Int. J. Anal. Appl. (2026), 24:238

International Journal of Analysis and Applications

Some New Integral Inequalities Using Quantum Integral Methods

Chanokgan Sahatsathatsana[1,*], **Sattra Sahatsathatsana**[2]

[1]*Department of Science and Mathematics, Faculty of Science and Health Technology, Kalasin University, Thailand*

[2]*Department of English, Faculty of Liberal Arts, Kalasin University, Thailand*

[*]*Corresponding author:* chanokgan.na@ksu.ac.th

Abstract. This paper presents new integral inequalities on finite intervals using analytic and elementary quantum integrals. The results extend several well-known classical inequalities as special cases. To support the theoretical findings, numerical examples are included to illustrate their use and provide a clearer understanding.

1. Introduction

Quantum calculus, also known as q-calculus, is a generalization of classical calculus in which the concept of limits is not required. Instead of the usual derivative, it employs the so-called q-difference operator, which allows the treatment of functions that may not be differentiable in the classical sense. The origins of this theory can be traced back to the pioneering work of Euler (1707–1783), particularly in his investigations of infinite series, which were themselves influenced by earlier ideas of Newton.

A systematic development of q-calculus was later carried out by Jackson in the early twentieth century [1,2]. In his work, the notions of the q-derivative and the q-integral were introduced for functions defined on the interval $(0,\infty)$. These constructions provide natural q-analogues of many classical operators and functions. An important feature of such q-analogues is that they reduce to their classical counterparts in the limiting case as $q \to 1$, thereby establishing a direct connection between classical analysis and its quantum counterpart.

In recent decades, q-calculus has received considerable attention due to its applicability in a variety of areas within both pure and applied mathematics, as well as in theoretical physics. In particular, it has been successfully applied in the study of special functions, difference equations,

Received: Apr. 19, 2026.

2020 *Mathematics Subject Classification.* 05A30, 26A51, 26D10, 26D15.

Key words and phrases. an integral inequalities; q-differentiable function; q-integral inequalities; q-calculus.

https://doi.org/10.28924/2291-8639-24-2026-238

ISSN: 2291-8639

approximation theory, and mathematical physics (see, for example, [3–19]). These developments demonstrate that q-calculus provides an effective framework for extending classical analytical techniques to discrete and non-classical settings. For a comprehensive treatment of the subject, including its foundational results and methods, we refer the reader to the monograph by Kac and Cheung [20], which serves as a standard reference in this area.

Motivated by the need to formulate q-calculus on bounded intervals, Tariboon and Ntouyas [21,22] introduced new definitions of q-derivatives and q-integrals for continuous functions defined on finite domains. Their approach extends the classical theory in a natural way while preserving many of its essential properties. They established several fundamental results concerning these operators and demonstrated their usefulness in applications. In particular, they studied the existence and uniqueness of solutions to initial value problems for certain classes of impulsive q-differential equations, providing insight into the qualitative behavior of such systems.

An important aspect of their work is the role played by inequalities in q-calculus. Such inequalities are indispensable tools in the analysis of functional properties and operator behavior, and they often serve as a foundation for further developments in the theory. This perspective has led to a growing body of research devoted to the establishment and refinement of various types of inequalities within the framework of q-calculus.

Integral inequalities constitute a fundamental tool in the study of differential and integral equations, providing essential techniques for estimating solutions and analyzing qualitative properties of mathematical models. Over the past two decades, substantial progress has been made in this area, particularly in the development of inequalities associated with fractional and quantum operators. In this context, fractional q-integral inequalities have emerged as an active and significant line of research, offering powerful methods for addressing complex problems arising in both theory and applications. To highlight the relevance of this topic and to place our work in an appropriate context, we briefly recall some representative results that have motivated the present study. In particular, Q. A. Ngô *et al.* [23] showed in 2006 that for any positive continuous function Γ defined on the interval $[0,1]$ satisfying

$$\int_x^1 \Gamma(\omega)\, d\omega \geq \int_x^1 \omega\, d\omega, \qquad \forall x \in [0,1], \tag{1.1}$$

it was shown that the following inequalities

$$\int_0^1 \Gamma^{\gamma+1}(\omega)\, d\omega \geq \int_0^1 \omega^{\gamma}\Gamma(\omega)\, d\omega, \tag{1.2}$$

and

$$\int_0^1 \Gamma^{\gamma+1}(\omega)\, d\omega \geq \int_0^1 \omega\, \Gamma^{\gamma}(\omega)\, d\omega, \tag{1.3}$$

hold for every real number $\gamma > 0$.

Inequalities of this form have attracted increasing interest due to their fundamental role in the study of integral inequalities and their connections to various problems in analysis. The condition (1.1) serves as a key assumption that ensures the validity of inequalities (1.2) and (1.3), which may be viewed as refined estimates involving weighted integral expressions. As a result, these inequalities have motivated extensive research in both classical and quantum settings.

In recent years, various extensions and applications of inequalities (1.2) and (1.3) have been extensively studied, leading to a clearer understanding of their underlying structure and analytical behavior. These developments highlight their relevance in several areas, such as functional inequalities, operator theory, and quantum calculus. For further results and discussions on related integral inequalities, we refer the reader to [24–29].

