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New Variant of Hermite-Hadamard-Type Inequality with Applications to Matrix and Bivariate Analysis

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Abstract. Convex analysis and mathematical inequalities are now essential to the development of numerous pure and applied scientific domains. In this article, first we introduce the concept of n-fractional polynomial $\mathtt{s}$-like $\mathtt{m}$-preinvex function. In addition, we introduce, the new variant of Hermite-Hadamard type inequality via newly introduced

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concept pertaining to the k-fractional operator. Furthermore, we demonstrate the applications to matrix and bivariat analysis via newly introduce Hermite-Hadamard type inequality. The study's conclusions present unique perspectives and contributions to the field, offering fresh and noteworthy improvements over previous research.

1. Introduction

Over the past few decades, convexity has produced an incredible and productive research field that addresses a wide range of issues in both the pure and applied sciences. Using a variety of instruments and techniques, the theory of convex functions has evolved in different directions. A rich framework for creating mathematical tools to address challenging issues is provided by convexity. Experts have been captivated by the profound connection between convexity and inequalities, leading to numerous extensions of the traditional Hermite–Hadamard (H-H) inequality. These have been extensively studied and are widely available in the literature. Convex functions have yielded remarkable applications and insightful results in all areas of applied sciences, particularly in fractional analysis. For the literature, see [1–10].

Due to their numerous applications in means and sums, numerical integration, and quantum calculus, inequalities have become a popular topic among researchers. Researchers make various refinements and generalizations. In mathematical analysis, inequalities play a significant role and are very helpful in nonlinear analysis. Convex function inequalities have received a lot of attention in the literature and are among the most effective tools in the development of many areas of mathematics. For the literature, see [11–20].

Fractional calculus, a field of mathematics that has garnered much attention in the past ten years, relies heavily on fractional operators. Studies of fractional integral inequalities are driven mainly by the aim of applying different fractional operators to extend classical inequalities, e.g. the Ostrowski, Simpson, and Hadamard inequalities. Researchers keep growing the field of fractional analysis by introducing new concepts and structures that challenge the limits of applied mathematics. Fractional analysis is used in many areas, such as financial modeling, fluid dynamics, image processing, weather forecasting, mathematical biology, medicine and engineering. Due to its numerous applications and implications, fractional calculus is a promising field of mathematics and science. The inequalities theory of fractional integral in particular has a high degree of application and implications on applied mathematics. For the literature, see [21–31].

This research explores a novel family of functions, namely n-fractional polynomial $\mathtt{s}$-like $\mathtt{m}$-preinvex functions, which extend existing convexity idea in a fractional approach. It gives a new H-H inequality obtained using the k-fractional operator, which gives a new approach to the classical inequalities. The research also generalizes these findings to practical uses of matrix analysis and the bivariate functions, which were not previously evident in literature. On the whole, the work makes its contribution to the unique theoretical developments and practical extensions, which makes it stand out of the previous studies in the field.

The goal of this study is to create a more diverse analytical framework by introducing n-fractional polynomial $\mathtt{s}$-like $\mathtt{m}$-preinvex functions and investigating their structural features. The study aims to refine the Hermite–Hadamard type inequality by employing a newly developed k-fractional operator, hence expanding conventional convex analysis conclusions. Another goal is to show how these new inequalities can be used effectively with matrix setups and bivariate functions. The goal of these developments is to provide greater understanding and substantive modifications to existing inequality theory.

Inspired by the broad literature in this field, this work is organized as follows. First, since all of these will be required in later sections, we include some well-known theorems, definitions, remarks, and corollaries in Section 2. In Section 3, we introduce the new concept of the n-fractional polynomial $\mathtt{s}$-like $\mathtt{m}$-preinvex mapping, and present examples with graphical verification. In Section 4, we explore the new modification of the H-H inequality through the newly discussed concept via the k-fractional operator. We also present new variants of the H-H type inequality in the form of corollaries. In Sections 5 and 6, we provide applications in matrix and bivariate analysis based on the obtained results. Finally, in Section 7, we present a conclusion.

2. Preliminaries

This section presents a review of key definitions and foundational concepts crucial for the forthcoming analysis, including $\mathtt{m}$-invexity, preinvex functions, generalized $\mathtt{m}$-preinvexity, and the extended Condition C. We also discuss the $\mathtt{s}$-like convex function, $\mathtt{s}$-like preinvexity, n-fractional polynomial convex function, the Riemann–Liouville fractional integral operator (RLFIO), and the k-fractional integral operator. The section concludes with a brief overview of the H-H inequality, all of which play a fundamental role in our study.

Definition 2.1 ([32]). *Consider a set $A \subseteq \mathbb{R}^n$ together with a mapping $\nabla : A \times A \times (0,1] \to \mathbb{R}^n$. The set A is termed* $\mathtt{m}$-invex *relative to ∇ whenever the condition*

$$\mathtt{m}\omega_x + u\nabla(\omega_y, \omega_x, \mathtt{m}) \in A$$

holds for every choice of $\omega_x, \omega_y \in A$, $\mathtt{m} \in (0,1]$, and $u \in [0,1]$.

Example 2.1 ([32]). *The following construction demonstrates that $\mathtt{m}$-invexity is strictly broader than classical convexity. Fix $\mathtt{m} = \frac{1}{4}$, and let the domains be*

$$A = \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \quad \mathbb{X} = (-1, 1)\,.$$

Assign the auxiliary mapping ∇ by the piecewise rule

$$\nabla(\omega_y,\omega_x,\mathtt{m}) = \begin{cases} \mathtt{m}\tan(\omega_y-\omega_x), & \text{if } \omega_x,\omega_y \in \left(0,\frac{\pi}{4}\right); \\ -\mathtt{m}\tan(\omega_y-\omega_x), & \text{if } \omega_x,\omega_y \in \left(-\frac{\pi}{4},0\right); \\ \mathtt{m}\tan(\omega_x), & \text{if } \omega_x \in \left(0,\frac{\pi}{4}\right),\ \omega_y \in \left(-\frac{\pi}{4},0\right); \\ -\mathtt{m}\tan(\omega_x), & \text{if } \omega_x \in \left(-\frac{\pi}{4},0\right),\ \omega_y \in \left(0,\frac{\pi}{4}\right). \end{cases}$$

A direct verification shows that $A = \left(-\frac{\pi}{4},\frac{\pi}{4}\right)$ *is* $\mathtt{m}$*-invex with respect to* ∇*, whereas it does not satisfy the standard definition of convexity.*

