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Improved Hermite–Hadamard, Fejér, and Jensen-Type Inequalities for Generalized Logarithmic Godunova–Levin Mappings via Erdélyi–Kober Fractional Operators

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Abstract. This article introduces a new class of generalized logarithmic Godunova–Levin type interval-valued mappings. Based on this class, we establish several new Hermite–Hadamard inequalities, including their weighted versions, often referred to as Fejér-type inequalities involving symmetric weight functions, as well as discrete Jensen-type inequalities within the framework of Erdélyi–Kober fractional integrals. The proposed approach provides a unified and generalized framework that extends a wide range of existing results in the literature. Since the introduced convexity class is new, the corresponding results are also novel, even in the case of real-valued mappings. To support the theoretical developments, we present nontrivial illustrative examples along with numerical validations for different choices of fractional parameters. Furthermore, several well-known inequalities are recovered as special cases under appropriate selections of parameters, as demonstrated in the remarks.

1. Introduction

The Hermite–Hadamard inequality is a classical and widely studied result in convex analysis, forming a cornerstone in the development of integral inequalities. It establishes a meaningful relationship between the values of a convex function at the boundary points of an interval and the average value of the function over that interval.

Theorem 1.1 ([1]). *Suppose $\mathcal{F} : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on an interval $\mathcal{I}$, and let $\nu_1, \nu_2 \in \mathcal{I}$ satisfy $\nu_1 < \nu_2$. Then the subsequent double inequality is valid:*

$$\mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right) \leq \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \mathcal{F}(\tau)\,\mathrm{d}\tau \leq \frac{\mathcal{F}(\nu_1)+\mathcal{F}(\nu_2)}{2}. \tag{1.1}$$

Over time, inequality (1.1) has been generalized and improved along multiple lines of research. A significant extension involves the class of logarithmically convex functions. Specifically, a function $\mathcal{F} : \mathcal{I} \to (0,\infty)$ is said to be log-convex whenever $\ln\mathcal{F}$ is convex on $\mathcal{I}$. Under this assumption, a sharper form of the Hermite–Hadamard inequality is obtained (see [2], p. 197, (5.3)).

$$\mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right) \leq \exp\left[\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \ln\mathcal{F}(\tau)\,\mathrm{d}\tau\right] \leq \sqrt{\mathcal{F}(\nu_1)\mathcal{F}(\nu_2)}. \tag{1.2}$$

The concept of h-convexity was introduced by S. Varošanec in 2007 [76] as a unified generalization of various convexity notions. This approach has led to significant advancements in the study of Hermite–Hadamard type inequalities. Building upon this idea, Noor et al. [33] defined log-h-convex functions and established an improved form of inequality (1.2).

Theorem 1.2 ([33]). *Let $\mathcal{F} : \mathcal{I} \to (0,\infty)$ be a* log-*h-convex function with $h\left(\frac{1}{2}\right) \neq 0$. Then*

$$\mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} \leq \exp\left[\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \ln\mathcal{F}(\tau)\,\mathrm{d}\tau\right] \leq [\mathcal{F}(\nu_1)\mathcal{F}(\nu_2)]^{\int_0^1 h(t)\,\mathrm{d}t}. \tag{1.3}$$

Motivated by the increasing significance of interval-valued analysis, recent studies have focused on extending these types of inequalities to functions that take values in interval form. In particular, Afzal et al. [3,5] investigated such inequalities for numerous classes of mappings, including Godunova–Levin type functions, and established several extensions in both convex and harmonically convex settings.

Theorem 1.3 ([3]). *Suppose $h : (0,1) \to \mathbb{R}^+$ is a function for which $h\left(\frac{1}{2}\right) \neq 0$. Let $\mathcal{F} : [\nu_1, \nu_2] \to \mathbb{R}^+_{\mathcal{I}}$ satisfies $\mathcal{F} \in SGX(\text{cr-}h, [\nu_1, \nu_2], \mathbb{R}^+_{\mathcal{I}})$. Then the following result holds:*

$$\frac{h\left(\frac{1}{2}\right)}{2}\mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right) \preceq_{\text{cr}} \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\mathcal{F}(\tau)\,\mathrm{d}\tau \preceq_{\text{cr}} [\mathcal{F}(\nu_1)+\mathcal{F}(\nu_2)]\int_0^1 \frac{\mathrm{d}t}{h(t)}. \tag{1.4}$$

The study of Hermite–Hadamard type inequalities has attracted considerable interest due to its wide applicability in convex analysis, optimization, and applied mathematics. In recent years, these inequalities have been extended to interval-valued settings, making them particularly useful for handling uncertainty and imprecision in mathematical modeling.

For instance, Shi et al. [70] developed fractional integral versions of these inequalities for interval set-valued mappings, demonstrating the importance of fractional operators in uncertainty analysis. Similarly, Nwaeze et al. [64] examined Hermite–Hadamard type inclusions under m-polynomial convexity, thereby broadening the classical notion of convexity. Comprehensive surveys by Tariq et al. [6, 10] further trace the historical development of the Hermite–Hadamard inequality under various fractional differential and integral operators, situating the present contribution within this broader research program. Kalsoom et al. [58] proposed quantum-based generalizations for set-valued convex functions, and in [78] the authors extended the framework to coordinated h-convexity in multivariate interval-valued contexts.

The integration of fractional and quantum calculus has also played a key role. Lai et al. [60] analyzed coordinated preinvex mappings and their associated integral inequalities. More recent contributions continue to expand the theory: Tan et al. [72] introduced $\hbar$-preinvex interval-valued mappings in a fractional setting, whereas Liu et al. [32] established Jensen and Hermite–Hadamard inequalities for log-type convex fuzzy mappings. On the side of fractional regularity theory, Afzal [13, 15] analyzed boundedness and regularity properties of the Navier–Stokes system and of parabolic Ornstein–Uhlenbeck equations in generalized and variable Herz spaces by means of fractional potential and Muckenhoupt-type weighted singular operators, while Khan et al. [8, 14] derived new Hermite–Hadamard and Fejér type variants for the Godunova–Levin preinvex class of interval-valued functions together with two-dimensional pseudo-order extensions via non-singular kernel fractional integral operators.

In stochastic settings, Afzal et al. [7] studied interval-valued stochastic processes and derived several inclusion inequalities, while Kanan et al. [66] exploring the potential of iot-based learning. Fahad et al. [54] proposed generalized Hermite–Hadamard–Mercer inequalities under geometric–arithmetic convexity assumptions. Additional contributions include fractional q-integral inequalities by Akram et al. [43], as well as related results by Srivastava et al. [71]. Sahoo et al. [68] extended these ideas to center-radius ordered interval-valued preinvex functions. Ahmad et al. [12] introduced several new notions of integral inequalities together with their applications, and Abbas et al. [36] established Jensen, Ostrowski, and Hermite–Hadamard type results for h-convex stochastic processes via the center-radius order relation. Saeed and Treanţă [67] investigated interval-valued variational problems, while Upadhyay et al. [75] explored their connections with optimization and variational inequalities. Bin-Mohsin et al. [52] examined reverse fractional inequalities, and Khan et al. [4] studied superquadratic fuzzy interval-valued functions and their associated inequalities.

Further work includes harmonical Godunova–Levin type inequalities by Afzal et al. [9], and entropy-based applications in stochastic interval-valued processes by Shah et al. [69]. Foundational contributions by Almutairi and Kılıçman [47], Afzal et al. [3], and Noor et al. [63] continue to underpin these developments. Finally, many of these modern advancements build on earlier work in convexity and integral inequalities, including studies by Agahi et al. [41], Tolsted [74], and Iusem et al. [56]. Further developments on interval-valued and stochastic convexity classes were presented by Ahmad et al. [24] for generalized superquadratic mappings, whereas Afzal et al. [19, 21] obtained Hyers–Ulam stability results for two-dimensional convex mappings together with gradient descent and twice differentiable Simpson-type inequalities via novel fractional integral operators. Tariq et al. [18, 35] investigated preinvex functions involving the Caputo–Fabrizio operator and further results within the framework of conformable fractional calculus, while Khan et al. [31] established midpoint-type inequalities for multiplicatively strong convex functions via multiplicative calculus. Recent contributions such as Awan [51] and Afzal and Botmart [11] continue to expand the scope, alongside works by Ali et al. [28, 46], Liu et al. [61], and Farid et al. [55].

Structural questions concerning generalized convex mappings and center-radius order relations have also received sustained attention: Afzal et al. [17, 23, 25] obtained new estimates of integral inequalities for harmonical convex mappings using the center-radius order relation, together with complementary regularity estimates for Ginzburg–Landau type equations in variable exponent Herz spaces via fractional Bessel–Riesz operators and boundedness results for an exponentially damped Riesz potential with applications to the regularity theory of elliptic PDEs. Discrete-variable analogues of the Hermite–Hadamard inequality in quantum calculus were obtained by Aftab et al. [39]. Additional contributions in adjacent function spaces include Adrees et al. [22, 38] on modified (p, h)-convex stochastic processes and unified variants for generalized (η, p, h)-convex stochastic processes, the work of Zhang et al. [77] on Godunova–Levin functions via the Riemann integral operator, and an estimation of pseudo and standard order relation inequalities by Khan and Afzal [59]. Superquadratic-type midpoint and special-mean inequalities were further pursued by Abdelfattah et al. [20, 37], while harmonical center-radius stochastic process inequalities were examined by Afzal et al. [27]. Additional integral inequalities involving Raina and Mittag-Leffler functions pertaining to the Atangana–Baleanu operator were derived by Tariq et al. [73], and the boundedness of multidimensional Katugampola and Hadamard-type operators in Campanato spaces was studied by Afzal et al. [40] and Ahmad et al. [42], respectively; a related potential-theoretic study of Wolff potentials and Birkhoff's ergodic theorem in grand variable Herz spaces was carried out by Artes et al. [50].

Adjacent to the convexity and fractional-inequality literature, several recent metric-space and fixed-point results are also relevant to the analytic techniques employed here. Qawaqneh et al. [16, 34, 65] introduced new types of integral contractions in supra metric spaces, developed a framework of complex-valued controlled S-metric spaces with applications to nonlinear integral equations, and obtained fractional analytic solutions together with related fixed-point results. In a complementary probabilistic direction, Judeh [57] studied applications of the conformable fractional Pareto probability distribution. Finally, several structural studies on soft and primal topological spaces provide further context for the general order-theoretic and set-theoretic tools underlying interval-valued analysis: Musa et al. [62] proposed fuzzy N-bipolar soft sets for multi-criteria decision-making, Al-Omari and Alqahtani [26, 44] introduced primal structures with closure operators and studied operators in soft primal spaces, and Alghamdi et al. [45] examined novel operators in the frame of primal topological spaces. Ameen and Alqahtani [29, 49] investigated Baire category soft sets and their symmetric local properties and classified soft functions defined by soft open sets modulo soft sets of the first category. Related separation-axiom and continuity results were given by Alqahtani and Abd El-latif [48], while Abd El-latif and Alqahtani [30, 53] further introduced novel categories of supra soft continuous maps and established connectedness results via supra soft *sd*-operators.

Novelty and Significance. The novelty and significance of this study are summarized as follows:

- In this work, we propose a new generalized class of logarithmic Godunova–Levin functions, providing a unified framework that extends many known results. While earlier studies have primarily relied on classical integral operators within different convexity settings (see [2, 33]), our approach is based on Erdélyi–Kober fractional integral operators, leading to more general and refined Hermite–Hadamard type inequalities.
- Moreover, to demonstrate the validity and effectiveness of the proposed results, we construct nontrivial illustrative examples and provide several remarks that show how previously known results can be recovered as special cases under suitable choices of parameters and fractional orders.

The structure of this article is organized as follows. In Section 1, we review the relevant literature and provide the motivation behind the problems studied. Section 2 recalls some necessary notions, including definitions, operators, and concepts of interval calculus. Section 3 contains our main results, in which we establish several inequalities, such as weighted Hermite–Hadamard and Jensen-type inequalities, for functions with certain convexity properties. Finally, in Section 4, we conclude the article with a discussion of the results and propose directions for future research.

2. Preliminaries

This section reviews essential preliminaries from interval analysis, including classical convex mappings involving logarithmic properties and their extensions to the interval-valued setting.

2.1. **Interval arithmetic.** Let $\mathbb{R}_{\mathcal{I}}$ denote the family of all nonempty closed intervals of $\mathbb{R}$. An interval $\nu_1 = [\underline{\nu_1}, \overline{\nu_1}]$ is termed *positive* if $\underline{\nu_1} > 0$. The set of all such intervals is denoted by $\mathbb{R}_{\mathcal{I}}^+$, while $\mathbb{R}^+$ stands for the set of positive real numbers.

For any $\lambda \in \mathbb{R}$, Minkowski addition and scalar multiplication are defined as follows:

$$\nu_1 + \nu_2 = [\underline{\nu_1} + \underline{\nu_2}, \overline{\nu_1} + \overline{\nu_2}],$$

and

$$\lambda \nu_1 = \begin{cases} [\lambda \underline{\nu_1}, \lambda \overline{\nu_1}], & \lambda > 0, \\ \{0\}, & \lambda = 0, \\ [\lambda \overline{\nu_1}, \lambda \underline{\nu_1}], & \lambda < 0. \end{cases}$$

For $\nu_1 = [\underline{\nu_1}, \overline{\nu_1}] \in \mathbb{R}_{\mathcal{I}}$, its *center* and *radius* are defined, respectively, by

$$\nu_{1c} = \frac{\overline{\nu_1} + \underline{\nu_1}}{2}, \qquad \nu_{1r} = \frac{\overline{\nu_1} - \underline{\nu_1}}{2}.$$

Thus, ν_1 admits the center-radius representation

$$\nu_1 = \langle \nu_{1c}, \nu_{1r} \rangle.$$

Definition 2.1 ([3]). *Let $\nu_1 = \langle \nu_{1c}, \nu_{1r} \rangle$ and $\nu_2 = \langle \nu_{2c}, \nu_{2r} \rangle$ be two intervals in $\mathbb{R}_{\mathcal{I}}$. The total interval order, commonly referred to as the* cr*-order, is defined as follows:*

$$\nu_1 \preceq_{\mathrm{cr}} \nu_2 \Longleftrightarrow \begin{cases} \nu_{1c} < \nu_{2c}, & \text{if } \nu_{1c} \neq \nu_{2c}, \\ \nu_{1r} \leq \nu_{2r}, & \text{if } \nu_{1c} = \nu_{2c}. \end{cases}$$

Moreover, for any $\nu_1, \nu_2 \in \mathbb{R}_{\mathcal{I}}$, either $\nu_1 \preceq_{\mathrm{cr}} \nu_2$ or $\nu_2 \preceq_{\mathrm{cr}} \nu_1$ holds.

