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Polynomial Product Weights, Ideal Comparison, and Weighted Wijsman Convergence for Triple Sequences of Closed Sets

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Abstract. This paper studies weighted Wijsman ideal convergence for triple sequences of closed sets in Menger probabilistic metric spaces with polynomial product weights $w(i,j,k) = i^{\alpha} j^{\beta} k^{\gamma}$. For this class of weights, we compute the weighted density of cylindrical, planar, and diagonal index sets and prove that different exponent triples yield distinct ideals. The map $(\alpha, \beta, \gamma) \mapsto \mathcal{I}_w$ is an injective order-reversing embedding into the lattice of ideals on $\mathbb{N}^3$. Under separability, property (AP3), and uniform equi-Wijsman-regularity, we prove that $\mathcal{I}_w^{\psi}$-convergence is equivalent to $\mathcal{I}_w^{*\psi}$-convergence. In the metric-induced Menger model, the regularity hypothesis holds for all nonempty closed sets, so the equivalence theorem applies directly in Euclidean spaces.

1. Introduction

We study weighted Wijsman convergence for triple sequences of closed sets on $\mathbb{N}^3$ equipped with the polynomial product weights $w(i,j,k) = i^{\alpha} j^{\beta} k^{\gamma}$.

Statistical convergence [4,16] and Wijsman convergence [18,19] are standard tools in generalized convergence theory. Their ideal-based interaction has been developed in stages: first for sequences of sets [8,11,17], then for double-index settings where rectangular asymptotics become relevant [1], and more recently for triple sequences and deferred or invariant variants [3,6,7,12]. What is still missing in that literature is a comparison theorem showing how the choice of weight changes the underlying ideal itself in the full three-parameter family $i^{\alpha} j^{\beta} k^{\gamma}$.

In the weighted setting, the exponents determine which subsets of $\mathbb{N}^3$ have zero density. In the unweighted case, sets with different growth rates can have the same ordinary density. Product weights separate these behaviors by assigning a distinct scale to each coordinate, so the triple (α, β, γ) enters directly into the convergence mechanism.

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Theorem 3.4 shows that the exponent triple determines $\mathcal{I}_w$ and that the map $(\alpha,\beta,\gamma) \mapsto \mathcal{I}_w$ is injective and order-reversing. Consequently, the unweighted triple theory [6] does not distinguish ideals such as $\mathcal{I}_{ijk}$ and $\mathcal{I}_{i^2j^2k^2}$, but the weighted densities studied here do.

The proof uses a three-coordinate comparison principle based on sparse-scale cylinders and their planar unions. These test sets isolate each exponent separately and then in pairs, allowing us to recover the full partial order on exponent triples from weighted upper density alone. Theorem 3.3 and Theorem 3.4 together connect the explicit density formulas of Section 3 to the order-theoretic description of $\mathcal{I}_w$.

We also prove that, under property (AP3) and uniform equi-Wijsman-regularity, $\mathcal{I}_w^{\psi}$-convergence coincides with $\mathcal{I}_w^{*\psi}$-convergence. For triple indices, property (AP3) requires finite unions of coordinate slices rather than finite sets, so the one-dimensional property (AP) of [9] does not suffice.

Two features of the triple setting are noteworthy. First, the geometry of exceptional sets differs from that of double sequences: finite unions of coordinate slices replace rectangular blocks. This affects both the formulation of property (AP3) and the convergence theory. Second, the density formulas of Theorem 3.3 do not determine $\mathcal{I}_w$ directly, because an ideal is defined by all zero-density sets, not just a few examples. Theorem 3.4 completes the argument by converting these model calculations into an exact description of ideal inclusion.

The star-equivalence theorem also reflects the three-dimensional index set. We first establish property (AP3) for polynomial product weights, then combine slice-based exceptional sets with a dense-set reduction in the base-point space. Proposition 2.1 shows that uniform equi-Wijsman-regularity holds automatically in the metric-induced Menger model, so the equivalence theorem applies directly in Euclidean spaces.

For double sequences, weighted statistical convergence [2,5,10] shows that the choice of weight affects the asymptotic size of exceptional sets. The present paper combines this effect with three-coordinate geometry, Wijsman convergence of closed sets, and an ideal comparison theorem for the family $i^\alpha j^\beta k^\gamma$. Related work includes [3,7,12,13].

Section 3 defines weighted density, proves the three-coordinate comparison theorem (Theorem 3.3), and derives the order-reversing embedding (Theorem 3.4). Section 4 proves star-equivalence (Theorem 4.2). The appendix contains the dyadic proof of Theorem 3.4(i).

2. Preliminaries

We record the notation and auxiliary facts used later.

2.1. **Menger Probabilistic Metric Spaces.**

A distribution function is a nondecreasing left-continuous $\psi : \mathbb{R} \to [0,1]$ with $\inf \psi = 0$ and $\sup \psi = 1$; let $\mathcal{D}^+$ denote those vanishing on $(-\infty, 0]$. A continuous t-norm is a continuous, commutative, associative, nondecreasing map $* : [0,1]^2 \to [0,1]$ with $a * 1 = a$; the standard examples are $a * b = \min(a,b)$, $a * b = ab$, and $a * b = \max(0, a+b-1)$.

Definition 2.1 ([14]). *A Menger probabilistic metric space is a triple $(X, \psi, *)$ where X is a nonempty set, $*$ is a continuous t-norm, and $\psi : X \times X \to \mathcal{D}^+$. Writing $\psi_{\xi,\zeta}(t)$ for the value at $t > 0$, the following hold for all $\xi, \zeta, \eta \in X$ and $t, s > 0$:*

- (PM1) $\psi_{\xi,\zeta}(t) = 1$ *for all* $t > 0$ *iff* $\xi = \zeta$.
- (PM2) $\psi_{\xi,\zeta}(t) = \psi_{\zeta,\xi}(t)$.
- (PM3) $\psi_{\xi,\zeta}(t+s) \geq \psi_{\xi,\eta}(t) * \psi_{\eta,\zeta}(s)$.

Throughout this paper we assume the t-norm $*$ satisfies:

$$a * b \geq a + b - 1 \qquad \text{for all } a, b \in [0,1]. \tag{2.1}$$

This is the Lukasiewicz condition. It holds for the minimum, product, and Lukasiewicz t-norms, and fails for degenerate cases such as the drastic t-norm.

