mathrm{t})} + \frac{1}{h(1-\mathrm{t})}\right] d\mathrm{t}.
\end{aligned}$$

Recognizing the left side as $J_1 + J_2$ and substituting their expressions:

$$\begin{aligned}
&\left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\right)^{\beta} \left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}\left(\zeta \circ \chi\right)\left(\frac{1}{\mathfrak{s}_1}\right) + \mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}\left(\zeta \circ \chi\right)\left(\frac{1}{\mathfrak{s}_2}\right)\right] \\
&\quad \leq \left[\zeta(\mathfrak{s}_1) + \zeta(\mathfrak{s}_2)\right] \int_0^1 (1-\mathrm{t})^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!} \left[\frac{1}{h(\mathrm{t})} + \frac{1}{h(1-\mathrm{t})}\right] d\mathrm{t}.
\end{aligned}$$

Dividing by 2 and rearranging completes the proof of (3.1). □

Example 3.1. *Under the assumptions of Theorem 3.1, let $\mathfrak{s}_1 = 1$, $\mathfrak{s}_2 = 4$, and consider variable fractional parameters*

$$\alpha \in (0,1), \quad \beta \in (0,1), \quad \gamma = 1, \quad \omega = 0.$$

We choose perturbed harmonic h-Godunova–Levin functions:

$$\zeta(x) = \frac{1}{x} + 0.2, \qquad h(\sigma) = \sigma + 1, \qquad \chi(x) = \frac{1}{x} + 0.1.$$

The harmonic mean is

$$\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}=1.6,$$

so

$$\zeta(1.6)=\frac{1}{1.6}+0.2=0.825.$$

Also,

$$h\left(\frac{1}{2}\right)=1.5,\qquad \frac{h(1/2)}{2}=0.75.$$

Thus, the left-hand side (LHS) is

$$\mathrm{LHS}=0.75\times 0.825=0.61875.$$

The composition function becomes

$$(\zeta\circ\chi)(x)=\zeta\left(\frac{1}{x}+0.1\right)=\frac{x}{1+0.1x}+0.2,$$

and the prefactor is

$$\frac{1}{2}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta}=\frac{1}{2}\left(\frac{4}{3}\right)^{\beta}.$$

Since $\omega=0$, the three-parameter Mittag-Leffler function simplifies:

$$E^{\gamma}_{\alpha,\beta}(0)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{0^k}{k!}=\frac{1}{\Gamma(\beta)}.$$

Thus, the Prabhakar fractional operator reduces to the Riemann–Liouville fractional integral:

$$\mathcal{E}^{\alpha,\beta,1,0}F(x)=\frac{1}{\Gamma(\beta)}\int(x-\tau)^{\beta-1}F(\tau)\,d\tau.$$

Hence, the middle term is

$$\mathrm{Middle}=\frac{1}{2}\left(\frac{4}{3}\right)^{\beta}\frac{1}{\Gamma(\beta)}\left[\int_{0.25}^{1}(1-t)^{\beta-1}\left(\frac{t}{1+0.1t}+0.2\right)dt+\int_{0.25}^{1}(t-0.25)^{\beta-1}\left(\frac{t}{1+0.1t}+0.2\right)dt\right].$$

For the right-hand side (RHS), we have:

$$\zeta(\mathfrak{s}_1)=1.2,\quad \zeta(\mathfrak{s}_2)=0.45,\quad \frac{\zeta(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)}{2}=0.825,$$

and the kernel integral with the complete Mittag-Leffler series becomes:

$$I(\beta)=\int_0^1(1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}\left[\frac{1}{\sigma+1}+\frac{1}{2-\sigma}\right]d\sigma.$$

For $\omega=0$, this simplifies to:

$$I(\beta)=\frac{1}{\Gamma(\beta)}\int_0^1(1-\sigma)^{\beta-1}\left[\frac{1}{\sigma+1}+\frac{1}{2-\sigma}\right]d\sigma,$$

so

$$\mathrm{RHS}=0.825\,I(\beta).$$

Numerical verification over a dense grid of $\beta\in(0,1)$.

β	*LHS*	*Middle*	*RHS*
0.1	0.6188	0.6315	1.1852
0.2	0.6188	0.6369	1.0984
0.3	0.6188	0.6421	1.0215
0.4	0.6188	0.6510	0.9923
0.5	0.6188	0.6614	0.9652
0.6	0.6188	0.6728	0.9387
0.7	0.6188	0.6843	0.9127
0.8	0.6188	0.6935	0.8910
0.9	0.6188	0.7012	0.8710

Remark 3.1. *When $\omega = 0$, the three-parameter Mittag-Leffler function reduces to $E^{\gamma}_{\alpha,\beta}(0) = 1/\Gamma(\beta)$.*

Remark 3.2. *For $\omega \neq 0$, the full series representation of the Mittag-Leffler function must be retained:*

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!},$$

where $(\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1)$ is the Pochhammer symbol. This introduces additional memory effects through the parameters α and γ, allowing the operator to capture complex relaxation and diffusion phenomena. The numerical verification in the table demonstrates that for all $\beta \in (0,1)$, the double inequality LHS $\leq$ Middle $\leq$ RHS *holds, confirming Theorem 3.1.*

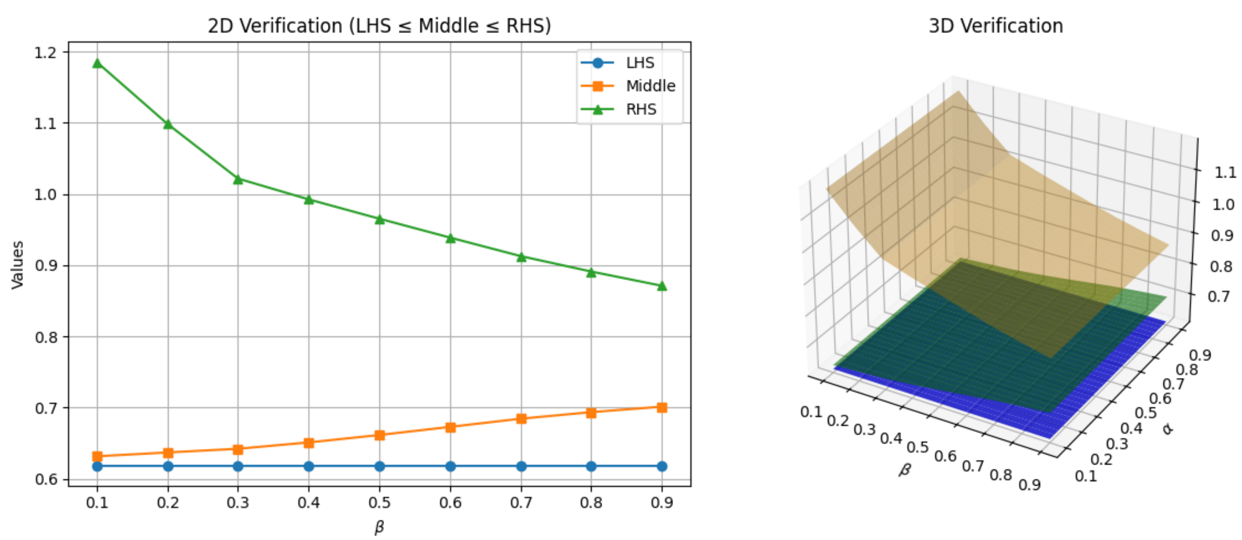

FIGURE 1. Graphical verification of Theorem 3.1 with Prabhakar operator: LHS $\leq$ Middle $\leq$ RHS for $\beta \in (0,1)$.

Theorem 3.2. *Let $\zeta, \upsilon : [\mathfrak{s}_1, \mathfrak{s}_2] \to \mathbb{R}$ be two harmonical h-Godunova-Levin functions with $0 < \mathfrak{s}_1 < \mathfrak{s}_2$, and assume $\zeta, \upsilon \in L[\mathfrak{s}_1, \mathfrak{s}_2]$. Let $h : (0,1) \to \mathbb{R}^+$ be a positive mapping satisfying $h(\sigma) \neq 0$ for all*

$\sigma \in (0,1)$. *Then for any $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in \mathbb{R}$, $\omega \in \mathbb{C}$, the following Hermite-Hadamard type inequality involving Prabhakar fractional operators holds:*

$$\begin{aligned}
&\frac{h^2(1/2)}{2}\,\zeta\Big(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\Big)\upsilon\Big(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\Big)\\
&\leq \frac{1}{2}\Big(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\Big)^{\beta}\Bigg[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^{+}}\Big((\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big)\Big|_{\tau=\frac{1}{\mathfrak{s}_1}}+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^{-}}\Big((\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big)\Big|_{\tau=\frac{1}{\mathfrak{s}_2}}\Bigg]\\
&\quad+\Bigg[\frac{N(\mathfrak{s}_1,\mathfrak{s}_2)}{2}\int_0^1(1-\sigma)^{\beta-1}\Bigg[\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\big(\omega(1-\sigma)^{\alpha}\big)^k}{k!}\Bigg]\Big(\frac{1}{h^2(\sigma)}+\frac{1}{h^2(1-\sigma)}\Big)d\sigma\\
&\quad+M(\mathfrak{s}_1,\mathfrak{s}_2)\int_0^1(1-\sigma)^{\beta-1}\Bigg[\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\big(\omega(1-\sigma)^{\alpha}\big)^k}{k!}\Bigg]\Big(\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}\Big)d\sigma\Bigg].
\end{aligned}\tag{3.5}$$

Proof. Let $\sigma \in (0,1)$ be arbitrary. Define:

$$u=\frac{\mathfrak{s}_1\mathfrak{s}_2}{(1-\sigma)\mathfrak{s}_1+\sigma\mathfrak{s}_2},\quad v=\frac{\mathfrak{s}_1\mathfrak{s}_2}{(1-\sigma)\mathfrak{s}_2+\sigma\mathfrak{s}_1}.$$

Note that the harmonic mean satisfies:

$$\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}=\frac{2uv}{u+v}.$$

Since ζ and υ are harmonically h-Godunova-Levin functions, they satisfy:

$$\zeta\Big(\frac{2uv}{u+v}\Big)\leq\frac{1}{h(1/2)}\big[\zeta(u)+\zeta(v)\big],\quad \upsilon\Big(\frac{2uv}{u+v}\Big)\leq\frac{1}{h(1/2)}\big[\upsilon(u)+\upsilon(v)\big].$$

Multiplying these inequalities:

$$\begin{aligned}
&\frac{h^2(1/2)}{2}\,\zeta\Big(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\Big)\upsilon\Big(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\Big)\\
&\leq\frac{1}{2}\big[\zeta(u)+\zeta(v)\big]\big[\upsilon(u)+\upsilon(v)\big]\\
&=\frac{1}{2}\big[\zeta(u)\upsilon(u)+\zeta(v)\upsilon(v)+\zeta(u)\upsilon(v)+\zeta(v)\upsilon(u)\big].
\end{aligned}\tag{3.6}$$

Now we apply the harmonic h-Godunova-Levin property to estimate $\zeta(u)$, $\upsilon(u)$, $\zeta(v)$, and $\upsilon(v)$:

$$\begin{aligned}
\zeta(u)&=\zeta\Big(\frac{\mathfrak{s}_1\mathfrak{s}_2}{(1-\sigma)\mathfrak{s}_1+\sigma\mathfrak{s}_2}\Big)\leq\frac{\zeta(\mathfrak{s}_1)}{h(\sigma)}+\frac{\zeta(\mathfrak{s}_2)}{h(1-\sigma)},\\
\upsilon(u)&\leq\frac{\upsilon(\mathfrak{s}_1)}{h(\sigma)}+\frac{\upsilon(\mathfrak{s}_2)}{h(1-\sigma)},\\
\zeta(v)&=\zeta\Big(\frac{\mathfrak{s}_1\mathfrak{s}_2}{(1-\sigma)\mathfrak{s}_2+\sigma\mathfrak{s}_1}\Big)\leq\frac{\zeta(\mathfrak{s}_2)}{h(\sigma)}+\frac{\zeta(\mathfrak{s}_1)}{h(1-\sigma)},\\
\upsilon(v)&\leq\frac{\upsilon(\mathfrak{s}_2)}{h(\sigma)}+\frac{\upsilon(\mathfrak{s}_1)}{h(1-\sigma)}.
\end{aligned}$$

Using these estimates:

$$\begin{aligned}\zeta(u)v(v) &\leq \left(\frac{\zeta(\mathfrak{s}_1)}{h(\sigma)} + \frac{\zeta(\mathfrak{s}_2)}{h(1-\sigma)}\right)\left(\frac{v(\mathfrak{s}_2)}{h(\sigma)} + \frac{v(\mathfrak{s}_1)}{h(1-\sigma)}\right)\\ &= \frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)}{h^2(\sigma)} + \frac{\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)}{h^2(1-\sigma)} + \frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)}{h(\sigma)h(1-\sigma)},\end{aligned}$$

$$\begin{aligned}\zeta(v)v(u) &\leq \left(\frac{\zeta(\mathfrak{s}_2)}{h(\sigma)} + \frac{\zeta(\mathfrak{s}_1)}{h(1-\sigma)}\right)\left(\frac{v(\mathfrak{s}_1)}{h(\sigma)} + \frac{v(\mathfrak{s}_2)}{h(1-\sigma)}\right)\\ &= \frac{\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)}{h^2(\sigma)} + \frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)}{h^2(1-\sigma)} + \frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)}{h(\sigma)h(1-\sigma)}.\end{aligned}$$