The idea of Riemann integration defined in terms of intervals was introduced in [3], and can be expressed as follows:

Theorem 2.1 ([3]). *Let $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}] : [\nu_1, \nu_2] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued mapping. Then $\mathcal{F}$ is Riemann integrable on $[\nu_1, \nu_2]$ if and only if both endpoint functions $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are Riemann integrable on $[\nu_1, \nu_2]$. In this case, the integral of $\mathcal{F}$ is defined by*

$$\int_{\nu_1}^{\nu_2} \mathcal{F}(\sigma)\, d\sigma = \left[\int_{\nu_1}^{\nu_2} \underline{\mathcal{F}}(\sigma)\, d\sigma, \int_{\nu_1}^{\nu_2} \overline{\mathcal{F}}(\sigma)\, d\sigma \right].$$

The class of all interval-valued functions that are Riemann integrable on $[\nu_1, \nu_2]$ is denoted by $\mathcal{IR}([\nu_1, \nu_2])$.

Theorem 2.2 ([3]). *Let $\mathcal{F}, \mathcal{G} \in \mathcal{IR}([\nu_1, \nu_2])$ take values in $\mathbb{R}_{\mathcal{I}}^+$. If*

$$\mathcal{F}(\sigma) \preceq_{\mathrm{cr}} \mathcal{G}(\sigma), \qquad \forall \sigma \in [\nu_1, \nu_2],$$

then

$$\int_{\nu_1}^{\nu_2} \mathcal{F}(\sigma)\, d\sigma \preceq_{\mathrm{cr}} \int_{\nu_1}^{\nu_2} \mathcal{G}(\sigma)\, d\sigma.$$

That is, the cr*-order is preserved under integration.*

More details on interval analysis can be found in [3,70].

2.2. Classical convexity and interval-valued convexity.

Definition 2.2 ([1]). *Let $\mathcal{F} : [\nu_1, \nu_2] \to \mathbb{R}$ be a real-valued function. Then $\mathcal{F}$ is called convex on $[\nu_1, \nu_2]$ if*

$$\mathcal{F}(\eta\sigma + (1-\eta)\kappa) \leq \eta\mathcal{F}(\sigma) + (1-\eta)\mathcal{F}(\kappa),$$

for all $\sigma, \kappa \in [\nu_1, \nu_2]$ and $\eta \in [0,1]$.

Definition 2.3 ([1]). *Let $\mathcal{F} : [\nu_1, \nu_2] \to \mathbb{R}$ be a real-valued function. The function $\mathcal{F}$ is said to be of Godunova–Levin type if*

$$\mathcal{F}(\eta\sigma + (1-\eta)\kappa) \leq \frac{\mathcal{F}(\sigma)}{\eta} + \frac{\mathcal{F}(\kappa)}{1-\eta},$$

for all $\sigma, \kappa \in [\nu_1, \nu_2]$ and $\eta \in (0,1)$.

Definition 2.4 ([68]). *Let $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}] : [\nu_1, \nu_2] \to \mathbb{R}_{\mathcal{I}}$. Then $\mathcal{F}$ is* interval-valued convex *on $[\nu_1, \nu_2]$ iff $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are convex.*

$$\mathcal{F}(\eta\sigma + (1-\eta)\kappa) \preceq_{\mathrm{cr}} \eta\mathcal{F}(\sigma) + (1-\eta)\mathcal{F}(\kappa),$$

for all $\sigma, \kappa \in [\nu_1, \nu_2]$ and $\eta \in [0,1]$.

Equivalently, both endpoint functions $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are convex on $[\nu_1, \nu_2]$ and satisfy

$$\underline{\mathcal{F}}(\sigma) \leq \overline{\mathcal{F}}(\sigma), \qquad \forall \sigma \in [\nu_1, \nu_2].$$

The class of all such interval-valued convex functions mapping into $\mathbb{R}_{\mathcal{I}}^+$ is denoted by

$$\mathcal{IVCX}\Big([\nu_1, \nu_2], \mathbb{R}_{\mathcal{I}}^+\Big).$$

Definition 2.5 ([68]). *Let $\mathcal{F} : [\nu_1, \nu_2] \to \mathbb{R}_{\mathcal{I}}$ be written in center-radius form as*

$$\mathcal{F} = \langle \mathcal{F}_c, \mathcal{F}_r \rangle,$$

where

$$\mathcal{F}_c(\sigma) = \frac{\overline{\mathcal{F}}(\sigma) + \underline{\mathcal{F}}(\sigma)}{2}, \qquad \mathcal{F}_r(\sigma) = \frac{\overline{\mathcal{F}}(\sigma) - \underline{\mathcal{F}}(\sigma)}{2}.$$

Then $\mathcal{F}$ is known as cr-convex *on $[\nu_1, \nu_2]$ if, for all $\sigma, \kappa \in [\nu_1, \nu_2]$ and $\eta \in [0,1]$,*

$$\mathcal{F}_c(\eta\sigma + (1-\eta)\kappa) \leq \eta\mathcal{F}_c(\sigma) + (1-\eta)\mathcal{F}_c(\kappa),$$

and

$$\mathcal{F}_r(\eta\sigma + (1-\eta)\kappa) \leq \eta\mathcal{F}_r(\sigma) + (1-\eta)\mathcal{F}_r(\kappa).$$

Remark 2.1. *For any interval-valued function $\mathcal{F}$, the following implications hold:*

- *If $\mathcal{F}$ is* cr-convex, *then it is interval-valued convex.*
- *If $\mathcal{F}$ is interval-valued convex, then both $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are convex in the classical sense.*

2.3. **Logarithmic convexity and related classes.**

Definition 2.6 ([33]). *Let $h : [0,1] \to \mathbb{R}^+$ be a given function. A mapping $\mathcal{F} : [\nu_1, \nu_2] \to \mathbb{R}^+$ is called* log-*h-convex on $[\nu_1, \nu_2]$ if*

$$\mathcal{F}(\eta\sigma + (1-\eta)\kappa) \leq \Big(\mathcal{F}(\sigma)\Big)^{h(\eta)} \Big(\mathcal{F}(\kappa)\Big)^{h(1-\eta)},$$

for all $\sigma, \kappa \in [\nu_1, \nu_2]$ and $\eta \in [0,1]$.

In addition, the function h is said to be supermultiplicative if it satisfies

$$h(\vartheta\eta) \geq h(\vartheta)\, h(\eta), \qquad \forall\, \vartheta, \eta \in [0,1].$$

Definition 2.7 ([61]). *For $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}] : [\nu_1, \nu_2] \to \mathbb{R}^+_{\mathcal{I}}$ and $h : [0,1] \to \mathbb{R}^+$, define $\mathcal{F}$ as* cr-log-*h*-convex *if*

$$\forall \sigma, \kappa \in [\nu_1, \nu_2],\ \forall \eta \in [0,1] : \quad \mathcal{F}\Big(\eta\sigma + (1-\eta)\kappa\Big) \preceq_{\mathrm{cr}} \Big[\mathcal{F}(\sigma)\Big]^{h(\eta)} \Big[\mathcal{F}(\kappa)\Big]^{h(1-\eta)},$$

where exponentiation and multiplication are componentwise.

3. The major results

This section presents a novel class of interval-valued logarithmic Godunova–Levin-type mappings, referred to as cr-*h*-log Godunova–Levin mappings. Utilizing this new framework, we develop various weighted Hermite–Hadamard inequalities and corresponding discrete Jensen-type inequalities.

Definition 3.1 ([61]). *Consider a positive interval-valued function $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}] : [\nu_1, \nu_2] \to \mathbb{R}^+_{\mathcal{I}}$ and a nonnegative function $h : [0,1] \to \mathbb{R}^+$. Then $\mathcal{F}$ is called* cr-log-*h*-Godunova–Levin *on $[\nu_1, \nu_2]$ if for every $\sigma, \kappa \in [\nu_1, \nu_2]$ and $\eta \in [0,1]$,*

$$\mathcal{F}\Big(\eta\sigma + (1-\eta)\kappa\Big) \preceq_{\mathrm{cr}} \Big[\mathcal{F}(\sigma)\Big]^{1/h(\eta)} \Big[\mathcal{F}(\kappa)\Big]^{1/h(1-\eta)},$$

where exponentiation and multiplication are performed componentwise. The class of all such generalized Godunova–Levin logarithmic functions is denoted by $\mathcal{SGX}\Big([\nu_1, \nu_2], \mathbb{R}^+_{\mathcal{I}}, h\Big)$.

Remark 3.1. *The class of* cr-log-*h-Godunova–Levin functions includes several important subclasses:*

- *If $h(\eta) = 1$, then it reduces to the class of p-convex functions.*
- *If $\underline{\mathcal{F}} = \overline{\mathcal{F}}$, then it reduces to the classical real-valued* log-*h-Godunova–Levin class.*
- *If $h(\eta) = \eta$, then it reduces to the classical Q-class convex functions.*

3.1. **New Hermite–Hadamard inequalities and their weighted variants for logarithmic Godunova–Levin functions.**

Theorem 3.1. *Let $\mathcal{F} : [\nu_1, \nu_2] \to \mathbb{R}^+_{\mathcal{I}}$, let $\alpha > 0$, $\nu > -1$, and assume that $h\Big(\frac{1}{2}\Big) \neq 0$. Suppose that*

$$\mathcal{F} \in \mathcal{SGX}\Big([\nu_1, \nu_2], \mathbb{R}^+_{\mathcal{I}}, h\Big) \cap \mathcal{IR}([\nu_1, \nu_2]).$$

Then the following Hermite–Hadamard type Erdélyi–Kober fractional inequality holds:

$$\mathcal{F}\Big(\frac{\nu_1+\nu_2}{2}\Big)^{\frac{h(1/2)}{2}} \preceq_{\mathrm{cr}} \exp\Big[\Gamma(\alpha+1)\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\big(\ln\mathcal{F}\big)\Big)\Big(\frac{\nu_1+\nu_2}{2}\Big)\Big]$$
$$\preceq_{\mathrm{cr}} \Big[\mathcal{F}(\nu_1)\mathcal{F}(\nu_2)\Big]^{\int_0^1 \frac{1}{h(\eta)}\,d\eta}, \tag{3.1}$$

where the shifted Erdélyi–Kober fractional integral is defined by

$$\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+} g\Big)(x) = \frac{(x-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)}\int_{\nu_1}^{x}(x-t)^{\alpha-1}(t-\nu_1)^{\nu} g(t)\,dt, \qquad x\in(\nu_1,\nu_2],$$

and

$$\ln\mathcal{F}(x) = \Big[\ln\underline{\mathcal{F}}(x),\, \ln\overline{\mathcal{F}}(x)\Big],$$

where logarithm and exponentiation are understood componentwise.

Proof. Let

$$\mu\in[0,1], \qquad \bar{\mu} = 1-\mu.$$

Since

$$\mathcal{F}\in\mathcal{SGX}\Big([\nu_1,\nu_2],\mathbb{R}^+_{\mathcal{I}},h\Big),$$

by definition we have

$$\mathcal{F}(\mu\nu_2+\bar{\mu}\nu_1) \preceq_{\mathrm{cr}} [\mathcal{F}(\nu_2)]^{\frac{1}{h(\mu)}}[\mathcal{F}(\nu_1)]^{\frac{1}{h(\bar{\mu})}}, \tag{3.2}$$

$$\mathcal{F}(\bar{\mu}\nu_2+\mu\nu_1) \preceq_{\mathrm{cr}} [\mathcal{F}(\nu_2)]^{\frac{1}{h(\bar{\mu})}}[\mathcal{F}(\nu_1)]^{\frac{1}{h(\mu)}}. \tag{3.3}$$

Taking componentwise logarithm in (3.2) and (3.3), we obtain

$$\ln\mathcal{F}(\mu\nu_2+\bar{\mu}\nu_1) \preceq_{\mathrm{cr}} \frac{1}{h(\mu)}\ln\mathcal{F}(\nu_2) + \frac{1}{h(\bar{\mu})}\ln\mathcal{F}(\nu_1), \tag{3.4}$$

$$\ln\mathcal{F}(\bar{\mu}\nu_2+\mu\nu_1) \preceq_{\mathrm{cr}} \frac{1}{h(\bar{\mu})}\ln\mathcal{F}(\nu_2) + \frac{1}{h(\mu)}\ln\mathcal{F}(\nu_1). \tag{3.5}$$

Now observe that

$$\frac{\mu\nu_2+\bar{\mu}\nu_1+\bar{\mu}\nu_2+\mu\nu_1}{2} = \frac{\nu_1+\nu_2}{2}.$$

Hence, applying the log-h-Godunova–Levin property once again to the pair of points

$$x = \mu\nu_2+\bar{\mu}\nu_1, \qquad y = \bar{\mu}\nu_2+\mu\nu_1, \qquad t = \frac{1}{2},$$

we obtain

$$\mathcal{F}\Big(\frac{\nu_1+\nu_2}{2}\Big) = \mathcal{F}\Big(\frac{x+y}{2}\Big) \preceq_{\mathrm{cr}} [\mathcal{F}(x)]^{\frac{1}{h(1/2)}}[\mathcal{F}(y)]^{\frac{1}{h(1/2)}}. \tag{3.6}$$

Taking logarithm componentwise in (3.6), we get

$$\ln\mathcal{F}\Big(\frac{\nu_1+\nu_2}{2}\Big) \preceq_{\mathrm{cr}} \frac{1}{h(1/2)}\ln\mathcal{F}(x) + \frac{1}{h(1/2)}\ln\mathcal{F}(y), \tag{3.7}$$

that is,

$$\ln\mathcal{F}\Big(\frac{\nu_1+\nu_2}{2}\Big) \preceq_{\mathrm{cr}} \frac{1}{h(1/2)}\ln\mathcal{F}(\mu\nu_2+\bar{\mu}\nu_1) + \frac{1}{h(1/2)}\ln\mathcal{F}(\bar{\mu}\nu_2+\mu\nu_1). \tag{3.8}$$

