Int. J. Anal. Appl. (2026), 24:194

International Journal of Analysis and Applications

# Topological Characterization of $\alpha$-Filters in Paradistributive Latticoids

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**Abstract.** We study topological properties of the space of prime $\alpha$-filters of a paradistributive latticoid. Using the hull-kernel topology we characterize compactness, separation axioms ($T_0, T_1$, Hausdorff), and relate minimality of prime $\alpha$-filters to topological properties. Several equivalent conditions for minimality are obtained and relationships to annulets and direct factors are discussed. Examples and consequences for various classes of latticoids are indicated.

## 1. Introduction

Paradistributive latticoids (PDLs) were studied recently in the literature as a generalization of several distributive-like algebraic structures; see [1, 4]. Filters and ideals on such latticoids play an important role in understanding their algebraic and topological behaviour. In particular, special filters called $\alpha$-filters (introduced in [15]) capture a closure property relevant to duality-like correspondences and to the construction of prime spectrum spaces.

The aim of this paper is to investigate the topological space $Spec_\alpha(V)$ of prime $\alpha$-filters of a paradistributive latticoid $V$ equipped with the hull-kernel topology generated by sets of the form $K'(\mu) = \{P \in Spec_\alpha(V) : \mu \notin P\}$ for $\mu \in V$. We characterize compact open sets, show $T_0$-separation, and derive necessary and sufficient conditions for $T_1$ and Hausdorff properties; we also give

Received: Apr. 14, 2026.

2020 *Mathematics Subject Classification.* 06D99, 06D15.

*Key words and phrases.* $\alpha$-filter; paradistributive latticoid; prime filter; hull-kernel topology; minimal prime.

https://doi.org/10.28924/2291-8639-24-2026-194

ISSN: 2291-8639

several algebraic equivalent conditions ensuring that every prime $\alpha$-filter is minimal. Our work extends and refines results from [2,15] and connects them to classical ideas from lattice theory [5,9].

This paper is organized as follows. In Section 2, we recall basic definitions and notation, and summarize needed properties of $\alpha$-filters. Section 3, introduces the hull-kernel topology on $Spec_\alpha(V)$ and presents the main topological results (compactness of basic opens, characterization of compact-open sets, separation axioms). Also, we study minimal prime filters and give equivalent algebraic/topological conditions for minimality. We conclude with remarks and possible directions for future work.

## 2. Preliminaries

This section recalls fundamental notions and notations used throughout the paper, from [4,15].

**Definition 2.1.** *[4] An algebraic structure $(V, \vee, \wedge, 1)$ is a* Paradistributive Latticoid *(PDL) if, for all $\mu, \nu, \xi \in V$,*

$$\mu \vee (\nu \wedge \xi) = (\mu \vee \nu) \wedge (\mu \vee \xi), \quad (\mu \wedge \nu) \vee \xi = (\mu \vee \xi) \wedge (\nu \vee \xi),$$
$$(\mu \vee \nu) \wedge \nu = \nu, \quad (\mu \vee \nu) \wedge \mu = \mu, \quad \mu \vee (\mu \wedge \nu) = \mu, \quad \mu \vee 1 = 1.$$

We say that $\mu \leq \nu$ if $\mu \wedge \nu = \mu$ (equally $\mu \vee \nu = \nu$), and an element $\mu$ is *minimal*, whenever $\zeta \leq \mu \implies \zeta = \mu$.

**Example 2.1.** *[4] Fix $\mu_0 \in V$ and define*

$$\mu \vee \nu = \begin{cases} \mu, & \nu \neq \mu_0, \\ \mu_0, & \nu = \mu_0, \end{cases} \qquad \mu \wedge \nu = \begin{cases} \nu, & \nu \neq \mu_0, \\ \mu, & \nu = \mu_0. \end{cases}$$

*Then $(V, \vee, \wedge, \mu_0)$ is called a disconnected-PDL and $\mu_0$ is the greatest element.*

**Definition 2.2.** *[4] A non-void subset $F$ of $V$ is a* filter*, if*

$$\mu \wedge \nu \in F, \quad \xi \vee \mu \in F, \quad \forall\, \mu, \nu \in F, \xi \in V$$

$[\mu) = \{\nu \vee \mu : \nu \in V\}$ is the filter generated by $\mu$.

