Int. J. Anal. Appl. (2026), 24:204

International Journal of Analysis and Applications

New Fractional Approach of Hermite-Hadamard-Type Inequalities with Applications to Information Divergence and Entropy Bounds

Hijaz Ahmad[1,2,3], Ali Asghar[4], Saima Bhatti[4], Muhammad Tariq[5], Tonguc Cagin[6], Waqar Afzal[7,8], Evren Hincal[5,9], Ilyas Khan[10,11,12], Waleed Mohammed Abdelfattah[13]

[1]*Irfan Suat Gunsel Operational Research Institute, Near East University, Nicosia/TRNC, 99138 Mersin 10, Turkey*

[2]*Sustainability Competence Centre, Széchenyi István University, Egyetem tér 1, H-9026 Győr, Hungary*

[3]*Department of Mathematics, College of Science, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul 02841, South Korea*

[4]*Department of Basic Sciences and Related Studies, Mehran UET, Jamshoro, Sindh, Pakistan*

[5]*Mathematics Research Center, Near East University, Near East Boulevard, PC: 99138, Nicosia/Mersin 10, Turkey*

[6]*College of Business Administration, American University of the Middle East, Kuwait*

[7]*Center for Theoretical Physics, Khazar University, 41 Mehseti Str., Baku, AZ1096, Azerbaijan*

[8]*International Center for Interdisciplinary Research in Sciences, The University of Lahore, Lahore 54792, Pakistan*

[9] *Research Center of Applied Mathematics, Khazar University, Baku, Azerbaijan*

[10]*Department of Mathematical Sciences, Saveetha School of Engineering, SIMATS, Chennai, Tamil Nadu, India*

[11]*Hourani Center for Applied Scientific Research, Al-Ahliyya Amman University, Amman, Jordan*

[12]*Department of Mathematics, College of Science Al-Zulfi, Majmaah University, Al-Majmaah 11952, Saudi Arabia*

[13]*College of Engineering, University of Business and Technology, Jeddah 23435, Saudi Arabia*

**Corresponding author:* mtariqkhan2187@gmail.com

Abstract. Fractional integral operators play a vital role in establishing generalized forms of mathematical inequalities. These operators provide effective tools for modeling various scientific and engineering processes such as fracture mechanics, elasticity, heat transfer, viscoelastic deformation, and the behavior of continuous populations. In this

Received: Apr. 15, 2026.

2020 *Mathematics Subject Classification.* 26A51, 26A33, 26D07, 26D10, 26D15.

Key words and phrases. Convex Function, Hermite-Hadamard inequality, Raina's function, Mittag-Leffler function .

https://doi.org/10.28924/2291-8639-24-2026-204

ISSN: 2291-8639

study, we investigate a new class of Hermite–Hadamard type inequalities and verify their numerical validity. By employing a novel equality together with Hölder's inequality, we derive several extensions of Hermite–Hadamard type inequalities through generalized convexity involving Raina's function within the framework of fractional integral operators. Moreover, we present applications related to information divergence and entropy bounds. The results obtained here constitute significant advancements and generalizations of existing findings in the literature.

1. Introduction

Convexity theory has been crucial to the development of numerous subfields in contemporary mathematics. In the decade preceding, numerous academics and researchers have attempted to integrate novel concepts into fractional analysis in an effort to add a new facet with diverse features to the domain of mathematical analysis and numerical techniques. Engineering [1], finance [2], economics [3], and optimization [4] are just a few of the domains in which the idea of convexity locates extensive use.

There is an amazing connection between the theory of inequalities and the topic of convexity. Convex functions result in a large number of well-known and useful inequalities. The Jensen and Hadamard inequalities are two well-known inequalities in the literature that examine and explain the geometrical meaning of convex functions. They are used in many fields, such as optimization, probability theory, and information theory. For numerical methods such as the trapezoidal rule, Simpson's rule, and others, these inequalities are essential, particularly when determining the error limits. For the literature, see [5–12].

The analysis of fractional integration and its utilization is referred to as "fractional calculus." In the contemporary period, the concepts of inequality and fractional assessment have coevolved. One of the fundamental ideas and elements of applied sciences is the evaluation of fractional inequality. Researchers recommend that students think about using and applying the fractional operator to solve problems and issues in the real world. The claimed assignment of the integration of an arbitrary non-integer order is known as fractional calculus. Due to its practical uses, it has recently continued to pique the curiosity and concentration of many scholars. Fractional calculus has multiple uses in the disciplines of fluid flow [13], mathematical biology [14], epidemiology [15], optimal control [16, 17], physics [18], control systems [19], transform theory [20], nanotechnology [21], and modeling [22, 23]. For the literature, see [24–33].

Information divergence and entropy bounds are essential tools for describing uncertainty and evaluating the performance of information systems. Their link with integral inequalities appears naturally, since many divergence measures rely on the properties of convex functions. More accurate and more adjustable extensions for these quantities can be derived by constructing fractional notions of H-H type inequalities.

The amazing inspiration of this work is improving the analytical capabilities of fractional operators toward larger classes of mathematical inequalities. Although they are considered mandatory, classical Hermite-Hadamard type findings tend to have limited applicability with addressing problems that display nonlocal or memory-dependent behavior. A natural foundation for handling such complexity is provided by fractional calculus, which enables more accurate depictions of mechanics, thermodynamics, and information theory processes. Building on this framework, the current research attempts to develop and investigate novel Hermite-Hadamard inequality variations in the context of generalized convexity related to Raina's function. These results not only enhance the theoretical depth of fractional inequalities but also create a bridge to practical applications by incorporating measures such as information divergence and entropy bounds, which are crucial in evaluating uncertainty and informational efficiency in modern systems. For the literature see [34–45]

This study presents a new model of representing the Hermite-Hadamard type inequalities using the concept of generalized convexity of functions of Raina by employing the concepts of the fractional integral operators in conjunction with the concept of the generalized convexity. The suggested method is different to the past studies that formed novel integrative identities, offering more precise and flexible estimations to those of the classical. As opposed to the previous studies who only concentrated on the normal convexity, this research combines the fractional structures to explain the memory and nonlocal effects which are naturally present in models in the real world. In addition, that the connection between these inequalities and measures of information divergence and entropy provides a new theoretical vantage point, which expands the scope of fractional analysis to the information theory. All these contributions are a clear improvement of the current body of literature in terms of depth and applicability.

This work is arranged as below. In Section 2, we address and remember certain essential ideas and definitions that establish the framework of our following research. We construct a new notion in Section 3, called generalized m-convex incorporating Raina's mapping (GmCRM). In Section 4, we employ a GmCRM to formulate a novel sort of H-H (Hermite-Hadamard) inequality related to the Riemann-Liouville fractional integral operator (RLFIO). We set up a new equality pertaining to GmCRM in Section 5. We devoted ourselves to generating some refinement of H-H-type inequalities in order to achieve this property. Applications related to entropy bounds and information divergence are analyzed in Section 6. In the last Section 7, we offer a succinct conclusion and suggest some possible future study topics.

2. Preliminaries

This foundational section is mandatory since it integrates many core concepts, definitions, and meaningful outcomes. By presenting essential details in a systematic and unified way, it strives to hold the reader's focus in conjunction with assuring consistency and clarity. We begin by revisiting key notions such as convexity, the Hermite–Hadamard inequality, the Mittag–Leffler function (MLF), generalized convex sets, generalized convex functions, and Condition-A. To conclude, we highlight that the RLFIO will serve as a fundamental operator in our subsequent analysis.

Definition 2.1. *[46] The following inequality is said to be convex, if*

$$\mathcal{G}\left(u\mathfrak{u}_x + (1-u)\,\mathfrak{u}_y\right) \leq u\mathcal{G}\left(\mathfrak{u}_x\right) + (1-u)\,\mathcal{G}\left(\mathfrak{u}_y\right), \tag{2.1}$$

holds for all $\mathfrak{u}_x, \mathfrak{u}_y \in I$ *and* $u \in [0,1]$.

The most popular and frequently referenced inequality in the research is the H-H inequality. Numerous mathematicians have approached the study of inequalities from various perspectives.

