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Results on Reversible Properties in Skew Monoid Rings

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Abstract. This paper introduces the concept of σ-skew strongly M-nil-reversible rings as a unified generalization of strongly M-reversible and M-compatible rings. We then examine the reversibility properties of certain monoid ring extensions. Specifically, for an M-Armendariz and σ-skew Armendariz ring R, we show that R is σ-skew strongly M-nil-reversible. Furthermore, we establish necessary conditions under which the skew monoid ring $R * M$ inherits this property, particularly when R is an *NI*-ring and M is a *u.p.*-monoid. Additionally, we investigate the persistence of this condition under factor rings (over strictly ordered monoids) and classical right quotient rings (over Ore rings). Finally, an explicit counterexample of 2×2 matrices over a field F is provided to show that this property does not hold in general.

1. Introduction

In this paper, the symbols R and M represent an associative ring with identity and a monoid, respectively. The notion of a reversible ring was first introduced by Cohn in [1], defining a ring R as reversible if $bd = 0$ implies $db = 0$ for all $b, d \in R$. Anderson and Camillo [2] referred to this property as ZC_2. The study in [3] demonstrated that reversible rings do not necessarily result in reversible polynomial rings. A ring is considered reduced if it contains no non-zero nilpotent elements as defined in [3], meaning that $b^2 = 0$ implies $b = 0$ for all $b \in R$. In [4], a strongly reversible ring is defined as one where polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$ implies $g(x)f(x) = 0$. It should be noted that every reduced rings are strongly reversible, but the converse is not true. The concept of an Armendariz ring was introduced by Rege and Chhawchharia [6]. They defined a ring R to be Armendariz if the product of any two polynomials $f(x)a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$ in $R[x]$ equals zero only

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if the product of any two coefficients a_i and b_j equals zero. A ring is said to be semicommutative if for all elements b and d in the ring, their product being zero implies that their right and left annihilators are also zero. According to [26], a ring R is said to be an M-Armendariz if any nonzero element $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ in $R[M]$ satisfy $\psi\varphi = 0$, then for all i, j, it follows that $b_ia_j = 0$. Furthermore, a ring R is strongly M-reversible if for any nonzero elements $\varphi, \psi \in R[M]$ such that $\varphi\psi = 0$, it holds that $b_ia_j = 0$, then $\psi\varphi = 0$ for all $\varphi, \psi \in R[M]$ [7]. Recall from [24] that a ring R is skew strongly M-reversible, if for any $\varphi, \psi \in R * M$ the products of tow elements $\varphi\psi$ equal 0 implies $\psi\varphi = 0$. Recently, the Armendariz property of a ring was extended to skew polynomial rings but with skewed scalar multiplication in [5, 8, 9]. For a ring R, if σ is an endomorphism, then R is said to be σ-Armendariz (resp., σ-skew Armendariz) if for $p = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m, q = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$ in $R[x, \sigma]$, $pq = 0$ implies $a_ib_j = 0$ (resp., $a_i\sigma^i(b_j) = 0$) for all $0 \leq i \leq m$ and $0 \leq j \leq n$. The investigation of the composition of the collection of nilpotent elements in noncommutative constructions is a crucial and highly active field in noncommutative algebra, as evidenced by numerous studies such as [13], [14].

This paper is devoted various conditions under which a ring R is σ-skew strongly M-*nil*-reversible, where M is a monoid and σ is a monoid homomorphism from M to the automorphism group of R. We discuss the basic properties of skew monoid rings of the form $R * M$. We prove that, if R is M-Armendariz (σ-skew Armendariz, respectively), then R is σ-skew strongly M-*nil*-reversible. Moreover, we consider different conditions under which a skew monoid ring $R * M$ is σ-skew strongly M-*nil*-reversible when $\sigma : M \to Aut(R)$ is a monoid homomorphism. We also prove that if R is any NI-ring and M is a $u.p$-monoid, then R is σ-skew strongly M-*nil*-reversible. Furthermore, for a strictly ordered monoid M, if R/I is $\bar{\sigma}$-skew strongly M-*nil*-reversible, then R is σ-skew strongly M-*nil*-reversible. In addition, we demonstrate that if R is an Ore ring with classical right quotient ring Q, then R is σ-skew strongly M-*nil*-reversible if and only if Q is $\bar{\sigma}$-skew strongly M-*nil*-reversible. Finally, the paper provides example and discusses some results on the skew monoid ring $R * M$.

