

Modified Ulam-Hyers-Rassias Stability Analysis of Caputo Impulsive Fractional Delay Differential Equations

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Abstract. This article discuss the stability analysis of Caputo impulsive fractional delay differential equations using a modified Ulam-Hyers-Rassias stability approach. The effect of perturbations is intrinsically time-dependent due to its nonlocal and memory-dependent nature. The modified Ulam-Hyers-Rassias stability is crucial to fractional calculus. The real-valued function's integrability permits the inclusion of variable perturbations rather than a constant and guarantees that the stability inequality stays well-defined in the fractional situation. Therefore, compared to the classical Ulam-Hyers-Rassias stability technique, the modified approach offers a more realistic and general framework in fractional calculus. The findings contribute to improving the qualitative properties of fractional differential equations with delay factors, impulse disturbances, and memory effects.

1. INTRODUCTION

In order to describe memory and genetic characteristics present in many natural and engineering systems, fractional calculus has become a crucial tool. Fractional calculus offer a more flexible framework for modelling large dynamical systems whose future evolution depends on both their current state and their complete previous history, in contrast to standard integer-order derivatives. Due to their applications in physics, control theory, and biological systems. Impulsive fractional delay differential equations have attracted a lot of attention in recent years, many evolutionary processes have abrupt state shifts at particular times, which can be brought on by shocks, harvests, or natural disasters [1, 7]. The study of impulsive fractional differential equations therefore plays a vital role in understanding systems that exhibit both memory effects and instantaneous state

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changes. When these systems also involve delays, the complexity increases further, as time delays are often used to describe processes where the present rate of change depends on past states. This leads naturally to the study of impulsive fractional delay differential equations, which provide a realistic and flexible framework for modeling complex dynamical behaviors in population dynamics, epidemiology and neural networks.

In [8], the Riemann-Liouville fractional integral of order ν is defined as

$$({}_{t_0}I_t^\nu)u(t) = \frac{1}{\Gamma(\nu)} \int_{t_0}^t (t-\mu)^{\nu-1} u(\mu) d\mu, \quad (1.1)$$

where

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} dt \quad \nu > 0, \quad (1.2)$$

is the gamma function, and the Riemann-Liouville derivative of order ν has the following definition:

$$\begin{aligned} {}_{t_0}^{RL}D_t^\nu u(t) &= \\ \frac{d^n}{dt^n} {}_{t_0}I_t^{n-\nu} u(t) &= \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_{t_0}^t (t-\mu)^{n-\nu-1} u(\mu) d\mu. \end{aligned} \quad (1.3)$$

The Caputo fractional derivative of order ν is described in [9] as the modification of (1.3), which incorporates initial value problems in the formulation such as $u(t_0), u'(t_0)$ to provide a more physical interpretation.

$${}_{t_0}^C D_t^\nu u(t) = \frac{1}{\Gamma(n-\nu)} \int_{t_0}^t (t-\mu)^{n-\nu-1} D^n u(\mu) d\mu. \quad (1.4)$$

In a 1940 address at Wisconsin University, Ulam first discussed the stability of functional equations. Ulam posed the following question: Under what conditions does an approximately additive mapping exist near an additive mapping? In [2], Hyers gave the first answer to the Ulam question in the framework of Banach space in 1941. Let M_1, M_2 and $\epsilon > 0$ be two real Banach spaces. Furthermore, for every mapping $f : M_1 \rightarrow M_2$ that satisfies

$$\|f(u+h) - f(u) - f(h)\| \leq \epsilon$$

for all $u, h \in M_1$, there exists a unique additive mapping $g : M_1 \rightarrow M_2$ with the property

$$\|f(u) - g(u)\| \leq \epsilon$$

for every $u \in M_1$. This type of stability is then referred to as Ulam-Hyers stability. Rassias [3] (1978) provided an outstanding extension of the Ulam-Hyers stability of mapping by considering variables. The stability properties of various equations are of interest to many mathematicians. The study of Ulam-Hyers-Rassias stability in particular has drawn the attention of many mathematicians and is now one of the primary subjects in mathematical analysis. Meanwhile, it is noteworthy that [4–6] examines Ulam stability of fractional differential equations and derives a number of interesting and new stability discoveries.