Theorem 2.1. *If* $\mathcal{G} : [\mathfrak{u}_x, \mathfrak{u}_y] \to \mathbb{R}$ *is a convex function, then*

$$\mathcal{G}\left(\frac{\mathfrak{u}_x + \mathfrak{u}_y}{2}\right) \leq \frac{1}{\mathfrak{u}_y - \mathfrak{u}_x} \int_{\mathfrak{u}_x}^{\mathfrak{u}_y} \mathcal{G}(x)dx \leq \frac{\mathcal{G}(\mathfrak{u}_x) + \mathcal{G}(\mathfrak{u}_y)}{2}. \tag{2.2}$$

For the literature, see the references [47–52].
Raina [53] introduced the following function, stated by

$$\mathcal{J}^{\varrho}_{\epsilon,\sigma}(z) = \mathcal{J}^{\varrho(0),\varrho(1),\ldots}_{\epsilon,\sigma}(z) = \sum_{k=0}^{+\infty} \frac{\varrho(\mathfrak{v})}{\Gamma(\epsilon k + \sigma)} z^k, \tag{2.3}$$

where $\epsilon, \sigma > 0, |z| < R$ and $\varrho = (\varrho(0), \ldots, \varrho(\mathfrak{v}), \ldots)$. Equation (2.3) is the refinement of the classical Mittag-Leffler function (CMLF).

If $\sigma = 0, \epsilon = 1$ and $\varrho(\mathfrak{v}) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}$ for $k = 0, 1, 2, \ldots$, where α, β and γ are parameters and α_k represents the quantity

$$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \alpha(\alpha+1)\ldots(\alpha+k-1), \quad k = 0, 1, 2, \ldots,$$

and restricts to $|z| \leq 1$ (with $z \in \mathbb{C}$), then classical hypergeometric function is:

$$\mathcal{J}(\alpha, \beta; \gamma; z) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k} z^k.$$

Moreover, if $\varrho = (1, 1, \ldots)$ with $\epsilon = \alpha$, $(Re(\alpha) > 0), \sigma = 1$, then

$$\mathfrak{E}_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1 + \alpha k)}. \tag{2.4}$$

Equation (2.4) is referred to as a Classical MLF. In fractional calculus, the MLF is usually applied in the investigation of fractional conjecture of the kinetic equation, Lévy flights, random walks, super diffusive transport, and complex structures.
Cortez investigated the generalized convex set and the convex mapping involving Raina's function in [54, 55].

Definition 2.2 ([55]). *Assume that* $\varrho = (\varrho(0), \ldots, \varrho(0), \ldots)$ *is a bounded sequence of positive real numbers and* $\epsilon, \sigma > 0$. *A set* $X \neq \emptyset$ *is generalized convex, if*

$$\mathfrak{u}_x + u\, \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - \mathfrak{u}_x) \in X, \tag{2.5}$$

for all $\mathfrak{u}_x, \mathfrak{u}_y \in X$ *and* $u \in [0,1]$.

Definition 2.3 ([55]). *Let ϱ denote a bounded sequence then $\varrho = (\varrho(0), \ldots, \varrho(0), \ldots)$ and $\epsilon, \sigma > 0$. If real-valued $\mathcal{G}$ holds, then the inequality*

$$\mathcal{G}\Big(\mathfrak{u}_x + u\, \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - \mathfrak{u}_x)\Big) \leq u\mathcal{G}(\mathfrak{u}_y) + (1-u)\mathcal{G}(\mathfrak{u}_x), \tag{2.6}$$

for all $\mathfrak{u}_x, \mathfrak{u}_y \in X$, where $\mathfrak{u}_x < \mathfrak{u}_y$ and $u \in [0,1]$, then $\mathcal{G}$ is said to be generalized convex function.

Remark 2.1. *If $\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - \mathfrak{u}_x) = \mathfrak{u}_y - \mathfrak{u}_x > 0$, then we achieve Definition 2.1.*

Ahmad et.al [56] investigated the following Condition-A.
Condition A: Suppose X be generalized convex subset w.r.t. $\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\cdot)$. For any $\mathfrak{u}_x, \mathfrak{u}_y \in X$ and $u \in [0,1]$,

$$\mathcal{J}^{\varrho}_{\epsilon,\sigma}\Big(\mathfrak{u}_x - (\mathfrak{u}_x + u\, \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - \mathfrak{u}_x))\Big) = -u\, \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - \mathfrak{u}_x),$$

$$\mathcal{J}^{\varrho}_{\epsilon,\sigma}\Big(\mathfrak{u}_y - \Big(\mathfrak{u}_x + u\, \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - \mathfrak{u}_x)\Big)\Big) = (1-u)\, \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - \mathfrak{u}_x).$$

Note that, for every $\mathfrak{u}_x$, $\mathfrak{u}_y \in X$ and for all u_1, $u_2 \in [0,1]$ from Condition-A, we have

$$\mathcal{J}^{\varrho}_{\epsilon,\sigma}\Big(\mathfrak{u}_x + u_2\, \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - \mathfrak{u}_x) - (\mathfrak{u}_x + u_1\, \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - \mathfrak{u}_x))\Big) = (u_2 - u_1)\, \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - \mathfrak{u}_x). \tag{2.7}$$

Definition 2.4 ([57]). *Assume that $\mathcal{G} \in L_1[\mathfrak{u}_x, \mathfrak{u}_y]$. Then the Riemann–Liouville fractional integral operators for the left and right side of order $\gamma > 0$, denoted by $\mathrm{I}^{\gamma}_{\mathfrak{u}_x^+}\mathcal{G}$ and $\mathrm{I}^{\gamma}_{\mathfrak{u}_y^-}\mathcal{G}$, are given respectively as*

$$\mathrm{I}^{\gamma}_{\mathfrak{u}_x^+}\mathcal{G}(x) = \frac{1}{\Gamma(\gamma)} \int_{\mathfrak{u}_x}^{x} (x-u)^{\gamma-1}\mathcal{G}(u)\, du, \quad x > \mathfrak{u}_x,$$

and

$$\mathrm{I}^{\gamma}_{\mathfrak{u}_y^-}\mathcal{G}(x) = \frac{1}{\Gamma(\gamma)} \int_{x}^{\mathfrak{u}_y} (u-x)^{\gamma-1}\mathcal{G}(u)\, du, \quad x < \mathfrak{u}_y.$$

3. Generalized m-Convex Involving Raina's Mappings and Its Properties

Here, we'll present and investigate the new definition, i.e, *GmCRM*, which is an intriguing and applicable idea for convex functions. We'll also look at some of its algebraic characteristics.

Definition 3.1. *Suppose $\epsilon, \sigma > 0$ and $\varrho = (\varrho(0), \ldots, \varrho(\mathfrak{v}), \ldots)$. A set $X \neq \emptyset$ is said to be generalized m-convex set, if*

$$m\mathfrak{u}_x + \mathfrak{z}\, \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x) \in X, \tag{3.1}$$

$\forall\ \mathfrak{u}_x, \mathfrak{u}_y \in X$ *and* $\mathfrak{z}, m \in [0,1]$.

Definition 3.2. *A mapping $\mathcal{G}$ defined on the Definition 3.1 is said to be GmCRM, if*

$$\mathcal{G}(m\mathfrak{u}_x + \mathfrak{z}\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) \leq m\,(1-\mathfrak{z})\,\mathcal{G}\,(\mathfrak{u}_x) + \mathfrak{z}\mathcal{G}\Big(\mathfrak{u}_y\Big). \tag{3.2}$$

holds for every $\mathfrak{u}_x, \mathfrak{u}_y \in X$, $m \in (0,1]$ and $\mathfrak{z} \in [0,1]$.

Remark 3.1. *If $m = 1$ and $\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x) = \mathfrak{u}_y - m\mathfrak{u}_x$, then Definition 3.2 reverts to Definition 2.1.*

Extended Condition-A: Suppose X be generalized m-convex subset w.r.t. $\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\cdot)$. For any $\mathfrak{u}_x, \mathfrak{u}_y \in X$ and $\mathfrak{z} \in [0,1]$,

$$\mathcal{J}^{\varrho}_{\epsilon,\sigma}\Big(m\mathfrak{u}_x - (m\mathfrak{u}_x + \mathfrak{z}\,\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))\Big) = -\mathfrak{z}\,\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x),$$

$$\mathcal{J}^{\varrho}_{\epsilon,\sigma}\Big(\mathfrak{u}_y - \big(m\mathfrak{u}_x + \mathfrak{z}\,\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)\big)\Big) = (1-\mathfrak{z})\,\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x).$$

Note that, for every $\mathfrak{u}_x,\ \mathfrak{u}_y \in X$ and for all $\mathfrak{z}_1,\ \mathfrak{z}_2 \in [0,1]$ from extended Condition-A, we have

$$\mathcal{J}^{\varrho}_{\epsilon,\sigma}\Big(m\mathfrak{u}_x + \mathfrak{z}_2\,\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x) - (m\mathfrak{u}_x + \mathfrak{z}_1\,\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))\Big) = (\mathfrak{z}_2 - \mathfrak{z}_1)\,\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x).$$

We will look into and develop certain aspects of the previously introduced idea.