2.2. **Wijsman Convergence for Sets.** For a nonempty closed set $\phi \subseteq X$ and $\xi \in X$, define

$$\psi_{\xi,\phi}(t) := \sup_{\zeta \in \phi} \psi_{\xi,\zeta}(t), \qquad t > 0.$$

Every convergence concept below is formulated through the scalar convergence of $\psi_{\xi,\phi_{mnk}}(t)$ for each base point ξ and each parameter $t > 0$.

Definition 2.2 ([18,19]). *A sequence of closed sets $\{\phi_n\}$ is Wijsman convergent to ϕ if $\lim_{n\to\infty} \psi_{\xi,\phi_n}(t) = \psi_{\xi,\phi}(t)$ for every $\xi \in X$ and $t > 0$.*

For triple sequences, $\{\phi_{mnk}\}$ is Wijsman convergent to ϕ in the Pringsheim sense if for every $\xi \in X$, $t > 0$, and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|\psi_{\xi,\phi_{mnk}}(t) - \psi_{\xi,\phi}(t)| < \varepsilon$ whenever $m, n, k \geq N$.

2.3. **Weighted Ideals and Regularity Hypotheses.** Since the index set is $\mathbb{N}^3$, all density calculations are taken over rectangular boxes $[1, M] \times [1, N] \times [1, P]$ and limits are understood in the Pringsheim sense.

Definition 2.3. *A weight is a map $w : \mathbb{N}^3 \to (0, \infty)$. For $A \subseteq \mathbb{N}^3$ define*

$$W_{MNP} := \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{P} w(i,j,k), \qquad W_{MNP}(A) := \sum_{\substack{1 \leq i \leq M,\, 1 \leq j \leq N,\, 1 \leq k \leq P \\ (i,j,k) \in A}} w(i,j,k).$$

Set

$$\overline{d}_w(A) := \limsup_{M,N,P\to\infty} \frac{W_{MNP}(A)}{W_{MNP}}, \qquad \underline{d}_w(A) := \liminf_{M,N,P\to\infty} \frac{W_{MNP}(A)}{W_{MNP}}$$

(Pringsheim sense). If they coincide, write $d_w(A)$ for their common value.

Definition 2.4. *A weight w satisfies (W1) if $W_{MNP} \to \infty$ in the Pringsheim sense, and (W4) if $d_w(\{i = i_0\}) = d_w(\{j = j_0\}) = d_w(\{k = k_0\}) = 0$ for every fixed $i_0, j_0, k_0 \in \mathbb{N}$. Under (W1) and (W4), the family $\mathcal{I}_w := \{A \subseteq \mathbb{N}^3 : \overline{d}_w(A) = 0\}$ is a strongly admissible ideal on $\mathbb{N}^3$.*

Definition 2.5. *An ideal $\mathcal{I}$ on $\mathbb{N}^3$ has property* (AP3) *if for every sequence $\{A_j\}$ of pairwise disjoint sets in $\mathcal{I}$, there exist $B_j \subseteq \mathbb{N}^3$ such that each symmetric difference $A_j \triangle B_j$ is contained in a finite union of coordinate slices and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.*

In the one-dimensional setting, property (AP) *of [9] requires each $A_j \triangle B_j$ to be finite. Here the corresponding requirement is containment in a finite union of coordinate slices.*

Definition 2.6. *A closed set $\phi \subseteq X$ is Wijsman-regular if $u \mapsto \psi_{\xi,\phi}(u)$ is continuous on $(0,\infty)$ for every $\xi \in X$.*

A family $\mathcal{E}$ of nonempty closed subsets of X is uniformly equi-Wijsman-regular if for every $\xi \in X$, $t > 0$, and $\lambda > 0$, there exist $s \in (0,t)$ and $\eta \in (0,1)$ such that whenever $\psi_{\xi,\xi'}(s) > 1-\eta$,

$$|\psi_{\xi,E}(t) - \psi_{\xi',E}(t-s)| < \lambda \qquad \text{for every } E \in \mathcal{E}.$$

Example 2.1 (Metric-induced Menger model)**.** *Let (X,d) be a metric space and set*

$$\psi_{x,y}(t) := \begin{cases} t/(t+d(x,y)) & t > 0, \\ 0 & t \le 0. \end{cases}$$

Then $(X,\psi,\cdot)$ is a Menger probabilistic metric space with the product t-norm. Moreover $\psi_{x,E}(t) = t/(t+d(x,E))$ for every nonempty closed $E \subseteq X$, so every closed set is Wijsman-regular, and a direct estimate shows that the class of all nonempty closed sets is uniformly equi-Wijsman-regular.

Proposition 2.1 (Uniform regularity in the metric-induced model)**.** *In Example 2.1, the family of all nonempty closed subsets of X is uniformly equi-Wijsman-regular. More precisely, for every $x \in X$, $t > 0$, and $\lambda > 0$, choose $s \in (0,t)$ so that $s < t\min\{1/2, \lambda/4\}$, and then choose $\eta \in (0,1/2)$ so that $2s\eta/t < \lambda/4$. For these choices, the implication*

$$\psi_{x,x'}(s) > 1-\eta \quad \Longrightarrow \quad |\psi_{x,E}(t) - \psi_{x',E}(t-s)| < \lambda$$

holds for every nonempty closed $E \subseteq X$.

Proof. Let $x \in X$, $t > 0$, and $\lambda > 0$ be given, and choose s,η as stated. Put $a := d(x,E)$ and $b := d(x',E)$. Since distance-to-set maps are 1-Lipschitz, $|a-b| \le d(x,x')$. From $\psi_{x,x'}(s) = s/(s+d(x,x')) > 1-\eta$ we obtain $d(x,x') < s\eta/(1-\eta) < 2s\eta$. Therefore

$$|a-b| < 2s\eta.$$

Define $f_u(r) = u/(u+r)$ for $u > 0$ and $r \ge 0$. Then

$$|\psi_{x,E}(t) - \psi_{x',E}(t-s)| = |f_t(a) - f_{t-s}(b)| \le |f_t(a) - f_t(b)| + |f_t(b) - f_{t-s}(b)|.$$

By the mean value theorem,

$$|f_t(a) - f_t(b)| \le \sup_{r \ge 0} |f_t'(r)|\,|a-b| \le \frac{|a-b|}{t} < \frac{2s\eta}{t} < \frac{\lambda}{4}.$$

For the second term,

$$|f_t(b) - f_{t-s}(b)| = \frac{bs}{(t+b)(t-s+b)} \le \frac{s}{t-s} < \frac{2s}{t} < \frac{\lambda}{2}.$$