Example 2.2 ([33]). *The subsequent example exhibits* $\mathtt{m}$*-invexity on a domain that is not connected. Take* $\Delta = [0,2] \cup [4,6]$ *and introduce the mapping* ∇ *through the four-branch formula*

$$\nabla(\omega_y,\omega_x,\mathtt{m}) = \begin{cases} \omega_x + \mathtt{m}(\omega_y - 4) & \text{if } 4 \le \omega_y \le 6,\ \ 0 \le \omega_x \le 2, \\ \omega_x + \mathtt{m}(\omega_y - 2) & \text{if } 0 \le \omega_y \le 2,\ \ 0 \le \omega_x \le 2, \\ 4 + \mathtt{m}(\omega_y - \omega_x) & \text{if } 0 \le \omega_y \le 2,\ \ 4 \le \omega_x \le 6, \\ 2 + \mathtt{m}(\omega_y - \omega_x) & \text{if } 4 \le \omega_y \le 6,\ \ 4 \le \omega_x \le 6. \end{cases}$$

With this specification, the set Ψ *is an* $\mathtt{m}$*-invex set with respect to* ∇*, while failing to be convex.*

The pioneering work of Weir and Mond [34] in 1988 laid the groundwork for the systematic study of invex sets and preinvex functions, opening a new avenue in the generalization of convexity-based analysis.

Definition 2.2 ([34]). *Suppose* $A \subseteq \mathbb{R}^n$ *and* $\nabla : A \times A \to \mathbb{R}^n$ *be an auxiliary function. A function* Δ *is called* preinvex *relative to* ∇ *provided that*

$$\Delta\left(\omega_x + u\nabla(\omega_y,\omega_x)\right) \le (1-u)\,\Delta(\omega_x) + u\,\Delta(\omega_y),$$

whenever $\omega_x,\omega_y \in A$ *and* $u \in [0,1]$.

Extensive foundational material and subsequent advances in this direction may be consulted in [35–37]. Over the intervening years, the theory of preinvexity has been enriched considerably, with numerous investigators introducing generalizations and structural refinements tailored to specific analytical needs.

Notably, Kalsoom [33] extended this framework by introducing the concept of *generalized* $\mathtt{m}$*-preinvexity*, formulated as follows.

Definition 2.3. *Suppose* $A \subseteq \mathbb{R}^n$ *and* $\nabla : A \times A \times (0,1] \to \mathbb{R}^n$ *be a function. A function* Δ *is termed* generalized $\mathtt{m}$-preinvex *relative to* ∇ *if the inequality*

$$\Delta\left(\mathtt{m}\omega_x + u\nabla(\omega_y,\omega_x,\mathtt{m})\right) \le \mathtt{m}(1-u)\,\Delta(\omega_x) + u\,\Delta(\omega_y), \tag{2.1}$$

is satisfied for all $\omega_x,\omega_y \in A$*,* $\mathtt{m} \in (0,1]$*, and* $u \in [0,1]$.

Motivated by the above developments, Du [38] formulated an extension of the classical Condition C that is adapted to the $\mathtt{m}$-preinvex setting. The distinguishing aspect of this generalization lies in the explicit dependence of the mapping ∇ on the parameter $\mathtt{m} \in (0,1]$.

Definition 2.4 (Extended Condition C, [38]). *Suppose $A \subset \mathbb{R}$ be an open invex set w.r.t. a function*

$$\nabla : A \times A \times (0,1] \to \mathbb{R}.$$

For arbitrary elements $v_a, v_b \in A$, a scalar $u \in [0,1]$, and a parameter $\mathtt{m} \in (0,1]$, the mapping ∇ is required to satisfy the three identities

$$\begin{aligned} \nabla\Big(\omega_x,\, \mathtt{m}\omega_x + u\,\nabla(\omega_y,\omega_x,\mathtt{m}),\, \mathtt{m}\Big) &= -u\,\nabla(\omega_y,\omega_x,\mathtt{m}),\\ \nabla\Big(\omega_y,\, \mathtt{m}\omega_x + u\,\nabla(\omega_y,\omega_x,\mathtt{m}),\, \mathtt{m}\Big) &= (1-u)\,\nabla(\omega_y,\omega_x,\mathtt{m}),\\ \nabla(\omega_y,\omega_x,\mathtt{m}) &= -\nabla(\omega_x,\omega_y,\mathtt{m}). \end{aligned} \tag{2.2}$$

Definition 2.5. *[39, 40] Given $\mathsf{s} \in [0,1]$, a function $\Delta : X^o \to \mathbb{R}$ is designated s-like convex on X^o when*

$$\Delta\Big(\xi\omega_x + (1-\xi)\,\omega_y\Big) \le (1 - \mathsf{s}\,(1-\xi))\,\Delta\,(\omega_x) + (1-\mathsf{s}\xi)\,\Delta\Big(\omega_y\Big) \tag{2.3}$$

holds for every $\omega_x, \omega_y \in X^o$ and each $\xi \in [0,1]$.

Definition 2.6. *[41] Let $X \subset \mathbb{R}$ be a nonempty invex set admitting the auxiliary mapping $\nabla : X^o \times X^o \subset \mathbb{R} \to \mathbb{R}$. The function $\Delta : X \to \mathbb{R}$ is classified as s-like preinvex whenever*

$$\Delta\Big(\omega_y + \xi\nabla(\omega_x,\omega_y)\Big) \le (1 - \mathsf{s}\,(1-\xi))\,\Delta\,(\omega_x) + (1-\mathsf{s}\xi)\,\Delta\Big(\omega_y\Big). \tag{2.4}$$

The concept of n-fractional polynomial convexity was put forward by İşcan [42], whose definition reads as follows.

Definition 2.7. *For a given $n \in \mathbb{N}$, a function $\Delta : X^o \subset \mathbb{R} \to \mathbb{R}$ is referred to as an n-fractional polynomial convex function provided that*

$$\Delta\Big(\xi\omega_x + (1-\xi)\,\omega_y\Big) \le \frac{1}{n}\sum_{j=1}^{n} \xi^{\frac{1}{j}}\,\Delta\,(\omega_x) + \frac{1}{n}\sum_{j=1}^{n}(1-\xi)^{\frac{1}{j}}\,\Delta\Big(\omega_y\Big) \tag{2.5}$$

for all $\omega_x, \omega_y \in X^o$ and $\xi \in [0,1]$.

For completeness, we recall the left- and right-sided Riemann–Liouville fractional integral operators, which constitute a cornerstone of classical fractional analysis.