**Definition 2.3.** *[4] A non-void subset $I$ of $V$ is an* ideal*, if*

$$\mu \vee \nu \in I, \quad \mu \wedge \xi \in I, \quad \forall\, \mu, \nu \in I, \xi \in V$$

$(\mu] = \{\mu \wedge \nu : \nu \in V\}$ is the ideal generated by $\mu$.

**Definition 2.4.** *[15] A filter $F$ is said to be an $\alpha$-filter, if, $(\mu)^{\bullet\bullet} \subseteq F$, for all $\mu \in F$, where $(\mu)^{\bullet} = \{\nu \in V : \mu \vee \nu = 1\}$.*

Given a filter $F$ of $V$, $F^e = \{\mu \in V : (\nu)^{\bullet} \subseteq (\mu)^{\bullet},$ for some $\nu \in F\}$ [15] is the least $\alpha$-filter holding $F$. Moreover $F^e = F \iff F$ is an $\alpha$-filter.

## 3. Topological characterization of $\alpha$-filters

This main section develops the topological theory of $Spec_{\alpha}(V)$, the class of prime $\alpha$-filters of $V$. Using Zorn's lemma, Theorem 3.1 establishes the existence of a prime $\alpha$-filter extending any given $\alpha$-filter that is disjoint from an ideal. The hull-kernel topology is introduced via sets $K'(\mu) = \{P \in Spec_{\alpha}(V) : \mu \notin P\}$, and Lemma 3.5 shows that these sets form a basis. Subsequent theorems characterize compactness, compact-open subsets, and the $T_0$ property of $Spec_{\alpha}(V)$. Furthermore, minimal and maximal prime $\alpha$-filters are examined, proving that every maximal $\alpha$-filter is prime. Several equivalent algebraic and topological conditions for the minimality of prime $\alpha$-filters are derived (Theorem 3.13), linking them to the $T_1$ and Hausdorff separation properties. Theorems 3.14–3.18 further refine the relationships between compactness, separation, and the lattice structure of $V$. These results generalize earlier work on prime filters in PDLs [1,2] and connect with recent studies on filters in nearlattices [6,8,11].

**Theorem 3.1.** *If $I$ is an ideal and $F$ is an $\alpha$-filter of $V$ with $F \cap I = \emptyset$, then there is a prime $\alpha$-filter $P$ with $P \cap I = \emptyset$ and $F \subseteq P$.*

*Proof.* Consider $\Sigma := \{G : G \text{ is an } \alpha\text{-filter}, F \subseteq G, \text{ and } G \cap I = \emptyset\} \neq \emptyset$, since $F \in \Sigma$. Let $\{F_i\}_{i\in\triangle}$ be a chain in $\Sigma$. Then $\bigcup_{i\in\triangle} F_i$ is a filter, since each $F_i$ is a filter. Choose $\mu \in \bigcup_{i\in\triangle} F_i$. Then $\mu \in F_j$, for some $j \in \triangle$. Since $F_j$ is an $\alpha$-filter, $(\mu)^{\bullet\bullet} \subseteq F_j \subseteq \bigcup_{i\in\triangle} F_i$. Therefore $\bigcup_{i\in\triangle} F_i$ is an $\alpha$-filter holding $F$ and $\bigcup_{i\in\triangle} F_i \cap I = \emptyset$. By Zorn's Lemma, there is a maximal element $M$ (say) in $\Sigma$. Choose $\mu, \nu \in V$ such that $\mu \notin M$ and $\nu \notin M$. Then $\mu \notin M \Rightarrow M \subset M \vee [\mu) \subseteq (M \vee [\mu))^e \cap I \neq \emptyset$ (since $M$ is maximal). Similarly, $(M \vee [\nu))^e \cap I \neq \emptyset$, since $\nu \notin M$. Choose $\xi \in (M \vee [\mu))^e \cap I$ and $\nu \in (M \vee [\nu))^e \cap I$. Then $\xi \vee \nu \in I$, and $\xi \vee \nu \in (M \vee [\mu))^e \cap (M \vee [\nu))^e = (M \vee [\mu \vee \nu))^e$. Suppose $\mu \vee \nu \in M$. Then $\xi \vee \nu \in M^e = M$, which implies $\xi \vee \nu \in M \cap I$, and absurd. Thus $M$ is a prime $\alpha$-filter. □