Thus the first term is controlled by the choice of η, while the second is controlled by the choice of s; together they give $|\psi_{x,E}(t) - \psi_{x',E}(t-s)| < \lambda$, uniformly in the closed set E. □

Remark 2.1. *Proposition 2.1 shows that in the standard metric-induced Menger model the uniform equi-Wijsman-regularity hypothesis in Section 4 is automatic; it is not an extra assumption on the family of closed sets. Hence Theorem 4.2 applies directly in Euclidean spaces.*

Lemma 2.1 (Base-point shift from (PM3))**.** *If $\psi_{\xi,\xi'}(s) > 1-\eta$ then for every nonempty closed $E \subseteq X$,*

$$\psi_{\xi,E}(t) \geq \psi_{\xi',E}(t-s) - \eta \quad \textit{and} \quad \psi_{\xi',E}(t) \geq \psi_{\xi,E}(t-s) - \eta.$$

Proof. By (PM3) $\psi_{\xi,\zeta}(t) \geq \psi_{\xi,\xi'}(s) * \psi_{\xi',\zeta}(t-s)$ for every $\zeta \in E$. Applying condition (2.1) with $a = \psi_{\xi,\xi'}(s) > 1-\eta$ and $b = \psi_{\xi',\zeta}(t-s)$ gives $\psi_{\xi,\zeta}(t) \geq \psi_{\xi',\zeta}(t-s) - \eta$. Taking suprema over $\zeta \in E$ yields the first inequality, the second follows by symmetry. □

3. Weighted Density for Polynomial Product Weights

3.1. **Polynomial Product Weights and Property (AP3).** For $\rho \geq 0$ and $M \in \mathbb{N}$, set $S_\rho(M) = \sum_{m=1}^{M} m^\rho$.

Lemma 3.1 (Power-sum bounds)**.** *For every $\rho \geq 0$ and every $M \in \mathbb{N}$,*

$$\frac{M^{\rho+1}}{\rho+1} \leq S_\rho(M) \leq \frac{(M+1)^{\rho+1}}{\rho+1},$$

Then

$$\left(\frac{L}{M+1}\right)^{\rho+1} \leq \frac{S_\rho(L)}{S_\rho(M)} \leq \left(\frac{L+1}{M}\right)^{\rho+1} \qquad (1 \leq L \leq M).$$

Proof. The integral test gives $\int_0^M x^\rho\, dx \leq S_\rho(M) \leq \int_1^{M+1} x^\rho\, dx$, which is the first estimate. Hence

$$\frac{L^{\rho+1}}{(M+1)^{\rho+1}} \leq \frac{S_\rho(L)}{S_\rho(M)} \leq \frac{(L+1)^{\rho+1}}{M^{\rho+1}},$$

which is the stated pair of ratio bounds. □

Lemma 3.2 (Block dominance for sparse scales)**.** *Let $\rho \geq 0$, $\tau \in (0,1)$, and $\{N_r\}$ be strictly increasing with $N_{r+1} > N_r^2$. Set $E_\tau := \bigcup_{r=1}^{\infty}\{\lceil N_r^\tau \rceil, \dots, N_r\}$. Then for every $r \geq 2$,*

$$\frac{\displaystyle\sum_{m \in E_\tau,\, m<N_r} m^\rho}{S_\rho(N_r)} \leq (r-1)\left(\frac{N_{r-1}+1}{N_r}\right)^{\rho+1} < \frac{r-1}{N_{r-1}^{\rho+1}} \to 0.$$

Proof. The numerator is at most $\sum_{q=1}^{r-1} S_\rho(N_q) \leq (r-1)S_\rho(N_{r-1}) \leq (r-1)N_{r-1}^{\rho+1}/(\rho+1)$. Dividing by $S_\rho(N_r) \geq N_r^{\rho+1}/(\rho+1)$ and using $N_r > N_{r-1}^2$ gives the result. □

Lemma 3.3 (Polynomial weights are strongly admissible)**.** *Let $w(i,j,k) = i^\alpha j^\beta k^\gamma$ with $\alpha, \beta, \gamma \geq 0$. Then w satisfies (W1) and (W4), and $\mathcal{I}_w$ is a strongly admissible ideal on $\mathbb{N}^3$.*

Proof. $W_{MNP} = S_\alpha(M)S_\beta(N)S_\gamma(P) \to \infty$, so (W1) holds. For any fixed i_0,

$$\frac{W_{MNP}(\{i = i_0\})}{W_{MNP}} = \frac{i_0^\alpha}{S_\alpha(M)} \to 0,$$

so (W4) holds. Hence $\mathcal{I}_w$ is strongly admissible. □

Theorem 3.1 (Property (AP3) for polynomial weights)**.** *Let $w(i, j, k) = i^\alpha j^\beta k^\gamma$ with $\alpha, \beta, \gamma \geq 0$. Then $\mathcal{I}_w$ has property* (AP3)*.*

Proof. Let $\{A_j\}$ be pairwise disjoint members of $\mathcal{I}_w$. Since $\overline{d}_w(A_j) = 0$, for each j there exists $N_j \in \mathbb{N}$ such that

$$\frac{W_{MNP}(A_j \setminus [1, N_j]^3)}{W_{MNP}} < 2^{-j} \qquad \text{whenever } M, N, P \geq N_j.$$

Choose N_j inductively with $N_1 < N_2 < \cdots$ and set $B_j := A_j \setminus [1, N_j]^3$. Then $A_j \triangle B_j = A_j \cap [1, N_j]^3$ is contained in $\bigcup_{a=1}^{N_j} \{i = a\}$. Let $B = \bigcup_{j=1}^{\infty} B_j$. For any $\varepsilon > 0$, pick s with $\sum_{j \geq s} 2^{-j} < \varepsilon/2$; the finite union $B_1 \cup \cdots \cup B_{s-1}$ belongs to $\mathcal{I}_w$ because ideals are closed under finite unions, so there exists N_0 such that

$$\frac{W_{MNP}(B_1 \cup \cdots \cup B_{s-1})}{W_{MNP}} < \varepsilon/2 \qquad \text{whenever } M, N, P \geq N_0.$$

Take $M, N, P \geq N_0$ and let $J = J(M, N, P)$ be the largest index with $N_J \leq \min\{M, N, P\}$, taking $J = 0$ if no such index exists. Then

$$\frac{W_{MNP}(B \setminus (B_1 \cup \cdots \cup B_{s-1}))}{W_{MNP}} \leq \sum_{j=s}^{J} \frac{W_{MNP}(B_j)}{W_{MNP}} < \sum_{j=s}^{J} 2^{-j} \leq \sum_{j=s}^{\infty} 2^{-j} < \varepsilon/2.$$