Definition 2.8 ([43]). *For a function $\Delta \in L_1[\omega_x, \omega_y]$ and an order $\gamma > 0$, then $\mathtt{I}^{\gamma}_{\omega_x^+}\Delta$ and $\mathtt{I}^{\gamma}_{\omega_y^-}\Delta$ are given, respectively, by*

$$\mathtt{I}^{\gamma}_{\omega_x^+}\Delta(x) = \frac{1}{\Gamma(\gamma)}\int_{\omega_x}^{x} (x-u)^{\gamma-1}\Delta(u)\,du, \quad x > \omega_x,$$

and

$$\mathtt{I}^{\gamma}_{\omega_y^-}\Delta(x) = \frac{1}{\Gamma(\gamma)}\int_{x}^{\omega_y} (u-x)^{\gamma-1}\Delta(u)\,du, \quad x < \omega_y.$$

Incorporating an additional parameter $k > 0$ into the fractional integral framework, Mubeen et al. [44] introduced the k-fractional integral operators defined below.

Definition 2.9. *[44] For $\Delta \in \mathcal{L}[\omega_x, \omega_y]$ and parameters $\gamma > 0$, $k > 0$, the left-sided and right-sided k-fractional integrals $\zeta^{\gamma,k}_{\omega_x^-}\Delta(x)$ and $\zeta^{\gamma,k}_{\omega_y^+}\Delta(x)$ take the forms*

$$\zeta^{\gamma,k}_{\omega_x^-}\Delta(x) = \frac{1}{k\Gamma_k(\gamma)}\int_{\omega_x}^{x}(x-\xi)^{\frac{\gamma}{k}-1}\Delta(\xi)\,d\xi, \quad \omega_x < x, \tag{2.6}$$

and

$$\zeta^{\gamma,k}_{\omega_y^+}\Delta(x) = \frac{1}{k\Gamma_k(\gamma)}\int_{x}^{\omega_y}(\xi-x)^{\frac{\gamma}{k}-1}\Delta(\xi)\,d\xi, \quad x < \omega_y, \tag{2.7}$$

where the k-gamma function is defined by

$$\Gamma_k(\gamma) = \int_0^\infty \xi^{\gamma-1}e^{-\frac{\xi^k}{k}}\,d\xi$$

and fulfils the recurrence $\Gamma_k(\gamma+k) = \gamma\Gamma_k(\gamma)$ as well as the boundary value $\Gamma_k(k) = 1$. In the limiting case, $\zeta^{0,k}_{\omega_x^-}\Delta(x) = \zeta^{0,k}_{\omega_y^+}\Delta(x) = \Delta(x)$.

The most popular inequality in the research is the H-H inequality. Numerous mathematicians have approached the study of inequalities from various perspectives.

Theorem 2.1. *If $\Delta : [\omega_x, \omega_y] \to \mathbb{R}$ is a convex function, then*

$$\Delta\left(\frac{\omega_x+\omega_y}{2}\right) \leq \frac{1}{\omega_y-\omega_x}\int_{\omega_x}^{\omega_y}\Delta(x)dx \leq \frac{\Delta(\omega_x)+\Delta(\omega_y)}{2}. \tag{2.8}$$

For the literature, see [45–53].

The theorem stated below is attributed to Noor [54] and presents the Hermite–Hadamard inequality in the context of preinvex functions; it will emerge as a particular case of the main result derived in the present work.

Theorem 2.2. *Let $\Delta : X = [\omega_x, \omega_x + \nabla(\omega_y, \omega_x)] \to (0,\infty)$ be a preinvex mapping on X^o with $\nabla(\omega_y, \omega_x) > 0$. Then the following chain of inequalities holds:*

$$\Delta\left(\frac{2\omega_x+\nabla\left(\omega_y,\omega_x\right)}{2}\right) \leq \frac{1}{\nabla\left(\omega_y,\omega_x\right)}\int_{\omega_x}^{\omega_x+\nabla\left(\omega_y,\omega_x\right)}\Delta(x)\,dx \leq \frac{\Delta(\omega_x)+\Delta\left(\omega_y\right)}{2}. \tag{2.9}$$

3. n-Fractional Polynomial s-Like $\mathfrak{m}$-Preinvex Mapping

The idea of preinvexity extends the usual notion of convexity by allowing curves between points to be determined by a suitable mapping, rather than by straight-line segments. Because of this added flexibility, preinvex functions are valuable in situations where the domain is not convex or where a function fails to be convex in the classical sense but still exhibits similar behaviour along an invex direction. This framework has proved particularly useful in nonlinear optimization, including global and constrained problems, variational methods, and mathematical programming, where it often leads to stronger conditions for optimality and the existence of solutions. Beyond optimization, the concept has been employed in several applied areas such as control theory, economics (for modeling utility and production), engineering design, machine learning, fuzzy systems, and signal processing. In these settings, preinvexity helps address problems that cannot be handled effectively when ordinary convexity assumptions break down.

In this section, we examine the new concept of the n-fractional polynomial $\mathtt{s}$-like $\mathtt{m}$-preinvex function.

Definition 3.1. *Let $n \in \mathbb{N}$ and $\mathtt{m} \in (0,1]$. A non-negative mapping $\Delta : X^o \to \mathbb{R}$ is called an n-fractional polynomial $\mathtt{s}$-like $\mathtt{m}$-preinvex mapping if*

$$\Delta\left(\mathtt{m}\omega_x + \xi\,\nabla\left(\omega_y,\omega_x,\mathtt{m}\right)\right) \le \frac{\sum\limits_{j=1}^{n}\mu_j\left(1-\mathtt{s}\left(1-\xi\right)\right)^{\frac{1}{j}}}{\sum\limits_{j=1}^{n}\mu_j}\Delta\left(\mathtt{m}\omega_x\right) + \frac{\sum\limits_{j=1}^{n}\mu_j\left(1-\mathtt{s}\xi\right)^{\frac{1}{j}}}{\sum\limits_{j=1}^{n}\mu_j}\Delta\left(\omega_y\right), \tag{3.1}$$

for all $\mathtt{s} \in [0,1]$, $\xi \in [0,1]$, $\mathtt{m} \in (0,1]$, and $\omega_x, \omega_y \in X^o$, where $\mu_j \ge 0$ for $j = \overline{1,n}$ with $\sum_{j=1}^{n}\mu_j > 0$.

