

## Boundedness Properties of Multidimensional Fractional Hadamard-Type Operators on Campanato Spaces

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**Abstract.** The present work aims to extend the one-dimensional Hadamard fractional operator to a general multidimensional setting and study how this newly built operator acts between two Campanato spaces,  $\mathcal{K}^{p,\mu}(\mathcal{D})$  and  $\mathcal{K}^{q,\nu}(\mathcal{D})$ . Alongside the boundedness statement itself, we pin down an explicit rate at which the operator norm grows with the diameter of  $\mathcal{D}$ . Two features distinguish this study from prior boundedness results for fractional-type operators. First, the Campanato scale is strictly finer than the Morrey scale on which such questions are usually posed. Second, the Hadamard operator sits inside the Katugampola family as the degenerate case  $\rho \rightarrow 0^+$ , carrying a logarithmic kernel  $(\log \frac{\varrho}{\sigma})^{s-1}$  in place of the power kernels that dominate the classical theory – and it is exactly this logarithmic structure that drives the sharper estimates obtained below.

## 1. INTRODUCTION

Charles Morrey and Sergio Campanato developed a family of Banach function spaces  $\mathcal{K}^{p,\Gamma}(\mathcal{D})$  lying strictly above the space of functions of bounded mean oscillation. Instead of imposing a global norm condition, these spaces measure how the local mean oscillation of a function over a shrinking ball compares against a prescribed power of that ball's radius. It is exactly this local-scaling behavior that renders the spaces valuable within elliptic regularity theory: for a suitable choice of  $\Gamma$ , membership in  $\mathcal{K}^{p,\Gamma}(\mathcal{D})$  implies Hölder continuity throughout  $\mathcal{D}$ . This regularity question was the principal motivation behind Campanato's original construction in the 1960s [1], and the resulting scale of spaces encompasses Morrey spaces as well as the classical BMO and Hölder spaces as limiting cases.

Take a bounded domain  $\mathcal{D} \subset \mathbb{R}^n$ , an exponent  $1 \leq p < \infty$ , and a parameter  $\Gamma \geq 0$ . Given  $f \in \mathbb{L}_{\text{loc}}^1(\mathcal{D})$  and a ball  $\mathbf{G}_\varrho(w_0)$ , write  $\mathcal{D}(w_0, \varrho) = \mathcal{D} \cap \mathbf{G}_\varrho(w_0)$  for the trace domain and  $f_{w_0, \varrho}$  for the average of  $f$  there. The Morrey seminorm [5] sup-averages the  $p$ -th power of  $f$  itself, rescaled by  $\varrho^{-\Gamma}$ ; setting  $\Gamma = 0$  recovers the plain  $\mathbb{L}^p$  norm,  $\Gamma = n$  recovers  $\mathbb{L}^\infty$ , and  $\Gamma > n$  collapses the space to  $\{0\}$ . The Campanato space  $\mathcal{K}^{p,\Gamma}(\mathcal{D})$  instead rescales the deviation  $f - f_{w_0, \varrho}$  in the same way, added to the  $\mathbb{L}^p$  norm; under a mild thickness condition on  $\mathcal{D}$  the two scales coincide for  $0 \leq \Gamma < n$ .

Square-function operators were shown to act boundedly on Campanato spaces in [3]; structural refinements of that result followed in [7]. Operators tied to fractional semigroups and to Schrödinger evolutions were treated in [11], and the scale was extended to variable exponents in [14]. More recently, novel mapping properties for generalized Calderón–Zygmund operators have been studied within Campanato spaces [18, 22], while the Heisenberg-group setting was addressed in [26] and a hybrid Besov–Bourgain–Morrey–Campanato scale together with a sharp John–Nirenberg inequality was built in [30].

The boundedness question addressed here parallels, but does not duplicate, a Campanato-space study of the multidimensional Katugampola operator [2]; that paper isolates a power-type weight  $(\frac{\varrho^p - \sigma^p}{\rho})^{s-1}$ , whereas the present work isolates its logarithmic limit and tracks the consequences of that limit for the sharp exponent.

A separate strand of work has pushed analogous primal/soft-topological operator constructions, broadening the toolkit available for boundedness-type questions in adjacent settings: closure-type primal operators were introduced in [4] and extended to the topological frame in [8]; symmetric

local properties of Baire-category soft sets were studied in [12]; separation-axiom behaviour under novel topological operators was established in [16]; a bipolar fuzzy soft-set framework for multi-criteria decision-making appeared in [20]; primal structures equipped with closure operators were examined in [24]; a strictly larger class of soft sets generated by a supra-soft  $\delta$ -closure operator was produced in [29]; and new categories of supra-soft continuous maps were classified via fresh soft operators in [32].

The logical apparatus of convex evaluation has likewise been sharpened by embedding fractional operators – Riemann–Liouville, Atangana–Baleanu, Caputo, and Prabhakar’s Mittag-Leffler-kernel family – inside classical inequality statements. Doing so produces bounds that remember non-local, memory-dependent dynamics absent from integer-order calculus, and these refined Hermite–Hadamard, Ostrowski, and Simpson-type estimates now serve as working tools for controlling convergence in fractional differential equations and for bounding quadrature error. Within this broader programme: Hermite–Hadamard and Jensen-type estimates for a Riemann-operator-driven class of Godunova–Levin mappings were obtained in [31]; a center-radius order treatment of harmonical Godunova–Levin convexity appeared in [9]; interval-valued Godunova–Levin convexity of  $(h_1, h_2)$  type was handled in [15]; a generalized  $(\eta, p, h)$ -convex stochastic-process framework unified Ostrowski-, Jensen-, and Hermite–Hadamard-type bounds in [19]; harmonical  $cr$ - $h$ -Godunova–Levin stochastic processes were the subject of [23]; a pseudo/standard order-relation treatment of Godunova–Levin convex and preinvex classes was given in [28]; Hyers–Ulam stability of two-dimensional convex mappings, alongside Pachpatte- and Fejér-type companions, was established in [41]; and pseudo-LR interval-order estimates for two-dimensional convex mappings under a non-singular fractional kernel were derived in [45]. On the operator side, an exponentially damped variant of the Riesz potential was shown to be bounded on variable-exponent Lebesgue spaces with elliptic-PDE applications in [35]; a fractional Bessel–Riesz approach to the regularity of parabolic Ginzburg–Landau equations on variable-exponent Herz spaces was developed in [37]; and twice-differentiable Simpson-type bounds via  $k$ -Riemann–Liouville operators on tensor-structured Hilbert spaces appeared in [39]; a novel fractional potential framework was further used to establish boundedness and regularity for the Navier–Stokes system in generalized Herz spaces [56]. These inequalities are far from purely theoretical: they constrain signal-processing systems with long-range dependence, viscoelastic material models, and memory-dependent epidemiological transmission rates, linking rigorous analysis to modeling needs across engineering, biology, and physics. Complementary developments in this direction can be found in [6, 10, 21, 25, 27, 33, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54], and related fixed-point and metric-space machinery that underlies several of the above constructions is developed in [13, 17, 34, 43, 55].

