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On Some Topological Properties of the Structural Graph of Dynamical Systems

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Abstract. Within the scope of the qualitative study of dynamical systems via graph-based methods, we extend the notion of structural graph to dynamical systems generated by actions of monoids, as abstract algebraic structures. The setting considers the action of an ordered monoid (not necessarily $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{R}$) on a metric space in a general sense, and therefore includes a wider class of dynamical systems than those described solely by ordinary differential or difference equations. We then investigate several fundamental properties of the structural graph, such as connectedness and degeneracy. In particular, it is shown that in the finite case, the structural graph associated to a C^1-vector field on a closed differential manifold is necessarily connected. As illustrative examples, we consider the classical van der Pol oscillator on the Poincaré sphere, the mathematical pendulum on the phase plane and the nonanalytic case of the center-focus. And as a perspective, we suggest work about what could be considered as a new form of linearization of dynamical systems via their structural graph, and address the question about a possible use for investigating global qualitative properties such as structural stability.

1. Introduction

Since the pioneering work of Hadamard [12] and his original idea to use orbit coding for studying geodesics on surfaces of negative curvature, followed by Morse [18], who is recognized as the founder of symbolic dynamics, Birkhoff [4] and his method for coding trajectories near a homoclinic orbit, and more recently, Smale [24] and his famous 'horseshoe', the notion of symbolic image of a dynamical system, which naturally leads to the notion of structural graph, has become a powerful mathematical tool to investigate the behavior of extremely complex dynamical systems. In effective algebra, for example, there was perhaps no more fruitful research area in the last decade than the study of algebraic structures within graph theory: the introduction by Beck [3] of the zero-divisor graph of a commutative ring, enhanced by the work of Anderson and Livingston [1], have set off an avalanche of publications and points of view, all leading to interesting results in the subject, but leaving the problem of completeness of such a graph, as well as the general inverse

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problem of how to recover the underlying ring up to isomorphism, still unsolved. We do not give a survey of the interaction between research areas in mathematics, such as dynamical systems, geometry or algebra, and graph theory, as it would load these pages too heavily to present such material here. We rather emphasize the notion of the structural graph of a dynamical system, a term which is used throughout this paper in the sense of Osipenko [20] (see also references therein, and especially [6] for the notion of chain recurrence). While previous research works have mainly emphasized the effective construction of the structural graph via the notion of symbolic image of a dynamical system, there is currently a lack of comprehensive analysis on its topological properties (completeness, planarity, degree and type of connectivity, etc.), and their dynamical interpretation in terms of global qualitative properties of the underlying dynamical system. We do not claim to address this gap in the present study, but we just try to present some ideas to help starting exploration of this uncharted area of research. The reader is supposed to be acquainted with the rudiments of the qualitative theory of differential equations, and optionally, to be able to identify the phase portrait of a dynamical system to its transition graph, the latter being obviously a deductive system and a category in Lambek' sense [15], as it will be worth mentioning in the last section. For the structural graph, as defined in [20], 'a vertex i corresponds to a component Q_i of the chain recurrent set Q. Each edge $i \to j$ of the graph corresponds to the orbits which have the α-limit set in the component Q_i and the ω-limit set in the component Q_j, i.e. the edge $i \to j$ corresponds to the connection $Q_i \to Q_j$'. Then, based on the interesting paper of Giunti and Mazzola [9], where dynamical systems on monoids are presented as minimal mathematical models for the intuitive notion of deterministic dynamics, all the involved dynamical objects such as chain recurrent components (nodes of the graph), limit sets ('sources'and 'targets'of the links), will be (re)defined using just the action of an ordered monoid on a metric space, thus obtaining a general definition for the structural graph of a dynamical system. In this context, we proceed to examine certain aspects of the structural graph, with a particular emphasis on its connectedness and degeneracy (section 2). Finally, taking profitably the notion of structural graph, we suggest what could be seen as a new form of linearization for dynamical systems via the notion of signless Laplacian of a graph [10], with a possible use in a global qualitative study of dynamical systems, a promising subject that we want to be developed in future work.