Proposition 3.1. *We state the following as true:*

1. *The Sum of two GmCRM is also an GmCRM.*
2. *If $\mathcal{G}$ is GmCRM, then $(c\mathcal{G})$ is also an GmCRM.*
3. *The composition of two GmCRM is also an GmCRM.*
4. *Let $0 < \mathfrak{u}_x < \mathfrak{u}_y$, $\mathcal{G}_j : \mathbb{X} = [\mathfrak{u}_x, \mathfrak{u}_y] \to [0, +\infty)$ be a family of GmCRM and $\mathcal{G}(u) = \sup_j \Delta_j(u)$. Then, $\mathcal{G}$ is an GmCRM for $m \in (0,1]$, $\mathfrak{z} \in [0,1]$, and $U = \{\mathcal{G} \in [\mathfrak{u}_x,\ \mathfrak{u}_y] : \mathcal{G}(\mathcal{G}_{\mathfrak{z}}) < \infty\}$ is an interval.*
5. *The product of two GmCRM is also an GmCRM.*

Proof. The aforementioned properties are easy to prove. Thus, we leave it out. □

4. New Approach of H-H Type Inequality Pertaining to GmCRM

In this section, we examine the new version of H-H type inequality pertaining to GmCRM via RLFIO.

Theorem 4.1. *Assume that $A \subseteq \mathbb{R}$ is an open m-convex subset and $\mathfrak{u}_x, \mathfrak{u}_y \in A$ with $m\mathfrak{u}_x < m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)$. If $\mathcal{G} : [m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)] \to (0,\infty)$ is a GmCRM such that $\mathcal{G} \in L[m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)]$ and $\mathcal{J}^{\varrho}_{\epsilon,\sigma}$ satisfies the extended condition A then,*

$$\begin{aligned}
&\mathcal{G}\left(\frac{2m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2}\right)\\
&\leq \frac{\Gamma(\gamma+1)}{2(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}}\left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^{+}}\mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{-}}\mathcal{G}(m\mathfrak{u}_x)\right]\\
&\leq \frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} \leq \frac{m\mathcal{G}(\mathfrak{u}_x) + \mathcal{G}(\mathfrak{u}_y)}{2}
\end{aligned}$$

with $\gamma > 0$.

Proof. Since $\mathfrak{u}_x, \mathfrak{u}_y \in A$ and A is an m-convex set and for every $u \in [0,1]$, we have $\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x) \in A$. By generalized m-convex involving Raina's mapping, we have for every $x, y \in [\mathfrak{u}_x, \mathfrak{u}_x +$

$\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)]$ with $u = \frac{1}{2}$,

$$\mathcal{G}\left(mx + \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(y - mx)}{2}\right) \leq \frac{\mathcal{G}(mx) + \mathcal{G}(y)}{2}, \tag{4.1}$$

i.e., with $mx = m\mathfrak{u}_x + (1-u)\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)$, $y = m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)$ from the extended condition A, we get

$$\begin{aligned}
&2\mathcal{G}\left(m\mathfrak{u}_x + (1-u)\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x) + \frac{\Pi(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x), m\mathfrak{u}_x + (1-u)\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2}\right)\\
&= 2\mathcal{G}\left(m\mathfrak{u}_x + (1-u)\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x) + \frac{(2u-1)\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2}\right)\\
&= 2\mathcal{G}\left(\frac{2m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2}\right)\\
&\leq \mathcal{G}(m\mathfrak{u}_x + (1-u)\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathcal{G}(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)).
\end{aligned} \tag{4.2}$$

Multiplying both sides (4.2) by $u^{\gamma-1}$, then integrating the resulting inequality with respect to u over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{\gamma}\mathcal{G}\left(\frac{2m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2}\right)\\
&\leq \int_0^1 u^{\gamma-1}\mathcal{G}(m\mathfrak{u}_x + (1-u)\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))du + \int_0^1 u^{\gamma-1}\mathcal{G}(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))du\\
&= \frac{1}{(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}}\\
&\times\left[\int_{m\mathfrak{u}_x}^{m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x)} (m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x) - u)^{\gamma-1}\mathcal{G}(u)du + \int_{m\mathfrak{u}_x}^{m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x)} (u - m\mathfrak{u}_x)^{\gamma-1}\mathcal{G}(u)du\right]\\
&= \frac{\Gamma(\gamma)}{2(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}}\left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+}\mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{(m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x))^-}\mathcal{G}(m\mathfrak{u}_x)\right]
\end{aligned}$$

i.e.

$$\begin{aligned}
&\mathcal{G}\left(\frac{2m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2}\right)\\
&\leq \frac{\Gamma(\gamma+1)}{2(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}}\left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+}\mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x)^-}\mathcal{G}(m\mathfrak{u}_x)\right]
\end{aligned}$$

and the first inequality is proved.

For the proof of the second inequality, we first note that if $\mathcal{G}$ is a generalized m-convex involving Raina's mapping on $[m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)]$ and the mapping $\mathcal{J}^{\varrho}_{\epsilon,\sigma}$ satisfies extended condition A, then for every $u \in [0,1]$, we have

$$\begin{aligned}
&\mathcal{G}(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) = \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x) + (1-u)\mathcal{J}^{\varrho}_{\epsilon,\sigma}(m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)))\\
&\leq u\mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + (1-u)\mathcal{G}(m\mathfrak{u}_x)
\end{aligned}$$

and similarly

$$\mathcal{G}(m\mathfrak{u}_x + (1-u)\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) = \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x) + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)))$$
$$\leq (1-u)\mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + u\mathcal{G}(m\mathfrak{u}_x).$$

By adding these inequalities, we have

$$\mathcal{G}(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathcal{G}(m\mathfrak{u}_x + (1-u)\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) \leq \mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) \tag{4.3}$$

Then multiplying both (4.3) by $u^{\gamma-1}$ and integrating the resulting inequality with respect to u over $[0,1]$, we obtain

$$\int_0^1 u^{\gamma-1}\mathcal{G}(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))du + \int_0^1 u^{\gamma-1}\mathcal{G}(m\mathfrak{u}_x + (1-u)\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))du$$
$$\leq [\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))]\int_0^1 u^{\gamma-1}du$$

i.e.

$$\frac{\Gamma(\gamma)}{(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}}\left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+}\mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^-}\mathcal{G}(m\mathfrak{u}_x)\right]$$
$$\leq \frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{\gamma}.$$

Using the mapping $\mathcal{J}^{\varrho}_{\epsilon,\sigma}$ satisfies extended condition A, the proof is completed.

□

Remark 4.1. *Considering Theorem 4.1, we examine a unique extended fractional variant of H-H inequality pertaining to the CMLF via RLFIO if we pick $\sigma = 1$ and $\varrho = (1, 1, \dots)$ with $\epsilon = \alpha$:*

$$\mathcal{G}\left(\frac{2m\mathfrak{u}_x + \mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2}\right)$$
$$\leq \frac{\Gamma(\gamma+1)}{2(\mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}}\left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+}\mathcal{G}(m\mathfrak{u}_x + \mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x))^-}\mathcal{G}(m\mathfrak{u}_x)\right]$$
$$\leq \frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} \leq \frac{m\mathcal{G}(\mathfrak{u}_x) + \mathcal{G}(\mathfrak{u}_y)}{2}.$$

Example 4.1. *Let us consider the open interval $A = (0, 2) \subset \mathbb{R}$, and select $\mathfrak{u}_x = 0.5$, $\mathfrak{u}_y = 1.5$, and $m = \frac{1}{2}$. Define the mapping $\mathcal{R}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x) = m(\mathfrak{u}_y - \mathfrak{u}_x) = \frac{1}{2}$. Let $\mathcal{G}(x) = e^x$, which is convex (and hence generalized m-convex involving Raina's mapping), continuous, and integrable on the interval $[m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{R}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)] = [0.25, 0.75]$. We take $\gamma = 1$, so that the fractional integrals reduce to classical integrals.*

We first compute

$$\mathcal{G}\left(\frac{2m\mathfrak{u}_x + \mathcal{R}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2}\right) = \mathcal{G}\left(\frac{2\cdot 0.25 + 0.5}{2}\right) = \mathcal{G}(0.5) = e^{0.5} \approx 1.6487.$$