In this context, for a ring R and a $u.p.$-monoid M. It is assumed that M operates on R through a homomorphism that maps to the automorphism group of R. This homomorphism is denoted as $\sigma : M \to Aut(R)$, and for any given element $g \in M$, the image of an element $r \in R$ under σ_g is denoted as $\sigma_g(r)$. By using the monoid homomorphism σ, we can create a skew monoid ring denoted as $R * M$. This ring consists of finite formal combinations of elements in M, represented as $\Sigma_{g\in M}x_gg$. Multiplication in this ring is induced by the formula $(x_gg)(y_hh) = (x_g\sigma_g(y_h))(gh)$. Consequently, $R * M$ is a free left R-module with basis M as elaborated in [25]. A commonly accepted fact is that if a polynomial $f(x)$ is defined over a commutative ring, then it is nilpotent only if every coefficient of $f(x)$ is nilpotent. However, it should be noted that this statement does not hold true for noncommutative rings. Motivated by the results of [4], [7], [12], [17], [21], [23] and [24] this paper is devoted to introduce and study the concept of σ-skew strongly M-*nil*-reversible rings in monoid rings $R * M$, which is a generalization of strongly reversible, strongly

M-reversible and skew strongly M-reversible. To recall, a monoid M is called a unique product ($u.p.$)-monoid if whenever two subsets X and Y not finite of M, to fined tow element u belong to X and element v belong to Y such that their product uv is distinct from the product of any other pair $(b, d) \in X \times Y$, i.e., $(b, d) \neq (u, v)$ implies $bd \neq uv$. The element uv is termed as a $u.p.$-element of the set $XY = \{pq : p \in X, q \in Y\}$. Unique product monoids and groups have significant implications in ring theory, particularly in providing a positive solution to the zero-divisor problem for group rings, for more details see [10, 18]. The category of monoids that fall under the classification of $u.p.$-monoids is both extensive and significant. For a positive integer n, let $Mat_n(R)$ denote the full square matrices ring, n rows and n columns. We write $R[x]$ and $S_n(R)$, for the polynomial ring over a ring R and the subring consisting of all upper triangular matrices over a ring R with equal main diagonal entries. According to [7], A ring R is said to be strongly reversible relative to a monoid M, if the tow elements $\varphi, \psi \in R[M]$ satisfy $\varphi\psi = 0$, then $\psi\varphi = 0$. By Definition 2.3 [16], a monoid homomorphism $\sigma : M \to Aut(R)$ is called compatible if the ring R is σ_g-compatible for each $g \in M$, that is $bd = 0 \Leftrightarrow b\sigma_g(d) = 0$ for all $b, d \in R$. Now we have the following generalization of reversible rings.

2. Reversible Properties within Skew Monoid Rings

In this section, we introduce the concept of σ-skew strongly M-*nil*-reversible rings, which serves as a natural unification and generalization of several established classes of rings. Specifically, this notion extends the concepts of Extension of strongly α-reversible rings studied by Liang and X. Zhu [29], *nil*-reversible rings introduced by Subba and Subedi [28], and strongly Reversible Rings Relative to a Monoid rings investigated by A.B. Singh et. al [7]. We then explore various necessary and sufficient conditions for a ring to satisfy this property within the framework of skew monoid rings $R * M$. Furthermore, we examine the relationships between this new notion and several well-established properties in ring theory.