[9], [13–21, 25, 26] discuss the qualitative analysis of existence, uniqueness, and stability of the

solution of fractional order derivative.

We examine the stability of the Caputo impulsive fractional delay differential equation of the following form in our work:

$$\begin{cases} {}^C D_t^\nu u(t) = f(t, u_t), & t \in [t_0, T], & t \neq t_r \\ \Delta u(t_r) = I_r(t_r, u(t_r)), & t = t_r, & r = 1, 2, \dots, l \\ u(t) = \phi(t) & t \in [-\tau, 0] \end{cases} \quad (1.5)$$

where $\Delta u(t) = u(t_r^+) - u(t_r^-)$, $t_0 < t_1 < \dots < t_l < t_{l+1} = T$, $f : [t_0, T] \times \mathbb{H} \rightarrow \mathbb{R}^n$ is Lebesgue measurable with respect to $t \in [t_0, T]$ and $f(t, \phi)$ is continuous with respect to $\phi \in \mathbb{H}$ and $I_r : [t_0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous for $r = 1, \dots, l$, where $\mathbb{H} = C([-\tau, t_0], \mathbb{R}^n)$ is the space of continuous functions.

Let $t \in [t_0, T]$, $r > 0$ and let

$$\{t_0 < t_1 < \dots < t_l < t_{l+1} = T\}$$

be a fixed finite partition of $[t_0, T]$. We define the space

$$PC([-\tau, T], \mathbb{R}^n) = \{u : [-\tau, T] \rightarrow \mathbb{R}^n \mid u \in C((t_r, t_{r+1}], \mathbb{R}^n), u \in C([-\tau, t_0], \mathbb{R}^n), \\ u(t_r^-) = \lim_{t \rightarrow t_r^-} u(t) \text{ exists}, u(t_r^+) = \lim_{t \rightarrow t_r^+} u(t) \text{ exists}, u(t_r^-) = (t_r) \text{ (left continuity)}\}.$$

Therefore, $PC([-\tau, T], \mathbb{R}^n)$ denote a space of piecewise continuous functions from $[-\tau, T]$ into \mathbb{R}^n , which are left continuous at the impulse, endowed with the norm

$$\|u\|_{PC} = \sup_{t \in [-\tau, T]} |u(t)|.$$

Therefore $PC([-\tau, T], \mathbb{R}^n, \|\cdot\|_{PC})$ is a Banach space. Since the impulses occur at

$$\{t_0 < t_1 < \dots < t_l < t_{l+1} = T\}$$

with impulse maps

$$I_r : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n \quad r = 1, \dots, l$$

The jump condition is written as

$$\Delta u(t) = u(t_r^+) - u(t_r^-) = I_r(t_r, z(t_r)).$$

For a given solution r , we define the norm as

$$\|I(u)\| = \max_{1 \leq r \leq l} \|I_r(t_r, u(t_r))\|.$$

Since there are only finitely many impulse points $t_1 \dots, t_l$. For a finite set. So

$$\sup_{1 \leq r \leq l} \|I_r(t_r, u(t_r))\| = \max_{1 \leq r \leq l} \|I_r(t_r, u(t_r))\|$$

The modified Ulam-Hyers-Rassias stability definitions in [24] are used for this stability analysis, and the perturbation bound depends on a real-valued function $\theta(t)$ that is Lebesgue integrable in $L^{\frac{1}{\psi}}[t_0, T]$. Because the effect of perturbations is intrinsically time-dependent due to its nonlocal

and memory-dependent nature, this novel definition of Ulam-Hyers-Rassias stability in [24] is crucial to fractional calculus. In the fractional setting, the integrability of $\theta(t)$ guarantees that the stability inequality is well-defined and permits the inclusion of variable perturbations instead of a constant. Therefore, compared to the classical Ulam-Hyers-Rassias stability technique, the improvement made in [24] offers a more realistic and general foundation in fractional calculus.