For $j > J$, $B_j \cap ([1, M] \times [1, N] \times [1, P]) = \varnothing$ by the definition of B_j. Hence $\overline{d}_w(B) \leq \varepsilon$; since ε is arbitrary, $B \in \mathcal{I}_w$. □

Theorem 3.2 (Product-measure form of weighted density)**.** *Let $w(i, j, k) = i^\alpha j^\beta k^\gamma$ with $\alpha, \beta, \gamma \geq 0$. Define $\mu_{MNP}^{(\alpha,\beta,\gamma)} := \mu_M^{(\alpha)} \otimes \mu_N^{(\beta)} \otimes \mu_P^{(\gamma)}$ where $\mu_M^{(\alpha)}(\{i\}) := i^\alpha / S_\alpha(M)$ for $i \in \{1, \dots, M\}$. Then for every $A \subseteq \mathbb{N}^3$,*

$$\mu_{MNP}^{(\alpha,\beta,\gamma)}\big(A \cap ([1, M] \times [1, N] \times [1, P])\big) = \frac{W_{MNP}(A)}{W_{MNP}},$$

so $\overline{d}_w(A) = \limsup_{M,N,P\to\infty} \mu_{MNP}^{(\alpha,\beta,\gamma)}(A \cap ([1, M] \times [1, N] \times [1, P]))$. Because $\mu_{MNP}^{(\alpha,\beta,\gamma)}$ is a product measure, every density computation on sets in the algebra generated by cylinders $E \times \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times F \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \times G$ reduces to one-coordinate marginals and inclusion-exclusion.

Proof.

$$\mu_{MNP}^{(\alpha,\beta,\gamma)}\big(A \cap ([1, M] \times [1, N] \times [1, P])\big) = \sum_{(i,j,k) \in A \cap \text{box}} \frac{i^\alpha}{S_\alpha} \cdot \frac{j^\beta}{S_\beta} \cdot \frac{k^\gamma}{S_\gamma} = \frac{W_{MNP}(A)}{W_{MNP}}.$$

□

3.2. **Reduction Lemmas for Three Coordinate Geometries.**

Proposition 3.1 (Cylindrical reduction)**.** *For* $A_E := E \times \mathbb{N} \times \mathbb{N}$ *with* $E \subseteq \mathbb{N}$*,*

$$\overline{d}_w(A_E) = \limsup_{M\to\infty} \frac{\displaystyle\sum_{i\in E,\, i\le M} i^{\alpha}}{S_\alpha(M)}.$$

$d_w(A_E)$ *depends only on* α*.*

Proof. For all $M, N, P \in \mathbb{N}$,

$$W_{MNP}(A_E) = \Big(\sum_{\substack{i\in E\\ i\le M}} i^{\alpha}\Big) S_\beta(N) S_\gamma(P).$$

Since $W_{MNP} = S_\alpha(M) S_\beta(N) S_\gamma(P)$,

$$\frac{W_{MNP}(A_E)}{W_{MNP}} = \frac{\sum_{i\in E,\, i\le M} i^{\alpha}}{S_\alpha(M)},$$

which is independent of N and P. Thus the limsup reduces to the ordinary one-variable limsup in M. □

Proposition 3.2 (Planar diagonal reduction)**.** *For* $\Delta_c := \{(i, j, k) : i = j + c\}$ *with* $c \ge 0$*,* $d_w(\Delta_c) = 0$*.*

Proof. $W_{MNP}(\Delta_c) = S_\gamma(P) \sum_{j=1}^{\min(M,N)-c} (j + c)^{\alpha} j^{\beta}$. Since $j + c \le (1 + c)j$ for $j \ge 1$, the inner sum is bounded by

$$(1 + c)^{\alpha} \sum_{j=1}^{\min(M,N)} j^{\alpha+\beta} \le C_{\alpha,\beta,c} \min(M, N)^{\alpha+\beta+1}$$

for a constant $C_{\alpha,\beta,c} > 0$ by Lemma 3.1. By Lemma 3.1, $W_{MNP} \ge M^{\alpha+1} N^{\beta+1} P^{\gamma+1}/[(\alpha + 1)(\beta + 1)(\gamma + 1)]$. Since $\alpha + \beta + 1 < (\alpha + 1) + (\beta + 1)$, the ratio $W_{MNP}(\Delta_c)/W_{MNP}$ is bounded above by a constant multiple of

$$\frac{\min(M, N)^{\alpha+\beta+1}}{M^{\alpha+1} N^{\beta+1}} \le \min\Big\{\frac{1}{N}, \frac{1}{M}\Big\},$$

which tends to 0. □

Proposition 3.3 (Three-dimensional diagonal reduction)**.** *For the main diagonal* $D := \{(r, r, r) : r \in \mathbb{N}\}$*,* $d_w(D) = 0$*.*

Proof. $W_{MNP}(D) = \sum_{r=1}^{\min(M,N,P)} r^{\alpha+\beta+\gamma} \le (\min(M, N, P))^{\alpha+\beta+\gamma+1}/(\alpha + \beta + \gamma + 1)$ by Lemma 3.1. Since $(\alpha + \beta + \gamma + 1) < (\alpha + 1) + (\beta + 1) + (\gamma + 1)$ and $W_{MNP} \ge M^{\alpha+1} N^{\beta+1} P^{\gamma+1}/[(\alpha + 1)(\beta + 1)(\gamma + 1)]$, the ratio $W_{MNP}(D)/W_{MNP}$ is bounded above by a constant multiple of

$$\frac{\min(M, N, P)^{\alpha+\beta+\gamma+1}}{M^{\alpha+1} N^{\beta+1} P^{\gamma+1}} \le \frac{1}{NP},$$

after using $\min(M, N, P) \le M$, $\min(M, N, P) \le N$, and $\min(M, N, P) \le P$. Thus $W_{MNP}(D)/W_{MNP}$ is bounded by a constant multiple of $1/(NP)$ and tends to 0. □

3.3. **The Full Three-Coordinate Comparison Theorem.**

Theorem 3.3 (Three-coordinate comparison theorem)**.** *Fix $\tau \in (0,1)$ and a strictly increasing sequence $\{N_r\}$ such that $N_{r+1} > N_r^2$. Let*

$$E_\tau := \bigcup_{r=1}^{\infty} \{\lceil N_r^\tau \rceil, \dots, N_r\},$$

and define

$$A_\tau^{(1)} := E_\tau \times \mathbb{N} \times \mathbb{N}, \quad A_\tau^{(2)} := \mathbb{N} \times E_\tau \times \mathbb{N}, \quad A_\tau^{(3)} := \mathbb{N} \times \mathbb{N} \times E_\tau,$$
$$A_\tau^{(12)} := \{(i,j,k) : i \in E_\tau \text{ or } j \in E_\tau\}.$$

Then for $w(i,j,k) = i^\alpha j^\beta k^\gamma$:

(i) $\overline{d}_w(A_\tau^{(1)}) = 1 - \tau^{\alpha+1}$, $\overline{d}_w(A_\tau^{(2)}) = 1 - \tau^{\beta+1}$, $\overline{d}_w(A_\tau^{(3)}) = 1 - \tau^{\gamma+1}$.