Remark 3.1.
- *If $n = 1$ and $\mathtt{m} = 1$, then Definition 3.1 reduces to the definition in [39].*
- *If $n = 1$, $\mathtt{m} = 1$, and $\mathtt{s} = 1$, then Definition 3.1 reduces to the definition in [34].*
- *If $n = 1$, $\mathtt{m} = 1$, and $\nabla(\omega_y,\omega_x,\mathtt{m}) = \omega_y - \mathtt{m}\omega_x$, then Definition 3.1 reduces to the deeinition in [39].*

We now present examples illustrating the nature of the n-fractional polynomial $\mathtt{s}$-like $\mathtt{m}$-preinvex mapping.

Example 3.1. *Let $X^o = [0,1]$, $n = 1$, $\mu_1 = 1$, $\mathtt{s} = 0$, and $\mathtt{m} = 1$. For $\omega_x, \omega_y \in X^o$, define*

$$\nabla(\omega_y,\omega_x,\mathtt{m}) = \omega_y - \omega_x,$$

so that

$$\mathtt{m}\omega_x + \xi\,\nabla(\omega_y,\omega_x,\mathtt{m}) = (1-\xi)\omega_x + \xi\omega_y, \qquad \xi \in [0,1].$$

Consider the mapping $\Delta : X^o \to \mathbb{R}$ defined by $\Delta(x) = x^2$, $x \in X^o$. Since $\Delta(x) \ge 0$ on X^o, it is non-negative. Under the above choices, inequality (3.1) *becomes*

$$\Delta\Big((1-\xi)\omega_x + \xi\omega_y\Big) \le \Delta(\omega_x) + \Delta(\omega_y), \qquad \forall\,\xi \in [0,1].$$

Indeed,

$$\Big((1-\xi)\omega_x + \xi\omega_y\Big)^2 \le (1-\xi)^2\omega_x^2 + \xi^2\omega_y^2 \le \omega_x^2 + \omega_y^2,$$

which confirms that $\Delta(x) = x^2$ satisfies Definition 3.1. Hence, $\Delta(x) = x^2$ is an example of an n-fractional polynomial $\mathtt{s}$-like $\mathtt{m}$-preinvex mapping.

Example 3.2. *Suppose $\Delta(x) = x^2$ with conditions $\mathtt{s} = 0.4$, $n = 2$, $\omega_x = 1$, $\omega_y = 3$, $\mathtt{m} = 1$, and $\xi = 0.4$. According to* (3.1),

$$\Delta\left(\mathtt{m}\omega_x + \xi\,\nabla\left(\omega_y, \omega_x, \mathtt{m}\right)\right) \leq \frac{\sum_{j=1}^{2} \mu_j \left(1 - \mathtt{s}\left(1-\xi\right)\right)^{\frac{1}{j}}}{\sum_{j=1}^{2} \mu_j} \Delta\left(\mathtt{m}\omega_x\right) + \frac{\sum_{j=1}^{2} \mu_j \left(1 - \mathtt{s}\xi\right)^{\frac{1}{j}}}{\sum_{j=1}^{2} \mu_j} \Delta\left(\omega_y\right).$$

With $\mu_1 = 1$, $\mu_2 = 2$, the left-hand side gives

$$\Delta\left(1 + 0.4 \cdot \nabla\left(3, 1, 1\right)\right) = \Delta(1.8) = 1.8^2 = 3.24,$$

and the right-hand side evaluates to

$$\frac{1 \cdot (1 - 0.4 \cdot 0.6)^1 + 2 \cdot (1 - 0.4 \cdot 0.6)^{\frac{1}{2}}}{3} \cdot 1^2 + \frac{1 \cdot (1 - 0.4 \cdot 0.4)^1 + 2 \cdot (1 - 0.4 \cdot 0.4)^{\frac{1}{2}}}{3} \cdot 3^2 = 8.35,$$

so that $3.24 \leq 8.35$, *which confirms the inequality.*

Example 3.3. *Let $X^o = [0, 1]$, $n = 1$, $\mu_1 = 1$, $\mathtt{s} = 0$, $\mathtt{m} = 1$, $\omega_x = 0.2$, $\omega_y = 0.8$, and $\Delta(x) = x^2$. Then* $\nabla(\omega_y, \omega_x, \mathtt{m}) = \omega_y - \omega_x = 0.6$ *and*

$$\mathtt{m}\omega_x + \xi\,\nabla(\omega_y, \omega_x, \mathtt{m}) = (1 - \xi)\omega_x + \xi\omega_y = 0.2 + 0.6\xi.$$

With $n = 1$, $\mu_1 = 1$, and $\mathtt{s} = 0$, inequality (3.1) *reduces to*

$$\Delta\Big((1 - \xi)\omega_x + \xi\omega_y\Big) \leq \Delta(\omega_x) + \Delta(\omega_y), \qquad \xi \in [0, 1].$$

The constant right-hand side is

$$\Delta(\omega_x) + \Delta(\omega_y) = (0.2)^2 + (0.8)^2 = 0.04 + 0.64 = 0.68.$$

Sample values of the left-hand side:

ξ	$(1-\xi)\omega_x + \xi\omega_y$	$\Delta\Big((1-\xi)\omega_x + \xi\omega_y\Big)$	*Inequality holds?*
0	0.20	0.0400	$0.0400 \leq 0.68$ ✓
0.25	0.35	0.1225	$0.1225 \leq 0.68$ ✓
0.50	0.50	0.2500	$0.2500 \leq 0.68$ ✓
0.75	0.65	0.4225	$0.4225 \leq 0.68$ ✓
1	0.80	0.6400	$0.6400 \leq 0.68$ ✓

All computed left-hand-side values satisfy the inequality, confirming it numerically for the tested values of ξ (and in fact for all $\xi \in [0, 1]$ under these choices).

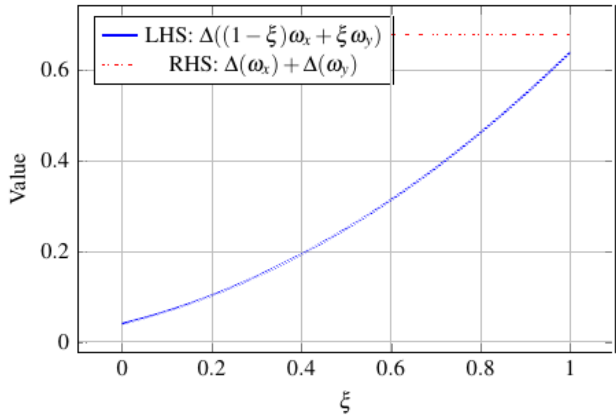

Figure 1. Graphical verification of the inequality $\Delta((1-\xi)\omega_x+\xi\omega_y)\leq\Delta(\omega_x)+\Delta(\omega_y)$.