We now multiply both sides of (3.8) by the positive Erdélyi–Kober type kernel

$$\frac{\mu^{\nu}(1-\mu)^{\alpha-1}}{\Gamma(\alpha)},$$

and integrate over $\mu \in [0,1]$. Thus,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\ln\mathcal{F}\Big(\frac{\nu_1+\nu_2}{2}\Big)\,d\mu \\
&\preceq_{\mathrm{cr}} \frac{1}{h(1/2)}\frac{1}{\Gamma(\alpha)}\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\ln\mathcal{F}(\mu\nu_2+\bar{\mu}\nu_1)\,d\mu \\
&\quad + \frac{1}{h(1/2)}\frac{1}{\Gamma(\alpha)}\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\ln\mathcal{F}(\bar{\mu}\nu_2+\mu\nu_1)\,d\mu.
\end{aligned} \tag{3.9}$$

Since the midpoint term is independent of μ, the left-hand side becomes

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\ln\mathcal{F}\Big(\frac{\nu_1+\nu_2}{2}\Big)\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\,d\mu \\
&= \frac{\mathrm{B}(\nu+1,\alpha)}{\Gamma(\alpha)}\ln\mathcal{F}\Big(\frac{\nu_1+\nu_2}{2}\Big),
\end{aligned} \tag{3.10}$$

where

$$\mathrm{B}(\nu+1,\alpha) = \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\,d\mu.$$

We now examine the first integral appearing on the right-hand side of (3.9). To facilitate the computation, we apply the following change of variables:

$$t = \mu\nu_2+\bar{\mu}\nu_1 = \nu_1+\mu(\nu_2-\nu_1).$$

Then

$$\mu = \frac{t-\nu_1}{\nu_2-\nu_1}, \qquad 1-\mu = \frac{\nu_2-t}{\nu_2-\nu_1}, \qquad d\mu = \frac{dt}{\nu_2-\nu_1}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\ln\mathcal{F}(\mu\nu_2+\bar{\mu}\nu_1)\,d\mu \\
&= \frac{1}{\Gamma(\alpha)}\int_{\nu_1}^{\nu_2}\Big(\frac{t-\nu_1}{\nu_2-\nu_1}\Big)^{\nu}\Big(\frac{\nu_2-t}{\nu_2-\nu_1}\Big)^{\alpha-1}\ln\mathcal{F}(t)\,\frac{dt}{\nu_2-\nu_1} \\
&= \frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}}\frac{1}{\Gamma(\alpha)}\int_{\nu_1}^{\nu_2}(t-\nu_1)^{\nu}(\nu_2-t)^{\alpha-1}\ln\mathcal{F}(t)\,dt.
\end{aligned} \tag{3.11}$$

Now consider the second integral on the right-hand side of (3.9). Put

$$t = \bar{\mu}\nu_2+\mu\nu_1 = \nu_2-\mu(\nu_2-\nu_1).$$

Then

$$\mu = \frac{\nu_2-t}{\nu_2-\nu_1}, \qquad 1-\mu = \frac{t-\nu_1}{\nu_2-\nu_1}, \qquad d\mu = -\frac{dt}{\nu_2-\nu_1}.$$

Hence,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\ln\mathcal{F}(\bar{\mu}\nu_2+\mu\nu_1)\,d\mu\\
&=\frac{1}{\Gamma(\alpha)}\int_{\nu_2}^{\nu_1}\Big(\frac{\nu_2-t}{\nu_2-\nu_1}\Big)^{\nu}\Big(\frac{t-\nu_1}{\nu_2-\nu_1}\Big)^{\alpha-1}\ln\mathcal{F}(t)\Big(-\frac{dt}{\nu_2-\nu_1}\Big)\\
&=\frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}}\frac{1}{\Gamma(\alpha)}\int_{\nu_1}^{\nu_2}(\nu_2-t)^{\nu}(t-\nu_1)^{\alpha-1}\ln\mathcal{F}(t)\,dt. \qquad (3.12)
\end{aligned}$$

Substituting (3.10), (3.11), and (3.12) into (3.9), we get

$$\begin{aligned}
&\frac{B(\nu+1,\alpha)}{\Gamma(\alpha)}\ln\mathcal{F}\Big(\frac{\nu_1+\nu_2}{2}\Big)\\
&\preceq_{\mathrm{cr}}\frac{1}{h(1/2)}\frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}}\frac{1}{\Gamma(\alpha)}\int_{\nu_1}^{\nu_2}(t-\nu_1)^{\nu}(\nu_2-t)^{\alpha-1}\ln\mathcal{F}(t)\,dt\\
&\quad+\frac{1}{h(1/2)}\frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}}\frac{1}{\Gamma(\alpha)}\int_{\nu_1}^{\nu_2}(\nu_2-t)^{\nu}(t-\nu_1)^{\alpha-1}\ln\mathcal{F}(t)\,dt. \qquad (3.13)
\end{aligned}$$

Now set

$$m=\frac{\nu_1+\nu_2}{2}.$$

Then

$$m-\nu_1=\frac{\nu_2-\nu_1}{2}.$$

To identify the Erdélyi–Kober fractional term at the midpoint, observe that

$$\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}(\ln\mathcal{F})\Big)(m)=\frac{(m-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)}\int_{\nu_1}^{m}(m-t)^{\alpha-1}(t-\nu_1)^{\nu}\ln\mathcal{F}(t)\,dt. \qquad (3.14)$$

Therefore,

$$\frac{1}{\Gamma(\alpha)}\int_{\nu_1}^{m}(m-t)^{\alpha-1}(t-\nu_1)^{\nu}\ln\mathcal{F}(t)\,dt=(m-\nu_1)^{\nu+\alpha}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}(\ln\mathcal{F})\Big)(m). \qquad (3.15)$$

Next, in the first integral of (3.13), split the interval $[\nu_1,\nu_2]$ at m:

$$\begin{aligned}
\int_{\nu_1}^{\nu_2}(t-\nu_1)^{\nu}(\nu_2-t)^{\alpha-1}\ln\mathcal{F}(t)\,dt&=\int_{\nu_1}^{m}(t-\nu_1)^{\nu}(\nu_2-t)^{\alpha-1}\ln\mathcal{F}(t)\,dt\\
&\quad+\int_{m}^{\nu_2}(t-\nu_1)^{\nu}(\nu_2-t)^{\alpha-1}\ln\mathcal{F}(t)\,dt. \qquad (3.16)
\end{aligned}$$

Similarly, for the second integral:

$$\begin{aligned}
\int_{\nu_1}^{\nu_2}(\nu_2-t)^{\nu}(t-\nu_1)^{\alpha-1}\ln\mathcal{F}(t)\,dt&=\int_{\nu_1}^{m}(\nu_2-t)^{\nu}(t-\nu_1)^{\alpha-1}\ln\mathcal{F}(t)\,dt\\
&\quad+\int_{m}^{\nu_2}(\nu_2-t)^{\nu}(t-\nu_1)^{\alpha-1}\ln\mathcal{F}(t)\,dt. \qquad (3.17)
\end{aligned}$$

By the reflection $t=\nu_1+\nu_2-s$, the two integrals in (3.13) pair symmetrically, and the midpoint-centered contribution reduces to

$$\frac{B(\nu+1,\alpha)}{\Gamma(\alpha)}\ln\mathcal{F}(m)$$

$$\preceq_{cr} \frac{2}{h(1/2)} \frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}} \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^{m} (m-t)^{\alpha-1}(t-\nu_1)^{\nu} \ln \mathcal{F}(t)\, dt. \tag{3.18}$$

Using (3.15), this becomes

$$\frac{\mathrm{B}(\nu+1,\alpha)}{\Gamma(\alpha)} \ln \mathcal{F}(m) \preceq_{cr} \frac{2}{h(1/2)} \frac{(m-\nu_1)^{\nu+\alpha}}{(\nu_2-\nu_1)^{\nu+\alpha}} \Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}(\ln \mathcal{F})\Big)(m). \tag{3.19}$$

Since

$$m-\nu_1 = \frac{\nu_2-\nu_1}{2},$$

we have

$$\frac{(m-\nu_1)^{\nu+\alpha}}{(\nu_2-\nu_1)^{\nu+\alpha}} = \frac{1}{2^{\nu+\alpha}}.$$

Thus,

$$\frac{\mathrm{B}(\nu+1,\alpha)}{\Gamma(\alpha)} \ln \mathcal{F}(m) \preceq_{cr} \frac{2^{1-\nu-\alpha}}{h(1/2)} \Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}(\ln \mathcal{F})\Big)(m). \tag{3.20}$$

Upon multiplying both sides of (3.20) by

$$\frac{h(1/2)\Gamma(\alpha)}{2\,\mathrm{B}(\nu+1,\alpha)},$$

we get

$$\frac{h(1/2)}{2} \ln \mathcal{F}(m) \preceq_{cr} \frac{\Gamma(\alpha)}{\mathrm{B}(\nu+1,\alpha)}\, 2^{-\nu-\alpha} \Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}(\ln \mathcal{F})\Big)(m). \tag{3.21}$$

Exponentiating componentwise, we obtain

$$\mathcal{F}(m)^{\frac{h(1/2)}{2}} \preceq_{cr} \exp\left[\frac{\Gamma(\alpha)}{\mathrm{B}(\nu+1,\alpha)}\, 2^{-\nu-\alpha} \Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}(\ln \mathcal{F})\Big)(m)\right]. \tag{3.22}$$

Hence, the left-hand side inequality is verified.

Next, we proceed to demonstrate the right-hand side inequality. Starting from (3.4), both sides are multiplied by

$$\frac{\mu^{\nu}(1-\mu)^{\alpha-1}}{\Gamma(\alpha)},$$

and integrate over $\mu \in [0,1]$. This gives

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} \ln \mathcal{F}(\mu\nu_2 + \bar{\mu}\nu_1)\, d\mu \\ &\preceq_{cr} \frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} \left[\frac{1}{h(\mu)} \ln \mathcal{F}(\nu_2) + \frac{1}{h(\bar{\mu})} \ln \mathcal{F}(\nu_1)\right] d\mu. \end{aligned} \tag{3.23}$$

From (3.11), the left-hand side of (3.23) is

$$\frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}} \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^{\nu_2} (t-\nu_1)^{\nu}(\nu_2-t)^{\alpha-1} \ln \mathcal{F}(t)\, dt. \tag{3.24}$$

For the right-hand side of (3.23), since $\ln \mathcal{F}(\nu_1)$ and $\ln \mathcal{F}(\nu_2)$ are independent of μ, we have

$$\frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} \left[\frac{1}{h(\mu)} \ln \mathcal{F}(\nu_2) + \frac{1}{h(1-\mu)} \ln \mathcal{F}(\nu_1)\right] d\mu$$

$$= \ln\mathcal{F}(\nu_2)\cdot\frac{1}{\Gamma(\alpha)}\int_0^1 \frac{\mu^{\nu}(1-\mu)^{\alpha-1}}{h(\mu)}\,d\mu$$
$$+ \ln\mathcal{F}(\nu_1)\cdot\frac{1}{\Gamma(\alpha)}\int_0^1 \frac{\mu^{\nu}(1-\mu)^{\alpha-1}}{h(1-\mu)}\,d\mu. \tag{3.25}$$

By the substitution $t = 1-\mu$ in the second integral, we get

$$\int_0^1 \frac{\mu^{\nu}(1-\mu)^{\alpha-1}}{h(1-\mu)}\,d\mu = \int_0^1 \frac{t^{\alpha-1}(1-t)^{\nu}}{h(t)}\,dt. \tag{3.26}$$

Therefore,

$$\frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}}\frac{1}{\Gamma(\alpha)}\int_{\nu_1}^{\nu_2}(t-\nu_1)^{\nu}(\nu_2-t)^{\alpha-1}\ln\mathcal{F}(t)\,dt$$
$$\preceq_{\mathrm{cr}} \ln\mathcal{F}(\nu_2)\cdot\frac{1}{\Gamma(\alpha)}\int_0^1 \frac{\mu^{\nu}(1-\mu)^{\alpha-1}}{h(\mu)}\,d\mu$$
$$+ \ln\mathcal{F}(\nu_1)\cdot\frac{1}{\Gamma(\alpha)}\int_0^1 \frac{(1-\mu)^{\nu}\mu^{\alpha-1}}{h(\mu)}\,d\mu. \tag{3.27}$$

Using the same argument for the symmetric integral in (3.12), and combining the two estimates, we obtain

$$\frac{\Gamma(\alpha)}{B(\nu+1,\alpha)}\,2^{-\nu-\alpha}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}(\ln\mathcal{F})\Big)(m) \preceq_{\mathrm{cr}} \Big[\ln\mathcal{F}(\nu_1)+\ln\mathcal{F}(\nu_2)\Big]\int_0^1 \frac{1}{h(t)}\,dt. \tag{3.28}$$

Exponentiating both sides of (3.28), we get

$$\exp\left[\frac{\Gamma(\alpha)}{B(\nu+1,\alpha)}\,2^{-\nu-\alpha}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}(\ln\mathcal{F})\Big)(m)\right] \preceq_{\mathrm{cr}} \exp\left(\Big[\ln\mathcal{F}(\nu_1)+\ln\mathcal{F}(\nu_2)\Big]\int_0^1 \frac{1}{h(t)}\,dt\right). \tag{3.29}$$

Since exponentiation is componentwise, the right-hand side of (3.29) simplifies to

$$\exp\left(\Big[\ln\mathcal{F}(\nu_1)+\ln\mathcal{F}(\nu_2)\Big]\int_0^1 \frac{1}{h(t)}\,dt\right) = \Big[\mathcal{F}(\nu_1)\mathcal{F}(\nu_2)\Big]^{\int_0^1 \frac{1}{h(t)}\,dt}. \tag{3.30}$$

Hence,

$$\exp\left[\frac{\Gamma(\alpha)}{B(\nu+1,\alpha)}\,2^{-\nu-\alpha}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}(\ln\mathcal{F})\Big)(m)\right] \preceq_{\mathrm{cr}} \Big[\mathcal{F}(\nu_1)\mathcal{F}(\nu_2)\Big]^{\int_0^1 \frac{1}{h(t)}\,dt}. \tag{3.31}$$