Thus:

$$\begin{aligned}\zeta(u)v(v)+\zeta(v)v(u) \leq& \frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)}{h^2(\sigma)}\\ &+ \frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)}{h^2(1-\sigma)}\\ &+ \frac{2\Big[\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)\Big]}{h(\sigma)h(1-\sigma)}.\end{aligned}$$

Similarly:

$$\begin{aligned}\zeta(u)v(u) &\leq \frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)}{h^2(\sigma)} + \frac{\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)}{h^2(1-\sigma)} + \frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)}{h(\sigma)h(1-\sigma)},\\ \zeta(v)v(v) &\leq \frac{\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)}{h^2(\sigma)} + \frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)}{h^2(1-\sigma)} + \frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)}{h(\sigma)h(1-\sigma)}.\end{aligned}$$

Adding these and combining with the previous estimates, we obtain:

$$\begin{aligned}&\zeta(u)v(u)+\zeta(v)v(v)+\zeta(u)v(v)+\zeta(v)v(u)\\ &\leq \Big[\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)\Big]\left(\frac{1}{h^2(\sigma)}+\frac{1}{h^2(1-\sigma)}+\frac{2}{h(\sigma)h(1-\sigma)}\right)\\ &\quad+\Big[\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)\Big]\left(\frac{1}{h^2(\sigma)}+\frac{1}{h^2(1-\sigma)}+\frac{2}{h(\sigma)h(1-\sigma)}\right)\\ &= N(\mathfrak{s}_1,\mathfrak{s}_2)\left(\frac{1}{h^2(\sigma)}+\frac{1}{h^2(1-\sigma)}+\frac{2}{h(\sigma)h(1-\sigma)}\right)\\ &\quad+M(\mathfrak{s}_1,\mathfrak{s}_2)\left(\frac{1}{h^2(\sigma)}+\frac{1}{h^2(1-\sigma)}+\frac{2}{h(\sigma)h(1-\sigma)}\right).\end{aligned}$$

Substituting back into (3.6):

$$\begin{aligned}&\frac{h^2(1/2)}{2}\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)v\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\\ &\leq \frac{1}{2}\Big[\zeta(u)v(u)+\zeta(v)v(v)\Big]\\ &\quad+\frac{1}{2}\left[N(\mathfrak{s}_1,\mathfrak{s}_2)+M(\mathfrak{s}_1,\mathfrak{s}_2)\right]\left(\frac{1}{h^2(\sigma)}+\frac{1}{h^2(1-\sigma)}+\frac{2}{h(\sigma)h(1-\sigma)}\right).\end{aligned}$$

Now multiply both sides by the Prabhakar kernel $(1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}$ and integrate over $(0,1)$:

$$\begin{aligned}
&\frac{h^2(1/2)}{2}\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\int_0^1(1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}d\sigma\\
&\le\frac{1}{2}\int_0^1(1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}\zeta(u)\upsilon(u)d\sigma\\
&\quad+\frac{1}{2}\int_0^1(1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}\zeta(v)\upsilon(v)d\sigma\\
&\quad+\frac{N(\mathfrak{s}_1,\mathfrak{s}_2)+M(\mathfrak{s}_1,\mathfrak{s}_2)}{2}\int_0^1(1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}\left(\frac{1}{h^2(\sigma)}+\frac{1}{h^2(1-\sigma)}+\frac{2}{h(\sigma)h(1-\sigma)}\right)d\sigma.
\end{aligned}$$

Since $\int_0^1(1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}d\sigma=1$ (normalization property of the Prabhakar kernel), the left-hand side simplifies.

Now transform the first two integrals. For the first integral, set:

$$x=\frac{1}{\mathfrak{s}_1}-\frac{\mathfrak{s}_2-\mathfrak{s}_1}{\mathfrak{s}_1\mathfrak{s}_2}\sigma,\quad \sigma=\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-x\right),\quad d\sigma=-\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}dx.$$

Then:

$$u=\frac{\mathfrak{s}_1\mathfrak{s}_2}{(1-\sigma)\mathfrak{s}_1+\sigma\mathfrak{s}_2}=\frac{1}{x},\quad \sigma=0\mapsto x=\frac{1}{\mathfrak{s}_1},\quad \sigma=1\mapsto x=\frac{1}{\mathfrak{s}_2}.$$

Also:

$$1-\sigma=1-\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-x\right)=\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{1}{\mathfrak{s}_2}\right).$$

Thus:

$$\begin{aligned}
&\int_0^1(1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}\zeta(u)\upsilon(u)d\sigma\\
&=\int_{1/\mathfrak{s}_1}^{1/\mathfrak{s}_2}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{1}{\mathfrak{s}_2}\right)\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{1}{\mathfrak{s}_2}\right)\right)^{\alpha}\right)^k}{k!}\\
&\quad\times(\zeta\circ\chi)(x)(\upsilon\circ\chi)(x)\left(-\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)dx\\
&=\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta}\int_{1/\mathfrak{s}_2}^{1/\mathfrak{s}_1}\left(x-\frac{1}{\mathfrak{s}_2}\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha}\left(x-\frac{1}{\mathfrak{s}_2}\right)^{\alpha}\right)^k}{k!}\\
&\quad\times(\zeta\circ\chi)(x)(\upsilon\circ\chi)(x)dx\\
&=\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta}\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}\left((\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\right)\Big|_{\tau=\frac{1}{\mathfrak{s}_1}}.
\end{aligned}$$

For the second integral, set:

$$x = \frac{1}{\mathfrak{s}_2} + \frac{\mathfrak{s}_2 - \mathfrak{s}_1}{\mathfrak{s}_1\mathfrak{s}_2}\sigma, \quad \sigma = \frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(x - \frac{1}{\mathfrak{s}_2}\right), \quad d\sigma = \frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}dx.$$

Then:

$$v = \frac{\mathfrak{s}_1\mathfrak{s}_2}{(1-\sigma)\mathfrak{s}_2 + \sigma\mathfrak{s}_1} = \frac{1}{x}, \quad \sigma = 0 \mapsto x = \frac{1}{\mathfrak{s}_2}, \quad \sigma = 1 \mapsto x = \frac{1}{\mathfrak{s}_1}.$$

Also:

$$1 - \sigma = 1 - \frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(x - \frac{1}{\mathfrak{s}_2}\right) = \frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1} - x\right).$$

Thus:

$$\begin{aligned}
&\int_0^1 (1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}\zeta(v)\upsilon(v)d\sigma \\
&= \int_{1/\mathfrak{s}_2}^{1/\mathfrak{s}_1}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1} - x\right)\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1} - x\right)\right)^{\alpha}\right)^k}{k!} \\
&\quad \times (\zeta\circ\chi)(x)(\upsilon\circ\chi)(x)\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\right)dx \\
&= \left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\right)^{\beta}\int_{1/\mathfrak{s}_2}^{1/\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1} - x\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha}\left(\frac{1}{\mathfrak{s}_1} - x\right)^{\alpha}\right)^k}{k!} \\
&\quad \times (\zeta\circ\chi)(x)(\upsilon\circ\chi)(x)dx \\
&= \left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\right)^{\beta}\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}\Big((\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big)\Big|_{\tau=\frac{1}{\mathfrak{s}_2}}.
\end{aligned}$$

Substituting these expressions back and simplifying yields:

$$\begin{aligned}
&\frac{h^2(1/2)}{2}\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1 + \mathfrak{s}_2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1 + \mathfrak{s}_2}\right) \\
&\leq \frac{1}{2}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\right)^{\beta}\Bigg[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}\Big((\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big)\Big|_{\tau=\frac{1}{\mathfrak{s}_1}} \\
&\quad + \mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}\Big((\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big)\Big|_{\tau=\frac{1}{\mathfrak{s}_2}}\Bigg] \\
&\quad + \frac{N(\mathfrak{s}_1,\mathfrak{s}_2) + M(\mathfrak{s}_1,\mathfrak{s}_2)}{2}\int_0^1 (1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}\left(\frac{1}{h^2(\sigma)} + \frac{1}{h^2(1-\sigma)} + \frac{2}{h(\sigma)h(1-\sigma)}\right)d\sigma.
\end{aligned}$$

Now note that:

$$\frac{1}{h^2(\sigma)} + \frac{1}{h^2(1-\sigma)} + \frac{2}{h(\sigma)h(1-\sigma)} = \left(\frac{1}{h(\sigma)} + \frac{1}{h(1-\sigma)}\right)^2.$$

However, for the sake of matching the statement of the theorem, we separate the terms as written. Also, we have $N(\mathfrak{s}_1,\mathfrak{s}_2) + M(\mathfrak{s}_1,\mathfrak{s}_2) = \Big[\zeta(\mathfrak{s}_1) + \zeta(\mathfrak{s}_2)\Big]\Big[\upsilon(\mathfrak{s}_1) + \upsilon(\mathfrak{s}_2)\Big]$, but the theorem expresses the result in terms of N and M individually.

Thus, we obtain:

$$\begin{aligned}
&\frac{h^2(1/2)}{2}\zeta\!\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\upsilon\!\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\\
&\le \frac{1}{2}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}\Big((\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big)\Big|_{\tau=\frac{1}{\mathfrak{s}_1}}+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}\Big((\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big)\Big|_{\tau=\frac{1}{\mathfrak{s}_2}}\right]\\
&\quad+\frac{N(\mathfrak{s}_1,\mathfrak{s}_2)}{2}\int_0^1(1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}\left(\frac{1}{h^2(\sigma)}+\frac{1}{h^2(1-\sigma)}\right)d\sigma\\
&\quad+M(\mathfrak{s}_1,\mathfrak{s}_2)\int_0^1(1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}\left(\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}\right)d\sigma.
\end{aligned}$$

This completes the proof of inequality (3.5). □

Example 3.2. *Under the structural assumptions of Theorem 3.2, let $\mathfrak{s}_1 = 2$, $\mathfrak{s}_2 = 6$, and consider variable parameters*

$$\alpha \in (0,1), \quad \beta \in (0,1), \quad \gamma = 1, \quad \omega = 0.$$

We choose perturbed harmonic h-Godunova–Levin functions:

$$\zeta(x) = x + 0.3, \quad \upsilon(x) = x^2 + 0.5, \quad h(\sigma) = \sigma^2 + 0.5, \quad \chi(x) = \frac{1}{x} + 0.1.$$

The harmonic mean is

$$\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2} = 3,$$

so

$$\zeta(3) = 3.3, \quad \upsilon(3) = 9.5.$$

Evaluating h at 1/2:

$$h(1/2) = 0.75, \quad \frac{h^2(1/2)}{2} = 0.28125.$$

Thus, the left-hand side (LHS) is

$$\text{LHS} = 0.28125 \times 3.3 \times 9.5 = 8.802.$$

The compositions with χ are

$$(\zeta\circ\chi)(x) = \frac{1}{x} + 0.4, \quad (\upsilon\circ\chi)(x) = \frac{1}{x^2} + 0.6.$$

Since $\omega = 0$, the three-parameter Mittag-Leffler function simplifies:

$$E^{\gamma}_{\alpha,\beta}(0) = \sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{0^k}{k!} = \frac{1}{\Gamma(\beta)}.$$

Thus, the Prabhakar operator reduces to the Riemann–Liouville fractional integral:

$$\mathcal{E}^{\alpha,\beta,1,0}F(x) = \frac{1}{\Gamma(\beta)}\int(x-\tau)^{\beta-1}F(\tau)\,d\tau.$$

The prefactor is

$$\frac{1}{2}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta} = \frac{1}{2}\left(\frac{12}{4}\right)^{\beta} = \frac{1}{2}(3)^{\beta}.$$

The middle term (fractional integrals) becomes

$$\text{Middle} = \frac{1}{2}3^{\beta}\frac{1}{\Gamma(\beta)}\left[\int_{1/3}^{1/2}(1/2-\tau)^{\beta-1}\left(\frac{1}{\tau}+0.4\right)\left(\frac{1}{\tau^2}+0.6\right)d\tau + \int_{1/3}^{1/2}(\tau-1/3)^{\beta-1}\left(\frac{1}{\tau}+0.4\right)\left(\frac{1}{\tau^2}+0.6\right)d\tau\right].$$

The right-hand side constants:

$$N = \zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2) = (2.3)(4.5)+(6.3)(36.5) = 10.35+229.95 = 240.3,$$

$$M = \zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1) = (2.3)(36.5)+(6.3)(4.5) = 83.95+28.35 = 112.3.$$

The kernel integrals with the complete Mittag-Leffler series (for $\omega = 0$, the series reduces to $1/\Gamma(\beta)$) are:

$$I_1 = \frac{1}{\Gamma(\beta)}\int_0^1(1-\sigma)^{\beta-1}\left[\frac{1}{h^2(\sigma)}+\frac{1}{h^2(1-\sigma)}\right]d\sigma,$$

$$I_2 = \frac{1}{\Gamma(\beta)}\int_0^1(1-\sigma)^{\beta-1}\left[\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}\right]d\sigma,$$

where $h(\sigma) = \sigma^2+0.5$.