2. PRELIMINARIES

The Ulam-Hyers-Rassias stability of the following Caputo impulsive fractional differential equation in [15]

$$\begin{cases} {}^C D_t^\nu u(t) = f(t, u(t)), & t \in [t_0, T], & t \neq t_r \\ \Delta u(t_r) = I_r(t_r, u(t_r)), & t = t_r, & r = 1, 2, \dots, l \\ u(t_0) = u_0 \end{cases} \quad (2.1)$$

Definition 2.1. We said that equation (2.1) is Ulam-Hyers stable if there exists a constant $C_{f,n} > 0$ such that for some $\epsilon > 0$ and for every solution $h \in PC([t_0, T], \mathbb{R}^n)$ of the inequality

$$\begin{cases} \| {}^C D_t^\nu h(t) - f(t, h(t)) \| \leq \epsilon \\ \| \Delta h(t) - I_r(t_r, h(t)) \| \leq \epsilon \end{cases} \quad (2.2)$$

there exists a unique solution $u(t) \in PC([t_0, T], \mathbb{R}^n)$ of equation (2.1) with

$$\|h(t) - u(t)\| \leq C_{f,n}\epsilon \quad (2.3)$$

Definition 2.2. We said that equation (2.1) is generalized Ulam-Hyers stable if there exists $\kappa_{f,n} \in PC([t_0, T], \mathbb{R}^+)$ with $\kappa_{f,n}(0) = 0$ such that for every solution $h \in PC([t_0, T], \mathbb{R}^n)$ of the inequality (2.2), there exists a unique solution $u \in PC([0, T], \mathbb{R}^n)$ of equation (2.1) which satisfies

$$\|h(t) - u(t)\| \leq \kappa_{f,n}(\epsilon) \quad (2.4)$$

Definition 2.3. We said that equation (2.1) is Ulam-Hyers-Rassias stable with respect to ϕ , $\theta \in C([t_0, T], \mathbb{R})$ if there exists a real number $\kappa_{f,\theta} > 0$ such that for each $\epsilon > 0$ and for every solution $h \in PC([t_0, T], \mathbb{R}^n)$ of the inequality

$$\begin{cases} \| {}^C D_t^\nu h(t) - f(t, h(t)) \| \leq \epsilon\theta(t) \\ \| \Delta h(t) - I_r(t_r, h(t)) \| \leq \epsilon\theta(t) \end{cases} \quad (2.5)$$

there exists a unique solution $u \in PC([t_0, T], \mathbb{R}^n)$ of equation (2.1) with

$$\|h(t) - u(t)\| \leq \epsilon\kappa_{f,\theta}\theta(t) \quad (2.6)$$

Definition 2.4. We said that equation (2.1) is generalized Ulam-Hyers-Rassias stable with respect to θ if there exists a real number $\kappa_{f,\theta} > 0$ such that for every solution $h \in PC([t_0, T], \mathbb{R}^n)$ of inequality

$$\begin{cases} \| {}^C D_t^\nu h(t) - f(t, h(t)) \| \leq \theta(t) \\ \| \Delta h(t) - I_r(t_r, h(t)) \| \leq \theta(t) \end{cases} \quad (2.7)$$

there exists a unique solution $u \in PC([t_0, T], \mathbb{R}^n)$ of equation (2.1) which satisfies

$$\|h(t) - u(t)\| \leq \kappa_{f,\theta}\theta(t) \tag{2.8}$$

Remark 2.1. We say that $h \in PC([t_0, T], \mathbb{R}^n)$ is a solution of the inequality (2.7) if there exists a function $h(t) \in PC([t_0, T], \mathbb{R}^n)$ and a sequence g_r which depend upon h such that