4. H-H-Type Inequality via n-Fractional Polynomial $\mathtt{s}$-Like $\mathtt{m}$-Preinvex Mapping

The H-H inequality is one of the classical results relating the value of a convex function at the midpoint of an interval to its integral mean over the entire interval. It asserts that the midpoint value is never greater than the integral average, which in turn does not exceed the average of the endpoint values. In recent years, researchers have extended this result to fractional integrals and functions of more than one variable, along with applications that produce sharper bounds. The inequality provides reliable tools for estimating and bounding function values in a wide range of applied settings, and it is now widely used in estimates.

We now establish a new extension of the H-H inequality for the n-fractional polynomial $\mathtt{s}$-like $\mathtt{m}$-preinvex mapping.

Theorem 4.1. *Let $A^o\subseteq\mathbb{R}$ be an open invex subset with respect to $\nabla: A^o\times A^o\times(0,1]\to\mathbb{R}$, and let $\omega_x,\omega_y\in A^o$ with $\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})\geq\mathtt{m}\omega_x$. Suppose that $\Delta:[\mathtt{m}\omega_x,\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})]\to\mathbb{R}$ is an n-fractional polynomial $\mathtt{s}$-like $\mathtt{m}$-preinvex mapping satisfying the extended Condition C, with $n\in\mathbb{N}$, $\mu_j\geq 0$ $(j=\overline{1,n})$, $\sum_{j=1}^{n}\mu_j>0$, $\mathtt{s}\in[0,1]$, $\gamma\in(0,1]$, and $k>0$. Then*

$$\begin{aligned}
&\frac{\sum_{j=1}^{n}\mu_j}{\sum_{j=1}^{n}\mu_j\left(1-\frac{\mathtt{s}}{2}\right)^{\frac{1}{j}}}\Delta\left(\frac{2\mathtt{m}\omega_x+\nabla\left(\omega_y,\omega_x,\mathtt{m}\right)}{2}\right)\\
&\leq\frac{\Gamma_k\left(\gamma+k\right)}{\nabla^{\frac{\gamma}{k}}\left(\omega_y,\omega_x,\mathtt{m}\right)}\left\{\zeta^{\gamma,k}_{\mathtt{m}\omega_x^+}\Delta\left(\mathtt{m}\omega_x+\nabla\left(\omega_y,\omega_x,\mathtt{m}\right)\right)+\zeta^{\gamma,k}_{\left(\mathtt{m}\omega_x+\nabla\left(\omega_y,\omega_x,\mathtt{m}\right)\right)^-}\Delta\left(\mathtt{m}\omega_x\right)\right\}\\
&\leq\left[\Delta\left(\mathtt{m}\omega_x\right)+\Delta\left(\mathtt{m}\omega_x+\nabla\left(\omega_y,\omega_x,\mathtt{m}\right)\right)\right]\\
&\times\int_0^1\xi^{\frac{\gamma}{k}-1}\left\{\frac{\sum_{j=1}^{n}\mu_j\left(1-\mathtt{s}\xi\right)^{\frac{1}{j}}}{\sum_{j=1}^{n}\mu_j}+\frac{\sum_{j=1}^{n}\mu_j\left(1-\mathtt{s}\left(1-\xi\right)\right)^{\frac{1}{j}}}{\sum_{j=1}^{n}\mu_j}\right\}d\xi. \qquad (4.1)
\end{aligned}$$

Proof. From Definition 3.1, for any $x, y \in A^o$, we have

$$\Delta\left(\mathtt{m}x + \frac{\nabla(y, x, \mathtt{m})}{2}\right) \leq \frac{\sum_{j=1}^{n} \mu_j \left(1 - \frac{\mathsf{s}}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} \Delta(\mathtt{m}x) + \frac{\sum_{j=1}^{n} \mu_j \left(1 - \frac{\mathsf{s}}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} \Delta(y)$$
$$= \frac{\sum_{j=1}^{n} \mu_j \left(1 - \frac{\mathsf{s}}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} \left[\Delta(\mathtt{m}x) + \Delta(y)\right]. \tag{4.2}$$

Substituting $x = \mathtt{m}\omega_x + (1-\xi)\nabla(\omega_y, \omega_x, \mathtt{m})$ and $y = \mathtt{m}\omega_x + \xi\,\nabla(\omega_y, \omega_x, \mathtt{m})$ into (4.2), and applying the extended Condition C, we obtain

$$\Delta\left(\frac{2\mathtt{m}\omega_x + \nabla(\omega_y, \omega_x, \mathtt{m})}{2}\right)$$
$$\leq \frac{\sum_{j=1}^{n} \mu_j \left(1 - \frac{\mathsf{s}}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} \left[\Delta\Big(\mathtt{m}\omega_x + (1-\xi)\nabla(\omega_y, \omega_x, \mathtt{m})\Big) + \Delta\Big(\mathtt{m}\omega_x + \xi\,\nabla(\omega_y, \omega_x, \mathtt{m})\Big)\right]. \tag{4.3}$$

Multiplying both sides by $\xi^{\frac{\gamma}{k}-1}$ and integrating over $\xi \in [0,1]$, we get

$$\frac{k}{\gamma}\Delta\left(\frac{2\mathtt{m}\omega_x + \nabla(\omega_y, \omega_x, \mathtt{m})}{2}\right) \leq \frac{\sum_{j=1}^{n} \mu_j \left(1 - \frac{\mathsf{s}}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j}$$
$$\times \frac{k\,\Gamma_k(\gamma)}{\nabla^{\frac{\gamma}{k}}(\omega_y, \omega_x, \mathtt{m})} \left\{\zeta^{\gamma,k}_{\mathtt{m}\omega_x^+} \Delta\Big(\mathtt{m}\omega_x + \nabla(\omega_y, \omega_x, \mathtt{m})\Big) + \zeta^{\gamma,k}_{(\mathtt{m}\omega_x + \nabla(\omega_y, \omega_x, \mathtt{m}))^-} \Delta(\mathtt{m}\omega_x)\right\}, \tag{4.4}$$

which establishes the left-hand inequality in (4.1).