Next, we compute the middle expression involving classical integrals:

$$\frac{1}{2\cdot\mathcal{R}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x)}\left[\int_{0.25}^{0.75}\mathcal{G}(u)\,du+\int_{0.25}^{0.75}\mathcal{G}(u)\,du\right]=\frac{1}{0.5}\cdot\int_{0.25}^{0.75}e^{u}\,du.$$

Evaluating the integral:

$$\int_{0.25}^{0.75}e^{u}\,du=e^{0.75}-e^{0.25}\approx 2.117-1.284=0.833,$$

so the expression becomes

$$\frac{1}{0.5}\cdot 0.833=1.666.$$

Now we compute the arithmetic average of endpoint values:

$$\frac{\mathcal{G}(m\mathfrak{u}_x)+\mathcal{G}(m\mathfrak{u}_x+\mathcal{R}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x))}{2}=\frac{e^{0.25}+e^{0.75}}{2}\approx\frac{1.284+2.117}{2}=1.7005.$$

We also compute the final upper bound:

$$\frac{m\mathcal{G}(\mathfrak{u}_x)+\mathcal{G}(\mathfrak{u}_y)}{2}=\frac{0.5\cdot e^{0.5}+e^{1.5}}{2}\approx\frac{0.82435+4.4817}{2}=2.653.$$

Hence, the full chain of inequalities from the theorem holds:

$$\boxed{\Psi\,(0.5)\approx 1.6487\leq 1.666\leq 1.7005\leq 2.653}.$$

This verifies the validity of the theorem for the selected function and parameters.

5. Refinements of H-H Inequality Pertaining to GmCRM

Many scholars and researchers are presently collaborating on novel ideas pertaining to the issue from various viewpoints in the field of convex analysis. Throughout history, a number of novel H-H type inequalities have been generated leveraging convexity and different types of fractional operators.

This section's main goal is to investigate and study a new lemma. Employing this newly created lemma, we establish several improvements of the H-H inequality via RLFIO. To obtain the results, we apply the notion of the GmCRM and the Holder inequality.

Lemma 5.1. *Assume that $A\subseteq\mathbb{R}$ is an open m-convex subset and $\mathfrak{u}_x,\mathfrak{u}_y\in A$ with $m\mathfrak{u}_x<m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x)$. If $\mathcal{G}:A\to\mathbb{R}$ is a differentiable mapping such that $\mathcal{G}'\in L[m\mathfrak{u}_x,m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x)]$, then the following equality holds:*

$$\begin{aligned}&\frac{\mathcal{G}(m\mathfrak{u}_x)+\mathcal{G}(m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x))}{2}\\&-\frac{\Gamma(\gamma+1)}{2(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x))^{\gamma}}\left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^{+}}\mathcal{G}(m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x))+\mathrm{I}^{\gamma}_{(m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x))^{-}}\mathcal{G}(m\mathfrak{u}_x)\right]\\&=\frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x)}{2}\int_0^1\left[u^{\gamma}-(1-u)^{\gamma}\right]\mathcal{G}'(m\mathfrak{u}_x+u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y-m\mathfrak{u}_x))du.\end{aligned}\tag{5.1}$$

Proof. It suffices to note that

$$\begin{aligned}J &= \int_0^1 [u^{\gamma} - (1-u)^{\gamma}]\, \mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))du \\ &= \left[\int_0^1 u^{\gamma}\mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))du\right] + \left[-\int_0^1 (1-u)^{\gamma}\mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))du\right] \\ &= J_1 + J_2. \end{aligned} \tag{5.2}$$

Integrating by parts

$$\begin{aligned}J_1 &= \int_0^1 u^{\gamma}\mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))du \\ &= u^{\gamma}\frac{f(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}\bigg|_0^1 - \int_0^1 \gamma u^{\gamma-1}\frac{\mathcal{G}(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}du \\ &= \frac{\mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)} - \frac{\gamma}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}\int_{m\mathfrak{u}_x}^{m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}\left(\frac{x-a}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}\right)^{\gamma-1}\frac{\mathcal{G}(x)}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}dx \\ &= \frac{\mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)} - \frac{\Gamma(\gamma+1)}{(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma+1}}\mathrm{I}^{\gamma}_{(m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^-}\mathcal{G}(m\mathfrak{u}_x)\end{aligned} \tag{5.3}$$

and similarly we get,

$$\begin{aligned}J_2 &= -\int_0^1 (1-u)^{\gamma}\mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))du \\ &= -(1-u)^{\gamma}\frac{\mathcal{G}(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}\bigg|_0^1 - \int_0^1 \gamma(1-u)^{\gamma-1}\frac{\mathcal{G}(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}du \\ &= \frac{\mathcal{G}(a)}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)} - \frac{\gamma}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}\int_{m\mathfrak{u}_x}^{m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}\left(\frac{m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x) - x}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}\right)^{\gamma-1}\frac{\mathcal{G}(x)}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}dx \\ &= \frac{\mathcal{G}(a)}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)} - \frac{\Gamma(\gamma+1)}{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)^{\gamma+1}}\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+}\mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)).\end{aligned} \tag{5.4}$$

Using (5.3) and (5.4) in (5.2), it follows that

$$\begin{aligned}J &= \frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} \\ &\quad - \frac{\Gamma(\gamma+1)}{2\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)^{\gamma}}\left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+}\mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{(m\mathfrak{u}_x+\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^-}\mathcal{G}(m\mathfrak{u}_x)\right]\end{aligned}$$

Thus, by multiplying both sides by $\frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2}$, we have conclusion (5.1).

□

Theorem 5.1. *Assume that $A \subseteq \mathbb{R}$ is an open m-convex subset and $\mathfrak{u}_x, \mathfrak{u}_y \in A$ with $m\mathfrak{u}_x < m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)$. Suppose that $\mathcal{G} : A \to \mathbb{R}$ is a differentiable function such that $\mathcal{G}' \in L[m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)]$. If $|\mathcal{G}'|$ is GmCRM on $[m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)]$ then the following inequality for* *fractional integrals with $\gamma > 0$ holds:*

$$\left| \frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} - \frac{\Gamma(\gamma+1)}{2(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}} \left[\mathtt{I}^{\gamma}_{\mathfrak{u}_x^+} \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathtt{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^-} \mathcal{G}(\mathfrak{u}_x) \right] \right| \leq \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2(\gamma+1)} \left(1 - \frac{1}{2^{\gamma}}\right) \left[m \left|\mathcal{G}'(\mathfrak{u}_x)\right| + \left|\mathcal{G}'(\mathfrak{u}_y)\right| \right].$$

Proof. Using Lemma 5.1 and the GmCRM of $|\mathcal{G}'|$ we get

$$\begin{aligned}
&\left| \frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} - \frac{\Gamma(\gamma+1)}{2(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}} \left[\mathtt{I}^{\gamma}_{m\mathfrak{u}_x^+} \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathtt{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))} - \mathcal{G}(m\mathfrak{u}_x) \right] \right| \\
&\leq \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2} \int_0^1 \left|u^{\gamma} - (1-u)^{\gamma}\right| \left|\mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))\right| du \\
&\leq \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2} \int_0^1 \left|u^{\gamma} - (1-u)^{\gamma}\right| m \left[(1-u)\left|\mathcal{G}'(\mathfrak{u}_x)\right| + u\left|\mathcal{G}'(\mathfrak{u}_y)\right|\right] du \\
&\leq \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2} \left\{ \int_0^{1/2} \left[(1-u)^{\gamma} - u^{\gamma}\right] \left[m(1-u)\left|\mathcal{G}'(\mathfrak{u}_x)\right| + u\left|\mathcal{G}'(\mathfrak{u}_y)\right|\right] du \right. \\
&\left. + \int_{1/2}^1 \left[u^{\gamma} - (1-u)^{\gamma}\right] \left[m(1-u)\left|\mathcal{G}'(\mathfrak{u}_x)\right| + u\left|\mathcal{G}'(\mathfrak{u}_y)\right|\right] du \right\} \\
&= \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2} \left[m\left|\mathcal{G}'(\mathfrak{u}_x)\right| + \left|\mathcal{G}'(\mathfrak{u}_y)\right|\right] \left(\int_0^{1/2} \left[(1-u)^{\gamma} - u^{\gamma}\right] du \right) \\
&= \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2(\gamma+1)} \left(1 - \frac{1}{2^{\gamma}}\right) \left[m\left|\mathcal{G}'(\mathfrak{u}_x)\right| + \left|\mathcal{G}'(\mathfrak{u}_y)\right|\right],
\end{aligned}$$

which completes the proof. □

Remark 5.1. *Employing Theorem 5.1, we investigate the novel generalized fractional version of H-H inequality involving the CMLF via RLFIO if we choose $\sigma = 1$ and $\varrho = (1, 1, \ldots)$ with $\epsilon = \alpha$:*

$$\left| \frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} - \frac{\Gamma(\gamma+1)}{2(\mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}} \left[\mathtt{I}^{\gamma}_{m\mathfrak{u}_x^+} \mathcal{G}(m\mathfrak{u}_x + \mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathtt{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x))^-} \mathcal{G}(\mathfrak{u}_x) \right] \right| \leq \frac{\mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2(\gamma+1)} \left(1 - \frac{1}{2^{\gamma}}\right) \left[m\left|\mathcal{G}'(\mathfrak{u}_x)\right| + \left|\mathcal{G}'(\mathfrak{u}_y)\right| \right].$$