**Corollary 3.1.** *If $F$ is an $\alpha$-filter and $\mu \notin F$, then, $F \subseteq P$ and $\mu \notin P$, for some prime $\alpha$-filter $P$.*

*Proof.* Clearly $(\mu] \cap F = \emptyset$. By Theorem 3.1, there is a prime $\alpha$-filter $P$ with $F \subseteq P$ and $(\mu] \cap P = \emptyset$. Therefore $\mu \notin P$. □

The following corollaries 3.2, 3.3 follows directly from Theorem 3.1.

**Corollary 3.2.** *Given an $\alpha$-filter $F$ of $V$,*

$$F = \bigcap \{P : F \subseteq P \text{ and } P \text{ is a prime } \alpha\text{-filter}\}$$

**Corollary 3.3.** *The intersection of prime $\alpha$-filters is equal to $\{1\}$.*

Let $Spec_{\alpha}(V)$ denote the class of prime $\alpha$-filters of $V$. Given $A \subseteq V$, define $K'(A) := \{P \in Spec_{\alpha}(V) : A \nsubseteq P\}$, and for $\mu \in V$, set $K'(\mu) := K'(\{\mu\})$.

**Lemma 3.1.** *For any $\mu, \nu \in V$,*

(1) $\bigcup_{\mu\in V} K'(\mu) = Spec_{\alpha}(V)$

(2) $K'(\mu) \cap K'(\nu) = K'(\mu \vee \nu)$
(3) $K'(\mu) \cup K'(\nu) = K'(\mu \wedge \nu)$
(4) $K'(\mu) = \emptyset \iff \mu \neq 1$
(5) $K'(0) = Spec_\alpha(V)$

From Lemma 3.1, it is evident that, the family $\{K'(\mu) : \mu \in V\}$ constitutes a basis for a topology on $Spec_\alpha(V)$, known as the hull-kernel topology.

**Theorem 3.2.** *In $V$, we have*

(1) *$K'(\mu)$ is compact in $Spec_\alpha(V)$, for all $\mu \in V$*
(2) *Every compact open set is in the form $K'(\mu)$, for some $\mu \in V$*
(3) *$Spec_\alpha(V)$ is a $T_0$-space*
(4) *The map $\mu \longmapsto K'(\mu)$ is an anti-homomorphism from $V$ onto the lattice of compact open subsets of $Spec_\alpha(V)$.*

*Proof.* (1). Suppose that $K'(\nu) \subseteq \bigcup_{\mu \in A} K'(\mu)$, where $\nu \in V$ and $A \subseteq V$. If $\nu \notin [A)^e$, where $[A)$ is the filter generated by $A$, then, by Theorem 3.1, there exits a prime $\alpha$-filter $P$ such that $[A)^e \subseteq P$ and $\nu \notin P$. Therefore $P \in K'(\nu) \subseteq \bigcup_{\mu \in A} K'(\mu)$. So that $\mu \notin P$, for some $\mu \in A$. Which is a contradiction. Therefore $\nu \in [A)^e$ and hence $\nu \in [\mu)^e$, for some $\mu \in [A)$ and $\mu = \mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n$, for some $\mu_1, \mu_2, \cdots, \mu_n \in A$. Thus $K'(\nu) \subseteq \bigcup_{i=1}^{n} K'(\mu_i)$, and it is compact.