The paper is organized as follows. Section 2 presents the basic preliminaries on q-derivatives and q-integrals required for our analysis. In Section 3, we establish Lemmas 3.2 and 3.3, along with several supporting results that are essential for deriving the main inequalities. The main results are given in Theorems 3.1, 3.2, 3.4, and 3.5. In Section 4, numerical examples are provided to demonstrate the applicability of the obtained results. Finally, Section 5 contains concluding remarks and discusses potential directions for future work.

2. Preliminaries

In this section, we recall several basic definitions and fundamental results from q-calculus that will be used throughout the paper. Unless otherwise specified, we assume that $0 < q < 1$ is a fixed real number. For a positive integer ϑ, the corresponding q-number is defined by

$$[\vartheta]_q := \frac{1-q^{\vartheta}}{1-q}, \qquad \vartheta \in \mathbb{N}.$$

Definition 2.1. *Let $\Phi : [\delta_1, \delta_2] \to \mathbb{R}$ be a continuous function. The q-derivative of Φ at a point $\xi \in (\delta_1, \delta_2]$ is defined by*

$$\begin{aligned} {}_{\delta_1}D_q\Phi(\xi) &= \frac{\Phi(\xi) - \Phi(q\xi + (1-q)\delta_1)}{(1-q)(\xi - \delta_1)}, \quad \xi \neq \delta_1, \\ {}_{\delta_1}D_q\Phi(\delta_1) &= \lim_{\xi \to \delta_1} {}_{\delta_1}D_q\Phi(\xi), \end{aligned} \tag{2.1}$$

provided that the above limit exists. The function Φ is said to be q-differentiable on the interval $[\delta_1, \delta_2]$ if ${}_{\delta_1}D_q\Phi(\xi)$ exists for all $\xi \in (\delta_1, \delta_2]$.

It is worth noting that, in the special case $\delta_1 = 0$, the operator ${}_{\delta_1}D_q\Phi(\xi)$ reduces to the standard q-derivative $D_q\Phi(\xi)$. In this case, (2.1) becomes

$$\begin{aligned} D_q\Phi(\xi) &= \frac{\Phi(\xi) - \Phi(q\xi)}{(1-q)\xi}, \quad \xi \neq 0, \\ D_q\Phi(0) &= \lim_{\xi \to 0} D_q\Phi(\xi), \end{aligned}$$

which coincides with the classical q-Jackson derivative. For further details, we refer the reader to [20, 21].

Definition 2.2. *Let $\Phi : [\delta_1, \delta_2] \to \mathbb{R}$ be a continuous function. The q-integral of Φ over the interval $[\delta_1, \xi]$, where $\xi \in (\delta_1, \delta_2]$, is defined by*

$$\int_{\delta_1}^{\xi} \Phi(t)_{\delta_1} d_q t = (1-q)(\xi - \delta_1) \sum_{n=0}^{\infty} q^n \Phi(q^n \xi + (1-q^n)\delta_1) . \tag{2.2}$$

In particular, when $\delta_1 = 0$, the q-integral defined in (2.2) reduces to

$$\int_0^{\xi} \Phi(t)_0 d_q t = \int_0^{\xi} \Phi(t)\, d_q x = (1-q)\xi \sum_{n=0}^{\infty} q^n \Phi(q^n \xi),$$

which is known as the q-Jackson integral; see [20,21] for further details.

Theorem 2.1. *Let $\Phi : [\delta_1, \delta_2] \to \mathbb{R}$ be a continuous function. Then the following statements hold:*

(i) ${}_{\delta_1}D_q \int_{\delta_1}^{\xi} \Phi(t)_{\delta_1} d_q t = \Phi(\xi)$,

(ii) $\int_c^{\xi} {}_{\delta_1}D_q \Phi(t)_{\delta_1} d_q t = \Phi(\xi) - \Phi(c)$ *for any* $c \in (\delta_1, \xi)$.

3. Main Results

Throughout this section, we present a preliminary result that serves as a key tool in deriving the main inequalities introduced later.

Lemma 3.1. *Let $\Gamma(x)$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 \Gamma(t)\, {}_x d_q t \geq \frac{1-x^2}{1+q}, \qquad \forall x \in [0,1]. \tag{3.1}$$

Assume further that

$$\Gamma(x) \geq x, \qquad \forall x \in [0,1].$$

Proof. Fix $x \in [0,1]$. By the definition of the q-integral over the interval $[x,1]$, we have

$$\int_x^1 \Gamma(t)\, {}_x d_q t = (1-q)(1-x) \sum_{n=0}^{\infty} q^n \Gamma(q^n + (1-q^n)x) .$$

Since $\Gamma(t) \geq t$ for all $t \in [0,1]$, it follows that

$$\Gamma(q^n + (1-q^n)x) \geq q^n + (1-q^n)x, \qquad \forall n \in \mathbb{N} \cup \{0\}.$$

Substituting this estimate into the above expression yields

$$\int_x^1 \Gamma(t)\, {}_x d_q t \geq (1-q)(1-x) \sum_{n=0}^{\infty} q^n \left(q^n + (1-q^n)x\right) .$$

Using standard identities for geometric series, we obtain

$$\begin{aligned}\int_x^1 \Gamma(t)\, {}_x d_q t &\geq (1-q)(1-x) \left[\frac{1}{1-q^2} + x\left(\frac{1}{1-q} - \frac{1}{1-q^2}\right)\right] \\ &= \frac{(1-x)(1+qx)}{1+q}.\end{aligned}$$