- (i) $|p_t| \leq \theta(t)$ and $|p_r| \leq \eta_r$
- (ii) ${}^C D_t^\gamma h(t) = f(t, h_t) + p_t$
- (iii) $\Delta h(t) = I_r(t_r, h(t)) + p_r$

[24] build on this foundation and modified Ulam-Hyers-Rassias stability concept and the modified approached was used to analyzed the stability of the following impulsive fractional delay differential equations involving the Atangana-Baleanu derivative given below

$$\begin{cases} {}^{ABC} D_t^\gamma u(t) = f(t, u_t), & t \in [t_0, T], \quad t \neq t_r \\ \Delta u(t_r) = I_r(t_r, u(t_r)), & t = t_r, \quad r = 1, 2, \dots, l \\ u_{t_0} = \phi_0 \in PC' \end{cases} \tag{2.9}$$

[24] obtained the following new definitions

Definition 2.5. We said that the solution of equation (2.9) is stable via modified Ulam-Hyers-Rassias stability with respect to $\theta(t) \in L^{\frac{1}{\psi}}[t_0, T]$ and $\eta_r < \infty$ if there exists a real number $\kappa_{f,\theta} > 0$ such that for every solution $h \in PC([t_0, T], \mathbb{R}^n)$ of the inequality

$$\begin{cases} \|{}^C D_t^\gamma h(t) - f(t, h_t)\| \leq \theta(t) & t \in [t_0, T] \\ \|\Delta h(t) - I_r(t_r, h(t))\| \leq \eta_r & r = 1, \dots, l \end{cases} \tag{2.10}$$

there exists a unique solution $u \in PC([t_0, T], \mathbb{R}^n)$ of equation (1.5) which satisfies

$$\|h(t) - u(t)\| \leq \kappa_{f,\theta}(\theta(t) + \eta_r) \quad t \in [t_0, T] \tag{2.11}$$

Remark 2.2. We say that $h \in PC([t_0, T], \mathbb{R}^n)$ is a solution of the inequality (2.10) if there exists a function $h(t) \in PC([t_0, T], \mathbb{R}^n)$ and a sequence p_r which depend upon h such that

- (i) $|p_t| \leq \theta(t) \in L^{\frac{1}{\psi}} \in [t_0, T]$ and $|p_r| \leq \eta_r$
- (ii) ${}^C D_t^\gamma h(t) = f(t, h_t) + p_t$
- (iii) $\Delta h(t) = I_r(t_r, h(t)) + p_r$

Remark 2.3. [22, 23] Let $f : [t_0, T] \times \mathbb{H} \rightarrow \mathbb{R}^n$ be a Caratheodory function and $\psi \in (0, \nu)$ be a constant, then a function $f(t, u_t)$ is a $L^{\frac{1}{\psi}}$ -Caratheodory if there exists a real-valued function $\beta(t) \in L^{\frac{1}{\psi}}([t_0, T])$ such that

$$\|f(t, u_t)\| \leq \beta(t) \quad a.e \quad t \in [t_0, T]$$

Remark 2.4. [22, 23] Let $f : [t_0, T] \times \mathbb{H} \rightarrow \mathbb{R}^n$, a constant $\gamma \in (0, \nu)$ and a real-valued function $\chi(t) \in L^{\frac{1}{\gamma}}([t_0, T])$ then the function $f(t, u_t)$ is $L^{\frac{1}{\gamma}}$ -Lipschitz if

$$\|f(t, u_t) - f(t, h_t)\| \leq \chi(t)\|u_t - h_t\| \quad a.e \quad t \in [t_0, T]$$

The existence and uniqueness of Solution of equation (1.5) was established in [9], we build on this solutions to analyzed the stability of impulsive fractional delay differential equations involving the Caputo derivative.