(2). If $C$ is a compact-open subset of $Spec_\alpha(V)$, then, it is in the form $\bigcup_{i=1}^{n} K'(\mu_i) = K'(\bigwedge_{i=1}^{n} \mu_i)$. Therefore $C = K'(\mu)$ for some $\mu \in V$.

(3). Suppose that $P, Q$ are two prime $\alpha$-filters, and $\mu \in P$ but $\mu \notin Q$. Then $P \notin K'(\mu)$ and $Q \in K'(\mu)$. Therefore $Spec_\alpha(V)$ is a $T_0$-space.

(4). It is trivial from by Lemma 3.1 (1), (2). □

It is well known that minimal prime filters always exist in $V$, and that each of them is precisely the set-theoretic complement of a maximal ideal of $V$. In what follows, we establish several equivalent conditions under which a prime $\alpha$-filter of $V$ possessing minimal elements becomes a minimal prime filter. To this end, we first present the following lemmas.

**Lemma 3.2.** *[2] A prime filter $P$ is minimal if and only if for each $\mu \in P$, there is $\nu \notin P$ with $\mu \vee \nu = 1$.*

**Lemma 3.3.** *Every minimal prime $\alpha$-filter is a minimal prime filter.*

*Proof.* Since every minimal prime filter is an $\alpha$-filter, it is obvious. □

**Lemma 3.4.** *The intersection of minimal prime filters is $\{1\}$.*

*Proof.* Let $\mu \neq 1$. Then there is a maximal ideal $M$ containing $\mu$. So, $V \setminus M$ is a minimal prime filter with $\mu \notin V \setminus M$. Hence the result holds. □

**Definition 3.1.** *A proper $\alpha$-filter $M$ of $V$ is called maximal, if there is no proper $\alpha$-filter $M_0$ with $M \subset M_0$.*

**Lemma 3.5.** *Every maximal $\alpha$-filter is prime.*

*Proof.* Let $M$ be a maximal $\alpha$-filter of $V$ such that $\mu \notin M$ and $\nu \notin M$. Then $M \subseteq M \vee [\mu) \subseteq (M \vee [\mu))^e$ and $M \subseteq M \vee [\nu) \subseteq (M \vee [\nu))^e$. Since $M$ is maximal, $(M \vee [\mu))^e = V$ and $(M \vee [\nu))^e = V$. Now, $V = (M \vee [\mu))^e \cap (M \vee [\nu))^e = ((M \vee [\mu)) \cap (M \vee [\nu)))^e = (M \vee [\mu \vee \nu))^e$. Which implies that $\mu \vee \nu \notin M$. Thus $\mu \vee \nu \notin M$, whenever $\mu, \nu \notin M$. Hence $M$ is prime. □

Given $A \subseteq V$, $H(A) = \{P \in Spec_\alpha(V) : A \subseteq P\}$ is a closed set in $Spec_\alpha(V)$ (since $H(A) = Spec_\alpha(V) \setminus K'(A)$). Moreover, every closed set is in the form $H(A)$, for some $A \subseteq V$.

**Theorem 3.3.** *For any $X \subseteq Spec_\alpha(V)$, $\overline{X} = H(\bigcap_{P \in X} P)$ .*

*Proof.* Let $X \subseteq Spec_\alpha(V)$ and $Q \in X$. Since $\bigcap_{P \in X} P \subseteq Q$, $Q \in H(\bigcap_{P \in X} P)$. So that $H(\bigcap_{P \in X} P)$ is a closed set containing $X$. Now, let $C$ be a closed set containing $X$. Then $C = H(A)$, for some $A \subseteq V$. Since $X \subseteq H(A)$, $A \subseteq P$, for every $P \in X$. Therefore $A \subseteq \bigcap_{P \in X} P$, which implies $H(\bigcap_{P \in X} P) \subseteq H(A) = C$. Hence $H(\bigcap_{P \in X} P)$ is the smallest closed set containing $X$. □