For the right-hand inequality, applying Definition 3.1 to each term gives

$$\Delta\Big(\mathtt{m}\omega_x + (1-\xi)\nabla(\omega_y, \omega_x, \mathtt{m})\Big)$$
$$\leq \frac{\sum_{j=1}^{n} \mu_j \left(1 - \mathsf{s}(1-\xi)\right)^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} \Delta\Big(\mathtt{m}\omega_x + \nabla(\omega_y, \omega_x, \mathtt{m})\Big) + \frac{\sum_{j=1}^{n} \mu_j (1 - \mathsf{s}\xi)^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} \Delta(\mathtt{m}\omega_x), \tag{4.5}$$

$$\Delta\Big(\mathtt{m}\omega_x + \xi\,\nabla(\omega_y, \omega_x, \mathtt{m})\Big)$$
$$\leq \frac{\sum_{j=1}^{n} \mu_j (1 - \mathsf{s}\xi)^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} \Delta\Big(\mathtt{m}\omega_x + \nabla(\omega_y, \omega_x, \mathtt{m})\Big) + \frac{\sum_{j=1}^{n} \mu_j \left(1 - \mathsf{s}(1-\xi)\right)^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} \Delta(\mathtt{m}\omega_x). \tag{4.6}$$

Adding these two inequalities yields

$$\begin{aligned}
&\Delta\left(\mathtt{m}\omega_x + (1-\xi)\nabla(\omega_y,\omega_x,\mathtt{m})\right) + \Delta\left(\mathtt{m}\omega_x + \xi\,\nabla(\omega_y,\omega_x,\mathtt{m})\right)\\
&\le \left[\Delta(\mathtt{m}\omega_x) + \Delta\left(\mathtt{m}\omega_x + \nabla(\omega_y,\omega_x,\mathtt{m})\right)\right]\left\{\frac{\sum\limits_{j=1}^{n}\mu_j(1-\mathsf{s}(1-\xi))^{\frac{1}{j}}}{\sum\limits_{j=1}^{n}\mu_j} + \frac{\sum\limits_{j=1}^{n}\mu_j(1-\mathsf{s}\xi)^{\frac{1}{j}}}{\sum\limits_{j=1}^{n}\mu_j}\right\}. \qquad (4.7)
\end{aligned}$$

Multiplying both sides by $\xi^{\frac{\gamma}{k}-1}$ and integrating over $\xi \in [0,1]$ gives

$$\begin{aligned}
&\frac{k\,\Gamma_k(\gamma)}{\nabla^{\frac{\gamma}{k}}(\omega_y,\omega_x,\mathtt{m})}\left\{\zeta^{\gamma,k}_{\mathtt{m}\omega_x^{+}}\Delta\left(\mathtt{m}\omega_x + \nabla(\omega_y,\omega_x,\mathtt{m})\right) + \zeta^{\gamma,k}_{(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m}))^{-}}\Delta(\mathtt{m}\omega_x)\right\}\\
&\le \left[\Delta(\mathtt{m}\omega_x) + \Delta\left(\mathtt{m}\omega_x + \nabla(\omega_y,\omega_x,\mathtt{m})\right)\right]\\
&\quad\times\int_0^1 \xi^{\frac{\gamma}{k}-1}\left\{\frac{\sum\limits_{j=1}^{n}\mu_j(1-\mathsf{s}(1-\xi))^{\frac{1}{j}}}{\sum\limits_{j=1}^{n}\mu_j} + \frac{\sum\limits_{j=1}^{n}\mu_j(1-\mathsf{s}\xi)^{\frac{1}{j}}}{\sum\limits_{j=1}^{n}\mu_j}\right\}d\xi. \qquad (4.8)
\end{aligned}$$

Combining (4.4) and (4.8) completes the proof of (4.1). □

Corollary 4.1. *If* $\mathsf{s} = 1$ *in Theorem 4.1, we obtain the following H-H type inequality pertaining to the n-fractional polynomial* s*-like* $\mathtt{m}$*-preinvex mapping and k-fractional integral operators:*

$$\begin{aligned}
&\frac{\sum\limits_{j=1}^{n}\mu_j}{\sum\limits_{j=1}^{n}\mu_j\left(\frac{1}{2}\right)^{\frac{1}{j}}}\Delta\left(\frac{2\mathtt{m}\omega_x + \nabla(\omega_y,\omega_x,\mathtt{m})}{2}\right)\\
&\le \frac{\Gamma_k(\gamma+k)}{\nabla^{\frac{\gamma}{k}}(\omega_y,\omega_x,\mathtt{m})}\left\{\zeta^{\gamma,k}_{\mathtt{m}\omega_x^{+}}\Delta\left(\mathtt{m}\omega_x + \nabla(\omega_y,\omega_x,\mathtt{m})\right) + \zeta^{\gamma,k}_{(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m}))^{-}}\Delta(\mathtt{m}\omega_x)\right\}\\
&\le \frac{\left[\Delta(\mathtt{m}\omega_x) + \Delta\left(\mathtt{m}\omega_x + \nabla(\omega_y,\omega_x,\mathtt{m})\right)\right]}{\sum\limits_{j=1}^{n}\mu_j}\int_0^1\sum_{j=1}^{n}\mu_j\,\xi^{\frac{\gamma}{k}-1}\left\{(1-\xi)^{\frac{1}{j}} + \xi^{\frac{1}{j}}\right\}d\xi.
\end{aligned}$$

Corollary 4.2. *If* $\mathsf{s} = \mathtt{m} = 1$ *in Theorem 4.1, we obtain the following H-H type inequality:*

$$\begin{aligned}
&\frac{\sum\limits_{j=1}^{n}\mu_j}{\sum\limits_{j=1}^{n}\mu_j\left(\frac{1}{2}\right)^{\frac{1}{j}}}\Delta\left(\frac{2\omega_x + \nabla(\omega_y,\omega_x)}{2}\right)\\
&\le \frac{\Gamma_k(\gamma+k)}{\nabla^{\frac{\gamma}{k}}(\omega_y,\omega_x)}\left\{\zeta^{\gamma,k}_{\omega_x^{+}}\Delta\left(\omega_x + \nabla(\omega_y,\omega_x)\right) + \zeta^{\gamma,k}_{(\omega_x+\nabla(\omega_y,\omega_x))^{-}}\Delta(\omega_x)\right\}\\
&\le \frac{\left[\Delta(\omega_x) + \Delta\left(\omega_x + \nabla(\omega_y,\omega_x)\right)\right]}{\sum\limits_{j=1}^{n}\mu_j}\int_0^1\sum_{j=1}^{n}\mu_j\,\xi^{\frac{\gamma}{k}-1}\left\{(1-\xi)^{\frac{1}{j}} + \xi^{\frac{1}{j}}\right\}d\xi.
\end{aligned}$$