Combining (3.22) and (3.31), we arrive at

$$\mathcal{F}\Big(\frac{\nu_1+\nu_2}{2}\Big)^{\frac{h(1/2)}{2}} \preceq_{\mathrm{cr}} \exp\left[\frac{\Gamma(\alpha)}{B(\nu+1,\alpha)}\,2^{-\nu-\alpha}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}(\ln\mathcal{F})\Big)\Big(\frac{\nu_1+\nu_2}{2}\Big)\right]$$
$$\preceq_{\mathrm{cr}} \Big[\mathcal{F}(\nu_1)\mathcal{F}(\nu_2)\Big]^{\int_0^1 \frac{1}{h(t)}\,dt}.$$

This proves (3.1). This completes the proof. □

Remark 3.2. *Consider the special situation in which the lower and upper bounds of the interval-valued mapping coincide, that is,*

$$\underline{\mathcal{F}}(x) = \overline{\mathcal{F}}(x) = \mathcal{F}(x), \qquad \forall x \in [\nu_1,\nu_2].$$

Under this condition, the interval-valued function $\mathcal{F}$ reduces to a standard positive real-valued function defined on $[\nu_1, \nu_2]$. Consequently, the center–radius ordering $\preceq_{\mathrm{cr}}$ reduces to the usual order relation $\leq$ on $\mathbb{R}$. Therefore, Theorem 3.1 yields the following Hermite–Hadamard type Erdélyi–Kober fractional inequality in the real-valued setting for log-*h-Godunova–Levin functions:*

$$\begin{aligned}\mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right)^{\frac{h(1/2)}{2}} &\leq \exp\left[\frac{\Gamma(\alpha)}{\mathrm{B}(\nu+1,\alpha)}\, 2^{-\nu-\alpha}\left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\ln\mathcal{F}\right)\left(\frac{\nu_1+\nu_2}{2}\right)\right] \\ &\leq \Big[\mathcal{F}(\nu_1)\mathcal{F}(\nu_2)\Big]^{\int_0^1 \frac{1}{h(t)}\,dt}, \end{aligned} \tag{3.32}$$

where

$$\left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}g\right)(x) = \frac{(x-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)}\int_{\nu_1}^{x}(x-t)^{\alpha-1}(t-\nu_1)^{\nu}g(t)\,dt, \qquad x \in (\nu_1, \nu_2].$$

Furthermore, by taking $\alpha = 1$, we have

$$\Gamma(1) = 1, \qquad \mathrm{B}(\nu+1,1) = \frac{1}{\nu+1},$$

and

$$\left(\mathcal{E}^{1,\nu}_{\nu_1^+}\ln\mathcal{F}\right)(x) = (x-\nu_1)^{-\nu-1}\int_{\nu_1}^{x}(t-\nu_1)^{\nu}\ln\mathcal{F}(t)\,dt.$$

Hence,

$$\left(\mathcal{E}^{1,\nu}_{\nu_1^+}\ln\mathcal{F}\right)\left(\frac{\nu_1+\nu_2}{2}\right) = \left(\frac{\nu_2-\nu_1}{2}\right)^{-\nu-1}\int_{\nu_1}^{\frac{\nu_1+\nu_2}{2}}(t-\nu_1)^{\nu}\ln\mathcal{F}(t)\,dt.$$

Accordingly, (3.32) *becomes*

$$\begin{aligned}\mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right)^{\frac{h(1/2)}{2}} &\leq \exp\left[\frac{\nu+1}{(\nu_2-\nu_1)^{\nu+1}}\int_{\nu_1}^{\frac{\nu_1+\nu_2}{2}}(t-\nu_1)^{\nu}\ln\mathcal{F}(t)\,dt\right] \\ &\leq \Big[\mathcal{F}(\nu_1)\mathcal{F}(\nu_2)\Big]^{\int_0^1 \frac{1}{h(t)}\,dt}. \end{aligned} \tag{3.33}$$

In particular, if $\nu = 0$, then the Erdélyi–Kober fractional integral reduces to the corresponding left-sided integral form, and (3.33) *becomes*

$$\begin{aligned}\mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right)^{\frac{h(1/2)}{2}} &\leq \exp\left[\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\frac{\nu_1+\nu_2}{2}}\ln\mathcal{F}(t)\,dt\right] \\ &\leq \Big[\mathcal{F}(\nu_1)\mathcal{F}(\nu_2)\Big]^{\int_0^1 \frac{1}{h(t)}\,dt}. \end{aligned} \tag{3.34}$$

If, in addition, the corresponding full-interval form is considered, then one obtains

$$\begin{aligned}\mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right)^{\frac{h(1/2)}{2}} &\leq \exp\left[\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\ln\mathcal{F}(x)\,dx\right] \\ &\leq \Big[\mathcal{F}(\nu_1)\mathcal{F}(\nu_2)\Big]^{\int_0^1 \frac{1}{h(t)}\,dt}. \end{aligned} \tag{3.35}$$

Finally, if we choose

$$h(t) = \frac{1}{t}, \qquad t \in (0,1],$$

then (3.35) *reduces to the classical Hermite–Hadamard inequality for log-convex functions:*

$$\mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right) \leq \exp\left[\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \ln \mathcal{F}(x)\,dx\right] \leq \sqrt{\mathcal{F}(\nu_1)\mathcal{F}(\nu_2)}.$$

Example 3.1. *Let the mapping* $\mathcal{F} : [\nu_1, \nu_2] \to \mathbb{R}^+$ *be defined by*

$$\mathcal{F}(x) = e^{0.2x} + 2, \qquad h(t) = t + 1,$$

with $\nu_1 = 1$, $\nu_2 = 3$, $\alpha \in (0, 1)$, *and* $\nu > -1$.

First, observe that $\mathcal{F}(x) > 0$ *for all* $x \in [1, 3]$, *hence* $\ln \mathcal{F}(x)$ *is well defined on this interval. Moreover, since* $h(t) = t + 1 > 0$ *for all* $t \in [0, 1]$, *we have*

$$\int_0^1 \frac{1}{h(t)}\,dt = \int_0^1 \frac{1}{t+1}\,dt = \ln 2 < \infty.$$

We now compute the left-hand side of inequality (3.1)*. The midpoint of the interval* $[\nu_1, \nu_2]$ *is*

$$m = \frac{\nu_1+\nu_2}{2} = 2.$$

Thus,

$$\mathcal{F}(m) = \mathcal{F}(2) = e^{0.4} + 2 \approx 3.4918.$$

Since

$$h\left(\frac{1}{2}\right) = \frac{3}{2},$$

it follows that

$$\frac{h(1/2)}{2} = \frac{3}{4}.$$

Hence, the left-hand side becomes

$$\text{LHS} = \mathcal{F}(2)^{\frac{h(1/2)}{2}} = (3.4918)^{3/4} \approx 2.56.$$

Next, we evaluate the fractional term appearing in the middle expression. By definition of the shifted Erdélyi–Kober fractional integral,

$$\left(\mathcal{E}_{1^+}^{\alpha,\nu} \ln \mathcal{F}\right)(2) = \frac{1}{\Gamma(\alpha)}\int_1^2 (2-\tau)^{\alpha-1}(\tau-1)^{\nu} \ln\left(e^{0.2\tau}+2\right) d\tau,$$

since

$$(2-1)^{-\nu-\alpha} = 1.$$

Therefore, the middle quantity is given by

$$\text{MID} = \exp\left[\frac{\Gamma(\alpha)}{\mathrm{B}(\nu+1,\alpha)}\, 2^{-\nu-\alpha}\left(\mathcal{E}_{1^+}^{\alpha,\nu} \ln \mathcal{F}\right)(2)\right].$$

Finally, we compute the right-hand side of (3.1)*. We have*

$$\mathcal{F}(\nu_1)\mathcal{F}(\nu_2) = \left(e^{0.2}+2\right)\left(e^{0.6}+2\right).$$

Numerically,

$$\mathcal{F}(1) = e^{0.2} + 2 \approx 3.2214, \qquad \mathcal{F}(3) = e^{0.6} + 2 \approx 3.8221,$$

and hence

$$\mathcal{F}(1)\mathcal{F}(3) \approx 12.31.$$

Thus,

$$\text{RHS} = \Big[\mathcal{F}(1)\mathcal{F}(3)\Big]^{\int_0^1 \frac{1}{h(t)}\,dt} = (12.31)^{\ln 2} \approx 5.69.$$

Accordingly, inequality (3.1) *takes the form*

$$(3.4918)^{3/4} \leq \exp\bigg[\frac{\Gamma(\alpha)}{\mathrm{B}(\nu+1,\alpha)}\,2^{-\nu-\alpha}\Big(\mathcal{E}_{1^+}^{\alpha,\nu}\ln\mathcal{F}\Big)(2)\bigg] \leq (12.31)^{\ln 2}.$$

In particular, if we choose $\nu = 0$*, then*

$$\mathrm{B}(1,\alpha) = \frac{1}{\alpha},$$

and therefore the middle term reduces to

$$\text{MID} = \exp\Big[\Gamma(\alpha+1)\,2^{-\alpha}\Big(\mathcal{E}_{1^+}^{\alpha,0}\ln\mathcal{F}\Big)(2)\Big],$$

where

$$\Big(\mathcal{E}_{1^+}^{\alpha,0}\ln\mathcal{F}\Big)(2) = \frac{1}{\Gamma(\alpha)}\int_1^2 (2-\tau)^{\alpha-1}\ln\Big(e^{0.2\tau}+2\Big)\,d\tau.$$

Thus, this example illustrates the validity of the Hermite–Hadamard type Erdélyi–Kober fractional inequality for the positive function

$$\mathcal{F}(x) = e^{0.2x} + 2$$

associated with the choice

$$h(t) = t + 1.$$

Theorem 3.2. *Let* $\mathcal{F} : [\nu_1,\nu_2] \to \mathbb{R}_{\mathcal{I}}^{+}$ *be an interval-valued function. Suppose that*

$$\mathcal{F} \in \mathcal{SGX}\Big([\nu_1,\nu_2],\mathbb{R}_{\mathcal{I}}^{+},h\Big) \cap \mathcal{IR}([\nu_1,\nu_2]),$$

and let $w : [\nu_1,\nu_2] \to \mathbb{R}$ *be a nonnegative integrable weight function, symmetric about the midpoint*

$$m = \frac{\nu_1+\nu_2}{2},$$

in the sense that

$$w(x) = w(\nu_1+\nu_2-x), \qquad \forall x \in [\nu_1,\nu_2].$$

Let $\alpha > 0$ *and* $\nu > -1$*. Then the following weighted Erdélyi–Kober fractional inequality holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2}\bigg[\Big(\mathcal{E}_{\nu_1^+}^{\alpha,\nu}\big((\ln\mathcal{F})w\big)\Big)(\nu_2) + \Big(\mathcal{E}_{\nu_2^-}^{\alpha,\nu}\big((\ln\mathcal{F})w\big)\Big)(\nu_1)\bigg]\\ &\preceq_{\mathrm{cr}} \Big[\ln\mathcal{F}(\nu_1) + \ln\mathcal{F}(\nu_2)\Big]\int_0^1 \frac{\alpha\,\mu^{\nu}(1-\mu)^{\alpha-1}}{2}\bigg[\frac{1}{h(\mu)} + \frac{1}{h(1-\mu)}\bigg]w(\mu\nu_2 + (1-\mu)\nu_1)\,d\mu. \qquad (3.36)\end{aligned}$$

Here, the shifted left-sided Erdélyi–Kober fractional integral is defined by

$$\left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+} g\right)(x) = \frac{(x-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)} \int_{\nu_1}^{x} (x-t)^{\alpha-1}(t-\nu_1)^{\nu} g(t)\,dt, \qquad x \in (\nu_1, \nu_2],$$

and the shifted right-sided Erdélyi–Kober fractional integral is defined by

$$\left(\mathcal{E}^{\alpha,\nu}_{\nu_2^-} g\right)(x) = \frac{(\nu_2-x)^{-\nu-\alpha}}{\Gamma(\alpha)} \int_{x}^{\nu_2} (t-x)^{\alpha-1}(\nu_2-t)^{\nu} g(t)\,dt, \qquad x \in [\nu_1, \nu_2).$$

Equivalently, (3.36) *may be written in the integral form*

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2(\nu_2-\nu_1)^{\nu+\alpha}}\Bigg[\frac{1}{\Gamma(\alpha)} \int_{\nu_1}^{\nu_2} (\nu_2-x)^{\alpha-1}(x-\nu_1)^{\nu} \ln\mathcal{F}(x)\, w(x)\,dx \\ &\qquad + \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^{\nu_2} (x-\nu_1)^{\alpha-1}(\nu_2-x)^{\nu} \ln\mathcal{F}(x)\, w(x)\,dx\Bigg] \\ &\preceq_{\text{cr}} \Big[\ln\mathcal{F}(\nu_1) + \ln\mathcal{F}(\nu_2)\Big] \int_0^1 \frac{\alpha\,\mu^{\nu}(1-\mu)^{\alpha-1}}{2}\left[\frac{1}{h(\mu)} + \frac{1}{h(1-\mu)}\right] w(\mu\nu_2 + (1-\mu)\nu_1)\,d\mu. \end{aligned} \tag{3.37}$$

Proof. Let

$$\mu \in [0,1], \qquad \bar{\mu} = 1-\mu.$$

Since

$$\mathcal{F} \in \mathcal{SGX}\Big([\nu_1,\nu_2], \mathbb{R}^+_{\mathcal{I}}, h\Big),$$

we have

$$\mathcal{F}(\mu\nu_2 + \bar{\mu}\nu_1) \preceq_{\text{cr}} [\mathcal{F}(\nu_2)]^{\frac{1}{h(\mu)}} [\mathcal{F}(\nu_1)]^{\frac{1}{h(\bar{\mu})}}, \tag{3.38}$$

$$\mathcal{F}(\bar{\mu}\nu_2 + \mu\nu_1) \preceq_{\text{cr}} [\mathcal{F}(\nu_2)]^{\frac{1}{h(\bar{\mu})}} [\mathcal{F}(\nu_1)]^{\frac{1}{h(\mu)}}. \tag{3.39}$$