2. Redefining the structural graph while minimizing the conditions in the definition of a dynamical system

Dynamical systems refer to mathematical models that describe how variables change over time, based on a set of rules or equations. These systems are used to study complex behaviors that arise from the interactions of multiple variables. Usually, dynamical systems are described by one or more differential equations that define their behavior over short time windows, or by difference equations. In the most abstract setting, we consider a dynamical system as the action of a group (practically the set of integer or real numbers) on a non-empty set. However, by endowing a dynamical system with a structure of graph and proving that such a graph is a Lambek-category if and only if the time model of the dynamical system is a monoid, Giunti and Mazzola [9] have shown that the minimal structure required on the time set is only that of a monoid, yet allowing to define a large family of meaningful dynamical concepts and deriving interesting results. This general approach for the standard notion of a dynamical system will be adopted in the present paper to introduce a quite general definition of the structural graph, while additional hypotheses such as a total order on the monoid or a metric on the phase space will be considered when necessary.

2.1. **Definition of a dynamical system on a monoid.**

Definition 2.1. *DS is a dynamical system on M if DS is a pair $(E, (\varphi^t)_{t\in T})$ and M is a pair $(T, +)$ such that*

1. *$M = (T, +)$ is a monoid. Any $t \in T$ is called a duration of the system, T is called its time set, and M its time model ;*
2. *E is a non-empty set. Any $x \in E$ is called a state of the system, and E is called its state or phase space ;*
3. *$(\varphi^t)_{t\in T}$ is a family indexed by T of functions from E to E. For any $t \in T$, the function φ^t is called the state transition of duration t (briefly, t-transition, or t-advance) of the system ;*
4. *for any $s, t \in T$, for any $x \in E$,*
   (i) *$\varphi^0(x) = x$, where 0 is the neutral element of M,*
   (ii) *$\varphi^{s+t}(x) = \varphi^t(\varphi^s(x))$.*

This section expands on the definition previously presented by redefining all involved dynamical objects, including recurrent points, nonwandering points, chain recurrent points, limit sets, and more. The redefinition is accomplished solely by using the action of an ordered monoid on a metric space for a given dynamical system. This approach will provide a generalized definition for the structural graph.

Definition 2.2.
1. *The orbit (or trajectory) of x is the set $orb(x) := \{\varphi^t(x), t \in T\}$.*
2. *γ is an orbit if for some $x \in E$, $\gamma = orb(x)$.*
3. *The phase portrait is the set $\{\gamma : \gamma$ is an orbit$\}$.*
4. *x is a fixed point if $orb(x) = \{x\}$.*
5. *A non-fixed point x is a periodic point if for some $t \in T$, $t \neq 0$, $\varphi^t(x) = x$. In this case, $orb(x)$ is called a periodic orbit. And if (E, d) is a metric space, and $M = (T, +, \leq)$ is a totally ordered monoid:*
6. *x is a recurrent point if for each neighborhood U of x and $t \in T$, there is $s > t$ (i.e. $s \geq t$ and $s \neq t$) so that $\varphi^s(x) \in U$.*

7. *x is a nonwandering point if for each neighborhood U of x and $t \in T$, there is $s > t$ so that $\varphi^s(U) \cap U \neq \emptyset$.*
8. *Given $\varepsilon > 0$, $t > 0$ and $x, y \in E$, an (ε, t)-chain from x to y is a pair of finite sequences $x = x_0, x_1, ..., x_{n-1}, x_n = y$ in E and $t_0, ..., t_{n-1}$ in T such that $\sum_{j=0}^{n-1} \sigma_j t_j \geq t$, the weight coefficients σ_j taking only the value 0 or 1 with at least one coefficient being nonzero, and $d(\varphi^{t_i}(x_i), x_{i+1}) < \varepsilon$ for $0 \leq i \leq n-1$.*
9. *Two points x and y are chain equivalent if for every $\varepsilon > 0$ and $t > 0$ there exists an (ε, t)-chain from x to y and their exists an (ε, t)-chain from y to x.*
10. *A point x is called chain recurrent if x is chain equivalent to itself.*

It is evident that a periodic point is recurrent, and a recurrent point is nonwandering. Likewise, a nonwandering point is chain recurrent. We can express the embedding inclusions as $Per \subset \Omega_0 \subset \Omega \subset Q$, where Per, Ω_0, Ω and Q represent the periodic, the recurrent, the nonwandering, and the chain recurrent set, respectively.

2.2. **Structural graph and some topological properties.** Consider a dynamical system $(E, (\varphi^t)_{t \in T})$ where (E, d) is a metric space and $M = (T, +)$ is a totally ordered monoid, Q referring to the set of chain recurrent points of the system.