Corollary 4.3. *If $k = 1$ in Corollary 4.2, we obtain the following H-H type inequality pertaining to Riemann–Liouville fractional integral operators:*

$$
\begin{aligned}
&\frac{\sum\limits_{j=1}^{n}\mu_j}{\sum\limits_{j=1}^{n}\mu_j\left(\frac{1}{2}\right)^{\frac{1}{j}}}\Delta\left(\frac{2\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})}{2}\right)\\
&\leq \frac{\Gamma(\gamma+1)}{\nabla^{\gamma}(\omega_y,\omega_x,\mathtt{m})}\left\{\zeta^{\gamma}_{\mathtt{m}\omega_x^{+}}\Delta\left(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})\right)+\zeta^{\gamma}_{(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m}))^{-}}\Delta(\mathtt{m}\omega_x)\right\}\\
&\leq \frac{\left[\Delta(\mathtt{m}\omega_x)+\Delta\left(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})\right)\right]}{\sum\limits_{j=1}^{n}\mu_j}\int_0^1\sum_{j=1}^{n}\mu_j\,\xi^{\gamma-1}\left\{(1-\xi)^{\frac{1}{j}}+\xi^{\frac{1}{j}}\right\}d\xi.
\end{aligned}
$$

Corollary 4.4. *If $k = \mathtt{m} = 1$ in Corollary 4.2, we obtain the following H-H type inequality:*

$$
\begin{aligned}
&\frac{\sum\limits_{j=1}^{n}\mu_j}{\sum\limits_{j=1}^{n}\mu_j\left(\frac{1}{2}\right)^{\frac{1}{j}}}\Delta\left(\frac{2\omega_x+\nabla(\omega_y,\omega_x)}{2}\right)\\
&\leq \frac{\Gamma(\gamma+1)}{\nabla^{\gamma}(\omega_y,\omega_x)}\left\{\zeta^{\gamma}_{\omega_x^{+}}\Delta\left(\omega_x+\nabla(\omega_y,\omega_x)\right)+\zeta^{\gamma}_{(\omega_x+\nabla(\omega_y,\omega_x))^{-}}\Delta(\omega_x)\right\}\\
&\leq \frac{\left[\Delta(\omega_x)+\Delta\left(\omega_x+\nabla(\omega_y,\omega_x)\right)\right]}{\sum\limits_{j=1}^{n}\mu_j}\int_0^1\sum_{j=1}^{n}\mu_j\,\xi^{\gamma-1}\left\{(1-\xi)^{\frac{1}{j}}+\xi^{\frac{1}{j}}\right\}d\xi.
\end{aligned}
$$

Remark 4.1. *Taking $\gamma = 1$, $\mathtt{m} = 1$, and $n = 1$ in Corollary 4.3 recovers inequality* (2.9).

5. Application to Matrix Analysis

The use of matrix inequalities has grown steadily, mainly because many modern mathematical models involve operators or matrix-valued functions rather than simple scalar expressions. Earlier work by Sababheh [55] showed that a certain mapping on the set of positive definite complex matrices is convex, a result that opened the door for extending classical inequalities to the matrix setting. Building on this observation and Theorem 4.1, it becomes possible to obtain new estimates for matrix expressions involving fractional powers. These developments are not merely theoretical; they arise naturally in numerical linear algebra, quantum information, control system design, and signal processing, where questions of stability, robustness, and optimization frequently rely on matrix bounds.

Application. Let $\mathbb{M}_n$ denote the algebra of $n \times n$ complex matrices and $\mathbb{M}_n^+$ the strictly positive matrices in $\mathbb{M}_n$, i.e., $\mathfrak{D} \in \mathbb{M}_n^+$ if $\langle \mathfrak{D}x, x\rangle > 0$ for all nonzero $x \in \mathbb{C}^n$.

In [55], Sababheh showed that the function

$$\Delta(u) = \left\|\mathfrak{D}^u X\mathfrak{B}^{1-u} + \mathfrak{D}^{1-u} X\mathfrak{B}^{u}\right\|, \quad \mathfrak{D}, \mathfrak{B} \in \mathbb{M}_n^+,\ X \in \mathbb{M}_n,$$

is convex on $[0,1]$. Applying Theorem 4.1 to this function yields

$$\begin{aligned}
&\frac{\sum_{j=1}^{n}\mu_j}{\sum_{j=1}^{n}\mu_j\left(1-\frac{\mathsf{s}}{2}\right)^{\frac{1}{j}}}\left\|\mathfrak{D}^{\frac{2\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})}{2}}X\mathfrak{B}^{1-\frac{2\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})}{2}}+\mathfrak{D}^{1-\frac{2\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})}{2}}X\mathfrak{B}^{\frac{2\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})}{2}}\right\|\\
&\leq\frac{\Gamma_k(\gamma+k)}{\nabla^{\frac{\gamma}{k}}(\omega_y,\omega_x,\mathtt{m})}\\
&\quad\times\left\{\zeta^{\gamma,k}_{\mathtt{m}\omega_x^+}\left\|\mathfrak{D}^{\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})}X\mathfrak{B}^{1-(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m}))}+\mathfrak{D}^{1-(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m}))}X\mathfrak{B}^{\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})}\right\|\right.\\
&\quad\left.+\zeta^{\gamma,k}_{(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m}))^-}\left\|\mathfrak{D}^{\mathtt{m}\omega_x}X\mathfrak{B}^{1-\mathtt{m}\omega_x}+\mathfrak{D}^{1-\mathtt{m}\omega_x}X\mathfrak{B}^{\mathtt{m}\omega_x}\right\|\right\}\\
&\leq\left[\left\|\mathfrak{D}^{\mathtt{m}\omega_x}X\mathfrak{B}^{1-\mathtt{m}\omega_x}+\mathfrak{D}^{1-\mathtt{m}\omega_x}X\mathfrak{B}^{\mathtt{m}\omega_x}\right\|\right.\\
&\quad\left.+\left\|\mathfrak{D}^{\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})}X\mathfrak{B}^{1-(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m}))}+\mathfrak{D}^{1-(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m}))}X\mathfrak{B}^{\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})}\right\|\right]\\
&\quad\times\int_0^1\xi^{\frac{\gamma}{k}-1}\left\{\frac{\sum_{j=1}^{n}\mu_j(1-\mathsf{s}\xi)^{\frac{1}{j}}}{\sum_{j=1}^{n}\mu_j}+\frac{\sum_{j=1}^{n}\mu_j(1-\mathsf{s}(1-\xi))^{\frac{1}{j}}}{\sum_{j=1}^{n}\mu_j}\right\}d\xi.
\end{aligned}$$

6. Applications to Bivariate Means

Bivariate analysis in the context of mathematical inequalities focuses on relationships between expressions involving two positive variables, such as special means. By examining quantities like the arithmetic and harmonic means and establishing well-known relationships between them, researchers can obtain useful bounds that support the development of more general inequalities.