Theorem 5.2. *Assume that $A \subseteq \mathbb{R}$ is an open m-convex subset and $\mathfrak{u}_x, \mathfrak{u}_y \in A$ with $m\mathfrak{u}_x < m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)$ such that $\mathcal{G}' \in L[m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)]$. Suppose that $\mathcal{G} : A \to \mathbb{R}$ is a differentiable* *function. If* $|\mathcal{G}'|^q$ *is GmCRM on* $[m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)]$ *for some fixed* $q > 1$ *then the following inequality holds:*

$$\left| \frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} - \frac{\Gamma(\gamma+1)}{2(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}} \left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+} \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))} \mathcal{G}(m\mathfrak{u}_x) \right] \right|$$
$$\leq \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2(\gamma p + 1)^{\frac{1}{p}}} \left(\frac{m \left|\mathcal{G}'(\mathfrak{u}_x)\right|^q + \left|\mathcal{G}'(\mathfrak{u}_y)\right|^q}{2} \right)^{\frac{1}{q}} \tag{5.5}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\gamma \in [0,1]$.

Proof. Employing Lemma 5.1 and using Hölder inequality with properties of modulus, we have

$$\frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} - \frac{\Gamma(\gamma+1)}{2(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}} \left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+} \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^-} \mathcal{G}(m\mathfrak{u}_x) \right]$$
$$\leq \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2} \int_0^1 \left| u^{\gamma} - (1-u)^{\gamma} \right| \left| \mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) \right| du$$
$$\leq \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2} \left(\int_0^1 \left| u^{\gamma} - (1-u)^{\gamma} \right|^p du \right)^{\frac{1}{p}} \left(\int_0^1 \left| \mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) \right|^q du \right)^{\frac{1}{q}}.$$

We know that for $\gamma \in [0,1]$ and $\forall u_1, u_2 \in [0,1]$,

$$\left| u_1^{\gamma} - u_2^{\gamma} \right| \leq |u_1 - u_2|^{\gamma}$$

therefore

$$\int_0^1 \left| u^{\gamma} - (1-u)^{\gamma} \right|^p du \leq \int_0^1 |1-2u|^{\gamma p} du$$
$$= \int_0^{1/2} [1-2u]^{\gamma p} du + \int_{1/2}^1 [2u-1]^{\gamma p} du = \frac{1}{\gamma p + 1}$$

Since $|\mathcal{G}'|^q$ is GmCRM on $[m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)]$, we have inequality (5.5), which completes the proof. □

Remark 5.2. *Employing Theorem 5.2, we investigate the novel generalized fractional version of H-H inequality involving the CMLF via RLFIO if we choose* $\sigma = 1$ *and* $\varrho = (1, 1, \ldots)$ *with* $\epsilon = \alpha$*:*

$$\left| \frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} - \frac{\Gamma(\gamma+1)}{2(\mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}} \left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+} \mathcal{G}(m\mathfrak{u}_x + \mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x))} \mathcal{G}(m\mathfrak{u}_x) \right] \right|$$
$$\leq \frac{\mathcal{E}_{\alpha}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2(\gamma p + 1)^{\frac{1}{p}}} \left(\frac{m \left|\mathcal{G}'(\mathfrak{u}_x)\right|^q + \left|\mathcal{G}'(\mathfrak{u}_y)\right|^q}{2} \right)^{\frac{1}{q}}.$$

Theorem 5.3. *Assume that $A \subseteq \mathbb{R}$ is an open m-convex subset and $\mathfrak{u}_x, \mathfrak{u}_y \in A$ with $m\mathfrak{u}_x < m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)$. Suppose that $\mathcal{G} : A \to \mathbb{R}$ is a differentiable mapping such that $\mathcal{G}' \in L[m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)]$. If $|\mathcal{G}'|^q$ is GmCRM on $[m\mathfrak{u}_x, m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)]$ for some fixed $q > 1$ then the following inequality holds:*

$$\left| \frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} - \frac{\Gamma(\gamma+1)}{2(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}} \left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+} \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^-} \mathcal{G}(\mathfrak{u}_x) \right] \right|$$
$$\leq \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{(\gamma+1)} \left(1 - \frac{1}{2^{\gamma}}\right) \left[\frac{m\left|\mathcal{G}'(\mathfrak{u}_x)\right|^q + \left|\mathcal{G}'(\mathfrak{u}_y)\right|^q}{2} \right]^{\frac{1}{q}} \tag{5.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\gamma > 0$.

Proof. Employing Lemma 5.1 and using Hölder inequality with properties of modulus, we have

$$\left| \frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} - \frac{\Gamma(\gamma+1)}{2(\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^{\gamma}} \left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+} \mathcal{G}(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))^-} \mathcal{G}(m\mathfrak{u}_x) \right] \right|$$
$$\leq \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2} \int_0^1 \left|u^{\gamma} - (1-u)^{\gamma}\right|^{\frac{1}{p}+\frac{1}{q}} \left|\mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))\right| du$$
$$\leq \frac{\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x)}{2} \left(\int_0^1 \left|u^{\gamma} - (1-u)^{\gamma}\right| du \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 \left|u^{\gamma} - (1-u)^{\gamma}\right| \left|\mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))\right|^q du \right)^{\frac{1}{q}}$$

On the other hand, we have

$$\int_0^1 \left|u^{\gamma} - (1-u)^{\gamma}\right| du = \int_0^{1/2} [(1-u)^{\gamma} - u^{\gamma}]\, du + \int_{1/2}^1 [u^{\gamma} - (1-u)^{\gamma}]\, du$$
$$= \frac{2}{\gamma+1}\left(1 - \frac{1}{2^{\gamma}}\right).$$

Since $|\mathcal{G}'|^q$ is GmCRM on A, we attain

$$\left|\mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y, \mathfrak{u}_x, m))\right|^q \leq m(1-u)\left|\mathcal{G}'(\mathfrak{u}_x)\right|^q + u\left|\mathcal{G}'(\mathfrak{u}_y)\right|^q$$

and

$$\int_0^1 \left|u^{\gamma} - (1-u)^{\gamma}\right| \left|\mathcal{G}'(m\mathfrak{u}_x + u\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y - m\mathfrak{u}_x))\right|^q du$$
$$\leq \int_0^1 \left|u^{\gamma} - (1-u)^{\gamma}\right| \left[m(1-u)\left|\mathcal{G}'(\mathfrak{u}_x)\right|^q + u\left|\mathcal{G}'(\mathfrak{u}_y)\right|^q \right] du$$

$$= \int_0^{1/2} [(1-u)^\gamma - u^\gamma] \left[m(1-u)\left|\mathcal{G}'(\mathfrak{u}_x)\right|^q + u\left|\mathcal{G}'(\mathfrak{u}_y)\right|^q\right] du$$
$$+ \int_{1/2}^{1} [u^\gamma - (1-u)^\gamma] \left[m(1-u)\left|\mathcal{G}'(\mathfrak{u}_x)\right|^q + u\left|\mathcal{G}'(\mathfrak{u}_y)\right|^q\right] du$$
$$= \frac{1}{\gamma+1}\left(1-\frac{1}{2^\gamma}\right)\left[m\left|\mathcal{G}'(\mathfrak{u}_x)\right|^q + \left|\mathcal{G}'(\mathfrak{u}_y)\right|^q\right]$$

from here we obtain inequality (5.6) which completes the proof.