Moreover, for $x \in [0,1]$, it is straightforward to verify that

$$(1-x)(1+qx) \geq 1-x^2.$$

Therefore,

$$\frac{(1-x)(1+qx)}{1+q} \geq \frac{1-x^2}{1+q}.$$

Combining the above estimates, we conclude that

$$\int_x^1 \Gamma(t)\, {}_x d_q t \geq \frac{1-x^2}{1+q},$$

which completes the proof. □

Remark 3.1. *Condition* (3.1) *may be equivalently expressed as*

$$\int_x^1 \Gamma(t)\, {}_x d_q t \geq \int_x^1 t\, {}_x d_q t, \qquad \forall x \in [0,1]. \tag{3.2}$$

Throughout the remainder of this section, we further assume that

$$\Gamma(x) \geq 0, \qquad \forall x \in [0,1]. \tag{3.3}$$

Lemma 3.2. *If condition* (3.1) *holds, then*

$$\int_0^1 \Gamma^2(x)\, d_q x \geq \int_0^1 x\Gamma(x)\, d_q x. \tag{3.4}$$

Proof. Observing that

$$\begin{aligned} 0 &\leq \int_0^1 (\Gamma(x)-x)^2\, d_q x \\ &= \int_0^1 \Gamma^2(x)\, d_q x - 2\int_0^1 x\Gamma(x)\, d_q x + \int_0^1 x^2\, d_q x, \end{aligned}$$

we obtain

$$\begin{aligned} \int_0^1 \Gamma^2(x)\, d_q x &\geq 2\int_0^1 x\Gamma(x)\, d_q x - \int_0^1 x^2\, d_q x \\ &= 2\int_0^1 x\Gamma(x)\, d_q x - \frac{1}{[3]_q}. \end{aligned}$$

On the other hand, using assumption (3.1), we have

$$\int_0^1 \left(\int_x^1 \Gamma(t)\, {}_x d_q t \right) d_q x \geq \int_0^1 \frac{1-x^2}{1+q}\, d_q x = \frac{q}{[3]_q}.$$

Applying q-integration by parts correctly, we obtain

$$\int_0^1 \left(\int_x^1 \Gamma(t)\, {}_x d_q t \right) d_q x = q\int_0^1 x\Gamma(x)\, d_q x.$$

Thus,

$$\int_0^1 x\Gamma(x)\, d_q x \geq \frac{1}{[3]_q}.$$

Combining the above estimates completes the proof. □

Lemma 3.3. *Let* $\Gamma(x)$ *be a continuous function on* $[0,1]$ *and let* $n \in \mathbb{N}$. *Assume that condition* (3.1) *holds. Then*

$$\int_0^1 x^{n+1}\Gamma(x)\,d_qx \geq \frac{1}{[n+3]_q}. \tag{3.5}$$

Proof. Let $n \in \mathbb{N}$ be fixed. We begin by applying the q-integration by parts formula. This yields

$$\int_0^1 x^n\left(\int_x^1 \Gamma(t)\,{}_xd_qt\right)d_qx = \frac{q^{n+1}}{[n+1]_q}\int_0^1 x^{n+1}\Gamma(x)\,d_qx.$$

Rearranging the above identity, we obtain

$$\int_0^1 x^{n+1}\Gamma(x)\,d_qx = \frac{[n+1]_q}{q^{n+1}}\int_0^1 x^n\left(\int_x^1 \Gamma(t)\,{}_xd_qt\right)d_qx.$$

Next, we use assumption (3.1), which ensures that

$$\int_x^1 \Gamma(t)\,{}_xd_qt \geq \frac{1-x^2}{1+q}, \qquad \forall x \in [0,1].$$

Substituting this estimate into the previous expression, we obtain

$$\int_0^1 x^{n+1}\Gamma(x)\,d_qx \geq \frac{[n+1]_q}{q^{n+1}}\int_0^1 x^n\left(\frac{1-x^2}{1+q}\right)d_qx.$$

We now evaluate the q-integral on the right-hand side. By linearity, we have

$$\int_0^1 x^n\left(\frac{1-x^2}{1+q}\right)d_qx = \frac{1}{1+q}\left(\int_0^1 x^n\,d_qx - \int_0^1 x^{n+2}\,d_qx\right).$$

Using the well-known formula for the q-integral of a power function, we obtain

$$\int_0^1 x^n\,d_qx = \frac{1}{[n+1]_q}, \qquad \int_0^1 x^{n+2}\,d_qx = \frac{1}{[n+3]_q}.$$

Hence,

$$\int_0^1 x^n\left(\frac{1-x^2}{1+q}\right)d_qx = \frac{1}{1+q}\left(\frac{1}{[n+1]_q} - \frac{1}{[n+3]_q}\right).$$

Substituting this back, we obtain

$$\int_0^1 x^{n+1}\Gamma(x)\,d_qx \geq \frac{[n+1]_q}{q^{n+1}}\cdot\frac{1}{1+q}\left(\frac{1}{[n+1]_q} - \frac{1}{[n+3]_q}\right).$$

A direct simplification of the above expression yields

$$\int_0^1 x^{n+1}\Gamma(x)\,d_qx \geq \frac{1}{[n+3]_q}.$$

This completes the proof. □

3.1. **The Case of Natural Numbers.**

Theorem 3.1. *Assume that conditions* (3.1) *and* (3.3) *are satisfied. Assume further that* $\Gamma(x) \geq x$ *for all* $x \in [0,1]$. *Then*

$$\int_0^1 \Gamma^{n+1}(x)\, d_q x \geq \int_0^1 x^n \Gamma(x)\, d_q x, \tag{3.6}$$

for every $n \in \mathbb{N}$.