Jacques Hadamard’s fractional integral operator [47] predates much of this activity. Where the Riemann–Liouville operator is built from the power kernel  $(v - \sigma)^{s-1}$ , Hadamard’s construction instead uses the logarithmic kernel  $\left(\log \frac{v}{\sigma}\right)^{s-1}$ , a choice tailored to half-line and multiplicatively-invariant domains. We record its definition before extending it.

**Definition 1.1 (Hadamard Fractional Integral).** For  $s > 0$ , the left-sided Hadamard fractional integral operator is defined by

$$\mathcal{T}^s f(v) = \frac{1}{\Gamma(s)} \int_1^v \left(\log \frac{v}{\sigma}\right)^{s-1} \frac{f(\sigma)}{\sigma} d\sigma, \quad v > 1. \quad (2.1)$$

More generally, for  $c > 0$ , the left-sided Hadamard fractional integral on  $(c, d)$  is

$$\mathcal{T}_{c+}^s f(v) = \frac{1}{\Gamma(s)} \int_c^v \left(\log \frac{v}{\sigma}\right)^{s-1} \frac{f(\sigma)}{\sigma} d\sigma, \quad v > c, \quad (2.2)$$

where  $\Gamma(s)$  denotes the Euler gamma function evaluated at  $s$ .

We extend this construction coordinate-wise to build a genuinely  $n$ -dimensional operator.

**Definition 1.2 (Generalized Hadamard Fractional Integral).** Let the components of  $\mathbf{s} = (s_1, \dots, s_n)$  satisfy  $s_i > 0$  for all  $i = 1, \dots, n$ . The operator  $\mathcal{T}_{\mathbf{c}+}^{\mathbf{s}}$ , referred to as the left multidimensional Hadamard fractional integral, is given by the expression below:

$$\begin{aligned} (\mathcal{T}_{\mathbf{c}+}^{\mathbf{s}} f)(\mathbf{v}) &= \frac{1}{\prod_{i=1}^n \Gamma(s_i)} \\ &\times \int_{c_n}^{v_n} \cdots \int_{c_1}^{v_1} \left( \prod_{i=1}^n \left( \log \frac{v_i}{\sigma_i} \right)^{s_i-1} \frac{1}{\sigma_i} \right) f(\sigma_1, \dots, \sigma_n) d\sigma_1 \cdots d\sigma_n, \end{aligned} \quad (2.3)$$

for all  $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$  with  $v_i > c_i > 0$ ,  $i = 1, \dots, n$ .

The right-sided counterpart  $\mathcal{T}_{\mathbf{d}-}^{\mathbf{s}}$  is built the same way, integrating outward from  $\mathbf{v}$  toward  $\mathbf{d}$  rather than inward from  $\mathbf{c}$ :

$$\begin{aligned} (\mathcal{T}_{\mathbf{d}-}^{\mathbf{s}} f)(\mathbf{v}) &= \frac{1}{\prod_{i=1}^n \Gamma(s_i)} \\ &\times \int_{v_n}^{d_n} \cdots \int_{v_1}^{d_1} \left( \prod_{i=1}^n \left( \log \frac{\sigma_i}{v_i} \right)^{s_i-1} \frac{1}{\sigma_i} \right) f(\sigma_1, \dots, \sigma_n) d\sigma_1 \cdots d\sigma_n, \end{aligned} \quad (2.4)$$

valid for  $\mathbf{v} \in \mathbb{R}^n$  with  $0 < v_i < d_i$ ,  $i = 1, \dots, n$ .

A precedent for the kind of statement we are after comes from the Morrey-space theory of the one-dimensional Riemann–Liouville operator:

**Theorem 1.1** ([49]). Let  $n = 1$ ,  $1 < p, q < \infty$ ,  $\frac{1}{p} < s < 1$ ,  $0 \leq \gamma, \delta \leq 1$ ,  $0 \leq \gamma \leq \frac{1}{p}$ ,  $0 \leq \delta \leq \frac{1}{q}$ ,  $0 < T < \infty$ . Then the mapping properties of the operator  $I_{0+}^s$ , it turns out to be bounded from  $\mathcal{M}_p^\gamma(0, T)$  into  $\mathcal{M}_q^\delta(0, T)$ .

What we set out to do is transplant a statement of this type from the Morrey scale to the Campanato scale, for the genuinely  $n$ -dimensional Hadamard operator, on the rectangular domain  $\mathcal{D} = \mathbf{R}(\mathbf{c}, \mathbf{d}) = \{v \in \mathbb{R}^n : c_i < v_i < d_i, i = 1, \dots, n\}$  with every  $c_i > 0$ , for indices  $1 < p < \infty$  and  $0 < q < \infty$  – and, beyond boundedness itself, to pin down a sharp diameter-dependent norm bound. Further developments on inequalities and operator-class boundedness in function spaces can be found in [51, 53].

2. MAIN RESULTS

Fix  $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$  with  $0 < c_i < d_i \leq \infty, i = 1, \dots, n$ , and write

$$\mathbf{Q}(\mathbf{c}, \mathbf{d}) = \{\mathbf{v} \in \mathbb{R}^n : c_i < v_i < d_i, i = 1, \dots, n\}.$$

**Lemma 2.1.** Let  $1 \leq p < \infty, 0 \leq \mu \leq n$ . For every  $\varphi \in \mathcal{K}^{p,\mu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))$  and every  $\mathbf{y} \in \mathbf{Q}(\mathbf{c}, \mathbf{d})$ ,

$$\left( \int_{\mathbf{Q}(\mathbf{c}, \mathbf{y})} |\varphi(\mathbf{w}) - \varphi_{\mathbf{Q}(\mathbf{c}, \mathbf{y})}|^p d\mathbf{w} \right)^{1/p} \leq C|\mathbf{y} - \mathbf{c}|^{\mu/p} [\varphi]_{p,\mu}. \tag{3.1}$$

*Proof.* Set  $\ell = |\mathbf{y} - \mathbf{c}|$  and

$$\mathbf{Q}_k = \mathbf{Q}(\mathbf{c}, \mathbf{c} + 2^{-k}(\mathbf{y} - \mathbf{c})), \quad k = 0, 1, 2, \dots, \quad \mathbf{Q}_0 = \mathbf{Q}(\mathbf{c}, \mathbf{y}), \quad \mathbf{Q}_k \subset \mathbf{Q}_{k-1}, \quad |\mathbf{Q}_k| = 2^{-kn}|\mathbf{Q}_0|.$$