[9] Let $0 < \nu < 1$ and $f : [t_0, T] \times \mathbb{H} \rightarrow \mathbb{R}^n$ be continuous, then the IVP

$$\begin{cases} {}^C D_t^\nu u(t) = f(t, u_t), & t \in [t_0, T], \\ u(t) = \phi(t) & t \in [-\tau, t_0] \end{cases} \quad (2.12)$$

is comparable to the following: Fractional integral Volterra with memory

$$\begin{cases} u_{t_0} = \phi_0 \\ u(t) = \phi_0(0) + \frac{1}{\Gamma(\nu)} \int_{t_0}^t f(\mu, u_\mu)(t - \mu)^{\nu-1} d\mu, & t \in [t_0, T]. \end{cases} \quad (2.13)$$

In other words, all solutions of (2.12) are also solutions of (2.13), and vice versa.

[9] A function $u \in PC[[t_0 - \tau, T], \mathbb{R}^n]$ is said to be a solution of (1.5) if u satisfies the equation ${}^C D_t^\nu u(t) = f(t, u_t)$ for almost everywhere $t \in [t_0, T]$; the equation $u(t_r^+) = u(t_r^-) + I_r(t_r, u(t_r^-))$ and the condition $u_{t_0} = \phi_0$ such that

$$u(t) = \begin{cases} \phi_0(0) + \frac{1}{\Gamma(\nu)} \int_{t_0}^t f(\mu, u_\mu)(t - \mu)^{\nu-1} d\mu, & t \in [t_0, T], \\ \phi_0(0) + \frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t_r - \mu)^{\nu-1} f(\mu, u_\mu) d\mu \\ + \frac{1}{\Gamma(\nu)} \int_{t_r}^t f(\mu, u_\mu)(t - \mu)^{\nu-1} d\mu \\ + \sum_{r=1}^l I_r(t_r, u(t_r)), & t \in [t_r, t_{r+1}], r = 1, \dots, l \end{cases} \quad (2.14)$$

Lemma 2.1. [9] Let $0 < \nu < 1$ and $f : [t_0, T] \times \mathbb{H} \rightarrow \mathbb{R}^n$ be Lebesgue measurable in t on $[t_0, T]$ and $f(t, \phi)$ is continuous with ϕ on PC' . A function $u \in PC[[t_0 - \tau, T], \mathbb{R}^n]$, is a solution of the IVP (1.5) if and only if $u \in PC[[t_0 - \tau, T], \mathbb{R}^n]$ satisfies the fractional integral equation (2.14)

Theorem 2.1. [9] Assume that the function $f : [t_0, T] \times \mathbb{H} \rightarrow \mathbb{R}^n$ satisfies the following conditions;
(G₁) There exists a real-valued function $\beta(t) \in L^{\frac{1}{\psi}}([t_0, T])$ such that $\|f(t, \phi)\| \leq \beta(t)$, for almost every $t \in [t_0, T]$ and all $\phi \in PC^1([t_0, T])$. Then, there exists a solution to the IVP (1.5) on $[t_0, T]$ provided

$$\frac{\|\phi(0)\| + \frac{1}{\Gamma(\nu)} \left(\frac{1-\psi}{\nu-\psi}\right)^{1-\psi} T^{\nu-\psi} B + lB^*}{b} < 1$$

Where $B = \left(\int_{t_0}^t (\beta(\mu))^{\frac{1}{\psi}} d\mu\right)^\psi$ and $B^* = \max\{\|I_r(t_r, u(t_r))\| : r = 1, \dots, l\}$

Theorem 2.2. [9] Assume that the conditions of Theorem 3 with the following conditions hold.