Proof. Since $\Gamma(x) \geq 0$ and $x \geq 0$, by the weighted AM–GM inequality, we observe that

$$\Gamma^{n+1}(x) + n x^{n+1} \geq (n+1) x^n \Gamma(x), \qquad \forall x \in [0,1].$$

Integrating both sides over $[0,1]$ with respect to the q-integral yields

$$\int_0^1 \Gamma^{n+1}(x)\, d_q x + n \int_0^1 x^{n+1}\, d_q x \geq (n+1) \int_0^1 x^n \Gamma(x)\, d_q x.$$

Furthermore, by Lemma 3.3 (applied with $n-1$), we have

$$\begin{aligned}(n+1) \int_0^1 x^n \Gamma(x)\, d_q x &= n \int_0^1 x^n \Gamma(x)\, d_q x + \int_0^1 x^n \Gamma(x)\, d_q x \\ &\geq \frac{n}{[n+2]_q} + \int_0^1 x^n \Gamma(x)\, d_q x.\end{aligned}$$

Combining the above inequalities gives

$$\int_0^1 \Gamma^{n+1}(x)\, d_q x \geq \int_0^1 x^n \Gamma(x)\, d_q x,$$

which completes the proof. □

Theorem 3.2. *Suppose that conditions* (3.1) *and* (3.3) *hold, and assume that* $\Gamma(x) \geq x$ *for all* $x \in [0,1]$. *Then*

$$\int_0^1 \Gamma^{n+1}(x)\, d_q x \geq \int_0^1 x\, \Gamma^n(x)\, d_q x, \tag{3.7}$$

for every $n \in \mathbb{N}$.

Proof. Since $\Gamma(x) \geq x$, we have

$$(\Gamma^n(x) - x^n)(\Gamma(x) - x) \geq 0, \qquad \forall x \in [0,1],$$

which implies

$$\Gamma^{n+1}(x) + x^{n+1} \geq x\Gamma^n(x) + x^n \Gamma(x), \qquad \forall x \in [0,1].$$

Integrating both sides over $[0,1]$ yields

$$\int_0^1 \Gamma^{n+1}(x)\, d_q x + \int_0^1 x^{n+1}\, d_q x \geq \int_0^1 x\Gamma^n(x)\, d_q x + \int_0^1 x^n \Gamma(x)\, d_q x.$$

By Lemma 3.3, we have

$$\int_0^1 \Gamma^{n+1}(x)\, d_q x + \frac{1}{[n+2]_q} \geq \int_0^1 x\Gamma^n(x)\, d_q x + \frac{1}{[n+2]_q},$$

which implies the desired result. □

Remark 3.2. *By an analogous argument, the conclusion of Lemma 3.3 remains valid when a positive real number replaces n. In particular,*

$$\int_0^1 x^{\alpha+1}\Gamma(x)\,d_qx \geq \frac{1}{[\alpha+3]_q}, \tag{3.8}$$

for every $\alpha > 0$.

3.2. **The Case of Real Numbers.**

Theorem 3.3. *Let $\alpha, \beta > 0$ satisfy $\alpha + \beta = 1$. Then, for all $x, y > 0$, the following inequality holds:*

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta}.$$

Theorem 3.4. *Assume that conditions (3.1) and (3.3) are satisfied, and assume that $\Gamma(x) \geq x$ for all $x \in [0,1]$. Then*

$$\int_0^1 \Gamma^{\alpha+1}(x)\,d_qx \geq \int_0^1 x^{\alpha}\Gamma(x)\,d_qx, \tag{3.9}$$

for any real number $\alpha > 0$.

Proof. Let $x \in [0,1]$ be arbitrary. We apply Theorem 3.3 with

$$\alpha_1 = \frac{1}{\alpha+1}, \qquad \beta_1 = \frac{\alpha}{\alpha+1},$$

so that $\alpha_1, \beta_1 > 0$ and $\alpha_1 + \beta_1 = 1$. Taking

$$X = \Gamma^{\alpha+1}(x), \qquad Y = x^{\alpha+1},$$

Theorem 3.3 yields

$$\frac{1}{\alpha+1}\Gamma^{\alpha+1}(x) + \frac{\alpha}{\alpha+1}x^{\alpha+1} \geq \left(\Gamma^{\alpha+1}(x)\right)^{\frac{1}{\alpha+1}} \left(x^{\alpha+1}\right)^{\frac{\alpha}{\alpha+1}}.$$

Simplifying, we obtain

$$\left(\Gamma^{\alpha+1}(x)\right)^{\frac{1}{\alpha+1}} = \Gamma(x), \qquad \left(x^{\alpha+1}\right)^{\frac{\alpha}{\alpha+1}} = x^{\alpha},$$

and hence

$$\frac{1}{\alpha+1}\Gamma^{\alpha+1}(x) + \frac{\alpha}{\alpha+1}x^{\alpha+1} \geq x^{\alpha}\Gamma(x), \qquad \forall x \in [0,1].$$