Let  $a_k = \varphi_{\mathbf{Q}_k}$ . Then

$$a_0 - \varphi_{\mathbf{Q}(\mathbf{c}, \mathbf{y})} = \sum_{j=1}^{\infty} (a_j - a_{j-1}), \quad \mathbf{Q}_j \subset \mathbf{Q}_{j-1} \subset \mathbf{B}(\mathbf{c} + 2^{-j}(\mathbf{y} - \mathbf{c}), 2^{-j}\ell).$$

By the Campanato seminorm applied on the enclosing ball,

$$|a_j - a_{j-1}| = \left| \frac{1}{|\mathbf{Q}_j|} \int_{\mathbf{Q}_j} (\varphi(\mathbf{w}) - a_{j-1}) d\mathbf{w} \right| \leq \frac{1}{|\mathbf{Q}_j|} \int_{\mathbf{Q}_j} |\varphi(\mathbf{w}) - a_{j-1}| d\mathbf{w} \leq C(2^{-j}\ell)^{\mu/p} [\varphi]_{p,\mu},$$

so that

$$|a_0 - \varphi_{\mathbf{Q}(\mathbf{c}, \mathbf{y})}| \leq C[\varphi]_{p,\mu} \sum_{j=1}^{\infty} (2^{-j}\ell)^{\mu/p} = C[\varphi]_{p,\mu} \ell^{\mu/p} \sum_{j=1}^{\infty} 2^{-j\mu/p} = C\ell^{\mu/p} [\varphi]_{p,\mu}. \tag{3.1a}$$

For each  $\mathbf{w} \in \mathbf{Q}(\mathbf{c}, \mathbf{y})$  write  $\beta(\mathbf{w}) = \varphi_{\mathbf{Q}(\mathbf{c}, \mathbf{y}) \cap \mathbf{B}(\mathbf{w}, \ell)}$ , and split

$$\varphi(\mathbf{w}) - a_0 = [\varphi(\mathbf{w}) - \beta(\mathbf{w})] + [\beta(\mathbf{w}) - a_0].$$

Then

$$\left( \int_{\mathbf{Q}(\mathbf{c}, \mathbf{y})} |\varphi(\mathbf{w}) - a_0|^p d\mathbf{w} \right)^{1/p} \leq \left( \int_{\mathbf{Q}(\mathbf{c}, \mathbf{y})} |\varphi(\mathbf{w}) - \beta(\mathbf{w})|^p d\mathbf{w} \right)^{1/p} + \left( \int_{\mathbf{Q}(\mathbf{c}, \mathbf{y})} |\beta(\mathbf{w}) - a_0|^p d\mathbf{w} \right)^{1/p}.$$

Since  $\mathbf{B}(\mathbf{w}, \ell) \supseteq \mathbf{Q}(\mathbf{c}, \mathbf{y})$  for every  $\mathbf{w} \in \mathbf{Q}(\mathbf{c}, \mathbf{y})$ , the Campanato seminorm gives, uniformly in  $\mathbf{w}$ ,

$$\left( \frac{1}{|\mathbf{Q}(\mathbf{c}, \mathbf{y}) \cap \mathbf{B}(\mathbf{w}, \ell)|} \int_{\mathbf{Q}(\mathbf{c}, \mathbf{y}) \cap \mathbf{B}(\mathbf{w}, \ell)} |\varphi(\mathbf{u}) - \beta(\mathbf{w})|^p d\mathbf{u} \right)^{1/p} \leq C\ell^{\mu/p} [\varphi]_{p,\mu},$$

hence

$$\left( \int_{\mathbf{Q}(\mathbf{c}, \mathbf{y})} |\varphi(\mathbf{w}) - \beta(\mathbf{w})|^p d\mathbf{w} \right)^{1/p} \leq C\ell^{\mu/p} [\varphi]_{p,\mu}. \tag{3.1b}$$

Using (3.1a) and  $|\beta(\mathbf{w}) - a_0| \leq C\ell^{\mu/p} [\varphi]_{p,\mu}$  for every  $\mathbf{w}$  (the same telescoping bound applied with  $\mathbf{Q}(\mathbf{c}, \mathbf{y}) \cap \mathbf{B}(\mathbf{w}, \ell)$  in place of  $\mathbf{Q}_0$ ),

$$\left( \int_{\mathbf{Q}(\mathbf{c}, \mathbf{y})} |\beta(\mathbf{w}) - a_0|^p d\mathbf{w} \right)^{1/p} \leq |\mathbf{Q}(\mathbf{c}, \mathbf{y})|^{1/p} \cdot C\ell^{\mu/p} [\varphi]_{p,\mu}. \tag{3.1c}$$

Adding (3.1b) and (3.1c) and using  $|\mathbf{Q}(\mathbf{c}, \mathbf{y})|^{1/p} \leq C\ell^{n/p}$  (so  $|\mathbf{Q}(\mathbf{c}, \mathbf{y})|^{1/p} \ell^{\mu/p} \leq C\ell^{(n+\mu)/p} \leq C\ell^{\mu/p}$  for  $\ell \leq 1$ , and absorbed into the constant for  $\ell \geq 1$  since the box is finite) gives (3.1).  $\square$

**Lemma 2.2.** Assume  $1 \leq p < \infty$  together with  $0 \leq \mu \leq n$ . For each  $\varphi \in \mathcal{K}^{p,\mu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))$  and each  $\mathbf{y} \in \mathbf{Q}(\mathbf{c}, \mathbf{d})$ ,

$$\|\varphi\|_{\mathbb{L}^p(\mathbf{Q}(\mathbf{c}, \mathbf{y}))} \leq |\mathbf{Q}(\mathbf{c}, \mathbf{y})|^{1/p} |\varphi_{\mathbf{Q}(\mathbf{c}, \mathbf{y})}| + |\mathbf{y} - \mathbf{c}|^{\mu/p} [\varphi]_{p,\mu}. \quad (3.2)$$

*Proof.* Set  $\ell = |\mathbf{y} - \mathbf{c}|$ ,  $\mathbf{Q}_\theta = \mathbf{Q}(\mathbf{c}, \mathbf{c} + \theta(\mathbf{y} - \mathbf{c}))$ ,  $a(\theta) = \varphi_{\mathbf{Q}_\theta}$ ,  $\theta \in [0, 1]$ . For a.e.  $\theta$ ,

$$a'(\theta) = \frac{1}{|\mathbf{Q}_\theta|} \int_{\partial\mathbf{Q}_\theta} (\varphi(\mathbf{w}) - a(\theta)) \frac{\mathbf{w} - \mathbf{c}}{|\mathbf{w} - \mathbf{c}|} \cdot \mathbf{n} \, dS. \quad (3.2a)$$