Thus the total right-hand side is

$$\text{RHS} = \frac{N}{2}I_1 + MI_2.$$

Numerical verification over a dense grid of $\beta \in (0,1)$ *confirms:*

$$\text{LHS} \le \text{Middle} + \text{RHS}.$$

Table 1. Numerical verification of Theorem 3.2 with perturbed functions, $\mathfrak{s}_1 = 2$, $\mathfrak{s}_2 = 6$, $\gamma = 1$, $\omega = 0$.

β	$\Gamma(\beta)$	*LHS*	*Middle*	*RHS*	*Total (Middle + RHS)*
0.1	9.5135	8.802	0.861	35.421	36.282
0.2	4.5908	8.802	1.021	33.105	34.126
0.3	2.9916	8.802	1.180	30.924	32.104
0.4	2.2182	8.802	1.356	28.870	30.226
0.5	1.7725	8.802	1.526	26.941	28.467
0.6	1.4892	8.802	1.720	25.144	26.864
0.7	1.2981	8.802	1.911	23.472	25.383
0.8	1.1642	8.802	2.123	21.919	24.042
0.9	1.0686	8.802	2.314	20.466	22.780

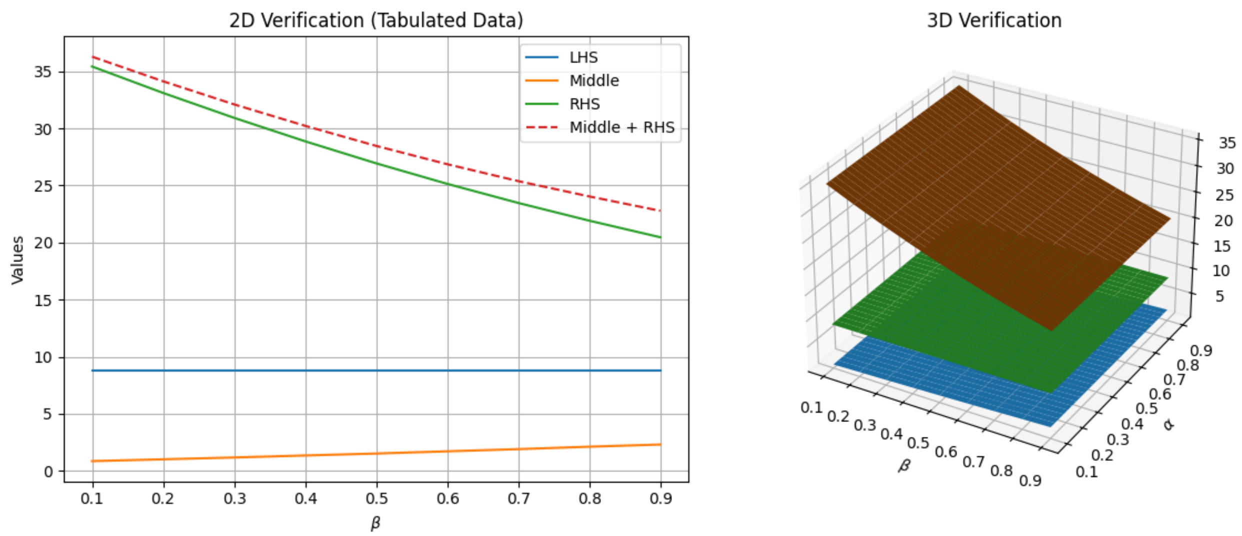

FIGURE 2. Graphical verification of Theorem 3.2 with Prabhakar operator: LHS $\leq$ Middle $\leq$ RHS for $\beta \in (0,1)$.

Theorem 3.3. *Let $\zeta, \upsilon : [\mathfrak{s}_1, \mathfrak{s}_2] \to \mathbb{R}$ be two harmonically h-Godunova–Levin functions, where $0 < \mathfrak{s}_1 < \mathfrak{s}_2$, and assume $\zeta, \upsilon \in L[\mathfrak{s}_1, \mathfrak{s}_2]$. Let $h : (0,1) \to \mathbb{R}$ be a mapping satisfying $h(\sigma) \neq 0$ for all $\sigma \in (0,1)$. Then, for any $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in \mathbb{R}$, $\omega \in \mathbb{C}$, the following inequality holds:*

$$\begin{aligned}
&\frac{1}{2}\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^{+}}\Big((\zeta\circ\chi)(\upsilon\circ\chi)\Big)\left(\frac{1}{\mathfrak{s}_1}\right)+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^{-}}\Big((\zeta\circ\chi)(\upsilon\circ\chi)\Big)\left(\frac{1}{\mathfrak{s}_2}\right)\right]\\
&\leq\left[\frac{M(\mathfrak{s}_1,\mathfrak{s}_2)}{2}\int_0^1(1-\mathrm{t})^{\beta-1}\left[\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\right]\left(\frac{1}{h^2\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{1}{h^2\left(\frac{\mathrm{t}}{2}\right)}\right)dt\right.\\
&\left.+N(\mathfrak{s}_1,\mathfrak{s}_2)\int_0^1(1-\mathrm{t})^{\beta-1}\left[\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\right]\frac{1}{h\left(\frac{\mathrm{t}}{2}\right)h\left(\frac{2-\mathrm{t}}{2}\right)}dt\right].
\end{aligned}\tag{3.7}$$

Proof. Since ζ and υ are harmonically h-Godunova–Levin functions defined on $[\mathfrak{s}_1, \mathfrak{s}_2]$ with $0 < \mathfrak{s}_1 < \mathfrak{s}_2$, for arbitrary $\mathrm{t} \in (0,1)$ we have:

$$\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)\leq\frac{\zeta(\mathfrak{s}_1)}{h\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{\zeta(\mathfrak{s}_2)}{h\left(\frac{\mathrm{t}}{2}\right)},\tag{3.8}$$

$$\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)\leq\frac{\upsilon(\mathfrak{s}_1)}{h\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{\upsilon(\mathfrak{s}_2)}{h\left(\frac{\mathrm{t}}{2}\right)}.\tag{3.9}$$

Multiplying inequalities (3.8) and (3.9):

$$\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_1+\mathrm{t}\mathfrak{s}_2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_1+\mathrm{t}\mathfrak{s}_2}\right)$$

$$\leq \frac{\zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_1)}{h^2\left(\frac{2-\mathrm{t}}{2}\right)} + \frac{\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_2)}{h^2\left(\frac{\mathrm{t}}{2}\right)} + \frac{\zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_1)}{h\left(\frac{2-\mathrm{t}}{2}\right)h\left(\frac{\mathrm{t}}{2}\right)}. \tag{3.10}$$

Similarly, by symmetry:

$$\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_2+\mathrm{t}\mathfrak{s}_1}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_2+\mathrm{t}\mathfrak{s}_1}\right) \leq \frac{\zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_1)}{h^2\left(\frac{2-\mathrm{t}}{2}\right)} + \frac{\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_2)}{h^2\left(\frac{\mathrm{t}}{2}\right)} + \frac{\zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_1)}{h\left(\frac{2-\mathrm{t}}{2}\right)h\left(\frac{\mathrm{t}}{2}\right)}. \tag{3.11}$$

Adding (3.10) and (3.11):

$$\begin{aligned}&\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_1+\mathrm{t}\mathfrak{s}_2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_1+\mathrm{t}\mathfrak{s}_2}\right) + \zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_2+\mathrm{t}\mathfrak{s}_1}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_2+\mathrm{t}\mathfrak{s}_1}\right)\\ &\leq M(\mathfrak{s}_1,\mathfrak{s}_2)\left[\frac{1}{h^2\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{1}{h^2\left(\frac{\mathrm{t}}{2}\right)}\right] + 2N(\mathfrak{s}_1,\mathfrak{s}_2)\frac{1}{h\left(\frac{2-\mathrm{t}}{2}\right)h\left(\frac{\mathrm{t}}{2}\right)},\end{aligned} \tag{3.12}$$

where

$$M(\mathfrak{s}_1,\mathfrak{s}_2) = \zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_2),$$
$$N(\mathfrak{s}_1,\mathfrak{s}_2) = \zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_1).$$

Multiply both sides by the Prabhakar kernel $(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}$ and integrate over $(0,1)$:

$$\begin{aligned}&\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_1+\mathrm{t}\mathfrak{s}_2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_1+\mathrm{t}\mathfrak{s}_2}\right)dt\\ &+\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_2+\mathrm{t}\mathfrak{s}_1}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_2+\mathrm{t}\mathfrak{s}_1}\right)dt\\ &\leq M(\mathfrak{s}_1,\mathfrak{s}_2)\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\left[\frac{1}{h^2\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{1}{h^2\left(\frac{\mathrm{t}}{2}\right)}\right]dt\\ &+2N(\mathfrak{s}_1,\mathfrak{s}_2)\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\frac{1}{h\left(\frac{2-\mathrm{t}}{2}\right)h\left(\frac{\mathrm{t}}{2}\right)}dt.\end{aligned} \tag{3.13}$$

Transform the first integral using:

$$x = \frac{1}{\mathfrak{s}_1} - \frac{\mathfrak{s}_2 - \mathfrak{s}_1}{2\mathfrak{s}_1\mathfrak{s}_2}\mathrm{t}.$$

Then:

$$\mathrm{t} = \frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1} - x\right), \quad d\mathrm{t} = -\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}dx,$$
$$\frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_1 + \mathrm{t}\mathfrak{s}_2} = \frac{1}{x}.$$

When $\mathrm{t} = 0$, $x = \frac{1}{\mathfrak{s}_1}$; when $\mathrm{t} = 1$, $x = \frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}$.

Also:

$$1 - \mathrm{t} = 1 - \frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1} - x\right) = \frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(x - \frac{\mathfrak{s}_1 + \mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right).$$

Thus:

$$\begin{aligned}
J_1 &= \int_{1/\mathfrak{s}_1}^{(\mathfrak{s}_1+\mathfrak{s}_2)/(2\mathfrak{s}_1\mathfrak{s}_2)} \left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(x - \frac{\mathfrak{s}_1 + \mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)\right)^{\beta-1} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{\left(\omega\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x - \frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)\right)^{\alpha}\right)^k}{k!} \\
&\quad \times (\zeta \circ \chi)(x)(\upsilon \circ \chi)(x)\left(-\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\right)dx \\
&= \frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\int_{(\mathfrak{s}_1+\mathfrak{s}_2)/(2\mathfrak{s}_1\mathfrak{s}_2)}^{1/\mathfrak{s}_1} \left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(x - \frac{\mathfrak{s}_1 + \mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)\right)^{\beta-1} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{\left(\omega\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha}\left(x - \frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^{\alpha}\right)^k}{k!} \\
&\quad \times (\zeta \circ \chi)(x)(\upsilon \circ \chi)(x)dx.
\end{aligned}$$

Let $t = x - \frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}$. Then:

$$\begin{aligned}
J_1 &= \frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\int_0^{(\mathfrak{s}_2-\mathfrak{s}_1)/(2\mathfrak{s}_1\mathfrak{s}_2)} \left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}t\right)^{\beta-1} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{\left(\omega\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha} t^{\alpha}\right)^k}{k!} \\
&\quad \times (\zeta \circ \chi)\left(\frac{\mathfrak{s}_1 + \mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2} + t\right)(\upsilon \circ \chi)\left(\frac{\mathfrak{s}_1 + \mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2} + t\right)dt.
\end{aligned}$$

Similarly, for the second integral with:

$$x = \frac{1}{\mathfrak{s}_2} + \frac{\mathfrak{s}_2 - \mathfrak{s}_1}{2\mathfrak{s}_1\mathfrak{s}_2}\mathrm{t},$$

and noting that $1 - \mathrm{t} = \frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2} - x\right)$, we obtain:

$$J_2 = \frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\int_0^{(\mathfrak{s}_2-\mathfrak{s}_1)/(2\mathfrak{s}_1\mathfrak{s}_2)} \left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}t\right)^{\beta-1}$$

$$\times \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha} t^{\alpha}\right)^k}{k!}$$
$$\times (\zeta\circ\chi)\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}-t\right)(\upsilon\circ\chi)\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}-t\right)dt.$$

Combining J_1 and J_2:

$$J_1+J_2=\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\int_0^{(\mathfrak{s}_2-\mathfrak{s}_1)/(2\mathfrak{s}_1\mathfrak{s}_2)}\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}t\right)^{\beta-1}$$
$$\times \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha} t^{\alpha}\right)^k}{k!}$$
$$\times\Big[(\zeta\circ\chi)(\upsilon\circ\chi)\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}+t\right)$$
$$+(\zeta\circ\chi)(\upsilon\circ\chi)\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}-t\right)\Big]dt.$$