(G₂) There exists a constant $\gamma \in (0, \nu)$ and a real-valued function $\chi(t) \in L^{\frac{1}{\gamma}}([t_0, T])$ such that $\|f(t, \phi) - f(t, \xi)\| \leq \gamma(t)\|\phi - \xi\|$, for almost every $t \in [t_0, T]$ and all $\phi, \gamma \in PC'([0, T])$

(G₃) There exists a positive constant χ^* such that $\|I_r(t_r, \phi(0)) - I_r(t_r, \xi(0))\| \leq \chi^* \|\phi(0) - \xi(0)\|$, for each $\phi, \xi \in PC([0, T])$ and $r = 1, 2, \dots, l$, then, the IVP (1.5) has a unique solution on $[t_0, T]$ provided

$$0 < \left[\frac{1}{\Gamma(\nu)} \left(\frac{1-\psi}{\nu-\gamma}\right)^{1-\gamma} (T-t_0)^{\nu-\gamma} L + l\chi^*\right] < 1 \quad (2.15)$$

where $L = \left(\int_{t_0}^t (\chi(\mu))^{\frac{1}{\gamma}} d\mu \right)^\gamma$.

3. RESULTS

3.1. Modified Ulam-Hyers-Rassias stability analysis of Caputo impulsive fractional delay differential equation.

Lemma 3.1. *Let $h \in PC([t_0, T], \mathbb{R})$ be a solution of the inequality (2.10) then h is a solution of the following integral inequality*

$$\begin{aligned} & \left| h(t) - \xi_0(0) - \frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t - \mu)^{\nu-1} f(\mu, h_\mu) d\mu - \frac{1}{\Gamma(\nu)} \int_{t_r}^t (t - \mu)^{\nu-1} f(\mu, h_\mu) d\mu - \sum_{r=1}^l I_r(t_r, h(t_r)) \right| \\ & \leq \frac{\theta^*}{\Gamma(\nu)} \left(\frac{1 - \psi}{\nu - \psi} \right)^{1-\psi} T^{\nu-\psi} + l\eta_r \end{aligned}$$

where $\left(\int_{t_0}^t (\theta(\mu) d\mu)^{\frac{1}{\psi}} \right)^\psi = \theta^*$

Proof. By remark 2.2 we have

$$\begin{aligned} h(t) = & \xi_0(0) + \frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t - \mu)^{\nu-1} f(\mu, h_\mu) d\mu + \frac{1}{\Gamma(\nu)} \int_{t_r}^t (t - \mu)^{\nu-1} f(\mu, h_\mu) d\mu + \sum_{r=1}^l I_r(t_r, h(t_r)) + \\ & \frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t - \mu)^{\nu-1} g(\mu) d\mu + \frac{\nu}{\Gamma(\nu)} \int_{t_r}^t (t - \mu)^{\nu-1} g(\mu) d\mu + \sum_{r=1}^l p_r \end{aligned}$$

from this we have

$$\begin{aligned} & \left| h(t) - \xi_0(0) - \frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t - \mu)^{\nu-1} f(\mu, h_\mu) d\mu - \frac{1}{\Gamma(\nu)} \int_{t_r}^t (t - \mu)^{\nu-1} f(\mu, h_\mu) d\mu - \sum_{r=1}^l I_r(t_r, h(t_r)) \right| \\ & = \frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t - \mu)^{\nu-1} p(\mu) d\mu + \frac{1}{\Gamma(\nu)} \int_{t_r}^t (t - \mu)^{\nu-1} g(\mu) d\mu + \sum_{r=1}^l p_r \Big| \\ & \leq \frac{1}{\Gamma(\nu)} \sum_{r=1}^l \left(\int_{t_{r-1}}^{t_r} ((t - \mu)^{\nu-1})^{\frac{1}{1-\psi}} d\mu \right)^{1-\psi} \left(\int_{t_{r-1}}^{t_r} (\|g(\mu)\|)^{\frac{1}{\psi}} d\mu \right)^\psi \\ & + \frac{1}{\Gamma(\nu)} \left(\int_{t_r}^t ((t - \mu)^{\nu-1})^{\frac{1}{1-\psi}} d\mu \right)^{1-\psi} \left(\int_{t_{r-1}}^{t_r} (\|p(\mu)\|)^{\frac{1}{\psi}} d\mu \right)^\psi + \sum_{r=1}^l \|p_r\| \\ & \leq \frac{1}{\Gamma(\nu)} \sum_{r=1}^l \left(\int_{t_{r-1}}^{t_r} ((t - \mu)^{\nu-1})^{\frac{1}{1-\psi}} d\mu \right)^{1-\psi} \left(\int_{t_{r-1}}^{t_r} (\theta(\mu))^{\frac{1}{\psi}} d\mu \right)^\psi \\ & + \frac{1}{\Gamma(\nu)} \left(\int_{t_r}^t ((t - \mu)^{\nu-1})^{\frac{1}{1-\psi}} d\mu \right)^{1-\psi} \left(\int_{t_{r-1}}^{t_r} (\theta(\mu))^{\frac{1}{\psi}} d\mu \right)^\psi + \sum_{r=1}^l \|p_r\| \\ & \leq \frac{1}{\Gamma(\psi)} \left(\int_{t_r}^t ((t - \mu)^{\nu-1})^{\frac{1}{1-\psi}} d\mu \right)^{1-\psi} \left(\int_{t_{r-1}}^{t_r} (\theta(\mu))^{\frac{1}{\psi}} d\mu \right)^\psi + \sum_{r=1}^l \|p_r\| \end{aligned}$$