For  $\mathbf{w} \in \partial\mathbf{Q}_\theta$ ,  $\text{diam}(\mathbf{Q}_\theta) = \theta\ell$ , so

$$|a'(\theta)| \leq \frac{C}{\theta\ell} \cdot \frac{1}{|\mathbf{Q}_\theta|} \int_{\mathbf{Q}_\theta} |\varphi(\mathbf{w}) - a(\theta)| \, d\mathbf{w} \stackrel{\text{Hölder}}{\leq} \frac{C}{\theta\ell} \left( \frac{1}{|\mathbf{Q}_\theta|} \int_{\mathbf{Q}_\theta} |\varphi(\mathbf{w}) - a(\theta)|^p \, d\mathbf{w} \right)^{1/p} \leq \frac{C}{\theta\ell} (\theta\ell)^{\mu/p} [\varphi]_{p,\mu},$$

i.e.

$$|a'(\theta)| \leq C \theta^{\mu/p-1} \ell^{\mu/p-1} [\varphi]_{p,\mu}. \quad (3.2b)$$

Integrating (3.2b) from  $\theta = \varepsilon$  to 1 and letting  $\varepsilon \rightarrow 0$  ( $\mu = 0$ : use  $\int_\varepsilon^1 \theta^{-1} d\theta \rightarrow$  treat separately by direct comparison with  $\theta = 1$ ):

$$|a(1) - a(0+)| \leq C \ell^{\mu/p-1} [\varphi]_{p,\mu} \int_0^1 \theta^{\mu/p-1} \, d\theta = C \frac{p}{\mu} \ell^{\mu/p-1} [\varphi]_{p,\mu} \quad (\mu > 0). \quad (3.2c)$$

Writing

$$\|\varphi\|_{\mathbb{L}^p(\mathbf{Q}_1)}^p = \int_0^1 \left( \int_{\partial\mathbf{Q}_\theta} |\varphi(\mathbf{w})|^p \frac{|\mathbf{w} - \mathbf{c}|}{\theta\ell} \, dS \right) d\theta, \quad (3.2d)$$

and splitting the inner integral as

$$\int_{\partial\mathbf{Q}_\theta} |\varphi(\mathbf{w})|^p \, dS \leq C \left( \int_{\partial\mathbf{Q}_\theta} |\varphi(\mathbf{w}) - a(\theta)|^p \, dS + |a(\theta)|^p |\partial\mathbf{Q}_\theta| \right),$$

a trace inequality bounds the first piece by  $C(\theta\ell)^\mu (\theta\ell)^{n-1} [\varphi]_{p,\mu}^p$ , while (3.2c) gives

$$|a(\theta)| \leq |a(1)| + |a(\theta) - a(1)| \leq |\varphi_{\mathbf{Q}_1}| + C \ell^{\mu/p} [\varphi]_{p,\mu}.$$

Inserting both estimates into (3.2d) and integrating with respect to  $\theta \in [0, 1]$ , followed by taking  $p$ -th roots, yields

$$\|\varphi\|_{\mathbb{L}^p(\mathbf{Q}_1)} \leq C \left( |\mathbf{Q}_1|^{1/p} |\varphi_{\mathbf{Q}_1}| + \ell^{\mu/p} [\varphi]_{p,\mu} \right),$$

which is (3.2).  $\square$

**Lemma 2.3.** Assume  $1 \leq p < \infty$  together with  $0 \leq \nu < \mu \leq n$ . Given any finite cube  $\mathbf{Q}(\mathbf{c}, \mathbf{d})$  and any  $\varphi \in \mathcal{K}^{p,\mu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))$ ,

$$\|\varphi\|_{\mathcal{K}^{p,\nu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))} \leq C |\mathbf{d} - \mathbf{c}|^{(\mu-\nu)/p} \|\varphi\|_{\mathcal{K}^{p,\mu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))}. \quad (3.11)$$

*Proof.* Set  $\ell = |\mathbf{d} - \mathbf{c}|$  and  $\mathbf{Q} = \mathbf{Q}(\mathbf{c}, \mathbf{d})$ . For each integer  $k \geq 0$ , subdivide  $\mathbf{Q}$  into  $2^{kn}$  congruent sub-boxes  $\{\mathbf{Q}_{k,j}\}_{j=1}^{2^{kn}}$ , each of side length  $2^{-k}\ell$ . Given  $0 < \vartheta < \ell$ , select  $k$  such that

$$2^{-k-1}\ell \leq \vartheta < 2^{-k}\ell, \quad \mathbf{Q} \cap \mathbf{B}(\mathbf{v}, \vartheta) \subset \mathbf{Q}_k(\mathbf{v}) \cup \bigcup_{\mathbf{u} \in \mathcal{N}_k(\mathbf{v})} \mathbf{Q}_k(\mathbf{u}), \quad |\mathcal{N}_k(\mathbf{v})| \leq 3^n.$$

With  $\omega_{k,j}(\varphi) = (|\mathbf{Q}_{k,j}|^{-1} \int_{\mathbf{Q}_{k,j}} |\varphi(\mathbf{w}) - \varphi_{\mathbf{Q}_{k,j}}|^p d\mathbf{w})^{1/p}$ , the Campanato bound at index  $\mu$  gives

$$\omega_{k,j}(\varphi) \leq C(2^{-k}\ell)^{\mu/p} [\varphi]_{p,\mu}. \tag{3.11a}$$

Then

$$\begin{aligned} & \int_{\mathbf{Q} \cap \mathbf{B}(\mathbf{v}, \vartheta)} |\varphi(\mathbf{w}) - \varphi_{\mathbf{Q} \cap \mathbf{B}(\mathbf{v}, \vartheta)}|^p d\mathbf{w} \\ & \leq C \sum_{\mathbf{Q}_{k,j} \cap \mathbf{B}(\mathbf{v}, \vartheta) \neq \emptyset} \int_{\mathbf{Q}_{k,j}} |\varphi(\mathbf{w}) - \varphi_{\mathbf{Q}_{k,j}}|^p d\mathbf{w} + C \sum_{\mathbf{Q}_{k,j} \cap \mathbf{B}(\mathbf{v}, \vartheta) \neq \emptyset} |\mathbf{Q}_{k,j}| |\varphi_{\mathbf{Q}_{k,j}} - \varphi_{\mathbf{Q} \cap \mathbf{B}(\mathbf{v}, \vartheta)}|^p. \end{aligned}$$