(ii) $\overline{d}_w(A_\tau^{(12)}) = 1 - \tau^{\alpha+\beta+2}$.

(iii) *Strict exponent comparison. For $(\alpha,\beta,\gamma) \le (\alpha',\beta',\gamma')$ with at least one strict inequality: $\overline{d}_w(A_\tau^{(1)}) \le \overline{d}_{w'}(A_\tau^{(1)})$, with equality iff $\alpha = \alpha'$; and similarly for $A_\tau^{(12)}$ with $\alpha+\beta$.*

Proof. (i) By Proposition 3.1,

$$\overline{d}_w\Big(A_\tau^{(1)}\Big) = \limsup_{M\to\infty} \frac{\sum_{i\in E_\tau,\, i\le M} i^\alpha}{S_\alpha(M)}.$$

At $M = N_r$, the dominant contribution is $S_\alpha(N_r) - S_\alpha(\lceil N_r^\tau \rceil - 1)$; earlier blocks are $o(S_\alpha(N_r))$ by Lemma 3.2. Using Lemma 3.1,

$$\frac{S_\alpha(N_r) - S_\alpha(N_r^\tau)}{S_\alpha(N_r)} \to 1 - \tau^{\alpha+1},$$

so

$$\limsup_{M\to\infty} \frac{\sum_{i\in E_\tau,\, i\le M} i^\alpha}{S_\alpha(M)} \ge 1 - \tau^{\alpha+1}.$$

For the reverse inequality, given M, choose r so that $N_r \le M < N_{r+1}$. Then

$$\sum_{i\in E_\tau,\, i\le M} i^\alpha \le \sum_{i\in E_\tau,\, i<N_r} i^\alpha + \sum_{i=\lceil N_r^\tau \rceil}^{M} i^\alpha.$$

After dividing by $S_\alpha(M)$ the first term tends to 0 by Lemma 3.2, and the second is maximized at $M = N_r$ within the rth block. Thus the limsup equals $1 - \tau^{\alpha+1}$. The other two cases follow by symmetry.

(ii) By inclusion-exclusion,

$$W_{MNP}\Big(A_\tau^{(12)}\Big) = W_{MNP}\Big(A_\tau^{(1)}\Big) + W_{MNP}\Big(A_\tau^{(2)}\Big) - W_{MNP}(E_\tau \times E_\tau \times \mathbb{N}).$$

By Theorem 3.2, the intersection ratio factors as $\mu^{(\alpha)}(E_\tau) \cdot \mu^{(\beta)}(E_\tau) \to (1-\tau^{\alpha+1})(1-\tau^{\beta+1})$, giving

$$\overline{d}_w(A_\tau^{(12)}) = (1-\tau^{\alpha+1}) + (1-\tau^{\beta+1}) - (1-\tau^{\alpha+1})(1-\tau^{\beta+1}) = 1 - \tau^{\alpha+\beta+2}.$$

(iii) $\alpha \mapsto 1 - \tau^{\alpha+1}$ is strictly increasing since $\log\tau < 0$. ◻

The parameter γ does not appear in the densities of $A_\tau^{(1)}$ (which depends only on α) or $A_\tau^{(12)}$ (which depends only on $\alpha+\beta$). These formulas also show that weighted density can separate sets with the same ordinary upper density: if $E_\tau \subseteq \mathbb{N}$ has ordinary upper density $1-\tau$, then the cylinder $E_\tau \times \mathbb{N} \times \mathbb{N}$ has weighted upper density $1-\tau^{\alpha+1}$. The next theorem identifies exactly when one polynomial product ideal is contained in another.

3.4. Ideal Inclusion and the Exponent Partial Order.

Theorem 3.4 (Ideal inclusion and order-reversing embedding)**.** *Let $w = i^\alpha j^\beta k^\gamma$ and $w' = i^{\alpha'} j^{\beta'} k^{\gamma'}$ with all exponents nonnegative.*

(i) *If $\alpha' \le \alpha$, $\beta' \le \beta$, $\gamma' \le \gamma$ then $\mathcal{I}_w \subseteq \mathcal{I}_{w'}$.*
(ii) *If $\mathcal{I}_w \subseteq \mathcal{I}_{w'}$ then $\alpha' \le \alpha$, $\beta' \le \beta$, $\gamma' \le \gamma$.*

$\mathcal{I}_w \subseteq \mathcal{I}_{w'}$ iff exponents of w dominate those of w' componentwise, $\mathcal{I}_w = \mathcal{I}_{w'}$ iff they coincide, and the map $\Phi(\alpha,\beta,\gamma) := \mathcal{I}_{i^\alpha j^\beta k^\gamma}$ is an injective order-reversing embedding of $[0,\infty)^3$ into the lattice of ideals on $\mathbb{N}^3$.

Proof. Part (i) is proved in Appendix A. Assume $\mathcal{I}_w \subseteq \mathcal{I}_{w'}$. If $\alpha' > \alpha$, then Appendix A applied with the roles of w and w' reversed gives $\mathcal{I}_{w'} \subseteq \mathcal{I}_w$, hence $\mathcal{I}_w = \mathcal{I}_{w'}$. But Theorem 3.3(i) gives $\overline{d}_w(A_\tau^{(1)}) = 1-\tau^{\alpha+1} \ne 1-\tau^{\alpha'+1} = \overline{d}_{w'}(A_\tau^{(1)})$, a contradiction. The same argument with $A_\tau^{(2)}$ and $A_\tau^{(3)}$ rules out $\beta' > \beta$ and $\gamma' > \gamma$. Therefore $\alpha' \le \alpha$, $\beta' \le \beta$, and $\gamma' \le \gamma$. □