**Theorem 3.4.** *The following are equivalent in $V$,*

(1) *Every prime $\alpha$-filter is maximal*
(2) *Every prime $\alpha$-filter is minimal.*

*Proof.* (1) $\Rightarrow$ (2) : Assume (1). Let $P \in Spec_\alpha(V)$. To show that $P$ is minimal, suppose that $Q \in Spec_\alpha(V)$ and $Q \subseteq P$. Since $Q$ is also a prime $\alpha$-filter, by the assumption of maximality, we must have $Q = P$ (otherwise $Q \subsetneq P$ contradicts the maximality of $Q$). Hence no prime $\alpha$-filter is properly contained in $P$, and so $P$ is minimal.

(2) $\Rightarrow$ (1) : Assume (2). Let $P \in Spec_\alpha(V)$ and suppose that $P \subseteq Q$ for some $Q \in Spec_\alpha(V)$. Since $Q$ is prime and minimal by assumption, it follows that $P = Q$ (otherwise $P \subsetneq Q$ contradicts the minimality of $Q$). Hence no prime $\alpha$-filter properly contains $P$, so $P$ is maximal. □

Although a prime $\alpha$-filter need not be minimal in general, we identify below a collection of identical conditions under which every prime $\alpha$-filter becomes minimal.

**Theorem 3.5.** *The following conditions are identical in $V$:*

(1) *Every prime $\alpha$-filter is minimal.*
(2) *$Spec_\alpha(V)$ is a $T_1$-space.*
(3) *$Spec_\alpha(V)$ is Hausdorff.*
(4) *For any $\mu, \nu \in V$, there is $\xi \in V$ with $\mu \vee \xi = 1$ and*

$$K'(\nu) \cap \Big(Spec_\alpha(V) \setminus K'(\mu)\Big) = K'(\nu \vee \xi).$$

*Proof.* (1) $\Rightarrow$ (2) : Assume (1). Let $P$ and $Q$ be two distinct prime $\alpha$-filters. Since $P$ and $Q$ are minimal (by (1)), $P \nsubseteq Q$ and $Q \nsubseteq P$. Choose $\mu \in P \setminus Q$ and $\nu \in Q \setminus P$. Then $Q \in K'(\mu) \setminus K'(\nu)$ and $P \in K'(\nu) \setminus K'(\mu)$. Therefore (2) holds.

$(2) \Rightarrow (3)$ : Assume (2). Let $P$ be a prime $\alpha$-filter of $V$. By Theorem 3.3, $\{P\} = \overline{\{P\}} = \{Q \in Spec_\alpha(V) : P \subseteq Q\}$. Therefore $P$ is maximal. Hence each prime $\alpha$-filer $P$ is minimal prime $\alpha$-filter. Let $P, Q \in Spec_\alpha(V)$ with $P \neq Q$. Choose $\mu \in P$ such that $\mu \notin Q$. Since $P$ is minimal, there is $\nu \notin P$ with $\mu \vee \nu = 1$. Therefore $P \in K'(\nu)$, $Q \in K'(\mu)$, and $K'(\mu) \cap K'(\nu) = K'(\mu \vee \nu) = \emptyset$. Thus (3) holds.

$(3) \Rightarrow (4)$ : Assume (3). Then $K'(\mu)$ is a compact subset of $Spec_\alpha(V)$ for each $\mu \in V$, and hence $K'(\mu)$ is clopen. Let $\mu, \nu \in V$ with $\mu \neq \nu$. Then $K'(\nu) \cap \big(Spec_\alpha(V) \setminus K'(\mu)\big)$ is a compact subset of the compact space $K'(\nu)$. Since $K'(\nu)$ is open in $Spec_\alpha(V)$, this intersection is a compact open subset of $Spec_\alpha(V)$. Hence, by Theorem 3.2(2), there is $\xi \in V$ with $K'(\xi) = K'(\nu) \cap \big(Spec_\alpha(V) \setminus K'(\mu)\big)$. Therefore, $K'(\nu) \cap \big(Spec_\alpha(V) \setminus K'(\mu)\big) = K'(\nu) \cap K'(\xi) = K'(\nu \vee \xi)$. Also, $K'(\mu \vee \xi) = K'(\mu) \cap K'(\xi) = \emptyset$, and hence $\mu \vee \xi = 1$.