We recall the following means of two positive numbers ω_x,ω_y with $\omega_x<\omega_y$ (see [**?**]):

(1) The arithmetic mean:
$$\mathcal{A}(\omega_x,\omega_y)=\frac{\omega_x+\omega_y}{2}.$$
(2) The harmonic mean:
$$\mathcal{H}(\omega_x,\omega_y)=\frac{2\omega_x\omega_y}{\omega_x+\omega_y}.$$

The following well-known relationship holds:
$$\mathcal{H}(\omega_x,\omega_y)\leq\mathcal{G}(\omega_x,\omega_y)\leq\mathcal{A}(\omega_x,\omega_y).$$

This kind of analysis is particularly useful when working with fractional integral operators, as many results in this area depend on how two-variable expressions behave under different conditions. By comparing various bivariate means, one can obtain tighter bounds and clearer estimates, which often lead to improvements over previously known inequalities. These comparisons also form a basic step in extending classical results to more general settings, and bivariate mean relations have become valuable tools for tackling problems that naturally appear in applied mathematics, physics, and engineering.

Proposition 6.1. *Suppose that* $0 < \omega_x < \omega_y$ *and* $\mathsf{s} \in [0,1]$. *Then*

$$\begin{aligned}
&\frac{\sum_{j=1}^{n} \mu_j}{\sum_{j=1}^{n} \mu_j \left(1-\frac{\mathsf{s}}{2}\right)^{\frac{1}{j}}} \mathcal{A}\Big(2\mathtt{m}\omega_x,\, \nabla(\omega_y,\omega_x,\mathtt{m})\Big) \\
&\le \frac{\Gamma_k(\gamma+k)}{\nabla^{\frac{\gamma}{k}}(\omega_y,\omega_x,\mathtt{m})} \Big\{ \zeta^{\gamma,k}_{\mathtt{m}\omega_x^+} \Big[2\mathcal{A}\Big(\mathtt{m}\omega_x,\, \nabla(\omega_y,\omega_x,\mathtt{m})\Big)\Big] + \zeta^{\gamma,k}_{(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m}))^-}(\mathtt{m}\omega_x) \Big\} \\
&\le 2\mathcal{A}\Big(\mathtt{m}\omega_x,\, \mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})\Big) \int_0^1 \xi^{\frac{\gamma}{k}-1} \left\{ \frac{\sum_{j=1}^{n} \mu_j (1-\mathsf{s}\xi)^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} + \frac{\sum_{j=1}^{n} \mu_j (1-\mathsf{s}(1-\xi))^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} \right\} d\xi.
\end{aligned}$$

Proof. If $\Delta(u) = u$ for $u > 0$ in Theorem (4.1), then we obtain the Proposition 6.1. □

Proposition 6.2. *Suppose that* $0 < \omega_x < \omega_y$ *and* $\mathsf{s} \in [0,1]$. *Then*

$$\begin{aligned}
&\frac{\sum_{j=1}^{n} \mu_j}{\sum_{j=1}^{n} \mu_j \left(1-\frac{\mathsf{s}}{2}\right)^{\frac{1}{j}}} \mathcal{A}^{-1}\Big(2\mathtt{m}\omega_x,\, \nabla(\omega_y,\omega_x,\mathtt{m})\Big) \\
&\le \frac{\Gamma_k(\gamma+k)}{\nabla^{\frac{\gamma}{k}}(\omega_y,\omega_x,\mathtt{m})} \Big\{ \zeta^{\gamma,k}_{\mathtt{m}\omega_x^+} \Big[2\mathcal{A}^{-1}\Big(\mathtt{m}\omega_x,\, \nabla(\omega_y,\omega_x,\mathtt{m})\Big)\Big] + \zeta^{\gamma,k}_{(\mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m}))^-}(\mathtt{m}\omega_x^{-1}) \Big\} \\
&\le 2\mathcal{H}^{-1}\Big(\mathtt{m}\omega_x,\, \mathtt{m}\omega_x+\nabla(\omega_y,\omega_x,\mathtt{m})\Big) \\
&\quad \times \int_0^1 \xi^{\frac{\gamma}{k}-1} \left\{ \frac{\sum_{j=1}^{n} \mu_j (1-\mathsf{s}\xi)^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} + \frac{\sum_{j=1}^{n} \mu_j (1-\mathsf{s}(1-\xi))^{\frac{1}{j}}}{\sum_{j=1}^{n} \mu_j} \right\} d\xi.
\end{aligned}$$

Proof. If $\Delta(u) = \frac{1}{u}$ for $u > 0$ in Theorem (4.1), then we obtain the Proposition 6.2. □

7. Conclusion

The field of fractional calculus offers great capabilities to attack problems in many fields of science. Important inequalities in physics, functional analysis, optimization, and the statistical theory are integral inequalities. Within this paper, we defined the n-fractional polynomial s-like $\mathtt{m}$-preinvex mapping and a new type of H-H inequality which is based on the operator of k-fractional operator. The findings were also demonstrated using the applications in the analysis of matrices and bivariate means. The findings can be used to provide new research directions in the study of fractional calculus and also the application of fractional calculus in mathematical modelling, optimization and applied sciences by extension of the known fractional integral operators.

Future Scope: The findings in this research leave a couple of bright avenues to the future investigations. Other generalized types of the fractional integrals like the AtanganaBaleanu, CaputoFabrizio and Hadamard types can be used to further study the newly introduced category of n-fractional polynomial s-like $\mathtt{m}$-preinvex, and respective HermiteHadamard type inequalities. Besides that, the gained inequalities could be extrapolated to the multidimensional situation and multivariate mappings that could enhance their theoretical utility. Alternative theme of interest is to examine these disparities with respect to other generalized forms of convexity, and apply exponentially convex, strongly preinvex, and stochastic preinvex functions. The results received can be also used in optimization theory whereby, in the theory, new optimum conditions and error bounds of nonlinear programming problem are determined. In addition to that, the inequalities can be employed to generate superior methods of numerical approximation and estimates of numerical integration errors. Future research would also be based on generalizing the findings to the operator inequalities on Hilbert and Banach spaces. Besides, to get additional revelations in the operator theory, one can use the applications of matrix analysis and functional analysis. The extensions can also be applied in dynamical system modeling and modeling of equations of the fractional form. Overall, the proposed framework may act as the handy foundation to further develop the theory of fractional calculus and inequalities and their use in the applied mathematics and other fields.

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