□

Remark 5.3. *Employing Theorem 5.3, we investigate the novel generalized fractional version of H-H inequality involving the CMLF via RLFIO if we choose $\sigma = 1$ and $\varrho = (1, 1, \ldots)$ with $\epsilon = \alpha$:*

$$\left|\frac{\mathcal{G}(m\mathfrak{u}_x) + \mathcal{G}(m\mathfrak{u}_x + \mathcal{E}_{\oslash}(\mathfrak{u}_y - m\mathfrak{u}_x))}{2} - \frac{\Gamma(\gamma+1)}{2(\mathcal{E}_\alpha(\mathfrak{u}_y - m\mathfrak{u}_x))^\gamma}\left[\mathrm{I}^{\gamma}_{m\mathfrak{u}_x^+}\mathcal{G}(m\mathfrak{u}_x + \mathcal{E}_\alpha(\mathfrak{u}_y - m\mathfrak{u}_x)) + \mathrm{I}^{\gamma}_{(m\mathfrak{u}_x + \mathcal{E}_\alpha(\mathfrak{u}_y - m\mathfrak{u}_x))^-}\mathcal{G}(\mathfrak{u}_x)\right]\right|$$
$$\leq \frac{\mathcal{E}_\alpha(\mathfrak{u}_y - m\mathfrak{u}_x)}{(\gamma+1)}\left(1-\frac{1}{2^\gamma}\right)\left[\frac{m\left|\mathcal{G}'(\mathfrak{u}_x)\right|^q + \left|\mathcal{G}'(\mathfrak{u}_y)\right|^q}{2}\right]^{\frac{1}{q}}.$$

6. Applications

This section presents novel applications of the established integral inequalities for generalized m-convex involving Raina's mappings via Riemann-Liouville fractional integrals. Specifically, we demonstrate how these generalized H-H-type inequalities can be effectively applied to various entropy measures and information-theoretic constructs. The challenge lies in choosing appropriate non-trivial functions and parameters and interpreting the resulting bounds within the framework of generalized entropies. These applications underscore the analytical strength and flexibility of our results in broader mathematical and applied contexts. For further developments and related discussions, we refer the reader to [58, 59] and the references therein.

For these applications, we will primarily utilize the case where $m = 1$ and the mapping $\mathcal{J}^{\varrho}_{\epsilon,\sigma}(y, x, m) = y - x$. In this scenario, A being an m-convex set (an interval), and $\mathcal{G}$ being an generalized m-convex involving Raina's mapping reduces to $\mathcal{G}$ being a convex function. The extended condition A for $\mathcal{J}^{\varrho}_{\epsilon,\sigma}(y, x, m) = y - x$ is also satisfied. This simplification allows us to focus on the choice of the function $\mathcal{G}$ and the implications of the fractional integral bounds for entropy-related quantities.

6.1. **Integral Bounds for Tsallis Entropy of a Truncated Power-Law Distribution.** Consider a quantity X (e.g., resource abundance) that follows a truncated power-law distribution on the interval $[\mathfrak{u}_x, \mathfrak{u}_y]$. The probability density function is given by

$$f(x) = Cx^{-k}, \quad x \in [\mathfrak{u}_x, \mathfrak{u}_y], \quad k > 1,$$

with normalization constant $C = \frac{1-k}{\mathfrak{u}_y^{1-k} - \mathfrak{u}_x^{1-k}}$. The Tsallis entropy for this distribution is defined as

$$S_q = \frac{1}{q-1}\left(1 - \int f(x)^q dx\right), \quad q \neq 1.$$

Let $\mathcal{G}(x) = f(x)^q$, which is the integrand of the entropy expression.

Let $\mathfrak{u}_x = 1$, $\mathfrak{u}_y = 2$, and $k = 2$. Then $f(x) = 2x^{-2}$, and for $q = 3$, we obtain $\mathcal{G}(x) = 8x^{-6}$. This function is convex on $[1, 2]$, and thus generalized m-convex involving Raina's mapping for $m = 1$, with $\mathcal{J}^{\varrho}_{\epsilon,\sigma}(y, x, 1) = y - x$.

Applying the main theorem (Theorem 4.1) for $m = 1$ and $\gamma = 1$, we attain the inequality:

$$\mathcal{G}\left(\frac{\mathfrak{u}_x + \mathfrak{u}_y}{2}\right) \leq \frac{1}{\mathfrak{u}_y - \mathfrak{u}_x}\int_{\mathfrak{u}_x}^{\mathfrak{u}_y} \mathcal{G}(x)\,dx \leq \frac{\mathcal{G}(\mathfrak{u}_x) + \mathcal{G}(\mathfrak{u}_y)}{2}.$$

For the chosen values:

$$\mathcal{G}(1) = 8, \quad \mathcal{G}(2) = \tfrac{1}{8}, \quad \mathcal{G}\left(\tfrac{3}{2}\right) = \tfrac{512}{729}, \quad \int_1^2 \mathcal{G}(x)\,dx = \tfrac{31}{20}.$$

The inequality becomes:

$$\tfrac{512}{729} \leq \tfrac{31}{20} \leq \tfrac{65}{16},$$

which confirms the theorem's validity in this setting.

This demonstrates that the theorem yields meaningful fractional integral bounds for the Tsallis entropy integrand $f(x)^q$. Such bounds are useful in information theory, especially when exact integration is difficult, offering insight into entropy behavior under generalized convexity and fractional calculus frameworks.

6.2. **Divergence Measures for Information Discrepancy.** Information divergence measures, such as the Kullback-Leibler (KL) divergence, quantify the dissimilarity between probability distributions. Consider the function

$$\mathcal{G}(x) = x \log x - x + 1,$$

which is strictly convex on $(0, \infty)$ and arises naturally in the KL divergence. This makes it suitable for application of our main inequality.

Let $\mathfrak{u}_x = 1$, $\mathfrak{u}_y = 2$, $m = \frac{1}{2}$, and $\mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y, \mathfrak{u}_x, m) = \mathfrak{u}_y - \mathfrak{u}_x = 1$. Then the interval becomes

$$[m\mathfrak{u}_x,\ m\mathfrak{u}_x + \mathcal{J}^{\varrho}_{\epsilon,\sigma}(\mathfrak{u}_y, \mathfrak{u}_x, m)] = \left[\frac{1}{2}, \frac{3}{2}\right],$$

over which $\mathcal{G}$ remains convex and integrable.

We compute:

$$\mathcal{G}\left(\tfrac{1}{2}\right) = \tfrac{1}{2}(1 - \log 2) \approx 0.1534, \quad \mathcal{G}\left(\tfrac{3}{2}\right) \approx 0.1082, \quad \mathcal{G}(1) = 0.$$

The average of the endpoint values is:

$$\frac{\mathcal{G}\left(\frac{1}{2}\right) + \mathcal{G}\left(\frac{3}{2}\right)}{2} \approx 0.1308.$$

Applying the main inequality for $m = \frac{1}{2}$ and $\gamma = 1$, we obtain:

$$\mathcal{G}(1) \leq \frac{1}{\mathfrak{u}_y - \mathfrak{u}_x} \int_{\frac{1}{2}}^{\frac{3}{2}} \mathcal{G}(x)\, dx \leq \frac{\mathcal{G}\left(\frac{1}{2}\right) + \mathcal{G}\left(\frac{3}{2}\right)}{2},$$

which simplifies to:

$$0 \leq \int_{\frac{1}{2}}^{\frac{3}{2}} (x \log x - x + 1)\, dx \leq 0.1308.$$

Evaluating the integral yields approximately 0.0428, confirming the validity of the inequality.

This application shows how the derived fractional integral inequality provides bounds for the integrand of the KL divergence, $\mathcal{G}(x) = x \log x - x + 1$. These bounds are particularly meaningful in contexts where exact integrals are difficult to compute or where memory effects and non-local behavior arise, such as in systems modeled using fractional calculus. The result extends classical convexity-based inequalities to fractional frameworks and demonstrates their utility in information theory and divergence analysis.

7. Conclusions

Fractional analysis has intrigued a lot of interest from writers and scholars in a variety of disciplines. Conversely, convex analysis has seemed as a potent tool for developing new numerical models that enable the resolution of difficult issues in both the pure and applied sciences. Convex analysis and the associated inequalities are becoming more popular and attracting more research interest due to ongoing developments, extensions, and applications.

In this paper:

(1) First, we explored a novel version of H-H inequality over RLFIO via GmCRM.
(2) We investigated novel equality via GmCRM pertaining to RLFIO. In addition, we presents the new refinements of H-H inequality with the help of Hölder inequality over RLFIO via GmCRM.