Remark 3.1. *Theorem 3.3 computes $\overline{d}_w(A_\tau^{(1)}) = 1-\tau^{\alpha+1}$ and $\overline{d}_w(A_\tau^{(12)}) = 1-\tau^{\alpha+\beta+2}$ for the test sets $A_\tau^{(1)} = E_\tau \times \mathbb{N} \times \mathbb{N}$ and $A_\tau^{(12)} = \{(i,j,k) : i \in E_\tau \text{ or } j \in E_\tau\}$. These formulas show that $\overline{d}_w$ distinguishes the exponents α and $\alpha+\beta$. However, an ideal $\mathcal{I}_w$ is defined by the collection of all sets with $\overline{d}_w = 0$, not merely by the densities of these examples. Theorem 3.4 closes this gap: it proves that $\mathcal{I}_w \subseteq \mathcal{I}_{w'}$ if and only if $\alpha' \le \alpha$, $\beta' \le \beta$, and $\gamma' \le \gamma$. Consequently, the test sets from Theorem 3.3 are already sufficient to recover the exponent triple (α,β,γ) uniquely.*

As an immediate consequence, if $\alpha' \le \alpha$, $\beta' \le \beta$, and $\gamma' \le \gamma$, then $\mathcal{I}_w^\psi$-convergence implies $\mathcal{I}_{w'}^\psi$-convergence.

Example 3.1 (A concrete Euclidean comparison)**.** *Take $X = \mathbb{R}^2$ with the usual metric and with the metric-induced Menger structure from Example 2.1. Fix the closed sets $F := \{(0,0)\}$ and $G := \{(1,0)\}$, and let $E_\tau \subseteq \mathbb{N}$ be the sparse-scale set used in Theorem 3.3. Define a triple sequence of closed sets by*

$$\phi_{mnk} := \begin{cases} G, & m \in E_\tau, \\ F, & m \notin E_\tau. \end{cases}$$

Then for $x = (0,0)$ and any $t > 0$,

$$\psi_{x,G}(t) = \frac{t}{t+1} < 1 = \psi_{x,F}(t),$$

so every index with $m \in E_\tau$ contributes the same fixed deviation from F, while the indices with $m \notin E_\tau$ contribute no deviation at all. Hence the exceptional set for convergence to F is exactly $E_\tau \times \mathbb{N} \times \mathbb{N}$, and *therefore*

$$\overline{d}_w\Big(\{(m,n,k) : |\psi_{x,\phi_{mnk}}(t) - \psi_{x,F}(t)| \geq \lambda\}\Big) = 1 - \tau^{\alpha+1}$$

for every $0 < \lambda \leq 1 - \frac{t}{t+1}$. *This example is intentionally one-coordinate: it isolates the role of the exponent* α *while leaving* β *and* γ *inactive. Thus the same sequence is* $\mathcal{I}_w^{\psi}$*-convergent for one exponent* α *but not* $\mathcal{I}_{w'}^{\psi}$*-convergent for another, consistent with Theorem 3.3.*

4. Weighted Wijsman Ideal Convergence

The uniform equi-Wijsman-regularity hypothesis in Theorem 4.2 is automatically satisfied in the metric-induced Menger model (Proposition 2.1) and more generally whenever the family $\{\phi_{mnk}\} \cup \{\phi\}$ consists of Wijsman-regular sets whose distance-like functions $\psi_{\cdot,\phi_{mnk}}(t)$ are uniformly equicontinuous in the base-point variable. In abstract Menger spaces, this is an independent condition; its necessity is illustrated by examples where star-convergence fails despite ideal convergence without it (see e.g., [15] for related phenomena in probabilistic normed spaces).

For a triple sequence $\{\phi_{mnk}\}$ of nonempty closed sets and a closed set ϕ, define the following weighted Wijsman notions.

Definition 4.1. *We say that* ϕ_{mnk} $\mathcal{I}_w^{\psi}$*-converges to* ϕ *if for every* $\xi \in X$, $\varepsilon > 0$, *and* $\lambda > 0$,

$$\{(m,n,k) : |\psi_{\xi,\phi_{mnk}}(\varepsilon) - \psi_{\xi,\phi}(\varepsilon)| \geq \lambda\} \in \mathcal{I}_w.$$

It $\mathcal{I}_w^{*\psi}$*-converges to* ϕ *if there exists* $A \in \mathcal{F}(\mathcal{I}_w)$ *such that for every* $\xi \in X$ *and* $\varepsilon > 0$,

$$\lim_{\substack{(m,n,k)\in A \\ m,n,k\to\infty}} \psi_{\xi,\phi_{mnk}}(\varepsilon) = \psi_{\xi,\phi}(\varepsilon).$$

4.1. **Star-Equivalence under Regularity.**

Theorem 4.1 (Star convergence implies ideal convergence)**.** $\phi_{mnk} \xrightarrow{\mathcal{I}_w^{*\psi}} \phi$ *implies* $\phi_{mnk} \xrightarrow{\mathcal{I}_w^{\psi}} \phi$.

Proof. Take $A \in \mathcal{F}(\mathcal{I}_w)$ witnessing $\mathcal{I}_w^{*\psi}$-convergence. For any $\xi, \varepsilon, \lambda$, choose N so that $|\psi_{\xi,\phi_{mnk}}(\varepsilon) - \psi_{\xi,\phi}(\varepsilon)| < \lambda$ whenever $(m,n,k) \in A$ and $m,n,k \geq N$. Then the exceptional set is contained in $(\mathbb{N}^3 \setminus A) \cup \{(m,n,k) : \min(m,n,k) < N\}$, which belongs to $\mathcal{I}_w$. □

Theorem 4.2 (Equivalence of ideal and star convergence)**.** *Let* X *be separable, let* $\mathcal{I}_w$ *have property* (AP3)*, let* ϕ *be Wijsman-regular, and suppose* $\{\phi_{mnk}\} \cup \{\phi\}$ *is uniformly equi-Wijsman-regular. Then*

$$\phi_{mnk} \xrightarrow{\mathcal{I}_w^{\psi}} \phi \iff \phi_{mnk} \xrightarrow{\mathcal{I}_w^{*\psi}} \phi.$$

The proof follows the same general line as [9], but the triple-index setting introduces two additional ingredients. One is the use of property (AP3), with coordinate slices in place of finite exceptional sets. The other is the need for uniform equi-Wijsman-regularity when one passes from convergence on a dense set of base points to convergence for all base points in the probabilistic metric setting. Proposition 2.1 shows that this condition is automatically satisfied in the metric-induced Menger model.