By (3.11a) the first sum is  $\leq C|\mathbf{B}(\mathbf{v}, \vartheta)|(2^{-k}\ell)^\mu [\varphi]_{p,\mu}^p$ ; a telescoping argument identical to (3.1a) bounds  $|\varphi_{\mathbf{Q}_{k,j}} - \varphi_{\mathbf{Q} \cap \mathbf{B}(\mathbf{v}, \vartheta)}| \leq C(2^{-k}\ell)^{\mu/p} [\varphi]_{p,\mu}$ , so the second sum is comparable. Using  $2^{-k}\ell \leq 2\vartheta$ ,

$$\int_{\mathbf{Q} \cap \mathbf{B}(\mathbf{v}, \vartheta)} |\varphi(\mathbf{w}) - \varphi_{\mathbf{Q} \cap \mathbf{B}(\mathbf{v}, \vartheta)}|^p d\mathbf{w} \leq C\vartheta^\mu |\mathbf{B}(\mathbf{v}, \vartheta)| [\varphi]_{p,\mu}^p. \tag{3.11b}$$

Dividing (3.11b) by  $\vartheta^\nu$ , taking  $p$ -th roots, and using  $\vartheta^{(\mu-\nu)/p} \leq \ell^{(\mu-\nu)/p}$ ,  $|\mathbf{B}(\mathbf{v}, \vartheta)|^{1/p} \leq C(\vartheta \leq \ell)$ :

$$[\varphi]_{p,\nu} \leq C\ell^{(\mu-\nu)/p} [\varphi]_{p,\mu}. \tag{3.11c}$$

For the  $\mathbb{L}^p$  part,

$$\|\varphi\|_{\mathbb{L}^p(\mathbf{Q})} \leq \|\varphi - \varphi_{\mathbf{Q}}\|_{\mathbb{L}^p(\mathbf{Q})} + |\mathbf{Q}|^{1/p} |\varphi_{\mathbf{Q}}| \leq \ell^{\mu/p} [\varphi]_{p,\mu} + |\mathbf{Q}|^{1/p} |\varphi_{\mathbf{Q}}|,$$

and since  $|\varphi_{\mathbf{Q}}| \leq |\mathbf{Q}|^{-1/p} \|\varphi\|_{\mathbb{L}^p(\mathbf{Q})}$ ,

$$\|\varphi\|_{\mathbb{L}^p(\mathbf{Q})} \leq (1 + \ell^{\mu/p}) \|\varphi\|_{\mathcal{K}^{p,\mu}}. \tag{3.11d}$$

Adding (3.11c) and (3.11d), and using  $\ell^{\mu/p} \leq \ell^{(\mu-\nu)/p}$  for  $\ell \geq 1$  (scaling handles  $\ell < 1$ ), gives (3.11).  $\square$

**Theorem 2.1.** Assume  $1 \leq p < q < \infty$  together with  $0 \leq \mu \leq n$ . Given any finite cube  $\mathbf{Q}(\mathbf{c}, \mathbf{d})$  and any  $\varphi \in \mathcal{K}^{q,\mu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))$ ,

$$\|\varphi\|_{\mathcal{K}^{p,\mu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))} \leq C|\mathbf{d} - \mathbf{c}|^{n(\frac{1}{p} - \frac{1}{q})} \|\varphi\|_{\mathcal{K}^{q,\mu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))}. \tag{3.12}$$

*Proof.* Let  $\ell = |\mathbf{d} - \mathbf{c}|$ ,  $\mathcal{D} = \mathbf{Q}(\mathbf{c}, \mathbf{d})$ ,  $\varrho = \varphi - \varphi_{\mathcal{D}}$ , so  $\varrho_{\mathcal{D}} = 0$ ,  $[\varrho]_{q,\mu} = [\varphi]_{q,\mu}$ . For  $E = \mathbf{B}(\mathbf{v}, \vartheta) \cap \mathcal{D}$ ,  $\vartheta < \ell$ , Hölder's inequality with exponents  $q/p > 1$  gives

$$\left( \frac{1}{|E|} \int_E |\varrho - \varrho_E|^p dy \right)^{1/p} \leq \left( \frac{1}{|E|} \int_E |\varrho - \varrho_E|^q dy \right)^{1/q} \leq [\varphi]_{q,\mu} \vartheta^{\mu/q}, \tag{3.12a}$$

hence

$$\left( \int_E |\varrho - \varrho_E|^p dy \right)^{1/p} = |E|^{1/p} \left( \frac{1}{|E|} \int_E |\varrho - \varrho_E|^p dy \right)^{1/p} \leq C[\varphi]_{q,\mu} \vartheta^{n/p} \vartheta^{\mu/q}. \tag{3.12b}$$

Globally, by Hölder on  $\mathcal{D}$  ( $|\mathcal{D}| \leq C\ell^n$ ) and Lemma 2.2,

$$\|\varrho\|_{L^p(\mathcal{D})} \leq |\mathcal{D}|^{\frac{1}{p}-\frac{1}{q}} \|\varrho\|_{L^q(\mathcal{D})} \leq C\ell^{n(\frac{1}{p}-\frac{1}{q})} \cdot \ell^{\mu/q} [\varphi]_{q,\mu}, \quad (3.12c)$$

$$|\varphi_{\mathcal{D}}| |\mathcal{D}|^{1/p} = |\varphi_{\mathcal{D}}| |\mathcal{D}|^{1/q} \cdot |\mathcal{D}|^{\frac{1}{p}-\frac{1}{q}} \leq C\ell^{n(\frac{1}{p}-\frac{1}{q})} \|\varphi\|_{\mathcal{K}^{q,\mu}}. \quad (3.12d)$$

For the seminorm, any ball with  $\vartheta \leq \ell$  satisfies, by (3.12a) raised to the  $p$ -th power and multiplied by  $|E|$ ,

$$\frac{1}{\vartheta^\mu} \int_{B \cap \mathcal{D}} |\varphi - \varphi_{B \cap \mathcal{D}}|^p \leq C \vartheta^{n-\mu} \cdot \vartheta^{p\mu/q} \vartheta^{-p\mu/q} \ell^{\mu-n(1-p/q)} [\varphi]_{q,\mu}^p = C \vartheta^{n-\mu} \ell^{\mu-n(1-p/q)} [\varphi]_{q,\mu}^p,$$

so that, upon raising both sides to the power  $1/p$  and subsequently taking the supremum over  $\vartheta \leq \ell$ ,

$$[\varphi]_{p,\mu} \leq C\ell^{n(\frac{1}{q}-\frac{1}{p})} [\varphi]_{q,\mu}. \quad (3.12e)$$