$(4) \Rightarrow (1)$ : Let $P$ be a prime $\alpha$-filter of $V$. Choose $\mu, \nu \in V$ with $\mu \in P$ and $\nu \notin P$. By (4), there is $\xi \in V$ with $\mu \vee \xi = 1$ and $K'(\nu) \cap \big(Spec_\alpha(V) \setminus K'(\mu)\big) = K'(\nu \vee \xi)$. Clearly, $P \in K'(\nu) \cap \big(Spec_\alpha(V) \setminus K'(\mu)\big) = K'(\nu \vee \xi)$. If $\xi \in P$, then $\nu \vee \xi \in P$, which contradicts $P \in K'(\nu \vee \xi)$. Therefore $\xi \notin P$ and hence, given $\mu \in P$, $\mu \vee \xi = 1$, for some $\xi \notin P$. Thus $P$ is minimal. □

Here, an additional equal condition is established for the space $Spec_\alpha(V)$ to be Hausdorff.

**Theorem 3.6.** *For any $\mu \in V$,*

(1) $K'([\mu)) = K'(\mu) = K'((\mu)^{\bullet\bullet})$
(2) $K'((\mu)^{\bullet\bullet}) \subseteq Spec_\alpha(V) \setminus K'((\mu)^{\bullet})$

*Proof.* (1). Let $P \in K'([\mu))$. Then $[\mu) \nsubseteq P$, that is, there exists $\nu \in [\mu)$ such that $\nu \notin P$. If $\mu \in P$, then $\nu = \nu \vee \mu \in P$, which is a contradiction. Therefore $\mu \notin P$ and then $P \in K'(\mu)$. Hence $K'([\mu)) \subseteq K'(\mu)$. On the other hand, let $P \in K'(\mu)$. Then $\mu \notin P$ and $[\mu) \nsubseteq P$. Therefore $P \in K'([\mu))$ and hence $K'([\mu)) = K'(\mu)$.

To prove that $K'([\mu)) = K'((\mu)^{\bullet\bullet})$. Let $P \in K'([\mu))$. Then $[\mu) \nsubseteq P$ and hence $(\mu)^{\bullet\bullet} \nsubseteq P$. Therefore $P \in K'((\mu)^{\bullet\bullet})$ and hence $K'([\mu)) \subseteq K'((\mu)^{\bullet\bullet})$. On the other hand, let $P \in K'((\mu)^{\bullet\bullet})$. Then $(\mu)^{\bullet\bullet} \nsubseteq P$. Since $P$ is an $\alpha$-filter, $\mu \notin P$. Therefore $P \in K'(\mu)$ and hence $K'((\mu)^{\bullet\bullet}) \subseteq K'(\mu)$.

(2). Let $P \in K'((\mu)^{\bullet\bullet})$. Then $(\mu)^{\bullet\bullet} \nsubseteq P$. Since $P$ is an $\alpha$-filter, $\mu \notin P$. Therefore $(\mu)^{\bullet} \subseteq P$ and hence $P \in Spec_\alpha(V) \setminus K'((\mu)^{\bullet})$. Thus $K'((\mu)^{\bullet\bullet}) \subseteq Spec_\alpha(V) \setminus K'((\mu)^{\bullet})$. □

It is evident that $Spec_\alpha(V)$ is a $T_0$-space. In what follows, we establish a necessary and sufficient condition for $Spec_\alpha(V)$ to be a $T_1$-space.