Definition 2.1. *We say that a ring R is σ-skew strongly M-nil-reversible (σ-skew strongly nil-reversible relative to a monoid M), if $\varphi\psi \in nil(R) * M$ implies that $b_i\sigma_{g_i}(\sigma_s(a_j)) \in nil(R)$, then $\psi\varphi \in nil(R) * M$, where $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n$, $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m \in R * M$ for all i, j and $s, g, h \in M$.*

If $M = (\mathbb{N} \cup \{0\}, +)$ and $\sigma_h = id_R$ for all $h \in M$, then a ring R is σ-skew strongly M-*nil*-reversible if and only if R is strongly *nil*-reversible. Also, if $M = \{e\}$ and $\sigma_h = id_R$ for all $h \in M$, then any ring R is σ-skew strongly M-*nil*-reversible. Let $\varphi = c_1h_1 + c_2h_2 + \cdots + c_nh_n \in R[M]$. The element $\varphi \in nil(R)[M]$ if and only if $c_i \in nil(R)$ for all $1 \leq i \leq n$. Also, by $\varphi \in nil(R) * M$ we mean φ is a nilpotent element in skew monoid ring $R * M$. For any $\varphi \in R * M$, we denote by C_φ the set of all coefficients of φ for more details see [16].

A monoid $(M, \leq)$ is considered strictly ordered if for any $\alpha_1, \alpha_2, \beta \in M$ where $\alpha_1 < \alpha_2$, it follows that $\alpha_1\beta < \alpha_2\beta$ and $\beta\alpha_1 < \beta\alpha_2$.

Definition 2.2. *We say that a ring R is strongly M-nil-reversible ring (strongly nil-reversible ring relative to a monoid M), if $\varphi\psi \in nil(R)[M]$, where $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m \in R[M]$, then $\psi\varphi \in nil(R)[M]$ for all $1 \leq i \leq n, 1 \leq j \leq m$.*

Theorem 2.3. *Let R be an NI-ring, M be a u.p.-monoid and $\sigma : M \to Aut(R)$ a monoid homomorphism. If R is a compatible, then R is σ-skew strongly M-nil-reversible.*

Proof. Let $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ be nonzero elements in $R * M$ such that $\varphi\psi \in nil(R) * M$. Then we have $C_{\varphi\psi} \in nil(R)$. We will show that $\psi\varphi \in nil(R) * M$. Now it is enough to show $a_j\sigma_{h_j}(\sigma_s(b_i)) \in nil(R)$ for each $1 \leq i \leq n, 1 \leq j \leq m$ and $s \in M$. We proceed by induction on both n and m. By using freely σ is compatible monoid homomorphism. Let $m = 1$ and hence $\psi = a_1h_1$. Since M is a *u.p.*-monoid then by Lemma 1.1 [15], $h_1g_i \neq h_1g_j$ for $i \neq j$. So $a_1\sigma_{h_1}(b_i) \in nil(R)$ for each i. The proof of the case $n = 1$ is similar. Assuming that M is a unique presentation monoid and $m, n \geq 2$, there exists a pair of indices i and j such that $1 \leq i \leq n$ and $1 \leq j \leq m$, and the product g_ih_j can be uniquely expressed by considering two subsets, $B = \{g_1, g_2, \ldots, g_n\}$ and $A = \{h_1, h_2, \ldots, h_m\}$ of the monoid M. It is safe to assume that, without affecting the overall outcome, we can consider the statement as follows: $i = n$ and $j = m$. Thus $a_m\sigma_{h_m}(b_n) \in nil(R)$. Hence there exist a positive integer $\gamma \geq 1$ such that $(a_m\sigma_{h_m}(b_n))^\gamma = 0$. Then $(a_m\sigma_{h_m}(b_n))(a_m\sigma_{h_m}(b_n)) \cdots (a_m\sigma_{h_m}(b_n)) = 0$. Now, since by hypothesis a compatibility, we have that $(a_m\sigma_{h_m}(b_n)) \cdots (a_m\sigma_{h_m}(b_n))(a_mb_n) = 0$ and then we conclude that $(a_m\sigma_{h_m}(b_n)) \cdots (a_m\sigma_{h_m}(b_n))\sigma_{h_m}(a_mb_n) = 0$. So $(a_m\sigma_{h_m}(b_n)) \cdots$ $a_m\sigma_{h_m}(b_na_mb_n) = 0$ and hence $(a_m\sigma_{h_m}(b_n)) \cdots (a_m\sigma_{h_m}(b_n))(a_mb_n)^2 = 0$. Carrying out this process further results that $(a_mb_n)^\gamma = 0$, consequently $a_m\sigma_{h_m}(b_n) \in nil(R)$. Then, we have $a_m(\psi - b_ng_n)\varphi = a_m\psi\varphi - a_mb_ng_n\varphi \in nil(R) * M$, since $nil(R)$ is an ideal of R. By induction hypothesis, we have $a_mb_ia_m \in nil(R)$ this leads $a_jb_n \in nil(R)$ for all j. Applying the preceding method repeatedly, we obtain $a_j\sigma_{h_j}(b_i) \in nil(R)$ for each i, j. Thus, $a_j\sigma_{h_j}(\sigma_s(b_i)) \in nil(R)$ as desired. Therefore, R is σ-skew strongly M-*nil*-reversible. □