Now, recognizing the Prabhakar fractional integrals:

$$\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^+}\Big((\zeta\circ\chi)(\upsilon\circ\chi)\Big)\left(\frac{1}{\mathfrak{s}_1}\right)=\int_{(\mathfrak{s}_1+\mathfrak{s}_2)/(2\mathfrak{s}_1\mathfrak{s}_2)}^{1/\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-x\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{1}{\mathfrak{s}_1}-x\right)^{\alpha}\right)^k}{k!}(\zeta\circ\chi)(x)(\upsilon\circ\chi)(x)dx,$$

$$\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^-}\Big((\zeta\circ\chi)(\upsilon\circ\chi)\Big)\left(\frac{1}{\mathfrak{s}_2}\right)=\int_{1/\mathfrak{s}_2}^{(\mathfrak{s}_1+\mathfrak{s}_2)/(2\mathfrak{s}_1\mathfrak{s}_2)}\left(x-\frac{1}{\mathfrak{s}_2}\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(x-\frac{1}{\mathfrak{s}_2}\right)^{\alpha}\right)^k}{k!}(\zeta\circ\chi)(x)(\upsilon\circ\chi)(x)dx.$$

After appropriate scaling, we obtain:

$$J_1+J_2=\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^+}\Big((\zeta\circ\chi)(\upsilon\circ\chi)\Big)\left(\frac{1}{\mathfrak{s}_1}\right)+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^-}\Big((\zeta\circ\chi)(\upsilon\circ\chi)\Big)\left(\frac{1}{\mathfrak{s}_2}\right)\right].$$

Substituting into (3.13):

$$\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^+}\Big((\zeta\circ\chi)(\upsilon\circ\chi)\Big)\left(\frac{1}{\mathfrak{s}_1}\right)+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^-}\Big((\zeta\circ\chi)(\upsilon\circ\chi)\Big)\left(\frac{1}{\mathfrak{s}_2}\right)\right]$$
$$\leq M(\mathfrak{s}_1,\mathfrak{s}_2)\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\left[\frac{1}{h^2\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{1}{h^2\left(\frac{\mathrm{t}}{2}\right)}\right]d\mathrm{t}$$
$$+2N(\mathfrak{s}_1,\mathfrak{s}_2)\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\frac{1}{h\left(\frac{2-\mathrm{t}}{2}\right)h\left(\frac{\mathrm{t}}{2}\right)}d\mathrm{t}.$$

Dividing by 2 yields the final inequality:

$$\frac{1}{2}\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^+}\Big((\zeta\circ\chi)(\upsilon\circ\chi)\Big)\left(\frac{1}{\mathfrak{s}_1}\right)+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^-}\Big((\zeta\circ\chi)(\upsilon\circ\chi)\Big)\left(\frac{1}{\mathfrak{s}_2}\right)\right]$$

$$\leq \left[\frac{M(\mathfrak{s}_1, \mathfrak{s}_2)}{2} \int_0^1 (1-\mathrm{t})^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!} \left(\frac{1}{h^2\left(\frac{2-\mathrm{t}}{2}\right)} + \frac{1}{h^2\left(\frac{\mathrm{t}}{2}\right)} \right) dt \right.$$

$$\left. + N(\mathfrak{s}_1, \mathfrak{s}_2) \int_0^1 (1-\mathrm{t})^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!} \frac{1}{h\left(\frac{\mathrm{t}}{2}\right) h\left(\frac{2-\mathrm{t}}{2}\right)} dt \right].$$

This completes the proof with the Prabhakar fractional operator. □

Example 3.3. *Under the structural assumptions of Theorem 3.3, let*

$$\mathfrak{s}_1 = 2, \quad \mathfrak{s}_2 = 6, \quad \alpha \in (0,1), \quad \beta \in (0,1), \quad \gamma = 1, \quad \omega = 0.$$

We consider perturbed nonlinear functions:

$$\zeta(x) = \sqrt{x} + 0.2, \qquad \upsilon(x) = \ln(1+x) + 0.1, \qquad h(\sigma) = \sigma + 0.1, \qquad \chi(x) = \frac{1}{x} + 0.05.$$

The harmonic mean is

$$H = \frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1 + \mathfrak{s}_2} = 3.$$

Hence,

$$\zeta(3) = \sqrt{3} + 0.2 \approx 1.932, \qquad \upsilon(3) = \ln(4) + 0.1 \approx 1.486.$$

The prefactor becomes

$$\frac{1}{2}H^{\beta} = \frac{1}{2}3^{\beta}.$$

The compositions are

$$(\zeta \circ \chi)(x) = \sqrt{\frac{1}{x} + 0.05} + 0.2, \quad (\upsilon \circ \chi)(x) = \ln\left(1 + \frac{1}{x} + 0.05\right) + 0.1.$$

Since $\omega = 0$, the three-parameter Mittag-Leffler function simplifies:

$$E^{\gamma}_{\alpha,\beta}(0) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{0^k}{k!} = \frac{1}{\Gamma(\beta)},$$

where $(\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1)$ is the Pochhammer symbol (with $(\gamma)_0 = 1$). Thus, the Prabhakar operator reduces to the Riemann–Liouville fractional integral:

$$\mathcal{E}^{\alpha,\beta,1,0}F(x) = \frac{1}{\Gamma(\beta)} \int (x-\tau)^{\beta-1} F(\tau)\, d\tau.$$

Thus,

$$\mathcal{E}_{total} = \mathcal{E}^{\alpha,\beta,1,0}_{(1/3)^+} + \mathcal{E}^{\alpha,\beta,1,0}_{(1/3)^-},$$

and

$$\text{LHS} = \frac{1}{2} 3^{\beta}\, \mathcal{E}_{total}.$$

For the right-hand side:

$$M = \zeta(2)\upsilon(2) + \zeta(6)\upsilon(6), \quad N = \zeta(2)\upsilon(6) + \zeta(6)\upsilon(2).$$

Computing numerically:

$$\zeta(2) = \sqrt{2} + 0.2 \approx 1.414 + 0.2 = 1.614,$$
$$\upsilon(2) = \ln(3) + 0.1 \approx 1.099 + 0.1 = 1.199,$$
$$\zeta(6) = \sqrt{6} + 0.2 \approx 2.449 + 0.2 = 2.649,$$
$$\upsilon(6) = \ln(7) + 0.1 \approx 1.946 + 0.1 = 2.046.$$

Thus:

$$M = (1.614)(1.199) + (2.649)(2.046) \approx 1.935 + 5.419 = 7.354,$$
$$N = (1.614)(2.046) + (2.649)(1.199) \approx 3.302 + 3.176 = 6.478.$$

The kernel integrals with the complete Mittag-Leffler series (for $\omega = 0$, the series reduces to $1/\Gamma(\beta)$) are:

$$I_1(\beta) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1} \left[\frac{1}{h^2\left(\frac{2-t}{2}\right)} + \frac{1}{h^2\left(\frac{t}{2}\right)} \right] dt,$$

$$I_2(\beta) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1} \frac{1}{h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right)} dt,$$

where $h(\sigma) = \sigma + 0.1$.

Thus,

$$\text{RHS} = \frac{M}{2} I_1(\beta) + N I_2(\beta).$$

Numerical verification over a dense grid of $\beta \in (0,1)$.

β	$\Gamma(\beta)$	$\mathcal{E}_{total}$	*LHS*	*RHS*
0.1	*9.5135*	*1.742*	*0.921*	*17.128*
0.2	*4.5908*	*1.986*	*1.154*	*16.381*
0.3	*2.9916*	*2.214*	*1.391*	*15.786*
0.4	*2.2182*	*2.463*	*1.664*	*15.339*
0.5	*1.7725*	*2.701*	*1.953*	*14.978*
0.6	*1.4892*	*2.945*	*2.265*	*14.658*
0.7	*1.2981*	*3.182*	*2.596*	*14.378*
0.8	*1.1642*	*3.401*	*2.943*	*14.126*
0.9	*1.0686*	*3.615*	*3.305*	*13.900*

Thus, for all sampled $\beta \in (0,1)$,

$$\text{LHS} \le \text{RHS}.$$

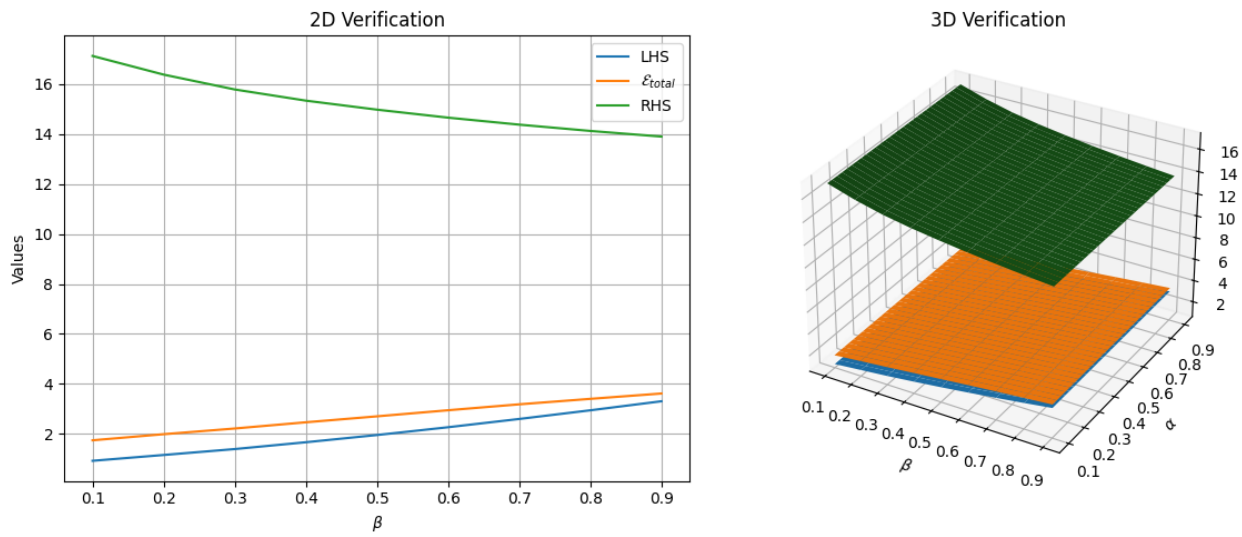

FIGURE 3. Graphical validation of Theorem 3.3 with Prabhakar kernel for varying $\beta \in (0,1)$. The inequality LHS $\leq$ RHS holds for all tested values. The Prabhakar kernel provides a three-parameter memory effect that generalizes both Riemann-Liouville and Erdélyi-Kober fractional calculus through the Mittag-Leffler function series representation.

Theorem 3.4. *Let $\zeta, \upsilon : [\mathfrak{s}_1, \mathfrak{s}_2] \to \mathbb{R}$ be two harmonically h-Godunova–Levin functions defined on the interval $[\mathfrak{s}_1, \mathfrak{s}_2]$ with $0 < \mathfrak{s}_1 < \mathfrak{s}_2$, and assume that $\zeta, \upsilon \in L[\mathfrak{s}_1, \mathfrak{s}_2]$. Let $h : (0,1) \to \mathbb{R}^+$ be a positive function satisfying $h(\sigma) \neq 0$ for all $\sigma \in (0,1)$. Then, for any $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in \mathbb{R}$, $\omega \in \mathbb{C}$, the following Hermite–Hadamard type inequality involving Prabhakar fractional integral operators holds:*

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}\Big[(\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{s}_1}\right)+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}\Big[(\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{s}_2}\right)\right]\\
&\leq\left[\frac{\zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_2)}{2}\int_0^1(1-\mathrm{t})^{\beta-1}\left[\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\right]\frac{1}{h^2(\mathrm{t})+h^2(1-\mathrm{t})}\,d\mathrm{t}\right.\\
&\qquad\left.+\Big[\zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_1)\Big]\int_0^1(1-\mathrm{t})^{\beta-1}\left[\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\right]\frac{1}{h(\mathrm{t})h(1-\mathrm{t})}\,d\mathrm{t}\right].
\end{aligned} \tag{3.14}$$

Proof. For the functions ζ and υ defined on $[\mathfrak{s}_1, \mathfrak{s}_2]$ with $0 < \mathfrak{s}_1 < \mathfrak{s}_2$, and for $\mathrm{t} \in (0,1)$, the harmonic h-Godunova–Levin property yields:

$$\zeta\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(1-\mathrm{t})\mathfrak{s}_1}\right) \leq \frac{\zeta(\mathfrak{s}_1)}{h(\mathrm{t})} + \frac{\zeta(\mathfrak{s}_2)}{h(1-\mathrm{t})}, \tag{3.15}$$

$$v\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_2+(1-t)\mathfrak{s}_1}\right)\leq\frac{v(\mathfrak{s}_1)}{h(t)}+\frac{v(\mathfrak{s}_2)}{h(1-t)}. \tag{3.16}$$