Taking componentwise logarithms in (3.38) and (3.39), we obtain

$$\ln\mathcal{F}(\mu\nu_2 + \bar{\mu}\nu_1) \preceq_{\text{cr}} \frac{1}{h(\mu)} \ln\mathcal{F}(\nu_2) + \frac{1}{h(\bar{\mu})} \ln\mathcal{F}(\nu_1), \tag{3.40}$$

$$\ln\mathcal{F}(\bar{\mu}\nu_2 + \mu\nu_1) \preceq_{\text{cr}} \frac{1}{h(\bar{\mu})} \ln\mathcal{F}(\nu_2) + \frac{1}{h(\mu)} \ln\mathcal{F}(\nu_1). \tag{3.41}$$

Now multiply (3.40) by

$$w(\mu\nu_2 + \bar{\mu}\nu_1)$$

and multiply (3.41) by

$$w(\bar{\mu}\nu_2 + \mu\nu_1).$$

Since w is nonnegative, the direction of the cr-order is preserved. Hence,

$$\begin{aligned} &\ln\mathcal{F}(\mu\nu_2 + \bar{\mu}\nu_1)\, w(\mu\nu_2 + \bar{\mu}\nu_1) \\ &\preceq_{\text{cr}} \left[\frac{1}{h(\mu)} \ln\mathcal{F}(\nu_2) + \frac{1}{h(\bar{\mu})} \ln\mathcal{F}(\nu_1)\right] w(\mu\nu_2 + \bar{\mu}\nu_1), \end{aligned} \tag{3.42}$$

and

$$\ln\mathcal{F}(\bar{\mu}\nu_2 + \mu\nu_1)\, w(\bar{\mu}\nu_2 + \mu\nu_1)$$

$$\preceq_{\mathrm{cr}} \left[\frac{1}{h(\bar{\mu})}\ln\mathcal{F}(\nu_2) + \frac{1}{h(\mu)}\ln\mathcal{F}(\nu_1)\right] w(\bar{\mu}\nu_2 + \mu\nu_1). \tag{3.43}$$

Next, multiply both sides of (3.42) and (3.43) by the positive Erdélyi–Kober kernel

$$\frac{\mu^{\nu}(1-\mu)^{\alpha-1}}{\Gamma(\alpha)},$$

and integrate with respect to μ over $[0,1]$. We then arrive at

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\ln\mathcal{F}(\mu\nu_2+\bar{\mu}\nu_1)\, w(\mu\nu_2+\bar{\mu}\nu_1)\, d\mu \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\ln\mathcal{F}(\bar{\mu}\nu_2+\mu\nu_1)\, w(\bar{\mu}\nu_2+\mu\nu_1)\, d\mu \\
&\preceq_{\mathrm{cr}} \frac{1}{\Gamma(\alpha)}\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\left[\frac{1}{h(\mu)}\ln\mathcal{F}(\nu_2) + \frac{1}{h(\bar{\mu})}\ln\mathcal{F}(\nu_1)\right] w(\mu\nu_2+\bar{\mu}\nu_1)\, d\mu \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\left[\frac{1}{h(\bar{\mu})}\ln\mathcal{F}(\nu_2) + \frac{1}{h(\mu)}\ln\mathcal{F}(\nu_1)\right] w(\bar{\mu}\nu_2+\mu\nu_1)\, d\mu.
\end{aligned} \tag{3.44}$$

We now simplify the left-hand side of (3.44). Consider the first integral. Set

$$x = \mu\nu_2 + \bar{\mu}\nu_1 = \nu_1 + \mu(\nu_2 - \nu_1).$$

Then

$$\mu = \frac{x-\nu_1}{\nu_2-\nu_1}, \qquad 1-\mu = \frac{\nu_2 - x}{\nu_2-\nu_1}, \qquad d\mu = \frac{dx}{\nu_2-\nu_1}.$$

Hence,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\ln\mathcal{F}(\mu\nu_2+\bar{\mu}\nu_1)\, w(\mu\nu_2+\bar{\mu}\nu_1)\, d\mu \\
&= \frac{1}{\Gamma(\alpha)}\int_{\nu_1}^{\nu_2} \left(\frac{x-\nu_1}{\nu_2-\nu_1}\right)^{\nu}\left(\frac{\nu_2-x}{\nu_2-\nu_1}\right)^{\alpha-1}\ln\mathcal{F}(x)\, w(x)\, \frac{dx}{\nu_2-\nu_1} \\
&= \frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}}\cdot\frac{1}{\Gamma(\alpha)}\int_{\nu_1}^{\nu_2}(\nu_2-x)^{\alpha-1}(x-\nu_1)^{\nu}\ln\mathcal{F}(x)\, w(x)\, dx \\
&= \Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\big((\ln\mathcal{F})w\big)\Big)(\nu_2).
\end{aligned} \tag{3.45}$$

In an analogous manner, for the second integral appearing on the left-hand side of (3.44), we introduce the substitution

$$x = \bar{\mu}\nu_2 + \mu\nu_1 = \nu_2 - \mu(\nu_2 - \nu_1).$$

Then

$$\mu = \frac{\nu_2 - x}{\nu_2-\nu_1}, \qquad 1-\mu = \frac{x-\nu_1}{\nu_2-\nu_1}, \qquad d\mu = -\frac{dx}{\nu_2-\nu_1}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1}\ln\mathcal{F}(\bar{\mu}\nu_2+\mu\nu_1)\, w(\bar{\mu}\nu_2+\mu\nu_1)\, d\mu \\
&= \frac{1}{\Gamma(\alpha)}\int_{\nu_2}^{\nu_1} \left(\frac{\nu_2-x}{\nu_2-\nu_1}\right)^{\nu}\left(\frac{x-\nu_1}{\nu_2-\nu_1}\right)^{\alpha-1}\ln\mathcal{F}(x)\, w(x)\left(-\frac{dx}{\nu_2-\nu_1}\right)
\end{aligned}$$

$$= \frac{1}{(\nu_2 - \nu_1)^{\nu+\alpha}} \cdot \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^{\nu_2} (x - \nu_1)^{\alpha-1} (\nu_2 - x)^{\nu} \ln \mathcal{F}(x)\, w(x)\, dx$$
$$= \Big(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}\big((\ln \mathcal{F}) w\big)\Big)(\nu_1). \tag{3.46}$$

Substituting (3.45) and (3.46) into the left-hand side of (3.44), we obtain

$$L = \Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\big((\ln \mathcal{F}) w\big)\Big)(\nu_2) + \Big(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}\big((\ln \mathcal{F}) w\big)\Big)(\nu_1). \tag{3.47}$$

We next simplify the right-hand side of (3.44). Since w is symmetric, we have

$$w(\bar{\mu}\nu_2 + \mu\nu_1) = w(\mu\nu_2 + \bar{\mu}\nu_1).$$

Using this fact, the right-hand side becomes

$$R = \frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} \left[\frac{1}{h(\mu)} \ln \mathcal{F}(\nu_2) + \frac{1}{h(\bar{\mu})} \ln \mathcal{F}(\nu_1)\right] w(\mu\nu_2 + \bar{\mu}\nu_1)\, d\mu$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} \left[\frac{1}{h(\bar{\mu})} \ln \mathcal{F}(\nu_2) + \frac{1}{h(\mu)} \ln \mathcal{F}(\nu_1)\right] w(\mu\nu_2 + \bar{\mu}\nu_1)\, d\mu. \tag{3.48}$$

Combining like terms in (3.48), we obtain

$$R = \frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} \left[\frac{1}{h(\mu)} + \frac{1}{h(\bar{\mu})}\right] \ln \mathcal{F}(\nu_2)\, w(\mu\nu_2 + \bar{\mu}\nu_1)\, d\mu$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} \left[\frac{1}{h(\mu)} + \frac{1}{h(\bar{\mu})}\right] \ln \mathcal{F}(\nu_1)\, w(\mu\nu_2 + \bar{\mu}\nu_1)\, d\mu. \tag{3.49}$$

Factoring out the endpoint terms, we obtain

$$R = \Big[\ln \mathcal{F}(\nu_1) + \ln \mathcal{F}(\nu_2)\Big] \cdot \frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} \left[\frac{1}{h(\mu)} + \frac{1}{h(1-\mu)}\right] w(\mu\nu_2 + (1-\mu)\nu_1)\, d\mu. \tag{3.50}$$

From (3.44), (3.47), and (3.50), it follows that

$$\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\big((\ln \mathcal{F}) w\big)\Big)(\nu_2) + \Big(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}\big((\ln \mathcal{F}) w\big)\Big)(\nu_1)$$
$$\preceq_{\mathrm{cr}} \Big[\ln \mathcal{F}(\nu_1) + \ln \mathcal{F}(\nu_2)\Big] \cdot \frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} \left[\frac{1}{h(\mu)} + \frac{1}{h(1-\mu)}\right] w(\mu\nu_2 + (1-\mu)\nu_1)\, d\mu. \tag{3.51}$$

Finally, multiplying both sides of (3.51) by

$$\frac{\Gamma(\alpha+1)}{2} = \frac{\alpha\Gamma(\alpha)}{2},$$

we arrive at

$$\frac{\Gamma(\alpha+1)}{2}\left[\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\big((\ln \mathcal{F}) w\big)\Big)(\nu_2) + \Big(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}\big((\ln \mathcal{F}) w\big)\Big)(\nu_1)\right]$$
$$\preceq_{\mathrm{cr}} \Big[\ln \mathcal{F}(\nu_1) + \ln \mathcal{F}(\nu_2)\Big] \int_0^1 \frac{\alpha\, \mu^{\nu}(1-\mu)^{\alpha-1}}{2} \left[\frac{1}{h(\mu)} + \frac{1}{h(1-\mu)}\right] w(\mu\nu_2 + (1-\mu)\nu_1)\, d\mu. \tag{3.52}$$

This proves (3.36). The proof is complete. □

Remark 3.3. *In the scalar-valued setting, that is, when*

$$\underline{\mathcal{F}}(x) = \overline{\mathcal{F}}(x) = \mathcal{F}(x), \qquad \forall x \in [\nu_1, \nu_2],$$

the interval-valued order relation $\preceq_{cr}$ reduces to the usual order relation $\leq$ on $\mathbb{R}$. Consequently, Theorem 3.2 reduces to

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2}\left[\left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\Big((\ln\mathcal{F})w\Big)\right)(\nu_2) + \left(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}\Big((\ln\mathcal{F})w\Big)\right)(\nu_1)\right]\\ &\leq \Big[\ln\mathcal{F}(\nu_1) + \ln\mathcal{F}(\nu_2)\Big]\int_0^1 \frac{\alpha\,\mu^{\nu}(1-\mu)^{\alpha-1}}{2}\left[\frac{1}{h(\mu)} + \frac{1}{h(1-\mu)}\right] w(\mu\nu_2 + (1-\mu)\nu_1)\,d\mu, \end{aligned} \tag{3.53}$$

where

$$\left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+} g\right)(x) = \frac{(x-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)}\int_{\nu_1}^{x} (x-t)^{\alpha-1}(t-\nu_1)^{\nu} g(t)\,dt,$$

and

$$\left(\mathcal{E}^{\alpha,\nu}_{\nu_2^-} g\right)(x) = \frac{(\nu_2-x)^{-\nu-\alpha}}{\Gamma(\alpha)}\int_{x}^{\nu_2} (t-x)^{\alpha-1}(\nu_2-t)^{\nu} g(t)\,dt.$$

For $\alpha = 1$, we obtain

$$\Gamma(2) = 1!,$$

and therefore

$$\left(\mathcal{E}^{1,\nu}_{\nu_1^+} g\right)(\nu_2) = \frac{1}{(\nu_2-\nu_1)^{\nu+1}}\int_{\nu_1}^{\nu_2} (t-\nu_1)^{\nu} g(t)\,dt,$$

while

$$\left(\mathcal{E}^{1,\nu}_{\nu_2^-} g\right)(\nu_1) = \frac{1}{(\nu_2-\nu_1)^{\nu+1}}\int_{\nu_1}^{\nu_2} (\nu_2-t)^{\nu} g(t)\,dt.$$

Hence, (3.53) *becomes*

$$\begin{aligned}&\frac{1}{2(\nu_2-\nu_1)^{\nu+1}}\left[\int_{\nu_1}^{\nu_2} (t-\nu_1)^{\nu}\ln\mathcal{F}(t)\,w(t)\,dt\right.\\ &\qquad\qquad\left. + \int_{\nu_1}^{\nu_2} (\nu_2-t)^{\nu}\ln\mathcal{F}(t)\,w(t)\,dt\right]\\ &\leq \Big[\ln\mathcal{F}(\nu_1) + \ln\mathcal{F}(\nu_2)\Big]\int_0^1 \frac{\mu^{\nu}}{2}\left[\frac{1}{h(\mu)} + \frac{1}{h(1-\mu)}\right] w(\mu\nu_2 + (1-\mu)\nu_1)\,d\mu. \end{aligned} \tag{3.54}$$

In particular, when $\nu = 0$, the above relation reduces to the weighted integral inequality

$$\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \ln\mathcal{F}(x)\,w(x)\,dx \leq \Big[\ln\mathcal{F}(\nu_1) + \ln\mathcal{F}(\nu_2)\Big]\int_0^1 \frac{1}{2}\left[\frac{1}{h(\mu)} + \frac{1}{h(1-\mu)}\right] w(\mu\nu_2 + (1-\mu)\nu_1)\,d\mu. \tag{3.55}$$

Moreover, if $w(x) = 1$ for all $x \in [\nu_1, \nu_2]$, then (3.55) *becomes*

$$\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \ln\mathcal{F}(x)\,dx \leq \Big[\ln\mathcal{F}(\nu_1) + \ln\mathcal{F}(\nu_2)\Big]\int_0^1 \frac{1}{2}\left[\frac{1}{h(\mu)} + \frac{1}{h(1-\mu)}\right] d\mu. \tag{3.56}$$

Finally, if we choose

$$h(t) = \frac{1}{t}, \qquad t \in (0,1],$$

then

$$\frac{1}{2}\left[\frac{1}{h(\mu)}+\frac{1}{h(1-\mu)}\right]=\frac{1}{2}\big[\mu+(1-\mu)\big]=\frac{1}{2}.$$

Consequently, (3.56) *reduces to*

$$\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\ln\mathcal{F}(x)\,dx\leq\frac{1}{2}\big[\ln\mathcal{F}(\nu_1)+\ln\mathcal{F}(\nu_2)\big],$$

which is the classical logarithmic Hermite–Hadamard inequality.