Corollary 2.4. *Let R be a ring with $nil(R)$ being an ideal of R. Suppose M is a u.p.-monoid, and $\sigma : M \to Aut(R)$ is a monoid homomorphism, then R is σ-skew strongly M-nil-reversible.*

Proof. The fact that $nil(R)$ is an ideal implies that R is an *NI*-ring. Thus, by applying Theorem 2.3, the proof follows. □

Corollary 2.5. (1) *Every σ-compatible NI-ring is σ-skew strongly $\mathbb{Z}$-nil-reversible.*
(2) *Every NI-ring is nil-reversible.*

Proof. (1) By setting $M = \{\ldots, x^{-2}, x^{-1}, 1, x^1, x^2, \ldots\}$ and defining $\sigma_{x^n}(r) = \sigma^n(r)$ for each $n \in \mathbb{Z}$ and any $r \in R$, we can verify an isomorphism between $R * M$ and $R[x, x^{-1}, \sigma]$. As a result, we can conclude from the theorem 2.3.
(2) By defining $M = \{1, x^1, x^2, \ldots\}$ and setting $\sigma_{x^n}(r) = r$ for each $n \in \mathbb{N} \cup \{0\}$ and any $r \in R$, we can create an isomorphism between $R * M$ and $R[x]$. Consequently, the result follows from the theorem 2.3. □

Corollary 2.6. *[7, Proposition 1] If R is a reduced ring and a unique product monoid M. Then, R is strongly reversible relative to a monoid M.*

Corollary 2.7. *If M be a strictly totally ordered monoid and R be an NI-ring, then R is σ-skew strongly M-nil-reversible.*

Proof. Since a *u.p.*-monoid is a strictly totally ordered monoid, we can apply Theorem 2.3 to prove the result. □

The following result investigates the descent of the *nil*-reversibility property from the skew monoid ring $R * M$ back to the base ring R. This serves as a natural generalization of the classic result by Sanjiv Subba and S, Subedi. T [28] on *nil*-reversible polynomial rings, as well as the descent conditions for strongly M-reversible rings established by A.B. Singh et. al [7]. By assuming the compatibility of R alongside monoids containing elements of infinite order, we establish that the N-*nil*-reversibility of $R * M$ strictly implies the σ-skew strongly M-*nil*-reversibility of R.