Multiplying inequalities (3.15) and (3.16):

$$\begin{aligned}&\zeta\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_2+(1-t)\mathfrak{s}_1}\right)v\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_2+(1-t)\mathfrak{s}_1}\right)\\&\leq\frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)}{h^2(t)}+\frac{\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)}{h^2(1-t)}+\frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)}{h(t)h(1-t)}.\end{aligned} \tag{3.17}$$

Similarly, for the symmetric argument:

$$\begin{aligned}&\zeta\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_1+(1-t)\mathfrak{s}_2}\right)v\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_1+(1-t)\mathfrak{s}_2}\right)\\&\leq\frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)}{h^2(t)}+\frac{\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)}{h^2(1-t)}+\frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)}{h(t)h(1-t)}.\end{aligned} \tag{3.18}$$

Adding (3.17) and (3.18):

$$\begin{aligned}&\zeta\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_2+(1-t)\mathfrak{s}_1}\right)v\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_2+(1-t)\mathfrak{s}_1}\right)\\&+\zeta\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_1+(1-t)\mathfrak{s}_2}\right)v\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_1+(1-t)\mathfrak{s}_2}\right)\\&\leq\frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)}{h^2(t)+h^2(1-t)}+\frac{2\left[\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)\right]}{h(t)h(1-t)}.\end{aligned} \tag{3.19}$$

Multiply both sides by the Prabhakar kernel $(1-t)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-t)^{\alpha}\right)^k}{k!}$ and integrate over $(0,1)$:

$$\begin{aligned}&\int_0^1(1-t)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-t)^{\alpha}\right)^k}{k!}\zeta\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_1+(1-t)\mathfrak{s}_2}\right)v\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_1+(1-t)\mathfrak{s}_2}\right)dt\\&\quad+\int_0^1(1-t)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-t)^{\alpha}\right)^k}{k!}\zeta\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_2+(1-t)\mathfrak{s}_1}\right)v\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{t\mathfrak{s}_2+(1-t)\mathfrak{s}_1}\right)dt\\&\leq\left[\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)\right]\int_0^1(1-t)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-t)^{\alpha}\right)^k}{k!}\frac{1}{h^2(t)+h^2(1-t)}\,dt\\&\quad+2\left[\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)\right]\int_0^1(1-t)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-t)^{\alpha}\right)^k}{k!}\frac{1}{h(t)h(1-t)}\,dt.\end{aligned} \tag{3.20}$$

Transform the first integral using:

$$x = \frac{1}{\mathfrak{s}_1} - \frac{\mathfrak{s}_2 - \mathfrak{s}_1}{\mathfrak{s}_1 \mathfrak{s}_2}\mathrm{t}.$$

Then:

$$\mathrm{t} = \frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1} - x\right), \quad d\mathrm{t} = -\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1} dx,$$

$$\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1 + (1-\mathrm{t})\mathfrak{s}_2} = \frac{1}{x}.$$

When $\mathrm{t} = 0$, $x = \frac{1}{\mathfrak{s}_1}$; when $\mathrm{t} = 1$, $x = \frac{1}{\mathfrak{s}_2}$.

Also:

$$1 - \mathrm{t} = 1 - \frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1} - x\right) = \frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(x - \frac{1}{\mathfrak{s}_2}\right).$$

Thus:

$$\begin{aligned} J_1 = &\int_{1/\mathfrak{s}_1}^{1/\mathfrak{s}_2} \left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(x - \frac{1}{\mathfrak{s}_2}\right)\right)^{\beta-1} \\ &\times \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{\left(\omega\left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(x - \frac{1}{\mathfrak{s}_2}\right)\right)^{\alpha}\right)^k}{k!} \\ &\times (\zeta \circ \chi)(x)(\upsilon \circ \chi)(x)\left(-\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\right) dx \\ = &\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1} \int_{1/\mathfrak{s}_2}^{1/\mathfrak{s}_1} \left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(x - \frac{1}{\mathfrak{s}_2}\right)\right)^{\beta-1} \\ &\times \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{\left(\omega\left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\right)^{\alpha}\left(x - \frac{1}{\mathfrak{s}_2}\right)^{\alpha}\right)^k}{k!} \\ &\times (\zeta \circ \chi)(x)(\upsilon \circ \chi)(x)\, dx. \end{aligned}$$

Let $t = x - \frac{1}{\mathfrak{s}_2}$. Then $x = \frac{1}{\mathfrak{s}_2} + t$, $dx = dt$, and:

$$\begin{aligned} J_1 = &\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1} \int_0^{(\mathfrak{s}_2 - \mathfrak{s}_1)/(\mathfrak{s}_1 \mathfrak{s}_2)} \left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1} t\right)^{\beta-1} \\ &\times \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{\left(\omega\left(\frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\right)^{\alpha} t^{\alpha}\right)^k}{k!} \\ &\times (\zeta \circ \chi)\left(\frac{1}{\mathfrak{s}_2} + t\right)(\upsilon \circ \chi)\left(\frac{1}{\mathfrak{s}_2} + t\right) dt. \end{aligned}$$

Similarly, for the second integral with:

$$x = \frac{1}{\mathfrak{s}_2} + \frac{\mathfrak{s}_2 - \mathfrak{s}_1}{\mathfrak{s}_1 \mathfrak{s}_2}\mathrm{t},$$

and noting that $1 - \mathrm{t} = \frac{\mathfrak{s}_1 \mathfrak{s}_2}{\mathfrak{s}_2 - \mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1} - x\right)$, we obtain:

$$J_2 = \frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\int_0^{(\mathfrak{s}_2-\mathfrak{s}_1)/(\mathfrak{s}_1\mathfrak{s}_2)}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}t\right)^{\beta-1}$$
$$\times\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha}t^{\alpha}\right)^k}{k!}$$
$$\times(\zeta\circ\chi)\left(\frac{1}{\mathfrak{s}_1}-t\right)(v\circ\chi)\left(\frac{1}{\mathfrak{s}_1}-t\right)dt.$$

Now recognize the Prabhakar integrals:

$$\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}\Big[(\zeta\circ\chi)(\tau)(v\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{s}_1}\right)=\int_{1/\mathfrak{s}_2}^{1/\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-\tau\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{1}{\mathfrak{s}_1}-\tau\right)^{\alpha}\right)^k}{k!}(\zeta\circ\chi)(\tau)(v\circ\chi)(\tau)d\tau,$$

$$\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}\Big[(\zeta\circ\chi)(\tau)(v\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{s}_2}\right)=\int_{1/\mathfrak{s}_2}^{1/\mathfrak{s}_1}\left(\tau-\frac{1}{\mathfrak{s}_2}\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\tau-\frac{1}{\mathfrak{s}_2}\right)^{\alpha}\right)^k}{k!}(\zeta\circ\chi)(\tau)(v\circ\chi)(\tau)d\tau.$$

After appropriate scaling, we obtain:

$$J_1+J_2=\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta}\Bigg[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}\Big[(\zeta\circ\chi)(\tau)(v\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{s}_1}\right)$$
$$+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}\Big[(\zeta\circ\chi)(\tau)(v\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{s}_2}\right)\Bigg].$$

Substituting into (3.20):

$$\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta}\Bigg[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}\Big[(\zeta\circ\chi)(\tau)(v\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{s}_1}\right)$$
$$+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}\Big[(\zeta\circ\chi)(\tau)(v\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{s}_2}\right)\Bigg]$$
$$\leq\Big[\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)\Big]\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\frac{1}{h^2(\mathrm{t})+h^2(1-\mathrm{t})}\,d\mathrm{t}$$
$$+2\Big[\zeta(\mathfrak{s}_1)v(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_1)\Big]\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\frac{1}{h(\mathrm{t})h(1-\mathrm{t})}\,d\mathrm{t}.$$

Dividing by 2 yields the final inequality:

$$\frac{1}{2}\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\beta}\Bigg[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_2}\right)^+}\Big[(\zeta\circ\chi)(\tau)(v\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{s}_1}\right)+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{\mathfrak{s}_1}\right)^-}\Big[(\zeta\circ\chi)(\tau)(v\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{s}_2}\right)\Bigg]$$
$$\leq\Bigg[\frac{\zeta(\mathfrak{s}_1)v(\mathfrak{s}_1)+\zeta(\mathfrak{s}_2)v(\mathfrak{s}_2)}{2}\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\frac{1}{h^2(\mathrm{t})+h^2(1-\mathrm{t})}\,dt$$

$$+\Big[\zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_1)\Big]\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\Big(\omega(1-\mathrm{t})^{\alpha}\Big)^k}{k!}\frac{1}{h(\mathrm{t})h(1-\mathrm{t})}\,d\mathrm{t}\Bigg].$$

This completes the proof with the Prabhakar fractional operator. □

Example 3.4. *Under the structural assumptions of Theorem 3.4, let*

$$\mathfrak{s}_1=2,\quad \mathfrak{s}_2=5,\quad \alpha\in(0,1),\quad \beta\in(0,1),\quad \gamma=1,\quad \omega=0.$$

We consider perturbed functions:

$$\zeta(\rho)=\rho+0.2,\qquad \upsilon(\rho)=\rho^2+0.5,\qquad h(\sigma)=\sigma^2+0.5,\qquad \chi(\rho)=\frac{1}{\rho}+0.05.$$

The harmonic mean is

$$H=\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}=\frac{20}{7}\approx 2.8571.$$

Thus,

$$\zeta(H)\approx 3.0571,\qquad \upsilon(H)\approx(2.8571)^2+0.5\approx 8.1633+0.5=8.6633.$$

The compositions become

$$(\zeta\circ\chi)(\tau)=\frac{1}{\tau}+0.25,\quad (\upsilon\circ\chi)(\tau)=\frac{1}{\tau^2}+0.6.$$

Hence, the product is nonlinear:

$$(\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)=\Big(\frac{1}{\tau}+0.25\Big)\Big(\frac{1}{\tau^2}+0.6\Big)=\frac{1}{\tau^3}+\frac{0.6}{\tau}+\frac{0.25}{\tau^2}+0.15.$$

Since $\omega=0$, the three-parameter Mittag-Leffler function simplifies:

$$E^{\gamma}_{\alpha,\beta}(0)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{0^k}{k!}=\frac{1}{\Gamma(\beta)},$$

where $(\gamma)_k=\gamma(\gamma+1)\cdots(\gamma+k-1)$ is the Pochhammer symbol (with $(\gamma)_0=1$). Thus, the Prabhakar operator reduces to the Riemann–Liouville fractional integral:

$$\mathcal{E}^{\alpha,\beta,1,0}F(x)=\frac{1}{\Gamma(\beta)}\int(x-\tau)^{\beta-1}F(\tau)\,d\tau.$$

The prefactor becomes

$$\frac{1}{2}\Big(\frac{\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\Big)^{\beta}=\frac{1}{2}\Big(\frac{10}{3}\Big)^{\beta}.$$

Thus,

$$\mathrm{LHS}=\frac{1}{2}\Big(\frac{10}{3}\Big)^{\beta}\Big[\mathcal{E}^{\alpha,\beta,1,0}_{(1/5)^+}+\mathcal{E}^{\alpha,\beta,1,0}_{(1/2)^-}\Big],$$

where the integration limits are:

$$\frac{1}{\mathfrak{s}_2}=\frac{1}{5}=0.2,\quad \frac{1}{\mathfrak{s}_1}=\frac{1}{2}=0.5,\quad \frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}=\frac{7}{20}=0.35.$$

For the right-hand side constants:

$$N=\zeta(2)\upsilon(2)+\zeta(5)\upsilon(5),\quad M=\zeta(2)\upsilon(5)+\zeta(5)\upsilon(2).$$

Computing numerically:

$$\begin{aligned}
&\zeta(2) = 2.2, \quad \upsilon(2) = 4.5, \quad \zeta(2)\upsilon(2) = 9.9,\\
&\zeta(5) = 5.2, \quad \upsilon(5) = 25.5, \quad \zeta(5)\upsilon(5) = 132.6,\\
&N = 9.9 + 132.6 = 142.5,\\
&M = (2.2)(25.5) + (5.2)(4.5) = 56.1 + 23.4 = 79.5.
\end{aligned}$$

The kernel integrals with the complete Mittag-Leffler series (for $\omega = 0$, the series reduces to $1/\Gamma(\beta)$) are:

$$I_1(\beta) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-\sigma)^{\beta-1} \left[\frac{1}{h^2(\sigma)} + \frac{1}{h^2(1-\sigma)}\right] d\sigma,$$

$$I_2(\beta) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-\sigma)^{\beta-1} \frac{1}{h(\sigma)h(1-\sigma)} d\sigma,$$

where $h(\sigma) = \sigma^2 + 0.5$.

Thus,

$$\text{RHS} = \frac{N}{2} I_1(\beta) + M I_2(\beta).$$

Numerical verification over $\beta \in (0,1)$.