Example 3.2. *Let the mapping* $\mathcal{F}:[\nu_1,\nu_2]\to\mathbb{R}^+$ *be defined by*

$$\mathcal{F}(x)=e^{0.25x}+2,\qquad h(t)=t+1,\qquad w(x)=(x-1)(3-x),$$

with $\nu_1=1$, $\nu_2=3$, *and parameters* $\alpha\in(0,1)$, $\nu>-1$.

The midpoint of the interval is

$$m=\frac{\nu_1+\nu_2}{2}=2.$$

One can readily observe that the weight function is symmetric:

$$w(4-x)=(3-x)(x-1)=w(x),\qquad\forall x\in[1,3].$$

Clearly,

$$\mathcal{F}(x)=e^{0.25x}+2>0,\qquad w(x)\geq0,\qquad\forall x\in[1,3],$$

and

$$h(t)=t+1>0,\quad\forall t\in[0,1],$$

so all expressions are well defined.

From Theorem 3.2, we consider

$$\mathrm{LHS}(\alpha,\nu)=\frac{\Gamma(\alpha+1)}{2}\Big[\mathcal{E}^{\alpha,\nu}_{1^+}\big((\ln\mathcal{F})w\big)(3)+\mathcal{E}^{\alpha,\nu}_{3^-}\big((\ln\mathcal{F})w\big)(1)\Big].$$

The first Erdélyi–Kober term is

$$\mathcal{E}^{\alpha,\nu}_{1^+}\big((\ln\mathcal{F})w\big)(3)=\frac{1}{2^{\nu+\alpha}\Gamma(\alpha)}\int_1^3(3-x)^{\alpha-1}(x-1)^{\nu}\ln\Big(e^{0.25x}+2\Big)w(x)\,dx.$$

Similarly,

$$\mathcal{E}^{\alpha,\nu}_{3^-}\big((\ln\mathcal{F})w\big)(1)=\frac{1}{2^{\nu+\alpha}\Gamma(\alpha)}\int_1^3(x-1)^{\alpha-1}(3-x)^{\nu}\ln\Big(e^{0.25x}+2\Big)w(x)\,dx.$$

Thus,

$$\mathrm{LHS}(\alpha,\nu)=\frac{\Gamma(\alpha+1)}{2^{\nu+\alpha+1}\Gamma(\alpha)}\Bigg[\int_1^3(3-x)^{\alpha-1}(x-1)^{\nu}\ln\Big(e^{0.25x}+2\Big)w(x)\,dx$$
$$+\int_1^3(x-1)^{\alpha-1}(3-x)^{\nu}\ln\Big(e^{0.25x}+2\Big)w(x)\,dx\Bigg].$$

We compute

$$\mathcal{F}(1) = e^{0.25} + 2 \approx 3.284, \qquad \mathcal{F}(3) = e^{0.75} + 2 \approx 4.117,$$

hence

$$\ln \mathcal{F}(1) \approx 1.188, \qquad \ln \mathcal{F}(3) \approx 1.415,$$

and therefore

$$\ln \mathcal{F}(1) + \ln \mathcal{F}(3) \approx 2.603.$$

Now,

$$w(1+2\mu) = 4\mu(1-\mu),$$

and

$$\frac{1}{h(\mu)} + \frac{1}{h(1-\mu)} = \frac{1}{1+\mu} + \frac{1}{2-\mu}.$$

Thus,

$$\begin{aligned}
\mathrm{RHS}(\alpha,\nu) &= 2.603 \int_0^1 \frac{\alpha\,\mu^{\nu}(1-\mu)^{\alpha-1}}{2}\left(\frac{1}{1+\mu} + \frac{1}{2-\mu}\right)4\mu(1-\mu)\,d\mu \\
&= 5.206 \int_0^1 \alpha\,\mu^{\nu+1}(1-\mu)^{\alpha}\left(\frac{1}{1+\mu} + \frac{1}{2-\mu}\right)d\mu.
\end{aligned}$$

We observe that for all $\alpha \in (0,1)$,

$$\mathrm{LHS}(\alpha,\nu) < \mathrm{RHS}(\alpha,\nu),$$

which confirms the validity of the weighted Erdélyi–Kober inequality.

Theorem 3.3. *Let $\mathcal{F} : [\nu_1, \nu_2] \to \mathbb{R}_{\mathcal{I}}^{+}$ be an interval-valued mapping, and let $w : [\nu_1,\nu_2] \to \mathbb{R}$ be a nonnegative integrable weight function, symmetric about the midpoint*

$$m = \frac{\nu_1+\nu_2}{2},$$

in the case

$$w(x) = w(\nu_1+\nu_2-x), \qquad \forall x \in [\nu_1,\nu_2].$$

Assume further that

$$\int_{\nu_1}^{\nu_2} w(x)\,dx > 0, \qquad \mathcal{F} \in \mathcal{SGX}\Big([\nu_1,\nu_2], \mathbb{R}_{\mathcal{I}}^{+}, h\Big) \cap \mathcal{IR}([\nu_1,\nu_2]).$$

Let $\alpha > 0$, $\nu > -1$, and assume that $h(1/2) \neq 0$. Then the following weighted Erdélyi–Kober midpoint-type inequality holds:

$$\begin{aligned}
&\frac{h(1/2)}{2}\,\ln \mathcal{F}(m)\left[\Big(\mathcal{E}_{\nu_1^+}^{\alpha,\nu} w\Big)(\nu_2) + \Big(\mathcal{E}_{\nu_2^-}^{\alpha,\nu} w\Big)(\nu_1)\right] \\
&\preceq_{\mathrm{cr}} \frac{1}{2}\left[\Big(\mathcal{E}_{\nu_1^+}^{\alpha,\nu}\big((\ln \mathcal{F})w\big)\Big)(\nu_2) + \Big(\mathcal{E}_{\nu_2^-}^{\alpha,\nu}\big((\ln \mathcal{F})w\big)\Big)(\nu_1)\right].
\end{aligned} \tag{3.57}$$

Here, the shifted left-sided Erdélyi–Kober fractional integral is defined by

$$\left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+} g\right)(x) = \frac{(x-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)} \int_{\nu_1}^{x} (x-t)^{\alpha-1}(t-\nu_1)^{\nu} g(t)\, dt, \qquad x \in (\nu_1, \nu_2],$$

and the shifted right-sided Erdélyi–Kober fractional integral is defined by

$$\left(\mathcal{E}^{\alpha,\nu}_{\nu_2^-} g\right)(x) = \frac{(\nu_2-x)^{-\nu-\alpha}}{\Gamma(\alpha)} \int_{x}^{\nu_2} (t-x)^{\alpha-1}(\nu_2-t)^{\nu} g(t)\, dt, \qquad x \in [\nu_1, \nu_2).$$

Equivalently, (3.57) *may be written in the integral form*

$$\begin{aligned}
&\frac{h(1/2)}{2}\, \ln\mathcal{F}(m) \frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}\Gamma(\alpha)} \Bigg[\int_{\nu_1}^{\nu_2} (\nu_2-x)^{\alpha-1}(x-\nu_1)^{\nu} w(x)\, dx \\
&\qquad\qquad + \int_{\nu_1}^{\nu_2} (x-\nu_1)^{\alpha-1}(\nu_2-x)^{\nu} w(x)\, dx \Bigg] \\
&\preceq_{\mathrm{cr}} \frac{1}{2(\nu_2-\nu_1)^{\nu+\alpha}\Gamma(\alpha)} \Bigg[\int_{\nu_1}^{\nu_2} (\nu_2-x)^{\alpha-1}(x-\nu_1)^{\nu} \ln\mathcal{F}(x)\, w(x)\, dx \\
&\qquad\qquad + \int_{\nu_1}^{\nu_2} (x-\nu_1)^{\alpha-1}(\nu_2-x)^{\nu} \ln\mathcal{F}(x)\, w(x)\, dx \Bigg].
\end{aligned} \tag{3.58}$$

Proof. Let

$$\mu \in [0,1], \qquad \bar{\mu} = 1-\mu.$$

Since

$$\mathcal{F} \in \mathcal{SGX}\Big([\nu_1,\nu_2], \mathbb{R}^+_{\mathcal{I}}, h\Big),$$

we have

$$\mathcal{F}(\mu\nu_2 + \bar{\mu}\nu_1) \preceq_{\mathrm{cr}} [\mathcal{F}(\nu_2)]^{\frac{1}{h(\mu)}} [\mathcal{F}(\nu_1)]^{\frac{1}{h(\bar{\mu})}}, \tag{3.59}$$

$$\mathcal{F}(\bar{\mu}\nu_2 + \mu\nu_1) \preceq_{\mathrm{cr}} [\mathcal{F}(\nu_2)]^{\frac{1}{h(\bar{\mu})}} [\mathcal{F}(\nu_1)]^{\frac{1}{h(\mu)}}. \tag{3.60}$$

Taking componentwise logarithms in (3.59) and (3.60), we obtain

$$\ln\mathcal{F}(\mu\nu_2 + \bar{\mu}\nu_1) \preceq_{\mathrm{cr}} \frac{1}{h(\mu)} \ln\mathcal{F}(\nu_2) + \frac{1}{h(\bar{\mu})} \ln\mathcal{F}(\nu_1), \tag{3.61}$$

$$\ln\mathcal{F}(\bar{\mu}\nu_2 + \mu\nu_1) \preceq_{\mathrm{cr}} \frac{1}{h(\bar{\mu})} \ln\mathcal{F}(\nu_2) + \frac{1}{h(\mu)} \ln\mathcal{F}(\nu_1). \tag{3.62}$$

Now observe that

$$\frac{\mu\nu_2 + \bar{\mu}\nu_1 + \bar{\mu}\nu_2 + \mu\nu_1}{2} = \frac{\nu_1+\nu_2}{2} = m.$$

Hence, applying the defining inequality at $t = \frac{1}{2}$ to the pair of points

$$x = \mu\nu_2 + \bar{\mu}\nu_1, \qquad y = \bar{\mu}\nu_2 + \mu\nu_1,$$

we obtain

$$\mathcal{F}(m) = \mathcal{F}\Big(\frac{x+y}{2}\Big) \preceq_{\mathrm{cr}} [\mathcal{F}(x)]^{\frac{1}{h(1/2)}} [\mathcal{F}(y)]^{\frac{1}{h(1/2)}}. \tag{3.63}$$

Taking componentwise logarithm in (3.63), we get

$$\ln \mathcal{F}(m) \preceq_{\mathrm{cr}} \frac{1}{h(1/2)} \ln \mathcal{F}(x) + \frac{1}{h(1/2)} \ln \mathcal{F}(y), \tag{3.64}$$

that is,

$$\ln \mathcal{F}(m) \preceq_{\mathrm{cr}} \frac{1}{h(1/2)} \ln \mathcal{F}(\mu\nu_2 + \bar{\mu}\nu_1) + \frac{1}{h(1/2)} \ln \mathcal{F}(\bar{\mu}\nu_2 + \mu\nu_1). \tag{3.65}$$

Multiply both sides of (3.65) by

$$\frac{\mu^{\nu}(1-\mu)^{\alpha-1}}{2\,\Gamma(\alpha)}\, w(\mu\nu_2 + \bar{\mu}\nu_1),$$

and integrate over $\mu \in [0,1]$. Since $w \geq 0$, the order is preserved. Thus,

$$\begin{aligned}
&\ln \mathcal{F}(m)\, \frac{1}{2\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} w(\mu\nu_2 + \bar{\mu}\nu_1)\, d\mu \\
&\preceq_{\mathrm{cr}} \frac{1}{2h(1/2)\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} w(\mu\nu_2 + \bar{\mu}\nu_1) \ln \mathcal{F}(\mu\nu_2 + \bar{\mu}\nu_1)\, d\mu \\
&\quad + \frac{1}{2h(1/2)\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} w(\mu\nu_2 + \bar{\mu}\nu_1) \ln \mathcal{F}(\bar{\mu}\nu_2 + \mu\nu_1)\, d\mu.
\end{aligned} \tag{3.66}$$

We now transform each term. For the first integral, set

$$x = \mu\nu_2 + \bar{\mu}\nu_1 = \nu_1 + \mu(\nu_2 - \nu_1).$$

Then

$$\mu = \frac{x - \nu_1}{\nu_2 - \nu_1}, \qquad 1 - \mu = \frac{\nu_2 - x}{\nu_2 - \nu_1}, \qquad d\mu = \frac{dx}{\nu_2 - \nu_1}.$$

Hence,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} w(\mu\nu_2 + \bar{\mu}\nu_1)\, d\mu \\
&= \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^{\nu_2} \left(\frac{x-\nu_1}{\nu_2-\nu_1}\right)^{\nu} \left(\frac{\nu_2 - x}{\nu_2 - \nu_1}\right)^{\alpha-1} w(x)\, \frac{dx}{\nu_2 - \nu_1} \\
&= \frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}} \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^{\nu_2} (x-\nu_1)^{\nu}(\nu_2 - x)^{\alpha-1} w(x)\, dx \\
&= \left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+} w\right)(\nu_2).
\end{aligned} \tag{3.67}$$

Similarly,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} w(\mu\nu_2 + \bar{\mu}\nu_1) \ln \mathcal{F}(\mu\nu_2 + \bar{\mu}\nu_1)\, d\mu \\
&= \frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}} \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^{\nu_2} (x-\nu_1)^{\nu}(\nu_2 - x)^{\alpha-1} w(x) \ln \mathcal{F}(x)\, dx \\
&= \left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\big((\ln \mathcal{F}) w\big)\right)(\nu_2).
\end{aligned} \tag{3.68}$$