Theorem 2.8. *Consider R be a compatible ring and σ be an automorphism. Let M be a monoid generated by an element μ such that μ has infinite order, and let N be any monoid that contains an element of infinite order. We define a homomorphism $\sigma : M \to Aut(R)$ by setting $\sigma_\mu = \psi$. If the skew monoid ring $R * M$ is N-nil-reversible, then R is σ-skew strongly M-nil-reversible.*

Proof. Let $\varphi = \sum_{i=0}^{m} b_i g_i, \psi = \sum_{j=0}^{n} a_j h_j \in R * M$ satisfy $\varphi\psi \in nil(R * M)[N]$. Then we have $b_i \sigma_{g_i}(a_j) \in nil(R)[N]$. Suppose that $\mu \in N$ such that μ has infinite order, define $\theta, \phi \in (R * M)[N]$ as in the following: $\theta = (a_0 e_M)e_N + (a_1 g)h + (a_2 g_2)h_2 + \cdots + (a_m g_m)h_m$ and $\phi = (b_0 e_M)e_N + (b_1 g)h + (b_2 g_2)h_2 + \cdots + (b_n g_n)h_n$. Since $\varphi = \sum_{i=0}^{m} b_i g_i$ and $\psi = \sum_{j=0}^{n} a_j h_j \in R * M$ so θ and $\phi \in (R * M)[M]$. So, by letting that $(\sum_{i=0}^{m} b_i g_i)(\sum_{j=0}^{n} a_j h_j) = 0$ and $\sigma_\mu = \psi$ is compatible, we conclude that $\theta\phi = 0$. Since by $R * M$ is strongly N-*nil*-reversible. Then $a_j b_i = 0$ whence $a_j \sigma_{h_j}(b_i) = 0$ for each $0 \le i \le n$ and $0 \le j \le m$ by a compatible automorphism. Therefore, R is σ-skew strongly M-*nil*-reversible. □

The example below shows that the statement in Theorem 2.3 stating that R is a reduced ring is indeed necessary.

Example 2.9. *Suppose M is a monoid with at least two elements, S is the set of 2×2 matrices over a field F, and $\sigma : M \to Aut(R)$ is a homomorphism that is a compatible monoid of M. Then, S is not σ-skew strongly M-nil-reversible.*

Solution. Take $e \neq g \in M$ and let $\varphi = E_{22}g, \psi = E_{12}e + E_{11}g \in S * M$. We see that $\varphi\psi \in nil(S * M)$. But, we have $\psi\varphi \neq 0$ which implies that S is not σ-skew strongly M-*nil*-reversible. □

The authors of [22] proved that for an *NI*-ring R and a *u.p.*-monoid M with $\mid M \mid \geq 2$, the equality $Nil(R * M) \neq Nil(R) * M$ does not hold. Using this result as a basis, we can state the following conclusion.

Theorem 2.10. *Suppose R be any ring, M be any monoid with at least two elements, and $\sigma : M \to Aut(R)$ is a homomorphism that is a compatible monoid of M such that $Nil(R * M) = Nil(R) * M$, then R is σ-skew strongly M-nil-reversible.*

Proof. Let $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ be nonzero elements in $R * M$ satisfy $\varphi\psi \in nil(R) * M$. Then, we fined $\lambda \in \mathbb{Z}^+$ satisfying $(\varphi\psi)^\lambda = \underbrace{(\varphi\psi)(\varphi\psi)\ldots(\varphi\psi)}_{\lambda\ times} = 0$. Now, since σ is compatible monoid homomorphism we have $b_i\sigma_{g_i}(a_j) \in nil(R)$. Thus, $a_j\sigma_{h_j}(b_i) \in nil(R)$. Therefore, $\psi\varphi \in nil(R) * M$. □

In [7, Proposition 3], it was shown that if I is a reduced ideal of a ring R such that R/I is strongly M-reversible, then R itself is strongly M-reversible. The following theorem extends this factor ring preservation property to the setting of nilpotent elements within the framework of σ-skew strongly M-*nil*-reversible rings.