**Theorem 3.7.** *The space $Spec_\alpha(V)$ is $T_1$ if and only if every prime $\alpha$-filter of $V$ is maximal.*

*Proof.* ($\Rightarrow$) Assume that $Spec_\alpha(V)$ is a $T_1$-space. Let $P$ be a prime $\alpha$-filter of $V$. Suppose, towards a contradiction, that $P$ is not maximal. Then there exists a proper $\alpha$-filter $Q$ such that $P \subset Q$. Select $\mu \in Q \setminus P$. Since $\mu \notin P$, we have $P \in K'(\mu)$, while $\mu \in Q$ implies $Q \notin K'(\mu)$. Thus every open set containing $P$ (for instance, $K'(\mu)$) fails to contain $Q$, whereas any open set containing $Q$ contains $P$ as well, because if $\nu \notin Q$ then $\nu \notin P$ (as $P \subset Q$), and hence $K'(\nu)$ contains both $P$ and $Q$. Therefore

$Q$ lies in the closure of $\{P\}$, so $\{P\}$ is not closed contradicting the $T_1$ property of $Spec_\alpha(V)$. Hence $P$ must be maximal.

($\Leftarrow$) Conversely, assume that the condition hods. Let $P, Q \in Spec_\alpha(V)$ with $P \neq Q$. Since they are distinct, there exists $\mu \in P \setminus Q$. Then $Q \in K'(\mu)$ but $P \notin K'(\mu)$, so $K'(\mu)$ is an open set containing $Q$ but not $P$. Similarly, as $P \neq Q$, there is $\nu \in Q \setminus P$, and thus $K'(\nu)$ contains $P$ but not $Q$. Therefore, each of the points $P$ and $Q$ has an open neighbourhood that excludes the other. Hence $Spec_\alpha(V)$ is a $T_1$-space. □

**Theorem 3.8.** *Assume that every annulet is a direct factor of $V$. Then the following conditions are equivalent:*

(1) *$Spec_\alpha(V)$ is a Hausdorff space.*
(2) *For any two distinct prime $\alpha$-filters $P$ and $Q$ in $V$, there is elements $\mu, \nu \in V$ with $(\mu)^\bullet \subseteq P$ and $(\nu)^\bullet \subseteq Q$, and moreover, there is no $R \in Spec_\alpha(V)$ with $\mu \vee \nu \notin R$.*

*Proof.* $(1) \Rightarrow (2)$ : Assume (1). Let $P, Q \in Spec_\alpha(V)$ such that $P \neq Q$. Since $Spec_\alpha(V)$ is Hausdorff, there are disjoint open sets $K'(\mu)$ and $K'(\nu)$ with $P \in K'(\mu)$ and $Q \in K'(\nu)$, with $K'(\mu) \cap K'(\nu) = \emptyset$. Since $P \in K'(\mu) = K'((\mu)^{\bullet\bullet})$, we have $(\mu)^{\bullet\bullet} \nsubseteq P$. Hence we may choose $\xi \in (\mu)^{\bullet\bullet}$ such that $\xi \notin P$. Because $(\mu)^\bullet \subseteq (\xi)^\bullet$ and $(\xi)^\bullet \subseteq P$, it follows that $(\mu)^\bullet \subseteq P$. By symmetry, we likewise obtain $(\nu)^\bullet \subseteq Q$. Now assume there exists $R \in Spec_\alpha(V)$ such that $\mu \vee \nu \notin R$. Then $R \in K'(\mu \vee \nu) = K'(\mu) \cap K'(\nu)$, contradicting $K'(\mu) \cap K'(\nu) = \emptyset$. Thus no such $R$ exists, completing the proof of (2).