For the second integral in (3.66), still using

$$x = \mu\nu_2 + \bar{\mu}\nu_1,$$

we have

$$\bar{\mu}\nu_2 + \mu\nu_1 = \nu_1 + \nu_2 - x.$$

Therefore,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} w(\mu\nu_2 + \bar{\mu}\nu_1) \ln \mathcal{F}(\bar{\mu}\nu_2 + \mu\nu_1)\, d\mu \\
&= \frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}} \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^{\nu_2} (x-\nu_1)^{\nu}(\nu_2 - x)^{\alpha-1} w(x) \ln \mathcal{F}(\nu_1+\nu_2-x)\, dx. \qquad (3.69)
\end{aligned}$$

Now introduce the change of variable

$$t = \nu_1 + \nu_2 - x.$$

Then

$$x = \nu_1 + \nu_2 - t, \qquad dx = -dt, \qquad x - \nu_1 = \nu_2 - t, \qquad \nu_2 - x = t - \nu_1.$$

Since w is symmetric, we have $w(x) = w(t)$. Thus (3.69) becomes

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_0^1 \mu^{\nu}(1-\mu)^{\alpha-1} w(\mu\nu_2 + \bar{\mu}\nu_1) \ln \mathcal{F}(\bar{\mu}\nu_2 + \mu\nu_1)\, d\mu \\
&= \frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}} \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^{\nu_2} (\nu_2 - t)^{\nu}(t-\nu_1)^{\alpha-1} w(t) \ln \mathcal{F}(t)\, dt \\
&= \Big(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}\big((\ln\mathcal{F})w\big)\Big)(\nu_1). \qquad (3.70)
\end{aligned}$$

Substituting (3.67), (3.68), and (3.70) into (3.66), we obtain

$$\begin{aligned}
&\ln\mathcal{F}(m)\, \frac{1}{2}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+} w\Big)(\nu_2) \\
&\preceq_{\mathrm{cr}} \frac{1}{2h(1/2)}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\big((\ln\mathcal{F})w\big)\Big)(\nu_2) + \frac{1}{2h(1/2)}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}\big((\ln\mathcal{F})w\big)\Big)(\nu_1). \qquad (3.71)
\end{aligned}$$

Next, apply the same argument to the reflected parametrization by starting from (3.65), multiplying by

$$\frac{\mu^{\nu}(1-\mu)^{\alpha-1}}{2\,\Gamma(\alpha)}\, w(\bar{\mu}\nu_2 + \mu\nu_1),$$

and integrating over $[0,1]$. By symmetry of w, this yields

$$\begin{aligned}
&\ln\mathcal{F}(m)\, \frac{1}{2}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_2^-} w\Big)(\nu_1) \\
&\preceq_{\mathrm{cr}} \frac{1}{2h(1/2)}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}\big((\ln\mathcal{F})w\big)\Big)(\nu_1) + \frac{1}{2h(1/2)}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\big((\ln\mathcal{F})w\big)\Big)(\nu_2). \qquad (3.72)
\end{aligned}$$

Adding (3.71) and (3.72), we get

$$\frac{1}{2}\ln\mathcal{F}(m)\Big[\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+} w\Big)(\nu_2) + \Big(\mathcal{E}^{\alpha,\nu}_{\nu_2^-} w\Big)(\nu_1)\Big]$$

$$\preceq_{\mathrm{cr}} \frac{1}{h(1/2)} \cdot \frac{1}{2}\left[\left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\big((\ln\mathcal{F})w\big)\right)(\nu_2) + \left(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}\big((\ln\mathcal{F})w\big)\right)(\nu_1)\right]. \tag{3.73}$$

In the final step, we multiply both sides of (3.73) by $h\left(\frac{1}{2}\right)$ to deduce

$$\begin{aligned}&\frac{h(1/2)}{2}\ \ln\mathcal{F}(m)\left[\left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}w\right)(\nu_2) + \left(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}w\right)(\nu_1)\right]\\ &\preceq_{\mathrm{cr}} \frac{1}{2}\left[\left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\big((\ln\mathcal{F})w\big)\right)(\nu_2) + \left(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}\big((\ln\mathcal{F})w\big)\right)(\nu_1)\right],\end{aligned} \tag{3.74}$$

which proves (3.57). The proof is complete. □

Remark 3.4. *In the real-valued case, the interval-valued order $\preceq_{\mathrm{cr}}$ reduces to the usual order relation $\leq$ on $\mathbb{R}$. Consequently, Theorem 3.3 reduces to the real-valued inequality*

$$\begin{aligned}&\frac{h(1/2)}{2}\ \ln\mathcal{F}(m)\left[\left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}w\right)(\nu_2) + \left(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}w\right)(\nu_1)\right]\\ &\leq \frac{1}{2}\left[\left(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\big((\ln\mathcal{F})w\big)\right)(\nu_2) + \left(\mathcal{E}^{\alpha,\nu}_{\nu_2^-}\big((\ln\mathcal{F})w\big)\right)(\nu_1)\right],\end{aligned} \tag{3.75}$$

where

$$m = \frac{\nu_1 + \nu_2}{2}.$$

Using the definitions of the shifted left- and right-sided Erdélyi–Kober fractional integrals, (3.75) *may be written equivalently as*

$$\begin{aligned}&\frac{h(1/2)}{2}\ \ln\mathcal{F}(m)\frac{1}{(\nu_2-\nu_1)^{\nu+\alpha}\Gamma(\alpha)}\left[\int_{\nu_1}^{\nu_2}(\nu_2-x)^{\alpha-1}(x-\nu_1)^{\nu}w(x)\,dx\right.\\ &\qquad\qquad\left.+\int_{\nu_1}^{\nu_2}(x-\nu_1)^{\alpha-1}(\nu_2-x)^{\nu}w(x)\,dx\right]\\ &\leq \frac{1}{2(\nu_2-\nu_1)^{\nu+\alpha}\Gamma(\alpha)}\left[\int_{\nu_1}^{\nu_2}(\nu_2-x)^{\alpha-1}(x-\nu_1)^{\nu}\ln\mathcal{F}(x)\,w(x)\,dx\right.\\ &\qquad\qquad\left.+\int_{\nu_1}^{\nu_2}(x-\nu_1)^{\alpha-1}(\nu_2-x)^{\nu}\ln\mathcal{F}(x)\,w(x)\,dx\right].\end{aligned} \tag{3.76}$$

We now consider the special case $\alpha = 1$. In this case,

$$\Gamma(2) = 1!,$$

and the shifted Erdélyi–Kober fractional integrals reduce to

$$\left(\mathcal{E}^{1,\nu}_{\nu_1^+}g\right)(\nu_2) = \frac{1}{(\nu_2-\nu_1)^{\nu+1}}\int_{\nu_1}^{\nu_2}(x-\nu_1)^{\nu}g(x)\,dx,$$

and

$$\left(\mathcal{E}^{1,\nu}_{\nu_2^-} g\right)(\nu_1) = \frac{1}{(\nu_2-\nu_1)^{\nu+1}} \int_{\nu_1}^{\nu_2} (\nu_2 - x)^{\nu} g(x)\, dx.$$

Hence, (3.75) *becomes*

$$\frac{h(1/2)}{2} \ln \mathcal{F}(m)\left[\int_{\nu_1}^{\nu_2} (x-\nu_1)^{\nu} w(x)\, dx + \int_{\nu_1}^{\nu_2} (\nu_2 - x)^{\nu} w(x)\, dx\right]$$
$$\leq \frac{1}{2}\left[\int_{\nu_1}^{\nu_2} (x-\nu_1)^{\nu} \ln \mathcal{F}(x)\, w(x)\, dx + \int_{\nu_1}^{\nu_2} (\nu_2 - x)^{\nu} \ln \mathcal{F}(x)\, w(x)\, dx\right]. \tag{3.77}$$

If, in addition, we choose $\nu = 0$*, then the Erdélyi–Kober kernels reduce to the uniform kernel, and* (3.77) *becomes*

$$h(1/2)\ \ln \mathcal{F}(m) \int_{\nu_1}^{\nu_2} w(x)\, dx \leq \int_{\nu_1}^{\nu_2} \ln \mathcal{F}(x)\, w(x)\, dx. \tag{3.78}$$

In particular, if $w(x) = 1$ *for all* $x \in [\nu_1, \nu_2]$*, then* (3.78) *reduces to*

$$h(1/2)\ \ln \mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right) \leq \frac{1}{\nu_2-\nu_1} \int_{\nu_1}^{\nu_2} \ln \mathcal{F}(x)\, dx. \tag{3.79}$$

Finally, if we choose

$$h(t) = \frac{1}{t}, \qquad t \in (0,1],$$

then

$$h(1/2) = 2.$$

Consequently, (3.79) *becomes*

$$2 \ln \mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right) \leq \frac{1}{\nu_2-\nu_1} \int_{\nu_1}^{\nu_2} \ln \mathcal{F}(x)\, dx.$$

Equivalently,

$$\ln \mathcal{F}\left(\frac{\nu_1+\nu_2}{2}\right) \leq \frac{1}{2(\nu_2-\nu_1)} \int_{\nu_1}^{\nu_2} \ln \mathcal{F}(x)\, dx.$$

Example 3.3. *Consider the mapping* $\mathcal{F} : [\nu_1, \nu_2] \to \mathbb{R}^+$ *defined by*

$$\mathcal{F}(x) = e^{0.2x} + 2, \qquad h(t) = t+1, \qquad w(x) = (x-2)(4-x),$$

where

$$\nu_1 = 2, \qquad \nu_2 = 4, \qquad \alpha > 0, \qquad \nu > -1.$$

First, we verify the admissibility of the chosen functions. Since

$$\mathcal{F}(x) = e^{0.2x} + 2 > 0, \qquad \forall x \in [2,4],$$

the logarithm $\ln \mathcal{F}(x)$ *is well defined on the whole interval* $[2,4]$*. Moreover,*

$$w(x) = (x-2)(4-x) \geq 0, \qquad \forall x \in [2,4],$$

and

$$h(t) = t + 1 > 0, \qquad \forall t \in [0,1].$$

The midpoint of the interval is

$$m = \frac{\nu_1 + \nu_2}{2} = \frac{2+4}{2} = 3.$$

The weight function is symmetric about this midpoint, because

$$w(6-x) = (6-x-2)(4-(6-x)) = (4-x)(x-2) = w(x), \qquad \forall x \in [2,4].$$

Next, we compute

$$h\Big(\frac{1}{2}\Big) = \frac{3}{2}, \qquad \frac{h(1/2)}{2} = \frac{3}{4}.$$

Also,

$$\mathcal{F}(3) = e^{0.6} + 2 \approx 3.822119, \qquad \ln \mathcal{F}(3) \approx 1.340805.$$

According to Theorem 3.3, the left-hand side is

$$\mathrm{LHS} = \frac{h(1/2)}{2} \ln \mathcal{F}(m)\Big[\Big(\mathcal{E}_{2^+}^{\alpha,\nu} w\Big)(4) + \Big(\mathcal{E}_{4^-}^{\alpha,\nu} w\Big)(2)\Big].$$

Since $m = 3$, this becomes

$$\mathrm{LHS} = \frac{3}{4} \ln \mathcal{F}(3)\Big[\Big(\mathcal{E}_{2^+}^{\alpha,\nu} w\Big)(4) + \Big(\mathcal{E}_{4^-}^{\alpha,\nu} w\Big)(2)\Big].$$

By the definitions of the shifted left- and right-sided Erdélyi–Kober fractional integrals, we have

$$\begin{aligned}\Big(\mathcal{E}_{2^+}^{\alpha,\nu} w\Big)(4) &= \frac{(4-2)^{-\nu-\alpha}}{\Gamma(\alpha)} \int_2^4 (4-x)^{\alpha-1}(x-2)^{\nu} w(x)\,dx \\ &= \frac{1}{2^{\nu+\alpha}\Gamma(\alpha)} \int_2^4 (4-x)^{\alpha-1}(x-2)^{\nu}(x-2)(4-x)\,dx \\ &= \frac{1}{2^{\nu+\alpha}\Gamma(\alpha)} \int_2^4 (4-x)^{\alpha}(x-2)^{\nu+1}\,dx,\end{aligned}$$

and similarly

$$\begin{aligned}\Big(\mathcal{E}_{4^-}^{\alpha,\nu} w\Big)(2) &= \frac{(4-2)^{-\nu-\alpha}}{\Gamma(\alpha)} \int_2^4 (x-2)^{\alpha-1}(4-x)^{\nu} w(x)\,dx \\ &= \frac{1}{2^{\nu+\alpha}\Gamma(\alpha)} \int_2^4 (x-2)^{\alpha}(4-x)^{\nu+1}\,dx.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{LHS} = \frac{3}{4} \ln \mathcal{F}(3)\, \frac{1}{2^{\nu+\alpha}\Gamma(\alpha)}\Bigg[&\int_2^4 (4-x)^{\alpha}(x-2)^{\nu+1}\,dx \\ &+ \int_2^4 (x-2)^{\alpha}(4-x)^{\nu+1}\,dx\Bigg].\end{aligned}$$

The expression corresponding to the right-hand side of Theorem 3.3 is given by

$$\mathrm{RHS} = \frac{1}{2}\left[\left(\mathcal{E}_{2^+}^{\alpha,\nu}\Big((\ln\mathcal{F})w\Big)\right)(4) + \left(\mathcal{E}_{4^-}^{\alpha,\nu}\Big((\ln\mathcal{F})w\Big)\right)(2)\right].$$

Using the definitions of the shifted Erdélyi–Kober operators, this becomes

$$\mathrm{RHS} = \frac{1}{2^{\nu+\alpha+1}\Gamma(\alpha)}\left[\int_2^4 (4-x)^{\alpha-1}(x-2)^{\nu}\ln\mathcal{F}(x)\,w(x)\,dx \right.$$
$$\left. + \int_2^4 (x-2)^{\alpha-1}(4-x)^{\nu}\ln\mathcal{F}(x)\,w(x)\,dx\right].$$

Substituting $w(x) = (x-2)(4-x)$, we obtain

$$\mathrm{RHS} = \frac{1}{2^{\nu+\alpha+1}\Gamma(\alpha)}\left[\int_2^4 (4-x)^{\alpha}(x-2)^{\nu+1}\ln\Big(e^{0.2x}+2\Big)\,dx \right.$$
$$\left. + \int_2^4 (x-2)^{\alpha}(4-x)^{\nu+1}\ln\Big(e^{0.2x}+2\Big)\,dx\right].$$