Theorem 2.11. *Let R be a ring, M is a monoid with a strict ordering relation and $\sigma : M \to Aut(R)$ is a homomorphism that is compatible with the monoid of M. Suppose I be an ideal of R contained in $nil(R)$. If R/I is $\bar{\sigma}$-skew strongly M-nil-reversible, then R is σ-skew strongly M-nil-reversible.*

Proof. Let $\phi, \psi \in R * M$ satisfying $\phi\psi \in nil(R) * M$. Since R/I is $\bar{\sigma}$-skew strongly M-*nil*-reversible, we have $\bar{\phi}\bar{\psi} \in nil(R/I) * M$, where $\phi = \Sigma_{i=1}^n c_i\ell_i$ and $\psi = \sum_{j=1}^m a_jh_j$ be nonzero elements in $R * M$. Since $nil(R)$ is an ideal of R, we write $\bar{R} = R/nil(R)$ and define $\bar{\sigma} : M \to Aut(\bar{R})$ by $\sigma_\ell(c + nil(R)) = \sigma_\ell(c) + nil(R)$, for each $\ell \in M$. We claim that $\bar{\sigma}$ is a compatible monoid homomorphism. For each $a, c \in R$, let $(c + nil(R))(a + nil(R)) = \bar{0}$. Then $ca \in nil(R)$ and hence $(c + nil(R))\bar{\sigma}(a + nil(R)) = \bar{0}$. Now we write $\ell_1 < \ell_2 < \ldots < \ell_n, h_1 < h_2 < \ldots < h_m$. We will use transfinite induction on the strictly totally ordered set $(M, \leq)$ to show that $a_j\sigma_{h_j}(\sigma_s(c_i)) \in nil(R)$. Since R/I is $\overline{\sigma}$-skew strongly M-*nil*-reversible and

$$\begin{aligned}\bar{0} = &\ (\bar{c}_1\ell_1 + \bar{c}_2\ell_2 + \cdots + \bar{c}_n\ell_n)(s)(\bar{a}_1h_1 + \bar{a}_2h_2 + \cdots + \bar{a}_mh_m)\\ = &\ (c_1 + I)\bar{\sigma}_{\ell_1}(a_1 + I) + (c_2 + I)\bar{\sigma}_{\ell_2}(a_2 + I) + \cdots + (c_n + I)\bar{\sigma}_{\ell_n}(a_m + I)\\ = &\ c_1\bar{\sigma}_{\ell_1}(a_1 + I) + c_2\bar{\sigma}_{\ell_2}(a_2 + I) + \cdots + c_n\bar{\sigma}_{\ell_n}(a_m + I) \in (R/I) * M\end{aligned}$$

for $s \in M$, so we have $c_i\sigma_{\ell_i}(a_j) \in I$ for all i, j. Since M is a strictly totally ordered monoid, we have $\ell_1h_1 < \ell_ih_1 \leq \ell_ih_j = \ell_1h_1$ for $i \neq 1$ or $j \neq 1$. It follows that $c_1\sigma_{\ell_1}(a_1) = 0$, i.e., $c_1\sigma_{\ell_1}(\sigma_s(a_1)) \in nil(R)$ since I is an ideal of R contained in $nil(R)$. Now suppose that $c_ia_j = 0$ for all $1 \leq i \leq n, 1 \leq j \leq m$ with $w \in M$ is such that for any ℓ_i and h_j, $\ell_ih_j < w$. We will show that $c_i\sigma_{\ell_i}(\sigma_s(a_j) \in nil(R)$ for any ℓ_i and h_j with $\ell_ih_j = w$. Set $\chi = \{(\ell_i, h_j)|\ell_ih_j = w\}$. Then χ is a finite set. We write χ as $\{(\ell_{i_\lambda}, h_{j_\lambda})|\lambda = 1, 2, \ldots, k\}$ such that $\ell_{i_1} < \ell_{i_2} < \ldots < \ell_{i_k}$. Since M is cancellative, $\ell_{i_1} = \ell_{i_2}$ and $\ell_{i_1}h_{j_1} = \ell_{i_2}h_{j_2} = w$ imply $h_{j_1} = h_{j_2}$. Since $\leq$ is a strict order, $\ell_{i_1} < \ell_{i_2}$ and $\ell_{i_1}h_{j_1} = \ell_{i_2}h_{j_2} = w$ imply $h_{j_2} < h_{j_1}$. Thus, we have $h_{j_k} < h_{j_{k-1}} < \ldots < h_{j_2} < h_{j_1}$. Now $\sum_{(\ell_i,h_j)\in X} c_i\sigma_{\ell_i}(\sigma_s(a_j)) = \sum_{\lambda=1}^k c_{i_\lambda}\sigma_{\ell_{i_\lambda}}(\sigma_s(a_{j_\lambda})) = 0$.