Definition 2.3. *A subset $q \subset Q$ refers to a component of the chain recurrent set if any two points from q are chain equivalent.*

Therefore, the chain recurrent set Q can be represented as the union of disjoint closed and positively invariant components Q_i: $Q = \bigcup_{i \in I} Q_i$, the set I being not necessarily finite or countable. Roughly speaking, we say for two components Q_i and Q_j that there is a connection $Q_i \to Q_j$ if there exists an orbit which "starts" at Q_i and "ends" at Q_j. Mathematically, we have the following:

Definition 2.4. *If Q_i and Q_j are two components of the chain recurrent set, we have the connection $Q_i \to Q_j$ iff there exists a point x such that $L_\alpha(x) \subset Q_i$ and $L_\omega(x) \subset Q_j$, where $L_\alpha(x)$ and $L_\omega(x)$ are, respectively, the α-limit set of x (or of $orb(x)$) and the ω-limit set defined by*

$$L_\alpha(x) := \bigcap_{t \leq 0} \overline{\{\varphi^s(x), s \leq t\}},$$

$$L_\omega(x) := \bigcap_{t \geq 0} \overline{\{\varphi^s(x), s \geq t\}},$$

0 being the neutral element of the monoid M and, for a subset F, $\overline{F}$ denotes the closure of F, i.e. the smallest (up to inclusion) closed set containing F.

Note that the definition of the α-limit set given above is not the only (and is perhaps the simplest and the natural) way to define such a concept for non-reversible dynamical systems. In the particular case of discrete dynamical systems defined by continuous maps on compact metric spaces, some refinements involving selected branches of backward orbits were proposed by Hero [13] and Balibrea et al. [2]. Besides, not only properties of such limit sets are relevant, such as emptiness, closeness or invariance, but different definitions can lead to different α-limit sets and then to different connections between the components of the chain recurrent set. In our case, if $\min\{t, t \in T\}$ exists and is equal to 0, we simply have $L_\alpha(x) = \{x\}$ and no connection can be established between the components of the chain recurrent set. This is the case, for example, when $T = \mathbb{N}$ endowed with the usual order. We invite the reader to think of examples where $T = \mathbb{N}$, but endowed with a Sharkovsky-like order, 0 being placed, for example, just after the odd numbers, whose the (usual) order has been reversed.

Now we are able to give a general definition of the structural graph of a dynamical system, the later being understood in the sense of Definition 2.1:

Definition 2.5. *The structural graph of a dynamical system DS is a directed graph $SG(DS)$ with vertices $\{i\}$ corresponding to components Q_i of the chain recurrent set Q and edges $\{i \to j\}$ corresponding to the orbits which have the α-limit set in the component Q_i and the ω-limit set in the component Q_j, i.e. the edge $\{i \to j\}$ corresponds to the connection $\{Q_i \to Q_j\}$.*

For additional details and examples of technical constructions of the structural graph in the classical continuous or discrete cases, please refer to the corresponding handbook [20]. In the continuous case, where a node in the structural graph represents a fixed point, we call to the fact that it is sometimes possible to learn about the stability of that point (in the Liapunov sense) simply by examining the direction of the arrow:

If the symbol '$\bullet \to$' is present, the corresponding fixed point is unstable, but it does not necessarily act as a repeller. For instance, it could represent a saddle. However, if the symbol '$\bullet \leftarrow$' appears, it is impossible to draw any definite conclusions about the nature of the fixed point. The stability of the point depends on its intrinsic properties. For example, if the fixed point is a focus or a node, it must be attractive, i.e. asymptotically stable. Additionally, it is important to note that a node of a structural graph can never be a center, or a fixed point with an elliptic sector.

In the following, we provide some results related to the structural graph of a dynamical system $DS = (E, (\varphi^t)_{t\in T})$, focusing on two significant aspects: connectivity and degeneracy.