$$= \frac{\theta^*}{\Gamma(\nu)} \left(\frac{1-\psi}{\nu-\psi} \right)^{1-\psi} T^{\nu-\psi} + l\eta_r$$

where $\left(\int_{t_0}^t (\theta(\mu) d\mu)^{\frac{1}{\psi}} \right)^{\psi} = \theta^*$ □

Theorem 3.1. Let $\chi \in PC([t_0, T], \mathbb{R}^n)$ be a unique solution of equation (1.5) and $\phi(t) \in L^{\frac{1}{\psi}}([t_0, T])$ be a nondecreasing Lebesgue integrable function. Let $\kappa_{f,\theta} > 0$ be a real number such that for every solution $h \in PC([t_0, T], \mathbb{R}^n)$ of the inequality (2.10) satisfying

$$|h - u| \leq \kappa_{f,\theta}(\theta(t) + \eta_r),$$

then the unique solution of equation(1.5) is modified Ulam-Hyers-Rassias stable.

Proof.

$$\begin{aligned} |h(t) - u(t)| &= \left\| \frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t - \mu)^{\nu-1} [f(\mu, \xi) - f(\mu, \phi)] d\mu \right\| \\ &+ \left\| \frac{1}{\Gamma(\nu)} \int_{t_r}^t (t - \mu)^{\nu-1} [f(\mu, \xi) - f(\mu, \phi)] d\mu \right\| + \left\| \sum_{r=1}^l I_r [(t_r, \xi) - (t_r, \phi)] \right\| + \\ &\frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t - \mu)^{\nu-1} p(\mu) d\mu + \frac{1}{\Gamma(\nu)} \int_{t_r}^t (t - \mu)^{\nu-1} p(\mu) d\mu + \sum_{r=1}^l p_r \left\| \right. \\ &\leq \frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t - \mu)^{\nu-1} \|f(\mu, \xi) - f(\mu, \phi)\| d\mu \\ &+ \frac{1}{\Gamma(\nu)} \int_{t_r}^t (t - \mu)^{\nu-1} \|f(\mu, \xi) - f(\mu, \phi)\| d\mu + \sum_{r=1}^l I_r \|(t_r, \xi) - (t_r, \phi)\| + \\ &\frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t - \mu)^{\nu-1} \|p(\mu)\| d\mu + \frac{1}{\Gamma(\nu)} \int_{t_r}^t (t - \mu)^{\nu-1} \|p(\mu)\| d\mu + \sum_{r=1}^l \|p_r\| \\ &\leq \frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t - \mu)^{\nu-1} \chi(\mu) \|\xi - \phi\| d\mu + \frac{1}{\Gamma(\nu)} \int_{t_r}^t (t - \mu)^{\nu-1} \chi(\mu) \|\xi - \phi\| d\mu \\ &\quad + \sum_{r=1}^l \chi^* \max_{t \in [t_r, T]} \|\xi(0) - \phi(0)\| + \\ &\frac{1}{\Gamma(\nu)} \sum_{r=1}^l \int_{t_{r-1}}^{t_r} (t - \mu)^{\nu-1} \|p(\mu)\| d\mu + \frac{1}{\Gamma(\nu)} \int_{t_r}^t (t - \mu)^{\nu-1} \|p(\mu)\| d\mu + \sum_{r=1}^l \|p_r\| \\ &\leq \frac{1}{\Gamma(\psi)} \sum_{0 < t_r < t} \left(\int_{t_{r-1}}^{t_r} ((t_r - \mu)^{\nu-1})^{\frac{1}{1-\gamma}} d\mu \right)^{1-\gamma} \left(\int_{t_r}^t (\chi(\mu))^{\frac{1}{\gamma}} d\mu \right)^{\gamma} \|h - u\|_{\infty} + \\ &\frac{1}{\Gamma(\nu)} \left(\int_{t_r}^t ((t - \mu)^{\nu-1})^{\frac{1}{1-\gamma}} d\mu \right)^{1-\gamma} \left(\int_{t_r}^t (\chi(\mu))^{\frac{1}{\gamma}} d\mu \right)^{\gamma} \|h - u\|_{\infty} + l\chi^* \|h - u\|_{\infty} \\ &\quad + \left(\frac{\theta^*}{\Gamma(\nu)} \left(\frac{1-\psi}{\nu-\psi} \right)^{1-\psi} T^{\nu-\psi} + l\eta_r \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\nu)} \left(\int_{t_0}^t ((t-\mu)^{\nu-1})^{\frac{1}{1-\gamma}} d\mu \right)^{1-\gamma} \left(\int_{t_0}^t (\chi(\mu))^{\frac{1}{\gamma}} d\mu \right)^\gamma \|h-u\|_\infty + l\chi^* \|h-u\|_\infty \\
&\quad + \left(\frac{\theta^*}{\Gamma(\nu)} \left(\frac{1-\psi}{\nu-\psi} \right)^{1-\psi} T^{\nu-\psi} + l\eta_r \right) \\
&\leq \left(\frac{L}{\Gamma(\nu)} \left(\frac{1-\gamma}{\nu-\gamma} \right)^{1-\gamma} T^{\nu-\gamma} + l\chi^* \right) \|h-u\|_\infty + \left(\frac{\theta^*}{\Gamma(\nu)} \left(\frac{1-\gamma}{\nu-\gamma} \right)^{1-\gamma} T^{\nu-\gamma} + l\eta_r \right) \\
&\quad \|h-u\| \leq \frac{1}{1-q} \left(\frac{\theta^*}{\Gamma(\nu)} \left(\frac{1-\gamma}{\nu-\gamma} \right)^{1-\gamma} T^{\nu-\gamma} + l\eta_r \right) \\
&\quad \|h-u\| \leq \kappa_{f,\theta}(\theta(t) + \eta_r)
\end{aligned}$$

where

$$q = 0 < \left(\frac{L}{\Gamma(\nu)} \left(\frac{1-\gamma}{\nu-\gamma} \right)^{1-\gamma} T^{\nu-\gamma} + l\chi^* \right) < 1.$$

$\left(\int_{t_0}^t (\chi(\mu) d\mu)^{\frac{1}{\gamma}} \right)^\gamma = L$ and $\left(\int_{t_0}^t (\theta(\mu) d\mu)^{\frac{1}{\psi}} \right)^\psi = \theta^*$. We conclude that the unique solution of Caputo impulsive fractional delay differential equation (1.5) is modified Ulam-Hyers-Rassias stable. \square

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