For any $\lambda \geq 2, \ell_{i_1}h_{j_\lambda} < \ell_{i_\lambda}h_{j_\lambda} = w$, and so $c_{i_1}\sigma_{\ell_{i_\lambda}}(\sigma_s(a_{j_\lambda}) = 0$ by induction hypothesis. Thus, $c_{i_1}a_{j_\lambda} = 0$ because R is M-compatible. Since I is reduced and $\sigma_{\ell_{i_\lambda}}(a_{j_\lambda})I(c_{i_1}) \subseteq I$, then we have $(a_{j_\lambda}Ic_{i_1})^2 = 0$ and I is reduced. Thus, for any $\lambda \geq 2, (c_{i_\lambda}a_{j_\lambda})(c_{i_1}a_{j_1})^2 = (c_{i_\lambda}a_{j_\lambda})(c_{i_1}a_{j_1})(c_{i_1}a_{j_1}) \in (c_{i_\lambda}a_{j_\lambda})I(c_{i_\lambda}a_{j_\lambda}) = (c_{i_\lambda}a_{j_\lambda})I(c_{i_\lambda}a_{j_\lambda}) = 0$, which implies that $(c_{i_\lambda}a_{j_\lambda})(c_{i_1}a_{j_1})^2 = 0$. Now multiplying $\sum_{\lambda=1}^k c_{i_\lambda}\sigma_{i_\lambda}(\sigma_s(a_{j_\lambda})) = 0$ on the right by $(c_{i_1}\sigma_{i_1}(a_{j_1}))^2$, we obtain

$$0 = (c_{i_\lambda}\sigma_{i_\lambda}(a_{j_\lambda}))(c_{i_1}\sigma_{i_1}(a_{j_1}))^2 = (c_{i_1}\sigma_{i_1}(a_{j_1}))^2(c_{i_1}\sigma_{i_1}(a_{j_1})) = (c_{i_1}\sigma_{i_1}(a_{j_1}))^3.$$

Since $c_{i_1}\sigma_{i_1}(a_{j_1}) \subseteq I$ and I is reduced and R is M-compatible, we have $c_{i_1}\sigma_{i_1}(a_{j_1})$ $= 0$. Thus, $\sum_{\lambda=2}^{k} c_{i_\lambda}\sigma_{i_\lambda}(a_{j_\lambda}) = 0$. Then, $c_{i_{t\lambda}}\sigma_{i_\lambda}(a_{j_\lambda}) \in nil(R)$ for $\lambda \geq 2$. Multiplying $(c_{i_2}\sigma_{i_2}(a_{j_2}))^2$ on $\sum_{\lambda=2}^{k} c_{i_\lambda}\sigma_{i_\lambda}(a_{j_\lambda}) = 0$ from the right-hand side, we obtain $c_{i_2}\sigma_{i_2}(a_{j_2}) = 0$. Thus, $c_{i_2}\sigma_{i_2}(a_{j_2}) \in nil(R)$ by the same way as the above. Continuing this process, we can prove $c_{i_\lambda}\sigma_{i_\lambda}(a_{j_\lambda}) = 0$ for $\lambda = 1, \ldots, k$. Thus $c_i\sigma_{\ell_i}(\sigma_s(a_j)) \in nil(R)$ because I is an ideal of R contained in $nil(R)$ for any i and j with $\ell_i h_j = w$. Therefore, by transfinite induction $c_i\sigma_{\ell_i}(\sigma_s(a_j)) \in nil(R)$ for any i and j. Thus, $a_j\sigma_{h_j}(\sigma_s(c_i)) \in nil(R)$. Therefore, R is σ-skew strongly M-*nil*-reversible. □