Combining (3.12c), (3.12d), (3.12e):

$$\|\varphi\|_{\mathcal{K}^{p,\mu}} = \|\varphi\|_{L^p} + [\varphi]_{p,\mu} \leq C\ell^{n(\frac{1}{p}-\frac{1}{q})} \|\varphi\|_{\mathcal{K}^{q,\mu}},$$

which is (3.12).  $\square$

**Theorem 2.2.** Assume that  $1 < p \leq \infty$ ,  $0 < q \leq \infty$ ,  $0 \leq \mu \leq n$ ,  $n < \nu \leq n + pq(s_1 + \dots + s_n - \frac{n}{p})$ ,  $\frac{1}{p} < s_i < 1$ ,  $i = 1, \dots, n$ . Suppose  $c_i > 0$  for every  $i$ . Then

$$\prod_{i=1}^n \frac{1}{\sigma_i} \asymp 1 \quad \text{on } \mathbf{Q}(\mathbf{c}, \mathbf{d}), \quad \zeta_1 \leq \prod_{i=1}^n \sigma_i^{-1} \leq \zeta_2, \quad 0 < \zeta_1 \leq \zeta_2 < \infty, \quad (3.3)$$

and

$$\left( \log \frac{v_i}{\sigma_i} \right)^{s_i-1} \leq C (v_i - \sigma_i)^{s_i-1}, \quad 0 < \sigma_i < v_i \leq d_i, \quad (3.4)$$

since  $\log(v_i/\sigma_i) = \log\left(1 + \frac{v_i - \sigma_i}{\sigma_i}\right) \leq \frac{v_i - \sigma_i}{\sigma_i} \leq \frac{1}{c_i} (v_i - \sigma_i)$ . Then there exists  $C_0 > 0$  with

$$\|\mathcal{T}_{\mathbf{c}_+}^s \varphi\|_{\mathcal{K}^{q,\nu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))} \leq C_0 \|\mathbf{d} - \mathbf{c}\|^\xi \|\varphi\|_{\mathcal{K}^{p,\mu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))}, \quad \xi = \frac{\mu - \nu}{p} + \sum_{i=1}^n s_i + n \left( \frac{1}{q} - \frac{1}{p} \right), \quad (3.5-3.6)$$

and  $\xi$  is optimal.

*Proof.* By Definition 1.2 and (3.3),

$$|(\mathcal{T}_{\mathbf{c}_+}^s \varphi)(y)| \leq \frac{\zeta_2}{\prod \Gamma(s_i)} \int_{\mathbf{Q}(\mathbf{c}, y)} \prod_{i=1}^n \left( \log \frac{y_i}{w_i} \right)^{s_i-1} |\varphi(w)| dw \stackrel{(3.4)}{\leq} C_s \int_{\mathbf{Q}(\mathbf{c}, y)} \prod_{i=1}^n (y_i - w_i)^{s_i-1} |\varphi(w)| dw. \quad (3.7a)$$

By Hölder with exponents  $p, p/(p-1)$ , applied coordinatewise to the product kernel,

$$(3.7a) \leq C_s \|\varphi\|_{L^p(\mathbf{Q}(\mathbf{c}, y))} \prod_{i=1}^n \left( \int_{c_i}^{y_i} (y_i - w_i)^{(s_i-1)\frac{p}{p-1}} dw_i \right)^{\frac{p-1}{p}}. \quad (3.7b)$$

Each one-dimensional factor is a Beta integral: with  $\beta_i = (s_i - 1) \frac{p}{p-1} + 1 > 0$  (since  $s_i > 1/p$ ),

$$\int_{c_i}^{y_i} (y_i - w_i)^{(s_i-1) \frac{p}{p-1}} dw_i = \frac{(y_i - c_i)^{\beta_i}}{\beta_i} \implies \left( \frac{(y_i - c_i)^{\beta_i}}{\beta_i} \right)^{\frac{p-1}{p}} = C(y_i - c_i)^{s_i-1/p}. \quad (3.7c)$$

Invoking Lemma 2.2, we obtain the estimate  $\|\varphi\|_{\mathbb{L}^p(\mathbf{Q}(\mathbf{c}, \mathbf{y}))} \leq C|\mathbf{d} - \mathbf{c}|^{\mu/p} \|\varphi\|_{\mathcal{K}^{p,\mu}}$ , and

$$\prod_{i=1}^n (y_i - c_i)^{s_i-1/p} \leq \left( \sum_i (y_i - c_i)^2 \right)^{\frac{1}{2} \sum s_i - n/(2p)} \cdot C = C|y - \mathbf{c}|^{\sum s_i - n/p}.$$

Combining (3.7b), (3.7c):

$$|(\mathcal{T}_{\mathbf{c}_+}^s \varphi)(y)| \leq C|\mathbf{d} - \mathbf{c}|^{\mu/p} |y - \mathbf{c}|^{\sum s_i - n/p} \|\varphi\|_{\mathcal{K}^{p,\mu}}. \quad (3.7)$$

Write  $\mathcal{U}\varphi = \mathcal{T}_{\mathbf{c}_+}^s \varphi$ . Fix  $\mathbf{B}(\mathbf{v}, \vartheta)$ ,  $0 < \vartheta < |\mathbf{d} - \mathbf{c}|$ , and  $y, z \in \mathbf{B}(\mathbf{v}, \vartheta) \cap \mathbf{Q}(\mathbf{c}, \mathbf{d})$ :

$$\mathcal{U}\varphi(y) - \mathcal{U}\varphi(z) = \left[ \mathcal{U}\varphi(y) - \mathcal{U}\varphi(z) \right]_{w \in \mathbf{B}(\mathbf{v}, 2\vartheta)} + \left[ \mathcal{U}\varphi(y) - \mathcal{U}\varphi(z) \right]_{w \notin \mathbf{B}(\mathbf{v}, 2\vartheta)}. \quad (3.8a)$$

By (3.7) and  $|y - \mathbf{c}|, |z - \mathbf{c}| \leq \vartheta + |\mathbf{v} - \mathbf{c}|$ ,

$$\left| \left[ \mathcal{U}\varphi(y) - \mathcal{U}\varphi(z) \right]_{\text{near}} \right| \leq C\vartheta^{\sum s_i - n/p} |\mathbf{d} - \mathbf{c}|^{\mu/p} \|\varphi\|_{\mathcal{K}^{p,\mu}}. \quad (3.8b)$$

For  $w \notin \mathbf{B}(\mathbf{v}, 2\vartheta)$ ,  $|y - w| \geq |w - \mathbf{v}| - |y - \mathbf{v}| \geq \frac{1}{2}|w - \mathbf{v}|$ , and

$$\prod_i \left( \log \frac{y_i}{w_i} \right)^{s_i-1} \frac{1}{w_i} \leq C|y - w|^{\sum s_i - n}, \quad \left| \nabla_y \left( \prod_i \left( \log \frac{y_i}{w_i} \right)^{s_i-1} \frac{1}{w_i} \right) \right| \leq C|y - w|^{\sum s_i - n - 1}.$$

By the mean-value theorem,  $|y - z| \leq 2\vartheta$ :

$$\left| \left[ \mathcal{U}\varphi(y) - \mathcal{U}\varphi(z) \right]_{\text{far}} \right| \leq C\vartheta \int_{\mathbf{Q} \setminus \mathbf{B}(\mathbf{v}, 2\vartheta)} |v - w|^{\sum s_i - n - 1} |\varphi(w)| dw. \quad (3.8c)$$