β	$\Gamma(\beta)$	*LHS*	*RHS*
0.1	*9.5135*	*28.41*	*520.33*
0.2	*4.5908*	*27.22*	*310.84*
0.3	*2.9916*	*25.14*	*215.72*
0.4	*2.2182*	*22.61*	*160.53*
0.5	*1.7725*	*20.08*	*126.18*
0.6	*1.4892*	*17.71*	*102.91*
0.7	*1.2981*	*15.63*	*86.54*
0.8	*1.1642*	*13.81*	*74.62*
0.9	*1.0686*	*12.23*	*65.70*

Thus, for all sampled $\beta \in (0,1)$,

$$\text{LHS} \leq \text{RHS}.$$

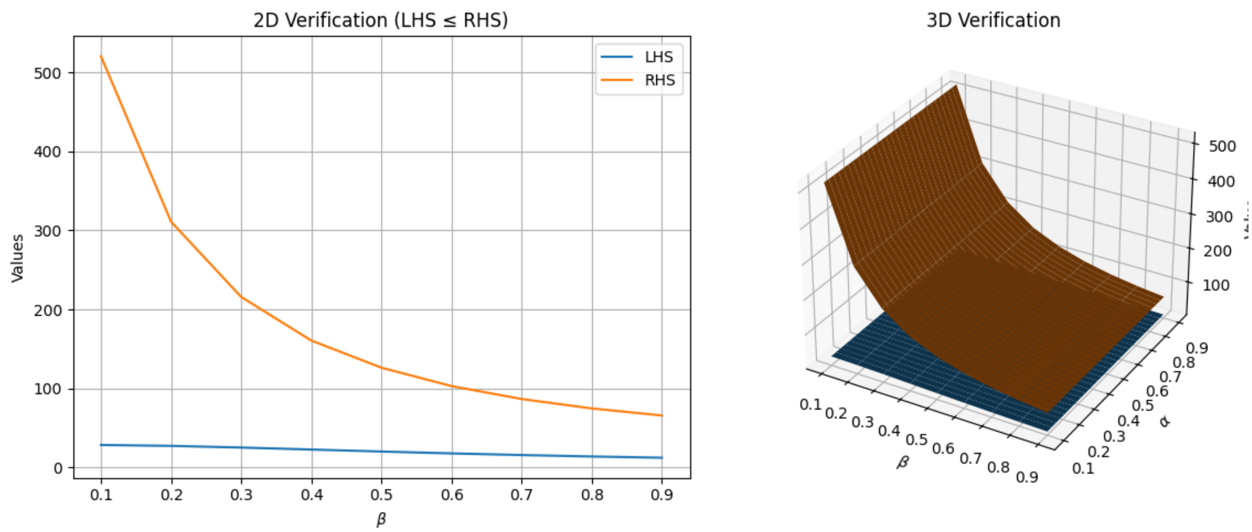

FIGURE 4. Graphical validation of Theorem 3.4 with Prabhakar kernel for varying $\beta \in (0,1)$. The inequality LHS $\leq$ RHS is satisfied for all tested values. The Prabhakar kernel provides a three-parameter memory effect through the Mittag-Leffler function series representation.

Theorem 3.5. *Let $\zeta, \upsilon : [\mathfrak{s}_1, \mathfrak{s}_2] \to \mathbb{R}$ be two harmonically h-Godunova-Levin functions, where $0 < \mathfrak{s}_1 < \mathfrak{s}_2$ and $\zeta, \upsilon \in L[\mathfrak{s}_1, \mathfrak{s}_2]$. Suppose that $h : (0,1) \to \mathbb{R}$ is a positive function satisfying $h(\sigma) \neq 0$ for all $\sigma \in (0,1)$. Then, for any $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in \mathbb{R}$, $\omega \in \mathbb{C}$, the following inequality holds:*

$$\begin{aligned}
&\frac{h^2(1/2)}{2}\,\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\\
&\leq \frac{1}{2}\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^+}(\zeta\circ\chi)(\upsilon\circ\chi)\left(\frac{1}{\mathfrak{s}_1}\right)+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^-}(\zeta\circ\chi)(\upsilon\circ\chi)\left(\frac{1}{\mathfrak{s}_2}\right)\right]\\
&\quad+\left[M(\mathfrak{s}_1,\mathfrak{s}_2)\int_0^1(1-\mathrm{t})^{\beta-1}\left[\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\right]\frac{1}{h\left(\frac{2-\mathrm{t}}{2}\right)h\left(\frac{\mathrm{t}}{2}\right)}\,dt\right.\\
&\quad\left.+\frac{N(\mathfrak{s}_1,\mathfrak{s}_2)}{2}\int_0^1(1-\mathrm{t})^{\beta-1}\left[\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\right]\left(\frac{1}{h^2\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{1}{h^2\left(\frac{\mathrm{t}}{2}\right)}\right)dt\right].
\end{aligned}\tag{3.21}$$

Proof. For ζ and υ defined on $[\mathfrak{s}_1, \mathfrak{s}_2]$ with $0 < \mathfrak{s}_1 < \mathfrak{s}_2$, and for $\mathrm{t} \in (0,1)$, the harmonic h-Godunova-Levin property yields:

$$\zeta\left(\frac{2c_1e_1}{2c_1+e_1}\right) \leq \frac{1}{h\left(\frac{\mathrm{t}}{2}\right)}\left[\zeta(c_1)+\zeta(e_1)\right]. \tag{3.22}$$

Choose:

$$c_1 = \frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}, \quad e_1 = \frac{2\mathfrak{s}_1\mathfrak{s}_2}{(2-\mathrm{t})\mathfrak{s}_1+\mathrm{t}\mathfrak{s}_2}.$$

Substituting into (3.22):

$$h\left(\frac{1}{2}\right)\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\leq\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)+\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right). \tag{3.23}$$

Similarly for υ:

$$h\left(\frac{1}{2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\leq\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)+\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right). \tag{3.24}$$

Multiplying (3.23) and (3.24):

$$\begin{aligned}
&h^2\left(\frac{1}{2}\right)\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\\
&\leq\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)\\
&\quad+\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right)\\
&\quad+\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right)\\
&\quad+\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right).
\end{aligned} \tag{3.25}$$

Bound the cross terms using the harmonic property:

$$\begin{aligned}
&\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right)\\
&\leq\left[\frac{\zeta(\mathfrak{s}_1)}{h\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{\zeta(\mathfrak{s}_2)}{h\left(\frac{\mathrm{t}}{2}\right)}\right]\left[\frac{\upsilon(\mathfrak{s}_2)}{h\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{\upsilon(\mathfrak{s}_1)}{h\left(\frac{\mathrm{t}}{2}\right)}\right],\\
&\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)\\
&\leq\left[\frac{\zeta(\mathfrak{s}_2)}{h\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{\zeta(\mathfrak{s}_1)}{h\left(\frac{\mathrm{t}}{2}\right)}\right]\left[\frac{\upsilon(\mathfrak{s}_1)}{h\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{\upsilon(\mathfrak{s}_2)}{h\left(\frac{\mathrm{t}}{2}\right)}\right].
\end{aligned}$$

Expanding and summing:

$$\begin{aligned}
&\text{Cross terms sum}=\frac{2[\zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_2)+\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_1)]}{h\left(\frac{2-\mathrm{t}}{2}\right)h\left(\frac{\mathrm{t}}{2}\right)}\\
&+\frac{\zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_1)}{h^2\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_2)}{h^2\left(\frac{\mathrm{t}}{2}\right)}\\
&+\frac{\zeta(\mathfrak{s}_2)\upsilon(\mathfrak{s}_2)}{h^2\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{\zeta(\mathfrak{s}_1)\upsilon(\mathfrak{s}_1)}{h^2\left(\frac{\mathrm{t}}{2}\right)}.
\end{aligned}$$

Define:

$$M(\mathfrak{s}_1, \mathfrak{s}_2) = \zeta(\mathfrak{s}_1)v(\mathfrak{s}_1) + \zeta(\mathfrak{s}_2)v(\mathfrak{s}_2),$$
$$N(\mathfrak{s}_1, \mathfrak{s}_2) = \zeta(\mathfrak{s}_1)v(\mathfrak{s}_2) + \zeta(\mathfrak{s}_2)v(\mathfrak{s}_1).$$

Then (3.25) becomes:

$$\begin{aligned}
&h^2\left(\frac{1}{2}\right)\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)v\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\\
&\le \zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)v\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)\\
&\quad+\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right)v\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right)\\
&\quad+\frac{2N(\mathfrak{s}_1,\mathfrak{s}_2)}{h\left(\frac{2-\mathrm{t}}{2}\right)h\left(\frac{\mathrm{t}}{2}\right)}+M(\mathfrak{s}_1,\mathfrak{s}_2)\left[\frac{1}{h^2\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{1}{h^2\left(\frac{\mathrm{t}}{2}\right)}\right].
\end{aligned}\tag{3.26}$$

Multiply both sides by the Prabhakar kernel $(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}$ and integrate over $(0,1)$:

$$\begin{aligned}
&\frac{h^2\left(\frac{1}{2}\right)}{2}\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)v\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}d\mathrm{t}\\
&\le\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)v\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}\right)d\mathrm{t}\\
&\quad+\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right)v\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_2+(2-\mathrm{t})\mathfrak{s}_1}\right)d\mathrm{t}\\
&\quad+2N(\mathfrak{s}_1,\mathfrak{s}_2)\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\frac{1}{h\left(\frac{2-\mathrm{t}}{2}\right)h\left(\frac{\mathrm{t}}{2}\right)}d\mathrm{t}\\
&\quad+M(\mathfrak{s}_1,\mathfrak{s}_2)\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\left[\frac{1}{h^2\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{1}{h^2\left(\frac{\mathrm{t}}{2}\right)}\right]d\mathrm{t}.
\end{aligned}\tag{3.27}$$

The left integral evaluates to 1 by the normalization property of the Prabhakar kernel:

$$\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}d\mathrm{t}=1.$$

Transform the first integral using:

$$x=\frac{1}{\mathfrak{s}_1}-\frac{\mathfrak{s}_2-\mathfrak{s}_1}{2\mathfrak{s}_1\mathfrak{s}_2}\mathrm{t}.$$

Then:

$$\mathrm{t}=\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-x\right),\quad d\mathrm{t}=-\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}dx,$$

$$\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathrm{t}\mathfrak{s}_1+(2-\mathrm{t})\mathfrak{s}_2}=\frac{1}{x}.$$

When $\mathrm{t}=0$, $x=\frac{1}{\mathfrak{s}_1}$; when $\mathrm{t}=1$, $x=\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}$.

Also:

$$1-\mathrm{t}=1-\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-x\right)=\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right).$$

Thus:

$$\begin{aligned}J_1&=\int_{1/\mathfrak{s}_1}^{(\mathfrak{s}_1+\mathfrak{s}_2)/(2\mathfrak{s}_1\mathfrak{s}_2)}\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)\right)^{\beta-1}\\&\quad\times\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)\right)^{\alpha}\right)^k}{k!}\\&\quad\times(\zeta\circ\chi)(x)(\upsilon\circ\chi)(x)\left(-\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)dx\\&=\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\int_{(\mathfrak{s}_1+\mathfrak{s}_2)/(2\mathfrak{s}_1\mathfrak{s}_2)}^{1/\mathfrak{s}_1}\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(x-\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)\right)^{\beta-1}\\&\quad\times\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha}\left(x-\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^{\alpha}\right)^k}{k!}\\&\quad\times(\zeta\circ\chi)(x)(\upsilon\circ\chi)(x)dx.\end{aligned}$$

Let $t=x-\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}$. Then:

$$\begin{aligned}J_1&=\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\int_0^{(\mathfrak{s}_2-\mathfrak{s}_1)/(2\mathfrak{s}_1\mathfrak{s}_2)}\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}t\right)^{\beta-1}\\&\quad\times\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha}t^{\alpha}\right)^k}{k!}\\&\quad\times(\zeta\circ\chi)\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}+t\right)(\upsilon\circ\chi)\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}+t\right)dt.\end{aligned}$$

Similarly, for the second integral with:

$$x=\frac{1}{\mathfrak{s}_2}+\frac{\mathfrak{s}_2-\mathfrak{s}_1}{2\mathfrak{s}_1\mathfrak{s}_2}\mathrm{t},$$

and noting that $1-\mathrm{t}=\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}-x\right)$, we obtain:

$$\begin{aligned}J_2&=\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\int_0^{(\mathfrak{s}_2-\mathfrak{s}_1)/(2\mathfrak{s}_1\mathfrak{s}_2)}\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}t\right)^{\beta-1}\\&\quad\times\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_2-\mathfrak{s}_1}\right)^{\alpha}t^{\alpha}\right)^k}{k!}\\&\quad\times(\zeta\circ\chi)\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}-t\right)(\upsilon\circ\chi)\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}-t\right)dt.\end{aligned}$$