Proof. The implication ($\Leftarrow$) is Theorem 4.1. For the reverse implication, assume $\phi_{mnk} \xrightarrow{\mathcal{I}_w^{\psi}} \phi$. Separability of X provides a countable dense set $D = \{\xi_\ell : \ell \in \mathbb{N}\} \subseteq X$. We use this countability together with rational parameters to reduce the problem to countably many exceptional sets, so that property (AP3) can be applied. Without separability the same argument would require an uncountable analogue of property (AP3), which lies outside the scope of this paper.

Fix the countable parameter set $\Gamma := D \times \mathbb{Q}_+ \times \mathbb{N}$, where $\mathbb{Q}_+$ denotes the positive rationals. For each $(\xi_\ell, \tau, r) \in \Gamma$, set

$$A_{\ell,\tau,r} := \{(m,n,k) : |\psi_{\xi_\ell,\phi_{mnk}}(\tau) - \psi_{\xi_\ell,\phi}(\tau)| \geq 1/r\}.$$

Because $\phi_{mnk} \xrightarrow{\mathcal{I}_w^{\psi}} \phi$, every set $A_{\ell,\tau,r}$ belongs to $\mathcal{I}_w$. Enumerate this family as $\{C_j : j \in \mathbb{N}\}$.

To bring property (AP3) into play, replace $\{C_j\}$ by a pairwise disjoint family. Define

$$D_1 := C_1, \qquad D_j := C_j \setminus \bigcup_{q<j} C_q \quad (j \geq 2).$$

Then the sets D_j are pairwise disjoint, each D_j belongs to $\mathcal{I}_w$, and for every j we have $C_j = D_j \cup \bigcup_{q<j}(C_j \cap D_q) \subseteq \bigcup_{q \leq j} D_q$. Apply property (AP3) to $\{D_j\}$. There exist sets $B_j \subseteq \mathbb{N}^3$ such that $D_j \triangle B_j$ is contained in a finite union of coordinate slices for all j, and

$$B := \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_w.$$

Set

$$A := \mathbb{N}^3 \setminus B \in \mathcal{F}(\mathcal{I}_w).$$

We claim that along A, each exceptional set C_j is eventually avoided in Pringsheim's sense.

Indeed, fix j. Since $C_j \subseteq \bigcup_{q \leq j} D_q$, we have

$$A \cap C_j \subseteq \bigcup_{q \leq j}(A \cap D_q) \subseteq \bigcup_{q \leq j}(D_q \setminus B_q) \subseteq \bigcup_{q \leq j}(D_q \triangle B_q).$$

For each $q \leq j$, the set $D_q \triangle B_q$ is contained in a finite union of coordinate slices. A finite union of such sets can intersect the tail $\{(m,n,k) : m,n,k \geq N\}$ only if one of the coordinates is fixed below N; therefore there exists $N(j) \in \mathbb{N}$ such that

$$(A \cap C_j) \cap \{(m,n,k) : m,n,k \geq N(j)\} = \varnothing.$$

Equivalently, whenever $(m,n,k) \in A$ and $m,n,k \geq N(j)$, we have $(m,n,k) \notin C_j$.

Now fix $(\xi_\ell, \tau, r) \in \Gamma$, and let j be its index in the enumeration $\{C_j\}$. For $(m,n,k) \in A$ with $m,n,k \geq N(j)$, the previous paragraph gives $(m,n,k) \notin C_j$, hence

$$|\psi_{\xi_\ell,\phi_{mnk}}(\tau) - \psi_{\xi_\ell,\phi}(\tau)| < 1/r.$$

Since r is arbitrary, it follows that along the set A,

$$\psi_{\xi_\ell,\phi_{mnk}}(\tau) \longrightarrow \psi_{\xi_\ell,\phi}(\tau) \qquad \text{for every } (\xi_\ell, \tau) \in D \times \mathbb{Q}_+,$$

where the limit is taken in the Pringsheim sense restricted to indices in A.

We next pass from rational parameters to an arbitrary $t > 0$ for points of the dense set D. Fix $\xi_\ell \in D$, $t > 0$, and $\lambda > 0$. By Wijsman regularity of ϕ, there exist rationals $u, v \in \mathbb{Q}_+$ with $u < t < v$ and

$$\psi_{\xi_\ell,\phi}(v) - \psi_{\xi_\ell,\phi}(u) < \lambda/3.$$

Because convergence already holds at the rational parameters u and v, there exists $N \in \mathbb{N}$ such that for all $(m, n, k) \in A$ with $m, n, k \geq N$,

$$|\psi_{\xi_\ell,\phi_{mnk}}(u) - \psi_{\xi_\ell,\phi}(u)| < \lambda/3, \qquad |\psi_{\xi_\ell,\phi_{mnk}}(v) - \psi_{\xi_\ell,\phi}(v)| < \lambda/3.$$

Since $u < t < v$ and each map $s \mapsto \psi_{\xi_\ell,E}(s)$ is nondecreasing, for such (m, n, k) we have

$$\psi_{\xi_\ell,\phi_{mnk}}(u) \leq \psi_{\xi_\ell,\phi_{mnk}}(t) \leq \psi_{\xi_\ell,\phi_{mnk}}(v).$$

Using the previous inequalities and the bound on $\psi_{\xi_\ell,\phi}(v) - \psi_{\xi_\ell,\phi}(u)$, we obtain

$$|\psi_{\xi_\ell,\phi_{mnk}}(t) - \psi_{\xi_\ell,\phi}(t)| < \lambda.$$

Hence convergence along A holds for every $\xi_\ell \in D$ and every real $t > 0$.

Finally, we pass from the dense set D to an arbitrary base point $\xi \in X$. Fix $\xi \in X$, $t > 0$, and $\lambda > 0$. Apply uniform equi-Wijsman-regularity to the family $\{\phi_{mnk}\} \cup \{\phi\}$ at the point ξ and tolerance $\lambda/3$. There exist $s \in (0, t)$ and $\eta \in (0, 1)$ such that whenever $\psi_{\xi,\xi'}(s) > 1 - \eta$,

$$|\psi_{\xi,E}(t) - \psi_{\xi',E}(t - s)| < \lambda/3 \qquad \text{for every } E \in \{\phi_{mnk} : m, n, k \in \mathbb{N}\} \cup \{\phi\}.$$

Choose $\xi_\ell \in D$ with $\psi_{\xi,\xi_\ell}(s) > 1 - \eta$. Since convergence along A already holds at the dense point ξ_ℓ and at the parameter $t - s > 0$, there exists $N \in \mathbb{N}$ such that for every $(m, n, k) \in A$ with $m, n, k \geq N$,

$$|\psi_{\xi_\ell,\phi_{mnk}}(t - s) - \psi_{\xi_\ell,\phi}(t - s)| < \lambda/3.$$