Integrating both sides over $[0,1]$ with respect to the q-integral, we obtain

$$\frac{1}{\alpha+1}\int_0^1 \Gamma^{\alpha+1}(x)\,d_qx + \frac{\alpha}{\alpha+1}\int_0^1 x^{\alpha+1}\,d_qx \geq \int_0^1 x^{\alpha}\Gamma(x)\,d_qx.$$

Using the known formula for the q-integral of a power function, we have

$$\int_0^1 x^{\alpha+1}\,d_qx = \frac{1}{[\alpha+2]_q}.$$

Thus,

$$\frac{1}{\alpha+1}\int_0^1 \Gamma^{\alpha+1}(x)\,d_qx + \frac{\alpha}{(\alpha+1)[\alpha+2]_q} \geq \int_0^1 x^{\alpha}\Gamma(x)\,d_qx.$$

Multiplying both sides by $(\alpha+1)$ gives

$$\int_0^1 \Gamma^{\alpha+1}(x)\, d_qx + \frac{\alpha}{[\alpha+2]_q} \geq (\alpha+1)\int_0^1 x^\alpha \Gamma(x)\, d_qx.$$

By inequality (3.8), we know that

$$\int_0^1 x^\alpha \Gamma(x)\, d_qx \geq \frac{1}{[\alpha+3]_q}.$$

Since $[\alpha+2]_q < [\alpha+3]_q$, it follows that

$$\frac{1}{[\alpha+2]_q} > \frac{1}{[\alpha+3]_q}.$$

Hence,

$$\int_0^1 x^\alpha \Gamma(x)\, d_qx > \frac{1}{[\alpha+2]_q}.$$

Therefore,

$$\alpha \int_0^1 x^\alpha \Gamma(x)\, d_qx \geq \frac{\alpha}{[\alpha+2]_q},$$

which implies

$$(\alpha+1)\int_0^1 x^\alpha \Gamma(x)\, d_qx - \frac{\alpha}{[\alpha+2]_q} \geq \int_0^1 x^\alpha \Gamma(x)\, d_qx.$$

Combining this with the previous inequality, we conclude that

$$\int_0^1 \Gamma^{\alpha+1}(x)\, d_qx \geq \int_0^1 x^\alpha \Gamma(x)\, d_qx,$$

which establishes (3.9) and completes the proof. □

Theorem 3.5. *Suppose that conditions* (3.1) *and* (3.3) *hold, and assume that* $\Gamma(x) \geq x$ *for all* $x \in [0,1]$. *Then*

$$\int_0^1 \Gamma^{\alpha+1}(x)\, d_qx \geq \int_0^1 x\, \Gamma^\alpha(x)\, d_qx, \tag{3.10}$$

for any real number $\alpha > 0$.

Proof. Let $x \in [0,1]$ be arbitrary. Under the assumption $\Gamma(x) \geq x$ and $\alpha > 0$, the function $t \mapsto t^\alpha$ is increasing on $[0,\infty)$. Hence,

$$\Gamma^\alpha(x) \geq x^\alpha, \qquad \forall x \in [0,1].$$

Thus, both $\Gamma^\alpha(x) - x^\alpha$ and $\Gamma(x) - x$ are nonnegative, and so

$$(\Gamma^\alpha(x) - x^\alpha)(\Gamma(x) - x) \geq 0, \qquad \forall x \in [0,1].$$

Expanding the above inequality, we obtain

$$\Gamma^{\alpha+1}(x) - x^\alpha\Gamma(x) - x\Gamma^\alpha(x) + x^{\alpha+1} \geq 0,$$

which implies that

$$\Gamma^{\alpha+1}(x) + x^{\alpha+1} \geq x\Gamma^\alpha(x) + x^\alpha\Gamma(x), \qquad \forall x \in [0,1].$$

Integrating both sides over $[0,1]$ with respect to the q-integral, we get

$$\int_0^1 \Gamma^{\alpha+1}(x)\,d_qx + \int_0^1 x^{\alpha+1}\,d_qx \geq \int_0^1 x\Gamma^{\alpha}(x)\,d_qx + \int_0^1 x^{\alpha}\Gamma(x)\,d_qx.$$

By applying inequality (3.8), we obtain

$$\int_0^1 x^{\alpha}\Gamma(x)\,d_qx \geq \frac{1}{[\alpha+3]_q}.$$

Moreover, it is well known that

$$\int_0^1 x^{\alpha+1}\,d_qx = \frac{1}{[\alpha+2]_q}.$$

Substituting these estimates, we have

$$\int_0^1 \Gamma^{\alpha+1}(x)\,d_qx + \frac{1}{[\alpha+2]_q} \geq \int_0^1 x\Gamma^{\alpha}(x)\,d_qx + \frac{1}{[\alpha+3]_q}.$$

Since $[\alpha+2]_q < [\alpha+3]_q$, it follows that

$$\frac{1}{[\alpha+2]_q} > \frac{1}{[\alpha+3]_q}.$$

Therefore, subtracting $\frac{1}{[\alpha+2]_q}$ from both sides yields

$$\int_0^1 \Gamma^{\alpha+1}(x)\,d_qx \geq \int_0^1 x\Gamma^{\alpha}(x)\,d_qx,$$

which completes the proof. □

4. Numerical Examples

Four numerical examples are provided to illustrate the applicability of Theorems 3.1–3.5. For each case, a continuous function $\Gamma(v)$ is specified together with suitable choices of the parameters n and α. The corresponding left-hand and right-hand sides of the inequalities are computed explicitly and visualized using MATLAB, providing a clear comparison between both quantities.