$(2) \Rightarrow (1)$ : Assume (2) holds. Let $P \neq Q$ in $Spec_\alpha(V)$. By (2), there are $\mu, \nu \in V$ with $(\mu)^\bullet \subseteq P$, $(\nu)^\bullet \subseteq Q$, and there exists no $R \in Spec_\alpha(V)$ with $\mu \vee \nu \notin R$. Since every annulet is a direct factor of $V$, both $(\mu)^\bullet$ and $(\nu)^\bullet$ are direct factors. Thus, $(\mu)^\bullet \vee (\mu)^{\bullet\bullet} = V$ and $(\nu)^\bullet \vee (\nu)^{\bullet\bullet} = V$. If $\mu \in P$, then, as $P$ is an $\alpha$-filter, $(\mu)^{\bullet\bullet} \subseteq P$, implying $V = (\mu)^\bullet \vee (\mu)^{\bullet\bullet} \subseteq P$, which is impossible. Hence $\mu \notin P$, and therefore $P \in K'(\mu)$. Similarly, $\nu \notin Q$, so $Q \in K'(\nu)$. To show that these open sets separate $P$ and $Q$, suppose that $K'(\mu) \cap K'(\nu) \neq \emptyset$. Then there exists some $R \in Spec_\alpha(V)$ with $R \in K'(\mu) \cap K'(\nu) = K'(\mu \vee \nu)$, which means $\mu \vee \nu \notin R$, contradicting the assumption in (2). Thus $K'(\mu)$ and $K'(\nu)$ are disjoint neighbourhoods of $P$ and $Q$, respectively. Hence $Spec_\alpha(V)$ is Hausdorff. □

**Theorem 3.9.** *For any $X(\neq \emptyset) \subseteq Spec_\alpha(V)$ with $\bigcap_{P \in X} P = \{1\}$, $X$ is Hausdorff if and only if for each $P \in X$, $P$ is the unique member in $X$ that containing $O^\bullet(P)$.*

*Proof.* Suppose that $X$ is a Hausdorff space. Let $P \in X$. Then $O^\bullet(P) \subseteq P$. Let $Q \in X$ such that $O^\bullet(P) \subseteq Q$ and $P \neq Q$. Since $X$ is Hausdorff, there are $\mu, \nu \in V$ with $P \in K'(\mu)$ and $Q \in K'(\nu)$ and $K'(\mu \vee \nu) = K'(\mu) \cap K'(b) \cap X = \emptyset$. Therefore $\mu \vee \nu \in R$, for all $R \in X$. Since $\bigcap_{P \in X} P = \{1\}$, $\mu \vee \nu = 1$. Since $\mu \in P$, $\nu \in O^\bullet(P) \subseteq Q$, which is a contradiction to the fact that $\nu \notin Q$. Therefore $P$ is the unique member in $X$ such that $O^\bullet(P) \subseteq P$.

Conversely, suppose that $P$ is the unique member in $X$ such that $O^\bullet(P) \subseteq P$. Let $P_1$ and $P_2$ be two distinct elements in $X$. By hypothesis, $O^\bullet(P_1) \nsubseteq P_2$. Choose $\xi \in O^\bullet(P_1)$ such that $\xi \notin P_2$. Since

$\xi \in O^{\bullet}(P_1)$, there exists $\nu \notin P_1$ such that $\xi \vee \nu = 1$. Therefore $P_1 \in K'(\nu)$ and $P_2 \in K'(\mu)$ and $K'(\xi) \cap K'(\nu) \cap X = K'(\xi \vee \nu) \cap X = \emptyset$. Hence $X$ is a Hausdorff space. □

## 4. Conclusions and future work

We have given a topological framework for studying prime $\alpha$-filters on paradistributive latticoids and obtained several equivalent algebraic and topological conditions for minimality and separation axioms. Future directions include: (1) studying spectral-like properties and possible representation theorems, (2) extending to related classes such as almost distributive lattices and nearlattices, and (3) investigating duality with appropriate ring- or space-like objects. Connections to the recent literature [6–8] suggest further avenues.

**Conflict of Interest.** The authors declare that there is no conflict of interest regarding the publication of this paper.

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