Proposition 2.1. *If E is finite, or if M admits a least element, $SG(DS)$ is totally disconnected.*

Proof. If E is finite, it is clear that in this case a component of the chain recurrent set consists of either a fixed point or a periodic orbit, and no connection can be established between such objects via a finite trajectory. If M admits a least element, say m, $L_\alpha(x) = \{\varphi^m(x)\}$ for all $x \in E$, reducing to a point p of the state space. So, if p is chain recurrent, say $p \in Q_i$ for an i, and as any component of the chain recurrent set is closed and positively invariant, necessarily we have $L_\omega(x) = L_\omega(p) \subset Q_i$, and no connection can be established between Q_i and another component Q_j. □

Proposition 2.2. *If DS is topologically transitive, that is, if for any pair of non-empty open sets U and V in E, $\exists\, t \geq 0$ such that $\varphi^t(U) \cap V \neq \emptyset$, then either E is an infinite set or E is reduced to a fixed point or E consists of the orbit of a single periodic point, resulting in a degenerate structural graph (a single vertex).*

Proof. Suppose that E is finite with at least two elements (the case where $E = \{x\}$ is trivial, x being necessarily a fixed point). Let

$$\delta := \min_{\substack{(x,y)\in E^2 \\ x\neq y}} d(x,y)$$

Since DS is topologically transitive, for any $(x,y) \in E^2$ with $x \neq y$, there is $z \in E$ and $t \geq 0$ such that

$$d(x,z) < \delta \text{ and } d(y, \varphi^t(z)) < \delta$$

As δ is the minimum distance between distinct elements of E, we have

$$x = z \text{ and } \varphi^t(z) = y$$

that is $\varphi^t(x) = y$, or equivalently, $y \in orb(x)$, and more precisely, since $x \neq y$,

$$y \in orb^+(x) := \left\{\varphi^t(x),\ t > 0\right\}$$

By the same argument, we have also $x \in orb^+(y)$, and then, by Definition 2.1 (4-ii),

$$x \in orb^+(x) = E \text{ for all } x \in E$$

The set E is reduced to a single periodic orbit and, by contraposition, if DS is topologically transitive and E is not reduced to a fixed point or to a single periodic orbit, then E is an infinite set. In both cases, E finite or not, every point of E is recurrent according to Definition 2.2 (6). We conclude that $E = Q$, with one component, thus leading to a degenerate structural graph. □

Note that degeneracy is the rule whenever the recurrent points are dense in E, thus an indecomposable state space, as for example stated by the Poincaré Recurrence Theorem in the case of a measure-preserving transformation from a finite measure space into itself. In the continuous case, one can consider the so-called irrational winding of the torus T^2 where the quasi-periodic trajectories are everywhere dense, and a classical example can be given in the scalar discrete case by the logistic function $f_\lambda(x) = \lambda x(1-x)$, $\lambda > 0$. We know that f_4 is chaotic on $[0,1]$ in the sense of Devaney [7], leading to a degenerate structural graph. Remember, for a system of n autonomous ordinary differential equations, and according to Poincaré-Bendixson Theorem, chaos can only appear if $n \geq 3$, but in the discrete case, (a subset of) the real line is large enough to house a chaotic behavior. Indeed, in the more complicated case where $\lambda > 4$ (and it is even considerably harder for $4 < \lambda \leq 2 + \sqrt{5}$), it is shown that f_λ is chaotic on the Cantor set

$$C := \left\{x \in [0,1] \ / \ f_\lambda^n(x) \in [0,1] \text{ for all } n \in \mathbb{N}\right\},$$

which acts as a hyperbolic repelling set. So, the structural graph of the dynamical system $(C, (f_\lambda^n)_{n\in\mathbb{N}})$ is degenerate for $\lambda > 4$.

Now, we move to the continuous case and focus on the question of connectedness of the structural graph. We prove the following when the number of components of the chain recurrent set is finite:

Proposition 2.3. *If E is a closed differential manifold (compact and connected), then the (finite) structural graph associated to the differential system $\dot{x} = f(x)$, where f is a C^1-vector field on E, is connected.*

Proof. First of all, since f is a C^1-vector field on a compact manifold, we know by Chillingworth's theorem (see for instance Guckenheimer and Holmes [11]) that f satisfies a global Lipschitz condition on E, thus defining a dynamical system in the sense that for each $x_0 \in E$, the initial value problem