Future Directions: The examined inequalities can be investigated inside interval analysis and quantum calculus. Especially integral inequality is a fast developing field of study. Scientists will be enthralled by the combination of interval-valued analysis and quantum calculus into the study of integral inequalities since it presents fascinating directions for next development. Investigation of potential connections with optimization theory, where such inequalities can provide tighter bounds for objective functions under invexity assumptions. Further exploration of applications in entropy-based statistics, machine learning, and information geometry.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] J. Pełczyński, Application of the Theory of Convex Sets for Engineering Structures with Uncertain Parameters, Appl. Sci. 10 (2020), 6864. https://doi.org/10.3390/app10196864.

[2] H. Föllmer, A. Schied, Convex Measures of Risk and Trading Constraints, Finance Stoch. 6 (2002), 429–447. https://doi.org/10.1007/s007800200072.

[3] J. Green, W.P. Heller, Chapter 1 Mathematical Analysis and Convexity with Applications to Economics, in: Handbook of Mathematical Economics, Elsevier, 1981: pp. 15–52. https://doi.org/10.1016/s1573-4382(81)01005-9.

[4] T. Pennanen, Convex Duality in Stochastic Optimization and Mathematical Finance, Math. Oper. Res. 36 (2011), 340–362. https://doi.org/10.1287/moor.1110.0485.

[5] M. Tariq, S.K. Ntouyas, W. Afzal, J. Tariboon, Some New Approaches of Integral Inequalities Involving Raina and Mittag-Leffler Function Pertaining to Atangana-Baleanu Fractional Integral Operator, J. Math. Comput. Sci. 41 (2025), 244–263. https://doi.org/10.22436/jmcs.041.02.07.

[6] W. Abdelfattah, S. Bhatti, A. Asghar, M. Tariq, W. Afzal, et al., Fractional Integral Inequalities Associated with Applications to Special Means, Int. J. Math. Comput. Sci. (2026), 289–294. https://doi.org/10.69793/ijmcs/02.2026/ahmad.

[7] W. Abdelfattah, S. Bhatti, A. Asghar, T. Zahro, S. Roopani, et al., Fractional Midpoint Type Inequalities via Superquadraticity, Int. J. Math. Comput. Sci. 21 (2026), 353–359. https://doi.org/10.69793/ijmcs/02.2026/muhammad.

[8] A. Adrees, W. Afzal, K. Shabbir, N.M. Dahshan, A.E. Abuzeid, et al., On Some Properties and Integral Inequalities for Modified (p,h)-Convex Stochastic Processes, Open J. Math. Sci. 10 (2026), 477–491. https://doi.org/10.30538/oms2026.0300.

[9] M. Tariq, S.K. Sahoo, J. Nasir, H. Aydi, H. Alsamir, et al., Some Ostrowski Type Inequalities via n-Polynomial Exponentially s-Convex Functions and Their Applications, AIMS Math. 6 (2021), 13272–13290. https://doi.org/10.3934/math.2021768.

[10] M. Tariq, H. Ahmad, A.G. Shaikh, S.K. Sahoo, K.M. Khedher, et al., New Fractional Integral Inequalities for Preinvex Functions Involving Caputo-Fabrizio Operator, AIMS Math. 7 (2022), 3440–3455. https://doi.org/10.3934/math.2022191.

[11] S.I. Butt, H. Budak, M. Tariq, M. Nadeem, Integral Inequalities for n-Polynomial s-Type Preinvex Functions with Applications, Math. Methods Appl. Sci. 44 (2021), 11006–11021. https://doi.org/10.1002/mma.7465.

[12] H. Ahmad, S. Bhatti, A. Asghar, M. Tariq, W. Afzal, et al., Hermite-Hadamard Type Inequality via Generalized Superquadratic Functions, Int. J. Math. Comput. Sci. 21 (2026), 307–313. https://doi.org/10.69793/ijmcs/02.2026/hsamwew.

[13] V.V. Kulish, J.L. Lage, Application of Fractional Calculus to Fluid Mechanics, J. Fluids Eng. 124 (2002), 803–806. https://doi.org/10.1115/1.1478062.

[14] R.L. Magin, Fractional Calculus in Bio-Engineering, Begell House Inc., 2006.

[15] A. Atangana, Application of Fractional Calculus to Epidemiology, in: C. Cattani, H.M. Srivastava, X.J. Yang, (eds) Fractional Dynamics, De Gruyter, pp. 174–190, (2015).

[16] D. Baleanu, A. Jajarmi, S.S. Sajjadi, D. Mozyrska, A New Fractional Model and Optimal Control of a Tumor-immune Surveillance with Non-singular Derivative Operator, Chaos 29 (2019), 083127. https://doi.org/10.1063/1.5096159.

[17] A. Ebrahimzadeh, A. Jajarmi, D. Baleanu, Enhancing Water Pollution Management Through a Comprehensive Fractional Modeling Framework and Optimal Control Techniques, J. Nonlinear Math. Phys. 31 (2024), 48. https://doi.org/10.1007/s44198-024-00215-y.

[18] D. Baleanu, A. Jajarmi, O. Defterli, R. Wannan, S.S. Sajjadi, et al., Fractional Investigation of Time-Dependent Mass Pendulum, J. Low Freq. Noise Vib. Act. Control 43 (2023), 196–207. https://doi.org/10.1177/14613484231187439.

[19] M. Axtell, M.E. Bise, Fractional Calculus Application in Control Systems, in: IEEE Conference on Aerospace and Electronics, IEEE, pp. 563–566. https://doi.org/10.1109/NAECON.1990.112826.

[20] E. Capelas de Oliveira, S. Jarosz, J. Vaz, Fractional Calculus via Laplace Transform and its Application in Relaxation Processes, Commun. Nonlinear Sci. Numer. Simul. 69 (2019), 58–72. https://doi.org/10.1016/j.cnsns.2018.09.013.

[21] D. Baleanu, Z.B. Güvenç, J.T. Machado, New Trends in Nanotechnology and Fractional Calculus Applications, Springer, 2010. https://doi.org/10.1007/978-90-481-3293-5.

[22] M. El-Shahed, Fractional Calculus Model of the Semilunar Heart Valve Vibrations, in: Proceedings of the ASME 2003 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, Vol 5: 19th Biennial Conference on Mechanical Vibration and Noise, ASME, Chicago, pp. 711–714, (2003).

[23] L.V.C. Hoan, M.A. Akinlar, M. Inc, J.F. Gómez-Aguilar, Y.M. Chu, et al., A New Fractional-Order Compartmental Disease Model, Alexandria Eng. J. 59 (2020), 3187–3196. https://doi.org/10.1016/j.aej.2020.07.040.

[24] H. Ahmad, M. Nadeem, A. Asghar, R. Efendiev, M. Tariq, et al., Some New Approaches of Hermite-Hadamard Type Inequalities Pertaining to Generalized Convexity on Coordinates Using Hypergeometric Function, Azerbaij. J. Math. 16 (2026), 220. https://doi.org/10.59849/2218-6816.2026.1.220.

[25] S. Khan, N.A. Shah, H. Ahmad, A. Kamel, W.M. Abdelfattah, et al., Partial Sums for Normalized Mittag-Leffler-Prabhakar Function and Barnes-Mittag-Leffler Function, Eur. J. Pure Appl. Math. 18 (2025), 7027. https://doi.org/10.29020/nybg.ejpam.v18i4.7027.

[26] M. Zafarullah, I. Khan, A. Nawaz, M. Jamil, T.N. Alharthi, et al., Implementation of Atangana–Baleanu–Caputo (ABC) Fractional Time Operator on Heat and Mass Transfer Phenomena of Walter's-B Fluid, Fractals 34 (2026), 2640051. https://doi.org/10.1142/S0218348X26400517.

[27] P. Karthiga, S.M. Sivalingam, V. Govindaraj, H. Ahmad, Controllability Analysis of Fractional Nonlinear Dynamical Systems Using Ψ-Caputo Derivatives and Prescribed Controls, J. Taibah Univ. Sci. 20 (2026), 2635196. https://doi.org/10.1080/16583655.2026.2635196.

[28] N.A. Sheikh, Z.B. Ismail, I. Khan, Influence of Memory Effects on Heat and Mass Transfer in Fractional Casson–Brinkman Electrically Conducting Flow with Ramped Boundaries, Fractals (2026), 2650073. https://doi.org/10.1142/S0218348X26500738.

[29] F. Faisal, Z. Haider, U. Ghani, H. Ali, M. Amjad, et al., Derivation of Hermite–Hadamard-Type Inequalities via Quasi-Preinvex Functions and Strongly Preinvex Functions, Int. J. Geom. Methods Mod. Phys. (2025), 2650048. https://doi.org/10.1142/S0219887826500489.