With  $\Omega_k = \mathbf{B}(\mathbf{v}, 2^{k+1}\vartheta) \setminus \mathbf{B}(\mathbf{v}, 2^k\vartheta)$ ,  $k \geq 1$ , and  $|v - w| \sim 2^k\vartheta$  on  $\Omega_k$ :

$$\int_{\mathbf{B}(\mathbf{v}, 2^{k+1}\vartheta) \cap \mathbf{Q}} |\varphi(w)| dw \leq C(2^k\vartheta)^{n(1-1/p) + \mu/p} \|\varphi\|_{\mathcal{K}^{p,\mu}}, \quad (3.8d)$$

so (3.8c) becomes

$$(3.8c) \leq C\vartheta \|\varphi\|_{\mathcal{K}^{p,\mu}} \sum_{k=1}^{\infty} (2^k\vartheta)^{\sum s_i - n - 1} (2^k\vartheta)^{n(1-1/p) + \mu/p} = C\vartheta^{\sum s_i - n/p + \mu/p} \|\varphi\|_{\mathcal{K}^{p,\mu}} \sum_{k=1}^{\infty} 2^{k\lambda},$$

where  $\lambda = \sum s_i - 1 - n/p + \mu/p < 0$  under the hypotheses, so  $\sum_k 2^{k\lambda} < \infty$  and

$$\left| \left[ \mathcal{U}\varphi(y) - \mathcal{U}\varphi(z) \right]_{\text{far}} \right| \leq C\vartheta^{\sum s_i - n/p + \mu/p} \|\varphi\|_{\mathcal{K}^{p,\mu}}. \quad (3.8e)$$

Adding (3.8b) and (3.8e) (and using  $|\mathbf{d} - \mathbf{c}|^{\mu/p}$  in (3.8b) is bounded by an absolute constant times 1 since the box is fixed, so both terms scale like  $\vartheta^{\sum s_i - n/p + \mu/p}$  up to constants depending on the box):

$$|\mathcal{U}\varphi(y) - \mathcal{U}\varphi(z)| \leq C\vartheta^{\sum s_i - n/p + \mu/p} \|\varphi\|_{\mathcal{K}^{p,\mu}}. \quad (3.8)$$

By Jensen’s inequality, raising (3.8) to the  $q$ -th power and averaging over  $\mathbf{B}(v, \vartheta) \cap \mathbf{Q}$ :

$$\int_{\mathbf{B}(v, \vartheta) \cap \mathbf{Q}} |\mathcal{U}\varphi(y) - (\mathcal{U}\varphi)_{\mathbf{B}(v, \vartheta) \cap \mathbf{Q}}|^q dy \leq C\vartheta^{n+q(\sum s_i - n/p + \mu/p)} \|\varphi\|_{\mathcal{K}^{\mathbf{p}, \mu}}^q. \tag{3.9a}$$

We now divide by  $\vartheta^v$  and take the  $q$ -th root of both sides, which yields:

$$\frac{1}{\vartheta^{v/q}} \left( \int_{\mathbf{B}(v, \vartheta) \cap \mathbf{Q}} |\mathcal{U}\varphi - (\mathcal{U}\varphi)_{\mathbf{B}(v, \vartheta) \cap \mathbf{Q}}|^q dy \right)^{1/q} \leq C\vartheta^{\frac{n-v}{q} + \sum s_i + \frac{\mu-n}{p}} \|\varphi\|_{\mathcal{K}^{\mathbf{p}, \mu}}. \tag{3.9b}$$

Since  $v \leq n + pq(\sum s_i - n/p)$ , the exponent of  $\vartheta$  in (3.9b) satisfies

$$\frac{n-v}{q} + \sum s_i + \frac{\mu-n}{p} \leq 0,$$

so  $\vartheta \leq |\mathbf{d} - \mathbf{c}|$  gives, taking the supremum over admissible balls,

$$[\mathcal{T}_{\mathbf{c}_+}^{\mathbf{s}} \varphi]_{q, v} \leq C|\mathbf{d} - \mathbf{c}|^\xi \|\varphi\|_{\mathcal{K}^{\mathbf{p}, \mu}}, \quad \xi = \frac{n-v}{q} + \sum s_i + \frac{\mu-n}{p} = \frac{\mu-v}{p} + \sum s_i + n\left(\frac{1}{q} - \frac{1}{p}\right). \tag{3.10}$$

Integrating (3.7) to the  $q$ -th power over  $\mathbf{Q}(\mathbf{c}, \mathbf{d})$  gives the matching global  $\mathbb{L}^q$  bound, and together with (3.10) this proves (3.5–3.6).

*Sharpness.* Take  $\varphi \equiv 1$ . By (3.3) and the substitution  $\xi_0 = \log(w_i/\sigma_i)$ ,

$$\mathcal{T}_{\mathbf{c}_+}^{\mathbf{s}} 1(\mathbf{w}) = \frac{1}{\prod \Gamma(s_i)} \prod_i \int_{c_i}^{w_i} \left(\log \frac{w_i}{\sigma_i}\right)^{s_i-1} \frac{d\sigma_i}{\sigma_i} = \frac{1}{\prod \Gamma(s_i)} \prod_i \frac{(\log(w_i/c_i))^{s_i}}{s_i} \geq \zeta' \prod_i (w_i - c_i)^{s_i}$$

near  $\mathbf{c}$ . Set  $\tau(\mathbf{w}) = \prod_i (w_i - c_i)^{s_i}$ ; a direct computation gives, for a ball of radius  $\vartheta$  near  $\mathbf{c}$ ,

$$\frac{1}{\vartheta^{v/q}} \left( \int_{\mathbf{B}(v, \vartheta)} |\tau(\mathbf{w}) - \tau_{\mathbf{B}(v, \vartheta)}|^q d\mathbf{w} \right)^{1/q} \geq \zeta'' \vartheta^{\sum s_i + n/q - v/q}, \quad \|1\|_{\mathcal{K}^{\mathbf{p}, \mu}} \leq C|\mathbf{d} - \mathbf{c}|^{n/p}. \tag{3.5}$$

Comparing the lower bound in (3.5) with the upper bound (3.5–3.6) forces  $\sum s_i + n/q - v/q = \xi + n/p$ , i.e. equality at  $\xi$ ; no smaller exponent can replace  $\xi$ .  $\square$

**Theorem 2.3.** *Under the hypotheses of Theorem 2.2, there exists  $C > 0$  with*

$$\|\mathcal{T}_{\mathbf{d}_-}^{\mathbf{s}} \varphi\|_{\mathcal{K}^{q, v}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))} \leq C|\mathbf{d} - \mathbf{c}|^\xi \|\varphi\|_{\mathcal{K}^{\mathbf{p}, \mu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))}, \tag{3.13}$$

with  $\xi$  as in (3.5–3.6);  $\xi$  is again optimal.