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

has a unique solution $x(t)$ which is defined for *all* $t \in \mathbb{R}$. In other words, and in terms of Definition 2.1, dynamical system's time set is equal to $\mathbb{R}$, the state transition φ^t corresponding to the flow associated to the underlying differential system. Suppose now that the corresponding structural graph is disconnected and, without loss of generality, we have two connected components, G_1 and G_2. In Lambek's notation [15], one can write for G_1, and similarly for G_2 by changing index 1 into 2,

$$G_1 = (\bigcup_i Q_i^1, A, \sigma, \tau)$$

where, for an arrow $a \in A$, the link $Q_i^1 \to_a Q_j^1$ means that the source $\sigma(a) = Q_i^1$ and the target $\tau(a) = Q_j^1$, that is, dynamically, there exist $x \notin Q_i^1 \cup Q_j^1$ such that $L_\alpha(x) \subset Q_i^1$ and $L_\omega(x) \subset Q_j^1$.

Let $\Gamma_{i,j}^1 := \{x \in E / L_\alpha(x) \subset Q_i^1 \text{ and } L_\omega(x) \subset Q_j^1\}$, $\Gamma_1 := \bigcup_{i,j} \Gamma_{i,j}^1$, and $E_1 := \bigcup_i Q_i^1 \cup \Gamma_1$ (respectively, for G_2, $E_2 := \bigcup_i Q_i^2 \cup \Gamma_2$).

In fact, E_1 (resp. E_2) is the union of the components of the chain recurrent set corresponding to the nodes of the subgraph G_1 (resp. G_2) with the set of all trajectories linking them. Obviously, E_1 and E_2 are disjoint invariant subsets of E. Furthermore, in the case of a finite number of components of the chain recurrent set, E_1 and E_2 are closed subsets of E as finite unions of closed sets (remember, the Q_i's are closed). In fact, for each point $a \in \overline{\Gamma_1} \backslash \bigcup_i Q_i^1$ (resp. $\overline{\Gamma_2} \backslash \bigcup_i Q_i^2$), thanks to the C^1-dependence of the (global) solutions on initial conditions, we have $L_\alpha(a) \subset \bigcup_i Q_i^1$ (resp. $\bigcup_i Q_i^2$) and idem for $L_\omega(a)$. As E is compact, we know that $L_\alpha(a)$ and $L_\omega(a)$ are non-empty, connected and compact subsets of E (see for instance Perko [22], p.193). Then, there is an i such that $L_\alpha(a) \subset Q_i^1$ (resp. Q_i^2) and a j such that $L_\omega(a) \subset Q_j^1$ (resp. Q_j^2), that is $a \in \Gamma_{i,j}^1 \subset \Gamma_1$ (resp. $\Gamma_{i,j}^2 \subset \Gamma_2$), obtaining the closeness of Γ_1 and Γ_2. Note that $i \neq j$, otherwise a would be chain recurrent. Let $x \in E \backslash (E_1 \cup E_2)$. Such a point exists since, E being connected, it cannot be the union of two closed sets. Moreover, we know that a limit point is nonwandering, then chain recurrent. So, by the connectedness of $L_\alpha(x)$ and $L_\omega(x)$, each one of the limit sets will be included in one of the components of the chain recurrent set, and consequently, it is included in E_1 or E_2. It is clear that one cannot have $L_\alpha(x) \subset E_1$ and $L_\omega(x) \subset E_2$ (or, $L_\alpha(x) \subset E_2$ and $L_\omega(x) \subset E_1$), otherwise, the structural graph would be connected. Then, either the two limit sets are included in E_1 or they are included in E_2. This is absurd because, as $x \notin E_1 \cup E_2$, $orb(x)$ is supposed to link no nodes, neither from the subgraph G_1 nor from G_2, and obviously, $orb(x)$ cannot link a node to itself, otherwise x would be chain recurrent. The structural graph is definitely connected. □

To illustrate the above proposition, we consider the behavior of the van der Pol system on the sphere S^2 using Poincaré's compactification, as presented below. We then provide simple examples where at least one of these (sufficient, but not necessary) conditions (finitness and closeness) fails, leading to a disconnected structural graph.