Example 4.1. *Consider the continuous function $\Gamma(v) = v^{\frac{1}{2}}$ on $[0,1]$ and take $n = 2$. From Theorem 3.1, the left-hand side of inequality* (3.6) *can be written as*

$$\int_0^1 \Gamma(v)^{n+1}\,d_qv = \int_0^1 \left(v^{\frac{1}{2}}\right)^3\,d_qv = \frac{1-q}{1-q^{\frac{5}{2}}},$$

whereas the right-hand side becomes

$$\int_0^1 v^n\Gamma(v)\,d_qv = \int_0^1 v^2v^{\frac{1}{2}}\,d_qv = \frac{1-q}{1-q^{\frac{7}{2}}}.$$

For comparison, both quantities are evaluated numerically using MATLAB, and the corresponding graphs are displayed in Figure 1. These results clearly indicate that inequality (3.6) *is satisfied.*

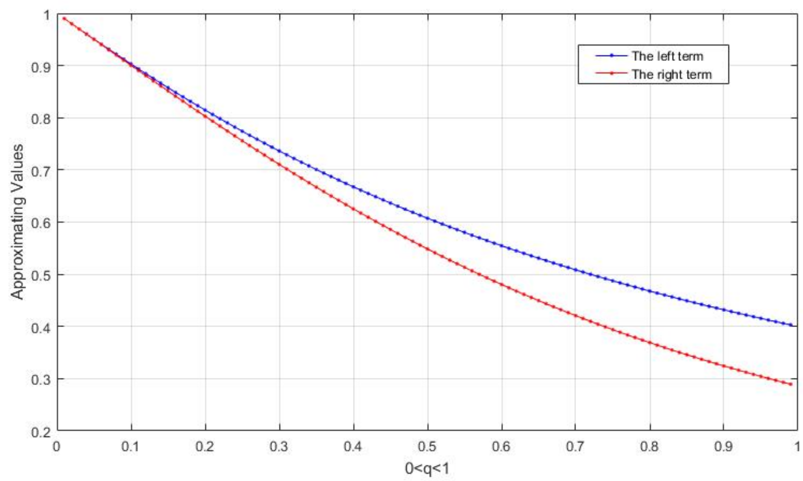

Figure 1. Comparison of both sides of inequality (3.6) for $\Gamma(v) = v^{\frac{1}{2}}$ with $n = 2$.

Example 4.2. *Let $\Gamma(v) = v^{\frac{1}{2}}$ on $[0,1]$ and fix $n = 2$. In view of Theorem 3.2, the left-hand side of inequality* (3.7) *is computed as*

$$\int_0^1 \Gamma(v)^{n+1}\, d_qv = \int_0^1 \left(v^{\frac{1}{2}}\right)^3 d_qv = \frac{1-q}{1-q^{\frac{5}{2}}},$$

while the right-hand side takes the form

$$\int_0^1 v\Gamma(v)^n\, d_qv = \int_0^1 v\left(v^{\frac{1}{2}}\right)^2 d_qv = \frac{1}{1+q+q^2}.$$

A numerical illustration obtained via MATLAB is presented in Figure 2, which supports the validity of inequality (3.7).

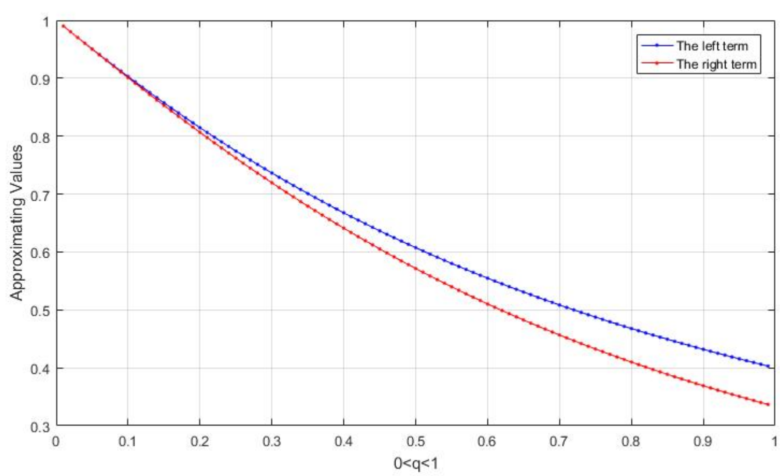

Figure 2. Comparison of both sides of inequality (3.7) for $\Gamma(v) = v^{\frac{1}{2}}$ with $n = 2$.