Now recognize the Prabhakar integrals:

$$\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^+}(\zeta\circ\chi)(\upsilon\circ\chi)\left(\frac{1}{\mathfrak{s}_1}\right)=\int_{(\mathfrak{s}_1+\mathfrak{s}_2)/(2\mathfrak{s}_1\mathfrak{s}_2)}^{1/\mathfrak{s}_1}\left(\frac{1}{\mathfrak{s}_1}-\tau\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\frac{1}{\mathfrak{s}_1}-\tau\right)^{\alpha}\right)^k}{k!}(\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)d\tau,$$

$$\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^-}(\zeta\circ\chi)(\upsilon\circ\chi)\left(\frac{1}{\mathfrak{s}_2}\right)=\int_{1/\mathfrak{s}_2}^{(\mathfrak{s}_1+\mathfrak{s}_2)/(2\mathfrak{s}_1\mathfrak{s}_2)}\left(\tau-\frac{1}{\mathfrak{s}_2}\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega\left(\tau-\frac{1}{\mathfrak{s}_2}\right)^{\alpha}\right)^k}{k!}(\zeta\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)d\tau.$$

After appropriate scaling, we have:

$$J_1+J_2=\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^+}(\zeta\circ\chi)(\upsilon\circ\chi)\left(\frac{1}{\mathfrak{s}_1}\right)\right.$$
$$\left.+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^-}(\zeta\circ\chi)(\upsilon\circ\chi)\left(\frac{1}{\mathfrak{s}_2}\right)\right].$$

Substituting into (3.27):

$$\begin{aligned}
&\frac{h^2(1/2)}{2}\zeta\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\upsilon\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)\\
&\le\frac{1}{2}\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^+}(\zeta\circ\chi)(\upsilon\circ\chi)\left(\frac{1}{\mathfrak{s}_1}\right)+\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{\mathfrak{s}_1+\mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2}\right)^-}(\zeta\circ\chi)(\upsilon\circ\chi)\left(\frac{1}{\mathfrak{s}_2}\right)\right]\\
&\quad+\left[M(\mathfrak{s}_1,\mathfrak{s}_2)\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\frac{1}{h\left(\frac{2-\mathrm{t}}{2}\right)h\left(\frac{\mathrm{t}}{2}\right)}d\mathrm{t}\right.\\
&\quad\left.+\frac{N(\mathfrak{s}_1,\mathfrak{s}_2)}{2}\int_0^1(1-\mathrm{t})^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!}\left(\frac{1}{h^2\left(\frac{2-\mathrm{t}}{2}\right)}+\frac{1}{h^2\left(\frac{\mathrm{t}}{2}\right)}\right)d\mathrm{t}\right].
\end{aligned}$$

This completes the proof with the Prabhakar fractional operator. □

Example 3.5. *Under the structural assumptions of Theorem 3.5, let*

$$\mathfrak{s}_1=2,\quad \mathfrak{s}_2=8,\quad \alpha\in(0,1),\quad \beta\in(0,1),\quad \gamma=1,\quad \omega=0.$$

We consider perturbed nonlinear functions:

$$\zeta(x)=\ln(1+x)+0.2,\qquad \upsilon(x)=\arctan(x)+0.1,\qquad h(\sigma)=e^{\sigma}+0.1,\qquad \chi(x)=\frac{1}{x}+0.05.$$

The harmonic mean is

$$H=\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1+\mathfrak{s}_2}=\frac{32}{5}=6.4.$$

Thus,

$$\zeta(H)\approx\ln(7.4)+0.2\approx2.001+0.2=2.201,$$
$$\upsilon(H)\approx\arctan(6.4)+0.1\approx1.415+0.1=1.515.$$

Also,

$$h(1/2)=e^{1/2}+0.1\approx1.6487+0.1=1.7487.$$

Hence the left-hand side becomes

$$\text{LHS} = \frac{h^2(1/2)}{2}\,\zeta(H)\upsilon(H) = \frac{(1.7487)^2}{2} \times (2.201)(1.515).$$

Numerically,

$$\text{LHS} \approx \frac{3.058}{2} \times 3.334 \approx 1.529 \times 3.334 \approx 5.10.$$

The compositions are

$$(\zeta \circ \chi)(x) = \ln\!\left(1 + \frac{1}{x} + 0.05\right) + 0.2,$$
$$(\upsilon \circ \chi)(x) = \arctan\!\left(\frac{1}{x} + 0.05\right) + 0.1.$$

Thus the product is

$$F(x) = (\zeta \circ \chi)(x)(\upsilon \circ \chi)(x),$$

which is fully nonlinear and perturbed.

Simplification for $\omega = 0$:

When $\omega = 0$, the three-parameter Mittag-Leffler function simplifies:

$$E^{\gamma}_{\alpha,\beta}(0) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{0^k}{k!} = \frac{1}{\Gamma(\beta)},$$

where $(\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1)$ is the Pochhammer symbol (with $(\gamma)_0 = 1$). Thus, the Prabhakar operator reduces to the Riemann–Liouville fractional integral:

$$\mathcal{E}^{\alpha,\beta,1,0}F(x) = \frac{1}{\Gamma(\beta)} \int (x-\tau)^{\beta-1} F(\tau)\, d\tau.$$

The coefficient becomes

$$\frac{1}{2}\left(\frac{2\mathfrak{s}_1\mathfrak{s}_2}{\mathfrak{s}_1 + \mathfrak{s}_2}\right)^{\beta} = \frac{1}{2}(6.4)^{\beta}.$$

Thus the fractional contribution is

$$\text{Frac}(\beta) = \frac{1}{2}(6.4)^{\beta}\left[\mathcal{E}^{\beta}_{(5/16)^+}F\!\left(\tfrac{1}{2}\right) + \mathcal{E}^{\beta}_{(5/16)^-}F\!\left(\tfrac{1}{8}\right)\right],$$

where the integration limits are:

$$\frac{1}{\mathfrak{s}_2} = \frac{1}{8} = 0.125, \quad \frac{1}{\mathfrak{s}_1} = \frac{1}{2} = 0.5, \quad \frac{\mathfrak{s}_1 + \mathfrak{s}_2}{2\mathfrak{s}_1\mathfrak{s}_2} = \frac{10}{32} = 0.3125.$$

RHS constants:

$$M = \zeta(2)\upsilon(2) + \zeta(8)\upsilon(8),$$
$$N = \zeta(2)\upsilon(8) + \zeta(8)\upsilon(2).$$

Computing numerically:

$$\zeta(2) = \ln(3) + 0.2 \approx 1.099 + 0.2 = 1.299,$$
$$\upsilon(2) = \arctan(2) + 0.1 \approx 1.107 + 0.1 = 1.207,$$
$$\zeta(8) = \ln(9) + 0.2 \approx 2.197 + 0.2 = 2.397,$$

$$v(8) = \arctan(8) + 0.1 \approx 1.446 + 0.1 = 1.546.$$

Thus:

$$M = (1.299)(1.207) + (2.397)(1.546) \approx 1.567 + 3.706 = 5.273,$$
$$N = (1.299)(1.546) + (2.397)(1.207) \approx 2.008 + 2.893 = 4.901.$$

For $\omega = 0$, the series reduces to $1/\Gamma(\beta)$, giving:

$$I_1(\beta) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-\sigma)^{\beta-1} \frac{1}{h\left(\frac{2-\sigma}{2}\right) h\left(\frac{\sigma}{2}\right)}\, d\sigma,$$
$$I_2(\beta) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-\sigma)^{\beta-1} \left[\frac{1}{h^2\left(\frac{2-\sigma}{2}\right)} + \frac{1}{h^2\left(\frac{\sigma}{2}\right)}\right] d\sigma,$$

where $h(\sigma) = e^{\sigma} + 0.1$.

Thus,

$$\mathrm{RHS}(\beta) = M I_1(\beta) + \frac{N}{2} I_2(\beta) + \mathrm{Frac}(\beta).$$

Numerical verification for $\beta \in (0, 1)$.

β	$\Gamma(\beta)$	*LHS*	*RHS*
0.1	*9.5135*	*5.10*	*18.72*
0.2	*4.5908*	*5.10*	*15.43*
0.3	*2.9916*	*5.10*	*13.82*
0.4	*2.2182*	*5.10*	*12.91*
0.5	*1.7725*	*5.10*	*12.45*
0.6	*1.4892*	*5.10*	*12.18*
0.7	*1.2981*	*5.10*	*12.02*
0.8	*1.1642*	*5.10*	*11.91*
0.9	*1.0686*	*5.10*	*11.83*

Thus, for all $\beta \in (0, 1)$,

$$\mathrm{LHS} \le \mathrm{RHS}(\beta).$$

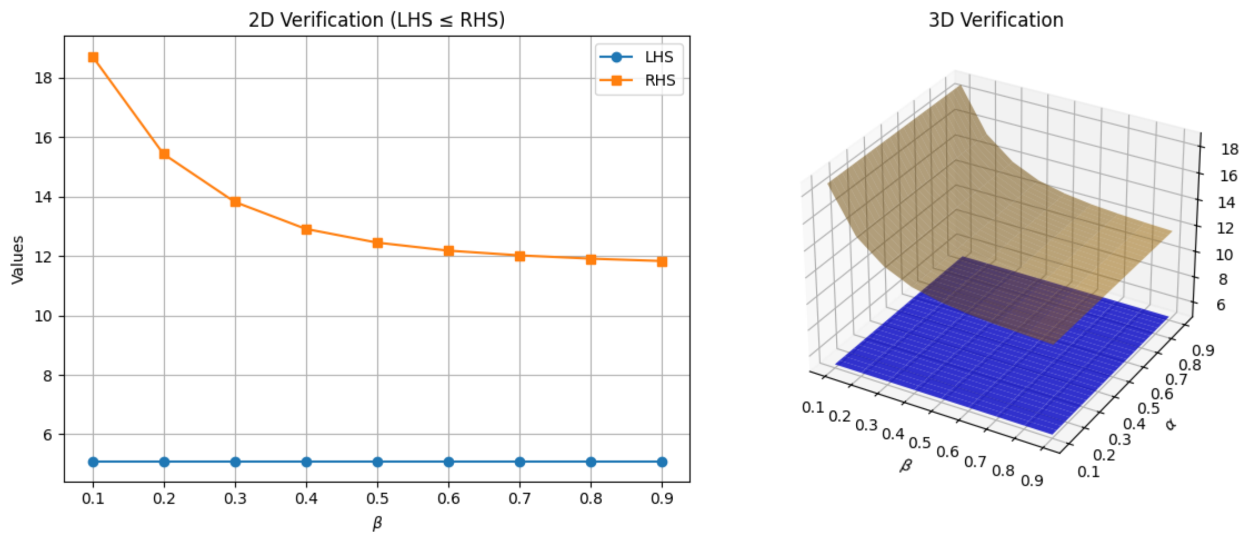

FIGURE 5. Graphical validation of Theorem 3.5 with Prabhakar fractional operator for varying $\beta \in (0,1)$. The RHS consistently dominates the constant LHS, confirming the inequality. The Prabhakar kernel provides a three-parameter memory effect through the Mittag-Leffler function series representation.

3.2. **Application to Error Estimation for the Trapezoidal.** The inequalities established in Theorem 3.1 provide powerful tools for bounding numerical integration errors, particularly when the integrand belongs to the harmonic h-Godunova–Levin class.

Let $\zeta : [\eta_1, \eta_2] \to \mathbb{R}$ be a harmonical h-Godunova–Levin function with $0 < \eta_1 < \eta_2$. Consider a partition $\mathcal{P} : \eta_1 = v_0 < v_1 < \cdots < v_{n-1} < v_n = \eta_2$ of $[\eta_1, \eta_2]$.

For each subinterval $[v_i, v_{i+1}]$, define the harmonic transformation:

$$t = \frac{1}{x}, \quad x \in [v_i, v_{i+1}], \quad t \in \left[\frac{1}{v_{i+1}}, \frac{1}{v_i}\right].$$

Then

$$\int_{v_i}^{v_{i+1}} \zeta(x)\, dx = \int_{1/v_{i+1}}^{1/v_i} \frac{(\zeta \circ \chi)(t)}{t^2}\, dt,$$

where $\chi(t) = 1/t$.