Example 2.1. *Let consider the van der Pol system:*

$$\begin{cases} \dot{x} = p(x,y) = y \\ \dot{y} = q(x,y) = \mu(1-x^2)y - x, \ \mu > 0, \end{cases}$$

evolving on the unit sphere $S^2 = \{(X,Y,Z) \in \mathbb{R}^3 \mid X^2 + Y^2 + Z^2 = 1\}$, *instead of the* (x,y)*-plane.*

To study the behaviour of trajectories 'at infinity', we follow Poincaré's approach [23], where we project from the center of the sphere S^2 *onto the* (x,y)*-plane tangent to* S^2 *at either the north or south pole. As will be seen, fixed points at infinity will be spread out along the equator of the sphere, but their analysis could be very complicated as it generally leads to a degenerate nonhyperbolic case with at least one zero eigenvalues of the leading matrix of the system.*

Projecting the upper hemisphere of S^2 *onto the* (x,y)*-plane defines* (x,y) *in terms of* (X,Y,Z)*, and reciprocally, as*

$$x = \frac{X}{Z}, \qquad y = \frac{Y}{Z}, \qquad X = \frac{x}{\sqrt{1+x^2+y^2}}, \quad Y = \frac{y}{\sqrt{1+x^2+y^2}}, \quad Z = \frac{1}{\sqrt{1+x^2+y^2}},$$

thus leading to a one-to-one correspondance between the upper hemisphere and the plane.

The van der Pol system can be written in the form of a single differential equation as

$$\frac{dy}{dx} = \frac{q(x,y)}{p(x,y)},$$

or, in differential form, as

$$q(x,y)\,dx - p(x,y)\,dy = 0,$$

where

$$dx = \frac{Z\,dX - X\,dZ}{Z^2}, \qquad dy = \frac{Z\,dY - Y\,dZ}{Z^2},$$

obtaining the equation

$$q(X/Z,\, Y/Z)\,(Z\,dX - X\,dZ) - p(X/Z,\, Y/Z)\,(Z\,dY - Y\,dZ) = 0$$

To eliminate Z *in the denominators, we multiply by* Z^3 *to get*

$$Zq^* \, dX - Zp^* \, dY + (Yp^* - Xq^*) \, dZ = 0$$

where

$$p^*(X,Y,Z) = Z^3 \, p(X/Z,\, Y/Z) = YZ^2 \text{ and } q^*(X,Y,Z) = Z^3 \, q(X/Z,\, Y/Z) = -XZ^2 + \mu YZ^2 - \mu YX^2$$

This equation defines the trajectories and the fixed points on the equator of S^2. More precisely, fixed points are solutions of the algebraic system, where $Z = 0$

$$\begin{cases} Yp^* - Xq^* = 0 \\ X^2 + Y^2 = 1, \end{cases}$$

that is

$$\begin{cases} YX^3 = 0 \\ X^2 + Y^2 = 1, \end{cases}$$

obtaining two fixed points, with their antipodals : $(\pm 1, 0)$ *and* $(0, \pm 1)$.

Note that the flow in neighborhoods of antipodal points is topologically equivalent, except that the direction of the flow may be reversed, which is not the case here as the maximum degree of the terms in p and q is an odd number. This flow is determined by projecting that neighborhood onto a plane tangent to S^2 at that point, obtaining for the point $(\pm 1, 0, 0)$ the system:

$$\begin{cases} \pm\dot{y} = yz^3 p(\frac{1}{z}, \frac{y}{z}) - z^3 q(\frac{1}{z}, \frac{y}{z}) \\ \pm\dot{z} = z^4 p(\frac{1}{z}, \frac{y}{z}) \end{cases}$$

or

$$\begin{cases} \pm\dot{y} = \mu y + z^2 - \mu y z^2 + y^2 z^2 \\ \pm\dot{z} = yz^3, \end{cases} \tag{1}$$

and, for the point $(0, \pm 1, 0)$, the system :

$$\begin{cases} \pm\dot{x} = xz^3 q(\frac{x}{z}, \frac{1}{z}) - z^3 p(\frac{x}{z}, \frac{1}{z}) \\ \pm\dot{z} = z^4 q(\frac{x}{z}, \frac{1}{z}), \end{cases}$$

or

$$\begin{cases} \pm\dot{x} = -z^2 - \mu x^3 + \mu x z^2 - x^2 z^2 \\ \pm\dot{z} = -xz^3 - \mu x^2 z + \mu z^3 \end{cases} \tag{2}$$

Both systems have to be studied near the origin, but as said before, linearization leads to a Jacobian matrix with at least a zero eigenvalue, which makes such a qualitative study difficult. Although it will not be the case here, such a situation generally leads to what Poincaré has qualified as a fixed point of the second species, and which could be referred to as a multiple fixed point, as it is shown that it can be made to split into a number of hyperbolic fixed points under a suitable perturbation of the system.