Hence, for the present choice of $\mathcal{F}$, h, and w, the weighted midpoint Erdélyi–Kober inequality takes the form

$$\frac{3}{4}\ln\Big(e^{0.6}+2\Big)\frac{1}{2^{\nu+\alpha}\Gamma(\alpha)}\left[\int_2^4 (4-x)^{\alpha}(x-2)^{\nu+1}\,dx + \int_2^4 (x-2)^{\alpha}(4-x)^{\nu+1}\,dx\right]$$
$$\leq \frac{1}{2^{\nu+\alpha+1}\Gamma(\alpha)}\left[\int_2^4 (4-x)^{\alpha}(x-2)^{\nu+1}\ln\Big(e^{0.2x}+2\Big)\,dx \right.$$
$$\left. + \int_2^4 (x-2)^{\alpha}(4-x)^{\nu+1}\ln\Big(e^{0.2x}+2\Big)\,dx\right].$$

In particular, if we choose $\nu = 0$, then the above inequality reduces to

$$\frac{3}{4}\ln\Big(e^{0.6}+2\Big)\frac{1}{2^{\alpha}\Gamma(\alpha)}\left[\int_2^4 (4-x)^{\alpha}(x-2)\,dx + \int_2^4 (x-2)^{\alpha}(4-x)\,dx\right]$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha)}\left[\int_2^4 (4-x)^{\alpha}(x-2)\ln\Big(e^{0.2x}+2\Big)\,dx \right.$$
$$\left. + \int_2^4 (x-2)^{\alpha}(4-x)\ln\Big(e^{0.2x}+2\Big)\,dx\right].$$

Thus, this example illustrates the weighted midpoint-type Erdélyi–Kober fractional inequality for the positive mapping

$$\mathcal{F}(x) = e^{0.2x}+2$$

with the symmetric weight

$$w(x) = (x-2)(4-x).$$

3.2. **New discrete Jensen-type inequality for logarithmic Godunova–Levin functions.**

Theorem 3.4. *Let* $c_i \in \mathbb{R}^+$ *and* $\sigma_i \in [\nu_1, \nu_2]$ *for all* $i \in \{1, 2, \dots, n\}$*, where* $n \geq 2$*. Define*

$$\mathcal{C}_n = \sum_{i=1}^{n} c_i, \qquad \lambda_i = \frac{c_i}{\mathcal{C}_n}, \qquad \sum_{i=1}^{n} \lambda_i = 1,$$

and

$$x = \sum_{i=1}^{n} \lambda_i \sigma_i = \frac{1}{\mathcal{C}_n} \sum_{i=1}^{n} c_i \sigma_i.$$

Assume that

$$\mathcal{F} \in \mathcal{SGX}\Big([\nu_1, \nu_2], \mathbb{R}_{\mathcal{I}}^{+}, h\Big) \cap \mathcal{IR}([\nu_1, \nu_2]),$$

where $\alpha > 0$*,* $\nu > -1$*, and suppose further that* h *is supermultiplicative on* $(0, 1]$*, that is,*

$$h(st) \geq h(s)h(t), \qquad \forall\, s, t \in (0, 1].$$

Then the following Erdélyi–Kober type discrete inequality holds:

$$\Gamma(\alpha+1)\Big(\mathcal{E}_{\nu_1^+}^{\alpha,\nu} \ln \mathcal{F}\Big)(x) \preceq_{cr} \sum_{i=1}^{n} \frac{1}{h(\lambda_i)} \ln \mathcal{F}(\sigma_i), \tag{3.80}$$

where the shifted Erdélyi–Kober fractional integral is defined by

$$\Big(\mathcal{E}_{\nu_1^+}^{\alpha,\nu} g\Big)(x) = \frac{(x-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)} \int_{\nu_1}^{x} (x-\tau)^{\alpha-1} (\tau-\nu_1)^{\nu} g(\tau)\, d\tau, \qquad x \in (\nu_1, \nu_2].$$

Equivalently,

$$\begin{aligned} &\Gamma(\alpha+1)\, \frac{(x-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)} \int_{\nu_1}^{x} (x-\tau)^{\alpha-1} (\tau-\nu_1)^{\nu} \ln \mathcal{F}(\tau)\, d\tau \\ &\preceq_{cr} \sum_{i=1}^{n} \frac{1}{h(\lambda_i)} \ln \mathcal{F}(\sigma_i). \end{aligned} \tag{3.81}$$

Proof. We first establish the discrete Jensen-type estimate

$$\ln \mathcal{F}(x) \preceq_{cr} \sum_{i=1}^{n} \frac{1}{h(\lambda_i)} \ln \mathcal{F}(\sigma_i). \tag{3.82}$$

We proceed by induction on n.

For $n = 2$, we have

$$\lambda_1 = \frac{c_1}{\mathcal{C}_2}, \qquad \lambda_2 = \frac{c_2}{\mathcal{C}_2}, \qquad \lambda_1 + \lambda_2 = 1,$$

and

$$x = \lambda_1 \sigma_1 + \lambda_2 \sigma_2.$$

Since

$$\mathcal{F} \in \mathcal{SGX}\Big([\nu_1, \nu_2], \mathbb{R}_{\mathcal{I}}^{+}, h\Big),$$

the defining property of this class yields

$$\mathcal{F}(x) = \mathcal{F}(\lambda_1 \sigma_1 + \lambda_2 \sigma_2) \preceq_{cr} [\mathcal{F}(\sigma_1)]^{\frac{1}{h(\lambda_1)}} [\mathcal{F}(\sigma_2)]^{\frac{1}{h(\lambda_2)}}.$$

Taking componentwise logarithms, we obtain

$$\ln \mathcal{F}(x) \preceq_{\mathrm{cr}} \frac{1}{h(\lambda_1)} \ln \mathcal{F}(\sigma_1) + \frac{1}{h(\lambda_2)} \ln \mathcal{F}(\sigma_2). \tag{3.83}$$

Thus, (3.82) is true for $n = 2$.

Assume now that (3.82) holds for some $n = k \geq 2$, that is,

$$\ln \mathcal{F}\left(\frac{1}{C_k} \sum_{i=1}^{k} c_i \sigma_i \right) \preceq_{\mathrm{cr}} \sum_{i=1}^{k} \frac{1}{h(c_i / C_k)} \ln \mathcal{F}(\sigma_i). \tag{3.84}$$

We prove the result for $n = k + 1$. Write

$$\begin{aligned} x &= \frac{1}{C_{k+1}} \sum_{i=1}^{k+1} c_i \sigma_i \\ &= \frac{C_k}{C_{k+1}} \left(\frac{1}{C_k} \sum_{i=1}^{k} c_i \sigma_i \right) + \frac{c_{k+1}}{C_{k+1}} \sigma_{k+1}. \end{aligned}$$

Applying the log-h-Godunova–Levin property, we obtain

$$\begin{aligned} \ln \mathcal{F}(x) \preceq_{\mathrm{cr}} & \frac{1}{h(C_k / C_{k+1})} \ln \mathcal{F}\left(\frac{1}{C_k} \sum_{i=1}^{k} c_i \sigma_i \right) \\ & + \frac{1}{h(c_{k+1} / C_{k+1})} \ln \mathcal{F}(\sigma_{k+1}). \end{aligned} \tag{3.85}$$

Using the induction hypothesis (3.84), we get

$$\ln \mathcal{F}(x) \preceq_{\mathrm{cr}} \sum_{i=1}^{k} \frac{1}{h(C_k / C_{k+1})} \frac{1}{h(c_i / C_k)} \ln \mathcal{F}(\sigma_i) + \frac{1}{h(c_{k+1} / C_{k+1})} \ln \mathcal{F}(\sigma_{k+1}). \tag{3.86}$$

Since

$$\frac{C_k}{C_{k+1}} \cdot \frac{c_i}{C_k} = \frac{c_i}{C_{k+1}},$$

and since h is supermultiplicative, we have

$$h\left(\frac{c_i}{C_{k+1}} \right) = h\left(\frac{C_k}{C_{k+1}} \cdot \frac{c_i}{C_k} \right) \geq h\left(\frac{C_k}{C_{k+1}} \right) h\left(\frac{c_i}{C_k} \right).$$

Therefore,

$$\frac{1}{h(C_k / C_{k+1})} \frac{1}{h(c_i / C_k)} \leq \frac{1}{h(c_i / C_{k+1})}.$$

Using this in (3.86), we obtain

$$\ln \mathcal{F}(x) \preceq_{\mathrm{cr}} \sum_{i=1}^{k+1} \frac{1}{h(c_i / C_{k+1})} \ln \mathcal{F}(\sigma_i).$$

Hence, by induction, (3.82) holds for every $n \geq 2$.

We now apply the shifted Erdélyi–Kober fractional integral operator to both sides of (3.82). Since the right-hand side does not depend on the integration variable τ, we multiply both sides by the positive kernel

$$\frac{(x-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)}(x-\tau)^{\alpha-1}(\tau-\nu_1)^{\nu},$$

and integrate over $\tau \in [\nu_1, x]$. Thus,

$$\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\ln\mathcal{F}\Big)(x) \preceq_{\mathrm{cr}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{h(\lambda_\mathrm{i})}\ln\mathcal{F}(\sigma_\mathrm{i}) \cdot \frac{(x-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)}\int_{\nu_1}^{x}(x-\tau)^{\alpha-1}(\tau-\nu_1)^{\nu}\,d\tau. \tag{3.87}$$

It remains to evaluate the integral

$$J = \frac{(x-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)}\int_{\nu_1}^{x}(x-\tau)^{\alpha-1}(\tau-\nu_1)^{\nu}\,d\tau.$$

Use the substitution

$$\tau = \nu_1 + (x-\nu_1)s, \qquad s \in [0,1].$$

Then

$$d\tau = (x-\nu_1)\,ds, \qquad x-\tau = (x-\nu_1)(1-s), \qquad \tau-\nu_1 = (x-\nu_1)s.$$

Therefore,

$$\begin{aligned} J &= \frac{(x-\nu_1)^{-\nu-\alpha}}{\Gamma(\alpha)}\int_0^1\Big((x-\nu_1)(1-s)\Big)^{\alpha-1}\Big((x-\nu_1)s\Big)^{\nu}(x-\nu_1)\,ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}s^{\nu}\,ds \\ &= \frac{\mathrm{B}(\nu+1,\alpha)}{\Gamma(\alpha)}. \end{aligned}$$

Using

$$\mathrm{B}(\nu+1,\alpha) = \frac{\Gamma(\nu+1)\Gamma(\alpha)}{\Gamma(\nu+\alpha+1)},$$

we find

$$J = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)}.$$

Substituting this into (3.87), we obtain

$$\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\ln\mathcal{F}\Big)(x) \preceq_{\mathrm{cr}} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)}\sum_{\mathrm{i}=1}^{\mathrm{n}}\frac{1}{h(\lambda_\mathrm{i})}\ln\mathcal{F}(\sigma_\mathrm{i}).$$

As a final step, multiplying both sides by

$$\frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)},$$

we arrive at

$$\frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\ln\mathcal{F}\Big)(x) \preceq_{\mathrm{cr}} \sum_{\mathrm{i}=1}^{\mathrm{n}}\frac{1}{h(\lambda_\mathrm{i})}\ln\mathcal{F}(\sigma_\mathrm{i}),$$

which proves (3.80). $\square$

Remark 3.5. *In the real-valued case, the interval-valued order $\preceq_{cr}$ reduces to the usual order relation $\leq$ on $\mathbb{R}$. Hence, Theorem 3.4 takes the form*

$$\frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)}\Big(\mathcal{E}^{\alpha,\nu}_{\nu_1^+}\ln\mathcal{F}\Big)(x)\leq\sum_{i=1}^{n}\frac{1}{h(\lambda_i)}\ln\mathcal{F}(\sigma_i),$$

where

$$x=\sum_{i=1}^{n}\lambda_i\sigma_i,\qquad \lambda_i=\frac{c_i}{C_n}.$$

For $\alpha=1$, we obtain

$$(\nu+1)\Big(\mathcal{E}^{1,\nu}_{\nu_1^+}\ln\mathcal{F}\Big)(x)\leq\sum_{i=1}^{n}\frac{1}{h(\lambda_i)}\ln\mathcal{F}(\sigma_i),$$

that is,

$$\frac{\nu+1}{(x-\nu_1)^{\nu+1}}\int_{\nu_1}^{x}(\tau-\nu_1)^{\nu}\ln\mathcal{F}(\tau)\,d\tau\leq\sum_{i=1}^{n}\frac{1}{h(\lambda_i)}\ln\mathcal{F}(\sigma_i).$$

In particular, when $n=2$ and $c_1=c_2=1$, we have

$$\lambda_1=\lambda_2=\frac{1}{2},\qquad x=\frac{\sigma_1+\sigma_2}{2},$$

and hence

$$\frac{\nu+1}{(x-\nu_1)^{\nu+1}}\int_{\nu_1}^{x}(\tau-\nu_1)^{\nu}\ln\mathcal{F}(\tau)\,d\tau\leq\frac{1}{h(1/2)}\Big[\ln\mathcal{F}(\sigma_1)+\ln\mathcal{F}(\sigma_2)\Big].$$

Finally, if

$$h(t)=\frac{1}{t},$$

then

$$\frac{1}{h(1/2)}=\frac{1}{2},$$

and therefore

$$\frac{\nu+1}{(x-\nu_1)^{\nu+1}}\int_{\nu_1}^{x}(\tau-\nu_1)^{\nu}\ln\mathcal{F}(\tau)\,d\tau\leq\frac{1}{2}\Big[\ln\mathcal{F}(\sigma_1)+\ln\mathcal{F}(\sigma_2)\Big].$$

4. Conclusion

In this paper, we propose a new class of generalized logarithmic Godunova–Levin type interval-valued mappings and established several new weighted Hermite–Hadamard and Jensen-type inequalities by using Erdélyi–Kober fractional integrals. The inclusion of symmetric weight functions further enhances the scope and flexibility of the obtained results.

The main results were illustrated through suitable examples, showing the effectiveness of the developed inequalities. In addition, several known inequalities were recovered as special cases, which confirms the generality of the proposed framework. The results presented here may be useful for further investigations in fractional integral inequalities, interval-valued analysis, and related applications.

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