Example 4.3. *Consider the function* $\Gamma(v) = v^{\frac{1}{2}}$ *defined on* $[0,1]$, *and set* $\alpha = \frac{1}{4}$. *From Theorem 3.4, we evaluate the left-hand side of inequality* (3.9) *as*

$$\int_0^1 \Gamma(v)^{\alpha+1}\, d_q v = \int_0^1 \left(v^{\frac{1}{2}}\right)^{\frac{5}{4}} d_q v = \frac{1-q}{1-q^{\frac{13}{8}}},$$

while the corresponding right-hand side is given by

$$\int_0^1 v^{\alpha}\Gamma(v)\, d_q v = \int_0^1 v^{\frac{1}{4}} v^{\frac{3}{4}}\, d_q v = \frac{1-q}{1-q^{\frac{7}{4}}}.$$

A graphical comparison of these expressions, obtained via MATLAB, is presented in Figure 3. The figure confirms that inequality (3.9) *is satisfied.*

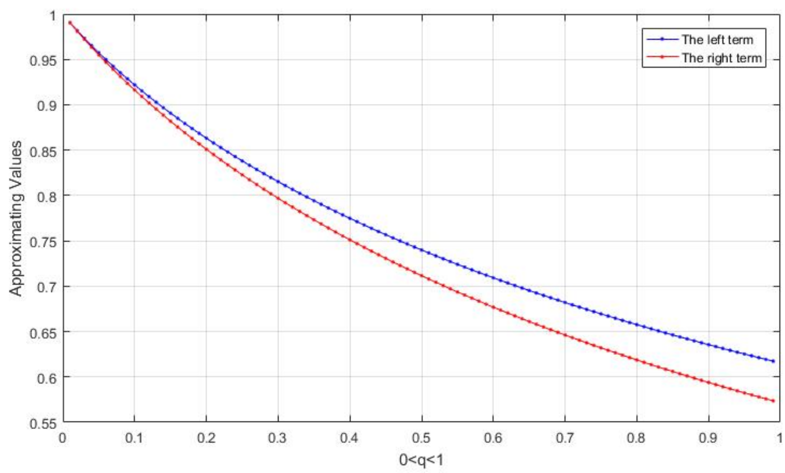

FIGURE 3. Comparison of both sides of inequality (3.9) for $\Gamma(v) = v^{\frac{1}{2}}$ with $\alpha = \frac{1}{4}$.

Example 4.4. *Let* $\Gamma(v) = v^{\frac{1}{2}}$ *on the interval* $[0,1]$, *and fix* $\alpha = \frac{1}{4}$. *In view of Theorem 3.5, the quantity on the left-hand side of* (3.10) *can be expressed as*

$$\int_0^1 \Gamma(v)^{\alpha+1}\, d_q v = \int_0^1 \left(v^{\frac{1}{2}}\right)^{\frac{5}{4}} d_q v = \frac{1-q}{1-q^{\frac{13}{8}}},$$

whereas the term on the right-hand side becomes

$$\int_0^1 v\Gamma(v)^{\alpha}\, d_q v = \int_0^1 v\left(v^{\frac{1}{2}}\right)^{\frac{1}{4}} d_q v = \frac{1-q}{1-q^{\frac{17}{8}}}.$$

The numerical behavior of both sides is illustrated in Figure 4, generated using MATLAB, which verifies the validity of inequality (3.10).

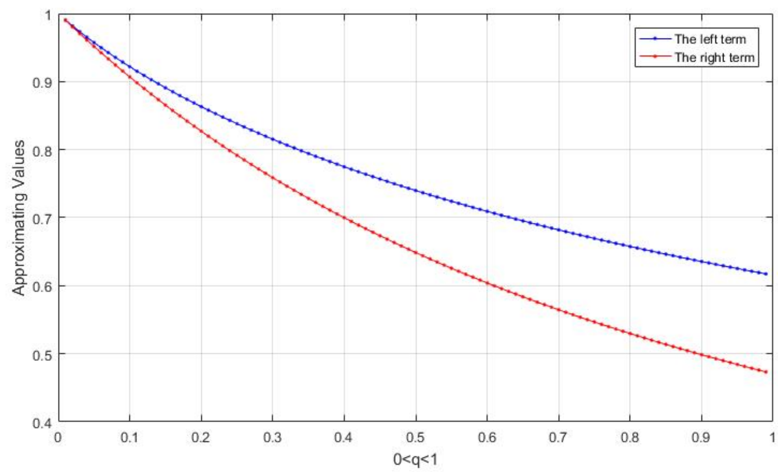

Figure 4. Graphical verification of inequality (3.10) for $\Gamma(v) = v^{\frac{1}{2}}$ and $\alpha = \frac{1}{4}$.

Remark 4.1. *These numerical examples demonstrate not only the validity of Theorems 3.1–3.5 but also their practical applicability. The explicit evaluations and graphical comparisons show that the proposed quantum integral inequalities are consistent with the analytical results and can be effectively verified through computational tools. Such an approach may be useful for further investigations involving more general classes of functions or related quantum integral inequalities.*

5. Concluding Remarks

In this paper, we have investigated Ngô-type integral inequalities within the setting of quantum calculus. By employing both q-derivatives and q-integrals, we established several integral inequalities on the interval $[0, 1]$ for the cases of natural numbers as well as positive real numbers. The obtained results extend and complement existing inequalities in the literature, and it is worth noting that our findings reduce to the results of K. Brahim [29] when $\alpha = 1$.

To illustrate the effectiveness and applicability of the proposed inequalities, a series of numerical examples were presented. These examples confirm the validity of the theoretical results and provide additional insight into their behavior. The methods and ideas developed in this work are expected to contribute to the further development of quantum integral inequalities and may inspire future research in related areas of quantum calculus and inequality theory.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] F.H. Jackson, On q-Definite Integrals, Q. J. Pure Appl. Math. 4 (1910), 193–203.

[2] F.H. Jackson, q-Difference Equations, Am. J. Math. 32 (1910), 305–314. https://doi.org/10.2307/2370183.

[3] G. Bangerezako, Variational q-Calculus, J. Math. Anal. Appl. 289 (2004), 650–665. https://doi.org/10.1016/j.jmaa.2003.09.004.