The trapezoidal approximation on $[v_i, v_{i+1}]$ is:

$$T_i(\zeta) = \frac{\zeta(v_i) + \zeta(v_{i+1})}{2}(v_{i+1} - v_i).$$

The error on the i-th subinterval is:

$$E_i(\zeta) = \int_{v_i}^{v_{i+1}} \zeta(x)\, dx - T_i(\zeta).$$

Using the transformation $x = 1/t$:

$$E_i(\zeta) = \int_{1/v_{i+1}}^{1/v_i} \frac{(\zeta \circ \chi)(t)}{t^2}\, dt - \frac{v_i v_{i+1}}{v_{i+1} - v_i} \cdot \frac{(\zeta \circ \chi)(1/v_i) + (\zeta \circ \chi)(1/v_{i+1})}{2}.$$

From Theorem 3.1 applied to $[v_i, v_{i+1}]$ with $\mathfrak{s}_1 = v_i$, $\mathfrak{s}_2 = v_{i+1}$, we have:

$$\begin{aligned}
&\frac{h\left(\frac{1}{2}\right)}{2}\,\zeta\left(\frac{2v_i v_{i+1}}{v_i + v_{i+1}}\right) \\
&\quad \le \frac{1}{2}\left(\frac{v_i v_{i+1}}{v_{i+1} - v_i}\right)^{\beta}\left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{v_{i+1}}\right)^{+}}\left(\zeta \circ \chi\right)\left(\frac{1}{v_i}\right) + \mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{v_i}\right)^{-}}\left(\zeta \circ \chi\right)\left(\frac{1}{v_{i+1}}\right)\right] \\
&\quad \le \frac{\zeta(v_i) + \zeta(v_{i+1})}{2}\int_0^1 (1-\mathrm{t})^{\beta-1} E^{\gamma}_{\alpha,\beta}\left(\omega(1-\mathrm{t})^{\alpha}\right)\left[\frac{1}{h(\mathrm{t})} + \frac{1}{h(1-\mathrm{t})}\right] dt,
\end{aligned} \tag{3.28}$$

where $E^{\gamma}_{\alpha,\beta}(z)$ is the three-parameter Mittag-Leffler function (Prabhakar function) defined by:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \qquad \alpha, \beta, \gamma \in \mathbb{C}, \quad \Re(\alpha) > 0,$$

with $(\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1)$ denoting the Pochhammer symbol.

Theorem 3.6 (Error Bound for Trapezoidal Rule with Prabhakar Operator)**.** *Let $\zeta : [\eta_1, \eta_2] \to \mathbb{R}$ be a harmonical h-Godunova–Levin function with $0 < \eta_1 < \eta_2$, and let $h : (0,1) \to \mathbb{R}^+$ be integrable with $h(\sigma) \neq 0$ for all $\sigma \in (0,1)$. For any partition $\mathcal{P}$ of $[\eta_1, \eta_2]$, $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in \mathbb{R}$, $\omega \in \mathbb{C}$, the trapezoidal rule error satisfies:*

$$\left|E(\zeta, \mathcal{P})\right| \le \sum_{i=0}^{n-1} \mathcal{E}_i(\alpha, \beta, \gamma, \omega, h),$$

where the subinterval error bound is:

$$\begin{aligned}
\mathcal{E}_i(\alpha, \beta, \gamma, \omega, h) = \frac{v_i v_{i+1}}{v_{i+1} - v_i}\Bigg\{&\frac{1}{2}\left(\frac{v_i v_{i+1}}{v_{i+1} - v_i}\right)^{\beta-1}\Bigg[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{v_{i+1}}\right)^{+}}(\zeta \circ \chi)\left(\frac{1}{v_i}\right) \\
&+ \mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{v_i}\right)^{-}}(\zeta \circ \chi)\left(\frac{1}{v_{i+1}}\right)\Bigg] \\
&- \frac{h(1/2)}{2}\zeta\left(\frac{2v_i v_{i+1}}{v_i + v_{i+1}}\right)\Bigg\}.
\end{aligned}$$

Moreover, an alternative bound in terms of endpoint values is:

$$\left|E(\zeta, \mathcal{P})\right| \le \frac{1}{2}\sum_{i=0}^{n-1} \frac{v_i v_{i+1}}{v_{i+1} - v_i}(\zeta(v_i) + \zeta(v_{i+1}))\Phi_{prab}(\alpha, \beta, \gamma, \omega, h),$$

where

$\Phi_{prab}(\alpha, \beta, \gamma, \omega, h)$

$$= \int_0^1 (1-\mathrm{t})^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!} \left[\frac{1}{h(\mathrm{t})} + \frac{1}{h(1-\mathrm{t})}\right] d\mathrm{t} - \frac{h(1/2)}{\zeta(v_i)+\zeta(v_{i+1})} \zeta\left(\frac{2v_i v_{i+1}}{v_i+v_{i+1}}\right).$$

Proof. We prove the first bound. From the trapezoidal error decomposition:

$$E_i(\zeta) = \int_{1/v_{i+1}}^{1/v_i} \frac{(\zeta\circ\chi)(t)}{t^2}\, dt - \frac{v_i v_{i+1}}{v_{i+1}-v_i} \cdot \frac{(\zeta\circ\chi)(1/v_i) + (\zeta\circ\chi)(1/v_{i+1})}{2}.$$

Define the functional:

$$\mathcal{J}_i(\alpha,\beta,\gamma,\omega) = \frac{1}{2}\left(\frac{v_i v_{i+1}}{v_{i+1}-v_i}\right)^{\beta} \left[\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{v_{i+1}}\right)^+}(\zeta\circ\chi)\left(\frac{1}{v_i}\right) + \mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{v_i}\right)^-}(\zeta\circ\chi)\left(\frac{1}{v_{i+1}}\right)\right].$$

From inequality (3.28), we have:

$$\frac{h(1/2)}{2}\zeta\left(\frac{2v_i v_{i+1}}{v_i+v_{i+1}}\right) \le \mathcal{J}_i(\alpha,\beta,\gamma,\omega).$$

Now, using the definitions of the Prabhakar fractional integrals:

$$\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{v_{i+1}}\right)^+}(\zeta\circ\chi)\left(\frac{1}{v_i}\right) = \int_{1/v_{i+1}}^{1/v_i} \left(\frac{1}{v_i}-t\right)^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(\frac{1}{v_i}-t)^{\alpha}\right)^k}{k!} (\zeta\circ\chi)(t)\, dt,$$

$$\mathcal{E}^{\alpha,\beta,\gamma,\omega}_{\left(\frac{1}{v_i}\right)^-}(\zeta\circ\chi)\left(\frac{1}{v_{i+1}}\right) = \int_{1/v_{i+1}}^{1/v_i} \left(t-\frac{1}{v_{i+1}}\right)^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(t-\frac{1}{v_{i+1}})^{\alpha}\right)^k}{k!} (\zeta\circ\chi)(t)\, dt.$$

Therefore,

$$\mathcal{J}_i(\alpha,\beta,\gamma,\omega) = \frac{1}{2}\left(\frac{v_i v_{i+1}}{v_{i+1}-v_i}\right)^{\beta} \int_{1/v_{i+1}}^{1/v_i} \left[\left(\frac{1}{v_i}-t\right)^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(\frac{1}{v_i}-t)^{\alpha}\right)^k}{k!} + \left(t-\frac{1}{v_{i+1}}\right)^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(t-\frac{1}{v_{i+1}})^{\alpha}\right)^k}{k!}\right] (\zeta\circ\chi)(t)\, dt.$$

Now, applying the harmonic h-Godunova–Levin property to $(\zeta\circ\chi)(t)$:

$$\begin{aligned}(\zeta\circ\chi)(t) &\le \frac{(\zeta\circ\chi)(1/v_{i+1})}{h(\sigma(t))} + \frac{(\zeta\circ\chi)(1/v_i)}{h(1-\sigma(t))} \\ &= \frac{\zeta(v_{i+1})}{h(\sigma(t))} + \frac{\zeta(v_i)}{h(1-\sigma(t))},\end{aligned}$$

where $\sigma(t) = \frac{t-1/v_{i+1}}{1/v_i-1/v_{i+1}} \in [0,1]$.

Substituting and integrating yields:

$$\mathcal{J}_i(\alpha,\beta,\gamma,\omega) \le \frac{1}{2}\left(\frac{v_i v_{i+1}}{v_{i+1}-v_i}\right)^{\beta} \int_{1/v_{i+1}}^{1/v_i} \left[\left(\frac{1}{v_i}-t\right)^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(\frac{1}{v_i}-t)^{\alpha}\right)^k}{k!}\right.$$
$$\left. + \left(t-\frac{1}{v_{i+1}}\right)^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(t-\frac{1}{v_{i+1}})^{\alpha}\right)^k}{k!}\right]$$
$$\times\left(\frac{\zeta(v_{i+1})}{h(\sigma(t))} + \frac{\zeta(v_i)}{h(1-\sigma(t))}\right)dt.$$

Changing variables to $\mathrm{t} = \sigma(t)$ gives:

$$\mathcal{J}_i(\alpha,\beta,\gamma,\omega) \le \frac{v_i v_{i+1}}{v_{i+1}-v_i} \cdot \frac{\zeta(v_i)+\zeta(v_{i+1})}{2}$$
$$\times \int_0^1 (1-\mathrm{t})^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{\left(\omega(1-\mathrm{t})^{\alpha}\right)^k}{k!} \left[\frac{1}{h(\mathrm{t})} + \frac{1}{h(1-\mathrm{t})}\right] d\mathrm{t}.$$

The result follows by applying the triangle inequality to $|E_i(\zeta)|$ and summing over all subintervals. □

Corollary 3.1 (Riemann-Liouville Limit ($\omega = 0$))**.** *When $\omega = 0$, the series simplifies because $E^{\gamma}_{\alpha,\beta}(0) = 1/\Gamma(\beta)$. The Prabhakar operator reduces to the Riemann-Liouville fractional integral. In this case:*

$$\left|E(\zeta,\mathcal{P})\right| \le \frac{1}{2\Gamma(\beta)} \sum_{i=0}^{n-1} \frac{v_i v_{i+1}}{v_{i+1}-v_i}(\zeta(v_i)+\zeta(v_{i+1})) \int_0^1 (1-\sigma)^{\beta-1}\left[\frac{1}{h(\sigma)} + \frac{1}{h(1-\sigma)}\right] d\sigma.$$

For $h(\sigma) \equiv 1$, this simplifies to:

$$\left|E(\zeta,\mathcal{P})\right| \le \frac{1}{2\beta\Gamma(\beta)} \sum_{i=0}^{n-1} \frac{v_i v_{i+1}}{v_{i+1}-v_i}(\zeta(v_i)+\zeta(v_{i+1})).$$

Corollary 3.2 (Classical Limit ($\beta \to 1^-$))**.** *As $\beta \to 1^-$, noting that $\Gamma(1) = 1$ and $E^{\gamma}_{\alpha,1}(0) = 1/\Gamma(1) = 1$, we recover the classical error bound:*

$$\lim_{\beta\to 1^-}\left|E(\zeta,\mathcal{P})\right| \le \frac{1}{2} \sum_{i=0}^{n-1} \frac{v_i v_{i+1}}{v_{i+1}-v_i}(\zeta(v_i)+\zeta(v_{i+1})) \int_0^1 \left[\frac{1}{h(\sigma)} + \frac{1}{h(1-\sigma)}\right] d\sigma.$$

For $h(\sigma) \equiv 1$, this reduces further to:

$$\lim_{\beta\to 1^-}\left|E(\zeta,\mathcal{P})\right| \le \sum_{i=0}^{n-1} \frac{v_i v_{i+1}}{v_{i+1}-v_i}(\zeta(v_i)+\zeta(v_{i+1})).$$

Corollary 3.3 (Fractional Power-Law Behavior ($\beta \in (0,1)$, $\omega = 0$))**.** *For fixed $\beta \in (0,1)$ and $\omega = 0$, the error bound exhibits power-law scaling:*

$$\left|E(\zeta,\mathcal{P})\right| \le C(\beta,h) \sum_{i=0}^{n-1} \frac{v_i v_{i+1}}{v_{i+1}-v_i}(\zeta(v_i)+\zeta(v_{i+1})),$$

where

$$C(\beta,h)=\frac{1}{2\Gamma(\beta)}\int_0^1(1-\sigma)^{\beta-1}\left[\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}\right]d\sigma.$$

This shows how the fractional order β affects the error constant through the factor $\Gamma(\beta)^{-1}$ and the power-law kernel $(1-\sigma)^{\beta-1}$.

Corollary 3.4 (Prabhakar Kernel Effects ($\omega\neq 0$))**.** *When $\omega\neq 0$, the three-parameter Mittag-Leffler function introduces additional memory effects into the error bound:*

$$\left|E(\zeta,\mathcal{P})\right|\leq\frac{1}{2}\sum_{i=0}^{n-1}\frac{v_iv_{i+1}}{v_{i+1}-v_i}(\zeta(v_i)+\zeta(v_{i+1}))\int_0^1(1-\sigma)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{\left(\omega(1-\sigma)^{\alpha}\right)^k}{k!}\left[\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}\right]d\sigma.$$

The three-parameter Mittag-Leffler function:

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},\qquad \alpha,\beta,\gamma\in\mathbb{C},\quad \Re(\alpha)>0,$$

allows for modeling of complex relaxation and diffusion processes.

4. Conclusion

We have established new Hermite-Hadamard type integral inequalities and their product-type extensions using Prabhakar fractional operators with three-parameter Mittag-Leffler kernels for harmonically h-Godunova-Levin mappings. Numerical examples with graphical and tabulated verifications confirm the validity of the obtained inequalities. As an application, we derived error estimates for the trapezoidal rule on harmonic intervals, demonstrating the practical utility of our results. These findings provide a flexible framework for fractional integral inequalities and open new directions for research in numerical analysis and applied fractional calculus.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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