For such (m, n, k),

$$\begin{aligned} &|\psi_{\xi,\phi_{mnk}}(t) - \psi_{\xi,\phi}(t)| \\ &\leq |\psi_{\xi,\phi_{mnk}}(t) - \psi_{\xi_\ell,\phi_{mnk}}(t - s)| + |\psi_{\xi_\ell,\phi_{mnk}}(t - s) - \psi_{\xi_\ell,\phi}(t - s)| \\ &\quad + |\psi_{\xi_\ell,\phi}(t - s) - \psi_{\xi,\phi}(t)| < \lambda. \end{aligned}$$

Therefore

$$\lim_{\substack{(m,n,k)\in A \\ m,n,k\to\infty}} \psi_{\xi,\phi_{mnk}}(t) = \psi_{\xi,\phi}(t)$$

for every $\xi \in X$ and every $t > 0$. Since $A \in \mathcal{F}(\mathcal{I}_w)$, this is exactly $\phi_{mnk} \xrightarrow{\mathcal{I}_w^{*\psi}} \phi$. □

5. Conclusion

For the polynomial product weights $w(i, j, k) = i^\alpha j^\beta k^\gamma$, the exponent triple determines the negligibility structure. The main point is not only that the densities of model sets change with the exponents, but that these changes are strong enough to recover the ideal itself.

Theorem 3.3 identifies the weighted sizes of cylinders, planar unions, and diagonals, while Theorem 3.4 turns those computations into an exact order-theoretic description of the family of ideals $\mathcal{I}_w$. In this sense, the paper shows that weighted density for triple sequences is sensitive enough to distinguish the full product family $i^\alpha j^\beta k^\gamma$ at the level of negligibility, not merely at the level of selected examples.

On the convergence side, Theorem 4.2 shows that under separability, property (AP3), and uniform equi-Wijsman-regularity, weighted ideal Wijsman convergence agrees with weighted star-Wijsman convergence. Proposition 2.1 makes this criterion concrete in the metric-induced Menger model, so the abstract theorem applies directly to Euclidean spaces and other standard metric settings.

Two directions remain especially natural. One is to remove the separability assumption from the star-equivalence theorem, which would require a usable uncountable analogue of property (AP3). The other is to study non-polynomial weights, such as deferred or lacunary weights, where the simple product-measure factorization used throughout Section 3 is no longer available.

Appendix A. Proof of Theorem 3.4(i)

For completeness, we include the dyadic argument proving Theorem 3.4(i).

Proof of Theorem 3.4(i). Let

$$T_{\alpha,\beta,\gamma}(M,N,P;A) := \sum_{\substack{1\le i\le M,\,1\le j\le N,\,1\le k\le P\\ (i,j,k)\in A}} i^\alpha j^\beta k^\gamma.$$

Assume $A \in \mathcal{I}_w$, so

$$\varepsilon(M,N,P) := \frac{T_{\alpha,\beta,\gamma}(M,N,P;A)}{S_\alpha(M)S_\beta(N)S_\gamma(P)} \longrightarrow 0$$

in the Pringsheim sense. We show that the analogous quotient with exponents (α',β',γ') also tends to 0.

Set $L_M := \lfloor \log_2 M\rfloor + 1$, $L_N := \lfloor \log_2 N\rfloor + 1$, $L_P := \lfloor \log_2 P\rfloor + 1$, and $I_a := (2^{a-1}, 2^a] \cap \mathbb{N}$. Decomposing $[1,M]\times[1,N]\times[1,P]$ into dyadic blocks,

$$T_{\alpha',\beta',\gamma'}(M,N,P;A) \le \sum_{a\le L_M}\sum_{b\le L_N}\sum_{c\le L_P} 2^{a(\alpha'-\alpha)}2^{b(\beta'-\beta)}2^{c(\gamma'-\gamma)} \cdot T_{\alpha,\beta,\gamma}(2^a,2^b,2^c;A),$$

because on $I_a\times I_b\times I_c$, $i^{\alpha'} j^{\beta'} k^{\gamma'} \le 2^{a(\alpha'-\alpha)}2^{b(\beta'-\beta)}2^{c(\gamma'-\gamma)} i^\alpha j^\beta k^\gamma$. Writing $\varepsilon_{abc} := \varepsilon(2^a,2^b,2^c)$ and using $T_{\alpha,\beta,\gamma}(2^a,2^b,2^c;A) = \varepsilon_{abc} S_\alpha(2^a)S_\beta(2^b)S_\gamma(2^c)$, Lemma 3.1 yields a constant $C > 0$ such that

$$T_{\alpha',\beta',\gamma'}(M,N,P;A) \le C\sum_{a\le L_M}\sum_{b\le L_N}\sum_{c\le L_P} \varepsilon_{abc}\, 2^{a(\alpha'+1)}2^{b(\beta'+1)}2^{c(\gamma'+1)}.$$

Since $S_{\alpha'}(M)S_{\beta'}(N)S_{\gamma'}(P) \geq c_0\, 2^{L_M(\alpha'+1)}2^{L_N(\beta'+1)}2^{L_P(\gamma'+1)}$ for a constant $c_0 > 0$, there is $C' > 0$ with

$$\frac{T_{\alpha',\beta',\gamma'}(M,N,P;A)}{S_{\alpha'}(M)S_{\beta'}(N)S_{\gamma'}(P)} \leq C' \sum_{a \leq L_M} \sum_{b \leq L_N} \sum_{c \leq L_P} \varepsilon_{abc} \cdot 2^{-(L_M-a)(\alpha'+1)}2^{-(L_N-b)(\beta'+1)}2^{-(L_P-c)(\gamma'+1)}.$$

The weights on the right form a summable product-kernel. Fix $\delta > 0$. Choose A_0, B_0, C_0 so that $\varepsilon_{abc} < \delta$ whenever $a \geq A_0$, $b \geq B_0$, and $c \geq C_0$; this is possible because $\varepsilon_{abc} \to 0$. Split the triple sum into the finitely many indices with at least one of $a < A_0$, $b < B_0$, $c < C_0$, and the tail where all three inequalities fail. The finite part tends to 0 as $L_M, L_N, L_P \to \infty$ because of the exponentially decaying kernel, while the tail is bounded by a constant multiple of δ. Since δ is arbitrary, the whole quotient tends to 0, proving $A \in \mathcal{I}_{w'}$. □

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