[4] H. Gauchman, Integral Inequalities in q-Calculus, Comput. Math. Appl. 47 (2004), 281–300. https://doi.org/10.1016/s0898-1221(04)90025-9.

[5] Y. Miao, F. Qi, Several q-Integral Inequalities, J. Math. Inequal. 1 (2009), 115–121. https://doi.org/10.7153/jmi-03-11.

[6] P. Raychev, R. Roussev, Y. Smirnov, The Quantum Algebra $SU_q(2)$ and Rotational Spectra of Deformed Nuclei, J. Phys. G: Nucl. Part. Phys. 16 (1990), L137–L141. https://doi.org/10.1088/0954-3899/16/8/006.

[7] A. Gavrilik, q-Serre Relations in $U_q(u_n)$ and q-Deformed Meson Mass Sum Rules, J. Phys. A: Math. Gen. 27 (1994), L91–L94. https://doi.org/10.1088/0305-4470/27/3/006.

[8] A. Aral, V. Gupta, R.P. Agarwal, Applications of q-Calculus in Operator Theory, Springer, New York, 2013. https://doi.org/10.1007/978-1-4614-6946-9.

[9] T. Ernst, A Method for q-Calculus, J. Nonlinear Math. Phys. 10 (2003), 487–525. https://doi.org/10.2991/jnmp.2003.10.4.5.

[10] B. Ahmad, S.K. Ntouyas, I.K. Purnaras, Existence Results for Nonlocal Boundary Value Problems of Nonlinear Fractional q-Difference Equations, Adv. Differ. Equ. 2012 (2012), 140. https://doi.org/10.1186/1687-1847-2012-140.

[11] H. Exton, q-Hypergeometric Functions and Applications, Halsted Press, New York, 1983.

[12] D.N. Page, Information in Black Hole Radiation, Phys. Rev. Lett. 71 (1993), 3743–3746. https://doi.org/10.1103/physrevlett.71.3743.

[13] A. Dobrogowska, A. Odzijewicz, Second Order q-Difference Equations Solvable by Factorization Method, J. Comput. Appl. Math. 193 (2006), 319–346. https://doi.org/10.1016/j.cam.2005.06.009.

[14] M. El-Shahed, H. Hassan, Positive Solutions of q-Difference Equation, Proc. Amer. Math. Soc. 138 (2009), 1733–1738. https://doi.org/10.1090/s0002-9939-09-10185-5.

[15] B. Ahmad, Boundary-Value Problems for Nonlinear Third-Order q-Difference Equations, Electron. J. Differ. Equ. 2011 (2011), 94.

[16] T. Ernst, A Comprehensive Treatment of q-Calculus, Springer, Basel, 2012. https://doi.org/10.1007/978-3-0348-0431-8.

[17] U. Ji, Quantum Extensions of Fourier-Gauss and Fourier-Mehler Transforms, J. Korean Math. Soc. 45 (2008), 1785–1801. https://doi.org/10.4134/jkms.2008.45.6.1785.

[18] C. Sahatsathatsana, S. Sahatsathatsana, On Quantum Montgomery and Ostrowski-Type Inequalities for Uniformly Convex Functions, Int. J. Anal. Appl. 24 (2026), 188. https://doi.org/10.28924/2291-8639-24-2026-188.

[19] C. Sahatsathatsana, P. Yotkaew, New Estimates of the q-Hermite–Hadamard Inequalities via Strong Convexity, Axioms 14 (2025), 576. https://doi.org/10.3390/axioms14080576.

[20] V.G. Kac, P. Cheung, Quantum Calculus, Springer, New York, 2000. https://doi.org/10.1007/978-1-4613-0071-7.

[21] J. Tariboon, S.K. Ntouyas, Quantum Calculus on Finite Intervals and Applications to Impulsive Difference Equations, Adv. Differ. Equ. 2013 (2013), 282. https://doi.org/10.1186/1687-1847-2013-282.

[22] J. Tariboon, S.K. Ntouyas, Quantum Integral Inequalities on Finite Intervals, J. Inequal. Appl. 2014 (2014), 121. https://doi.org/10.1186/1029-242x-2014-121.

[23] Q.A. Ngô, D.D. Thang, T.T. Dat, D.A. Tuan, Note on an Integral Inequality, J. Inequal. Pure Appl. Math. 7 (2006), 120.

[24] W.J. Liu, G.S. Cheng, C.C. Li, Further Development of an Open Problem Concerning an Integral Inequality, J. Inequal. Pure Appl. Math. 9 (2008), 14.

[25] L. Bougoffa, Note on an Open Problem, J. Inequal. Pure Appl. Math. 8 (2007), 58.

[26] K. Boukerrioua, A. Guezane-Lakoud, On an Open Question Regarding an Integral Inequality, J. Inequal. Pure Appl. Math. 8 (2007), 77.

[27] W.J. Liu, C.C. Li, J.W. Dong, On an Open Problem Concerning an Integral Inequality, J. Inequal. Pure Appl. Math. 8 (2007), 74.

[28] W.J. Liu, G.S. Cheng, C.C. Li, Further Development of an Open Problem, J. Inequal. Pure Appl. Math. 9 (2008), 108.

[29] K. Brahim, On Some q-Integral Inequalities, J. Inequal. Pure Appl. Math. 9 (2008), 106.