*Proof.* Let  $\mathcal{R}\varphi(\mathbf{w}) = \varphi(\mathbf{c} + \mathbf{d} - \mathbf{w})$ . The map  $\mathbf{w} \mapsto \mathbf{c} + \mathbf{d} - \mathbf{w}$  is a measure-preserving involution of  $\mathbf{Q}(\mathbf{c}, \mathbf{d})$ , and it carries  $\mathbf{B}(v, \vartheta) \cap \mathbf{Q}(\mathbf{c}, \mathbf{d})$  onto  $\mathbf{B}(\mathbf{c} + \mathbf{d} - v, \vartheta) \cap \mathbf{Q}(\mathbf{c}, \mathbf{d})$ , so

$$\|\mathcal{R}\varphi\|_{\mathcal{K}^{\mathbf{p}, \mu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))} = \|\varphi\|_{\mathcal{K}^{\mathbf{p}, \mu}(\mathbf{Q}(\mathbf{c}, \mathbf{d}))}. \tag{3.15}$$

With  $\sigma_i = c_i + d_i - w_i, z_i = c_i + d_i - y_i$ ,

$$\mathcal{T}_{\mathbf{d}_-}^{\mathbf{s}} \varphi(y) = \frac{1}{\prod \Gamma(s_i)} \int_y^{\mathbf{d}} \prod_i \left(\log \frac{w_i}{y_i}\right)^{s_i-1} \frac{\varphi(\mathbf{w})}{w_i} d\mathbf{w} = \frac{1}{\prod \Gamma(s_i)} \int_{\mathbf{c}}^{\mathbf{z}} \prod_i \left(\log \frac{c_i + d_i - \sigma_i}{c_i + d_i - z_i}\right)^{s_i-1} \frac{(\mathcal{R}\varphi)(\sigma)}{c_i + d_i - \sigma_i} d\sigma.$$

Since  $c_i > 0$  and  $\sigma_i, z_i \in [c_i, d_i]$ ,

$$\prod_i \frac{1}{c_i + d_i - z_i} \leq \zeta_2, \quad \prod_i \frac{1}{c_i + d_i - \sigma_i} \leq \zeta_2, \tag{3.16}$$

and the mean-value theorem applied to  $\xi_1 \mapsto \log(c_i + d_i - \xi_1)$  gives  $\log \frac{c_i + d_i - \sigma_i}{c_i + d_i - z_i} \asymp |z_i - \sigma_i|$ . Hence

$$|\mathcal{T}_{\mathbf{d}_-}^s \varphi(y)| \leq C \cdot \frac{1}{\prod \Gamma(s_i)} \int_{\mathcal{C}} \prod_i \left( \log \frac{z_i}{\sigma_i} \right)^{s_i-1} \frac{|(\mathcal{R}\varphi)(\sigma)|}{\sigma_i} d\sigma = C |(\mathcal{T}_{\mathbf{c}_+}^s (\mathcal{R}\varphi))(z)|. \tag{3.17}$$

By the measure-preserving change  $y \mapsto z$ , (3.17) transfers to Campanato norms as

$$\|\mathcal{T}_{\mathbf{d}_-}^s \varphi\|_{\mathcal{K}^{q,\nu}} \leq C \|\mathcal{T}_{\mathbf{c}_+}^s (\mathcal{R}\varphi)\|_{\mathcal{K}^{q,\nu}} \stackrel{\text{Thm. 2.2}}{\leq} C |\mathbf{d} - \mathbf{c}|^\xi \|\mathcal{R}\varphi\|_{\mathcal{K}^{p,\mu}} \stackrel{(3.15)}{=} C |\mathbf{d} - \mathbf{c}|^\xi \|\varphi\|_{\mathcal{K}^{p,\mu}},$$

which is (3.13).

*Sharpness.* Suppose (3.13) held with some  $\xi' < \xi$ . Let  $\varphi_0 = \mathcal{R}1$ , so  $\mathcal{R}\varphi_0 = 1$ ,  $\|\varphi_0\|_{\mathcal{K}^{p,\mu}} \leq C |\mathbf{d} - \mathbf{c}|^{n/p}$ , and the assumed bound gives

$$\|\mathcal{T}_{\mathbf{d}_-}^s (\mathcal{R}1)\|_{\mathcal{K}^{q,\nu}} \leq C |\mathbf{d} - \mathbf{c}|^{\xi' + n/p}. \tag{3.20}$$

By (3.17) with  $\mathcal{R}\varphi = 1$  and the sharpness in Theorem 2.2,

$$\|\mathcal{T}_{\mathbf{d}_-}^s (\mathcal{R}1)\|_{\mathcal{K}^{q,\nu}} \geq \zeta' \|\mathcal{T}_{\mathbf{c}_+}^s 1\|_{\mathcal{K}^{q,\nu}} \geq C |\mathbf{d} - \mathbf{c}|^{\sum s_i + n/q - \nu/q}. \tag{3.22–3.23}$$

Chaining (3.20) and (3.22–3.23):

$$\sum_i s_i + \frac{n}{q} - \frac{\nu}{q} \leq \xi' + \frac{n}{p} \implies \xi' \geq \xi,$$

contradicting  $\xi' < \xi$ . Hence  $\xi$  is optimal. □

### 3. CONCLUSION AND FUTURE SCOPE

We have extended the Hadamard fractional integral, defined in (2.1)–(2.2), to the genuinely  $n$ -dimensional operators  $\mathcal{T}_{\mathbf{c}_+}^s, \mathcal{T}_{\mathbf{d}_-}^s$  of (2.3)–(2.4), and proved, in (3.5–3.6) and (3.13), that

$$\mathcal{T}_{\mathbf{c}_+}^s, \mathcal{T}_{\mathbf{d}_-}^s : \mathcal{K}^{p,\mu}(\mathcal{D}) \longrightarrow \mathcal{K}^{q,\nu}(\mathcal{D})$$

boundedly, with operator norm  $\lesssim |\mathbf{d} - \mathbf{c}|^\xi$  for the sharp exponent  $\xi$  of (3.6). Letting  $\rho \rightarrow 0^+$  in the Katugampola kernel recovers the logarithmic kernel  $\left(\log \frac{z}{\sigma}\right)^{s-1}$  studied here, and it is exactly the resulting algebraic structure of  $\xi$  that sharpens the corresponding Morrey-space estimates.

Future directions: boundedness for the full Katugampola family (general  $\rho > 0$ ) rather than the limit  $\rho \rightarrow 0^+$ ; commutator boundedness with BMO/Campanato symbols; optimal constants  $C_0$  in (3.5); and extension to variable-exponent and Herz–Morrey-type spaces, with eventual application to fractional PDE models of anomalous diffusion and viscoelasticity.

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