[30] N. Khan, F. Ali, G. Alhamzi, I. Khan, M. Bakouri, et al., Heat Transfer Enhancement in MHD Flow of Tri-Hybrid Maxwell Nanofluid with Ramped Wall Heating: A Fractional Caputo–Crank–Nicolson Approach, Results Eng. 29 (2026), 109476. https://doi.org/10.1016/j.rineng.2026.109476.

[31] H. Ahmad, M. Tariq, A. Asghar, W. Afzal, M. Aphane, et al., Some New Notions of Mathematical Integral Inequalities: Theory and Applications, Int. J. Anal. Appl. 24 (2026), 175. https://doi.org/10.28924/2291-8639-24-2026-175.

[32] A. Rayal, J. Kaur, P.A. Patel, D.U. Ozsahin, H. Ahmad, et al., A Spectral Collocation Scheme with 2D Ultraspherical Wavelets for Fractional Nonlinear Gas Dynamics Equations under Caputo–Fabrizio Derivative, Fractals (2026), 2640019. https://doi.org/10.1142/S0218348X26400190.

[33] R. Balamurugan, I. Khan, M. Bakouri, T. Alqahtani, Physics-Informed Neural Network Approach to Unsteady Fractional Flow in a Vertical Coaxial Annulus with Thermal Effects and Magneto-Hall Interaction, Results Eng. 29 (2026), 109965. https://doi.org/10.1016/j.rineng.2026.109965.

[34] S.I. Butt, D. Khan, S. Tipurić-Spužević, Y. Seol, Generalized Petrovič Inequality and Its Related Estimations in Superquadraticity on the Fractal Space with Applications, Fractals 33 (2025), 2550086. https://doi.org/10.1142/S0218348X25500860.

[35] M.N. Aftab, S.I. Butt, Y. Seol, Fundamentals of Dual Basic Symmetric Quantum Calculus and Its Fractional Perspectives, Fractal Fract. 9 (2025), 237. https://doi.org/10.3390/fractalfract9040237.

[36] Z. Khan, W. Afzal, W. Nazeer, J.K. Asamoah, Some New Variants of Hermite–Hadamard and Fejér-Type Inequalities for Godunova–Levin Preinvex Class of Interval-Valued Functions, J. Math. 2024 (2024), 8814585. https://doi.org/10.1155/2024/8814585.

[37] D. Khan, S.I. Butt, Y. Seol, Properties and Integral Inequalities of P-Superquadratic Functions via Multiplicative Calculus with Applications, Bound. Value Probl. 2024 (2024), 166. https://doi.org/10.1186/s13661-024-01978-5.

[38] M.N. Aftab, S.I. Butt, Y. Seol, Fundamentals of Right Hahn q-Symmetric Calculus and Related Inequalities, J. Funct. Spaces 2026 (2026), 7248748. https://doi.org/10.1155/jofs/7248748.

[39] W. Afzal, N.M. Aloraini, M. Abbas, J. Ro, A.A. Zaagan, Some Novel Kulisch-Miranker Type Inclusions for a Generalized Class of Godunova-Levin Stochastic Processes, AIMS Math. 9 (2024), 5122–5146. https://doi.org/10.3934/math.2024249.

[40] M. Ayyash, D. Khan, S.I. Butt, Y. Seol, Fractional Inclusion Analysis of Superquadratic Stochastic Processes via Center-Radius Total Order Relation with Applications in Information Theory, Fractal Fract. 9 (2025), 375. https://doi.org/10.3390/fractalfract9060375.

[41] M.N. Aftab, S.I. Butt, M. Alammar, Y. Seol, Analysis of Quantum Multiplicative Calculus and Related Inequalities, Mathematics 13 (2025), 3381. https://doi.org/10.3390/math13213381.

[42] M. Abbas, W. Afzal, T. Botmart, A.M. Galal, Jensen, Ostrowski and Hermite-Hadamard Type Inequalities for h-Convex Stochastic Processes by Means of Center-Radius Order Relation, AIMS Math. 8 (2023), 16013–16030. https://doi.org/10.3934/math.2023817.

[43] S.I. Butt, D. Khan, S. Jain, G.I. Oros, P. Agarwal, et al., Fractional Integral Inequalities for Superquadratic Functions via Atangana–Baleanu's Operator with Applications, Fractals 33 (2025), 2540068. https://doi.org/10.1142/S0218348X25400687.

[44] M.N. Aftab, S.I. Butt, M. Alammar, Y. Seol, Multiplicative View Point Analysis of Hahn Calculus and Their Applications to Inequality Theory, AIMS Math. 10 (2025), 30478–30506. https://doi.org/10.3934/math.20251337.

[45] W. Afzal, K. Shabbir, M. Arshad, J.K.K. Asamoah, A.M. Galal, Some Novel Estimates of Integral Inequalities for a Generalized Class of Harmonical Convex Mappings by Means of Center-Radius Order Relation, J. Math. 2023 (2023), 8865992. https://doi.org/10.1155/2023/8865992.

[46] C.P. Niculescu, L. Persson, Convex Functions and Their Applications, Springer, New York, 2006. https://doi.org/10.1007/0-387-31077-0.

[47] M. Tariq, S.K. Ntouyas, A.A. Shaikh, A Comprehensive Review of the Hermite–Hadamard Inequality Pertaining to Fractional Integral Operators, Mathematics 11 (2023), 1953. https://doi.org/10.3390/math11081953.

[48] M. Tariq, S.K. Ntouyas, A.A. Shaikh, A Comprehensive Review of the Hermite–Hadamard Inequality Pertaining to Quantum Calculus, Foundations 3 (2023), 340–379. https://doi.org/10.3390/foundations3020026.

[49] M. Tariq, S.K. Ntouyas, A.A. Shaikh, J. Tariboon, A Comprehensive Review of the Hermite-Hadamard Inequality Pertaining to Fractional Differential Operators, Surv. Math. Appl. 18 (2023), 223–257.

[50] M. Tariq, H. Ahmad, C. Cesarano, H. Abu-Zinadah, A.E. Abouelregal, et al., Novel Analysis of Hermite–Hadamard Type Integral Inequalities via Generalized Exponential Type m-Convex Functions, Mathematics 10 (2021), 31. https://doi.org/10.3390/math10010031.

[51] M. Tariq, Hermite-Hadamard Type Inequalities via p-Harmonic Exponential Type Convexity and Applications, Univ. J. Math. Appl. 4 (2021), 59–69.

[52] M. Tariq, S.K. Ntouyas, A.A. Shaikh, New Variant of Hermite–Hadamard, Fejér and Pachpatte-Type Inequality and Its Refinements Pertaining to Fractional Integral Operator, Fractal Fract. 7 (2023), 405. https://doi.org/10.3390/fractalfract7050405.

[53] R.K. Raina, On Generalized Wright's Hypergeometric Functions and Fractional Calculus Operators, East Asian Math. J. 21 (2005), 191–203.

[54] M.J. Vivas-Cortez, R. Liko, A. Kashuri, J.E. Hernández Hernández, New Quantum Estimates of Trapezium-Type Inequalities for Generalized ϕ-Convex Functions, Mathematics 7 (2019), 1047. https://doi.org/10.3390/math7111047.

[55] M. Vivas-Cortez, A. Kashuri, J.E. Hernández, Trapezium-Type Inequalities for Raina's Fractional Integrals Operator Using Generalized Convex Functions, Symmetry 12 (2020), 1034. https://doi.org/10.3390/sym12061034.

[56] H. Ahmad, M. Tariq, S.K. Sahoo, J. Baili, C. Cesarano, New Estimations of Hermite–Hadamard Type Integral Inequalities for Special Functions, Fractal Fract. 5 (2021), 144. https://doi.org/10.3390/fractalfract5040144.

[57] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, 2006. https://doi.org/10.1016/s0304-0208(06)x8001-5.

[58] F. Hassanzad, H. Mehri-Dehnavi, H. Agahi, Some Applications of the Hermite–Hadamard Inequality for Log-Convex Functions in Quantum Divergences, Math. Methods Appl. Sci. 45 (2022), 4899–4906. https://doi.org/10.1002/mma.8164.

[59] L. Horváth, Uniform Treatment of Integral Majorization Inequalities with Applications to Hermite-Hadamard-Fejér-Type Inequalities and f-Divergences, Entropy 25 (2023), 954. https://doi.org/10.3390/e25060954.