For system (1), we used the Poincaré Code, a normal form Maple package written by F. Zinoun in the 90's and published later as a theoretical paper by Mikram et al. *[17], and particularly the* `Poincaré_Dulac_Walcher_NF` *command (implementing Walcher's normal form algorithm [25]), to unveil the topological nature of the origin, which has been found to be a (topological) saddle. The obtained normalized system till order* 5 *is given by:*

$$\begin{cases} \pm\dot{y} = \mu y - \mu y z^2 \\ \pm\dot{z} = -\dfrac{z^5}{\mu} \end{cases}$$

As for system (2), it is already in Poincaré-Dulac normal form as we have a zero leading matrix. We tried to make use of the renormalizing normal form command `Poincaré_Dulac_JNF` *(JNF for Joint Normal Form), but it requires the knowledge of a Lie-point symmetry of the system, at least to order 4, which is certainly not straightforward in our case. To the best of our knowledge, all possible types of phase portraits for homogenous quadratic systems have been classified by Lyagina [16] (see also Nemytsky and Stepanov [19] for more information on the topic), but the system we are dealing with presents a much more complicated case. So we have confined ourselves to drawing a numerical phase portrait near the origin for* $\mu = 1$*, which point has been found to be a (improper) node.*

Now, taking into account the well-known phase portrait on the plane, the whole picture of the global phase portrait of the van der Pol system on the Poincaré sphere is complete: two (unstable) foci on the poles, each of them is surrounded by a (stable) limit cycle, attracting all the trajectories of the corresponding hemisphere, except the point on the pole and the trajectories on the equator. The laters are either fixed points (nodes or saddles), or heteroclinic trajectories linking these points. The structural graph of the van der Pol system on the Poincaré hemisphere is then given by:

$$\begin{tikzcd}
 & F \arrow[d] & \\
 & LC & \\
S_1^\infty \arrow[ur] & & N_1^\infty \arrow[ul] \arrow[ll] \arrow[d] \\
N_2^\infty \arrow[u] \arrow[uur] \arrow[rr] & & S_2^\infty \arrow[uul]
\end{tikzcd}$$

where F, LC, $S_{1,2}^\infty$ *and* $N_{1,2}^\infty$ *refer, respectively, to the focus on the pole, the surrounding limit cycle, the antipodal saddles, and the antipodal nodes on the equator. Note that this is compatible with (a direct* *consequence of) the well-known Poincaré Index Theorem:*

$$n + f - s = 2(1 - p),$$

where n, f, and s are the number of (isolated) nodes, foci, and saddles respectively, p being the genus of the underlying analytic two-dimensional surface, equal to 0 *in our case.*

Still within the finite case, we consider the mathematical pendulum in the phase plane. Its phase portrait exhibits an infinite sequence of centers and saddle points, arising from the periodicity of the angular variable. Each center is surrounded by a continuous family of periodic orbits corresponding to oscillatory motions, whereas the saddle points are connected by heteroclinic orbits that separate oscillatory and rotational regimes. The rotational trajectories are unbounded in the angular variable. So, the associated structural graph consists of two nodes, one corresponding to the point at infinity and is endowed with a loop representing the unbounded trajectories, the other representing the continuous family of periodic orbits together with the separatrix cycles. Since there is no connection between these two nodes, this provides an example of a finite graph that is not connected.

As for the infinite case, we consider a center-focus point (say, the origin 0) corresponding to a family of isolated periodic orbits on the phase plane (limit cycles) accumulating at the origin. More precisely, there is a sequence of periodic orbits C_n, with C_{n+1} in the interior of C_n, such that, in a metric sense, $C_n \longrightarrow 0$ as $n \longrightarrow \infty$, and such that every orbit between C_n and C_{n+1} spirals toward C_n or C_{n+1} as time tends to $\pm\infty$. Note that a center-focus cannot occur in the analytic case. This is a consequence of the well-known Dulac's Theorem [8] about the finitness of the number of limit cycles of an analytic planar system, which is related to the famous and still unsolved Hilbert's 16th Problem for planar polynomial systems. An example is given by the following planar system, taken from Perko [22]:

$$\begin{cases} \dot{x} = -y + x\sqrt{x^2+y^2}\,\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) \\ \dot{y} = x + y\sqrt{x^2+y^2}\,\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) \end{cases}$$

for $x^2 + y^2 \neq 0$, where the origin is considered as a fixed point. In polar coordinates, we have

$$\begin{cases} \dot{r} = r^2 \sin\left(\frac{1}{r}\right) \\ \dot{\theta} = 1 \end{cases}$$

for $r > 0$ with $\dot{r} = 0$ at $r = 0$. Clearly, $\dot{r} = 0$ for $r = 1/n\pi$; i.e., each of the circles $r = 1/n\pi$ represents a trajectory of the system. Furthermore, for

$$n\pi < \frac{1}{r} < (n+1)\pi,$$

$\dot{r} < 0$ if n is odd and $\dot{r} > 0$ if n is even; i.e., the trajectories between the circles $r = 1/n\pi$ spiral inward or outward to one of these circles. So, the origin is a center-focus and the corresponding structural graph, which the nodes are given by the C_n's, the origin and the point at infinity, linked by spiraling orbits, is obviously not connected as there is no finite path between the center-focus and any of the surrounding limit cycles.

3. Concluding remarks

A general definition of the structural graph of a dynamical system on a monoid has been introduced, starting from the basic notion of state transition, and arriving to the notion of chain equivalence and chain recurrence. Connectedness and degeneracy of the resulting structural graph have been investigated in some special situations, leaving a general characterization of such properties, among others, still at an embryonic stage of realization. For example, it would be worth to seek for a necessary and sufficient condition for connectedness in the general case. Besides, it is easy to check that the well-known transition graph (for two nodes x and y, which represent two states of the system, we have the connection $x \longrightarrow y$ if there is a $t \in T$ such that $\varphi^t(x) = y$), which is obviously a category in Lambek' sense [15], is complete (at least according to one orientation, and it will be the case in the strict sense if M is a group) if restricted to an orbit, but we know nearly nothing about the completeness of the structural graph. From a pure mathematical point of view, we know that the structural graph is not transitive, otherwise, if not directed, it would be complete when connected. For instance, a van der Pol-like structural graph, where the limit cycle is semi-stable ($F \longleftarrow LC \longleftarrow \infty$), lacks a connection between the point at infinity and the focus F. The orbits from infinity are unable to reach the critical point by penetrating the limit cycle: this is what we call the Fundamental Theorem of Existence and Unicity of Solutions. However, both numerically (due to round-off errors) and physically (due to external disturbances), a system starting from an initial state, represented by an out-of-cycle point in the phase plane, will inevitably reach the equilibrium F. Consequently, the idea of a physical structural graph of a nonlinear dynamical system arises, and it would be interesting to define a corresponding deductive system in order to endow the structural graph with a structure of category in Lambek' sense.

Another path to explore is the following: inspired by models arising in the study of viral capsids [5], and following a work by Giunti and Perri [10], we know that from any given graph $G = (V, E)$, with node set V (of cardinal n) and edge set E, one can define a linear gradient system of the form

$$\dot{x} = -\nabla U(x)$$

where $U : \mathbb{R}^n \to \mathbb{R}$ is a potential function describing the connection between nodes and depending on the adjacency matrix of the graph and the diagonal matrix of vertex degrees. Without going into details, the fundamental result in [10] is that, after analyzing a large class of graphs, with different types of connectivity or degree characteristics, and using the signless Laplacian matrix, *the more connected the graph is, the stronger under perturbation the system is.* A natural idea is then, if such a graph stands for the structural graph of a nonlinear dynamical system, with a finite node set, the corresponding gradient system will define a new form of linearization which has nothing to do with, for example, the linearization around a hyperbolic fixed point via Hartman-Grobman Theorem, or the well-known Carleman Linearization Procedure :

(Nonlinear Dynamical System → Structural Graph → Linear Gradient System)

Keeping in mind this fact and taking into account the nature of the nodes (remember, these are the components of the chain recurrent set) and their connectivity, to what extent the degree of stability of the linear system (that is, the distribution of the eigenvalues in the complex plane) can inform about the structural stability of the nonlinear dynamical system ? This is an exiting question that could be first addressed in the planar case within, for example, Peixoto's [21] or Kotus, Krych, and Nitecki's Theorems [14] on structural stability.

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