Proposition 2.12. *Consider a monoid M with a strict ordering relation and $\sigma : M \to Aut(R)$ is a homomorphism that is a compatible monoid of M. If R is finite subdirect product of σ-skew strongly M-nil-reversible, then R is σ-skew strongly M-nil-reversible.*

Proof. Suppose $I_k (k = 1, \ldots, l)$ denote ideals of R satisfy R/I_k is $\bar{\sigma}$-skew strong M-*nil*-reversibility and $\bigcap_{k=1}^{l} I_k = 0$. Suppose φ and ψ be in $R * M$ with $\varphi\psi \in nil(R) * M$. Clearly $\bar{\varphi}\bar{\psi} \in nil(R/I_k) * M$. Since R/I_k is $\bar{\sigma}$-skew strongly M-*nil*-reversible, we have $(b_i\sigma_{g_i}(a_j))^\ell \in I_k$, for some positive integer ℓ. So $(b_i\sigma_{g_i}(a_j))^\ell \in \bigcap_{k=1}^{l} I_k = 0$. Hence $(b_i\sigma_{g_i}(a_j))^\ell \in nil(R)$. Thus, $a_j\sigma_{h_j}(b_i) \in nil(R)$ since σ is compatible. Therefore, $\psi\varphi \in nil(R) * M$ and we are done. □

3. Some Extensions Satisfying Skew Monoid Ring Conditions

In this section, we systematically investigate the preservation of σ-skew strongly M-*nil*-reversibility across several fundamental classes of ring extensions. First, we focus on classical right quotient rings. Recall that a ring R is a right (resp., left) Ore ring if, for any $b, d \in R$ with d regular (i.e., a non-zero-divisor), there exist $b_1, d_1 \in R$ such that d_1 is regular and $bd_1 = db_1$ (resp., $d_1 b = b_1 d$). By a well-known result of Kraemer [20, Theorem 6.2], R admits a classical right quotient ring $Q = Q_{cl}^r(R)$ if and only if the set of regular elements of R satisfies the right Ore condition. Under this setting, any automorphism σ of R uniquely extends to an automorphism $\bar{\sigma}\colon Q \to Q$ defined by $\bar{\sigma}(bd^{-1}) = \sigma(b)\sigma(d)^{-1}$ for all $bd^{-1} \in Q$.

Building upon these foundational constructions, we analyze how the property of being σ-skew strongly M-*nil*-reversible transfers between R and its quotient ring Q. Furthermore, we explore the structural implications of this notion on monoid ring extensions, motivated by the result of [23, Lemma 2.11], which establishes that strong M-α-reversibility of R implies the semicommutativity of the monoid ring $R[M]$.

Lemma 3.1. *Let R be a ring, M be a u.p.-monoid and $\sigma : M \to Aut(R)$ a compatible monoid homomorphism. If R is σ-skew strongly M-nil-reversibility, then $R * M$ is semicommutativity.*

Proof. Let $\phi = c_1\ell_1 + c_2\ell_2 + \cdots + c_n\ell_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ be nonzero elements in $R * M$ satisfy $\phi\psi = 0$. Since R is σ-skew strongly M-*nil*-reversible, we have $c_i\sigma_{\ell_i}(d_\lambda a_j) = 0$ for any $\theta = d_1g_1 + d_2g_2 + \cdots + d_kg_k \in R * M$ for each $1 \leq i \leq n, 1 \leq j \leq m, 1 \leq \lambda \leq k$. Hence, $c_i d_k a_j = 0$ by compatibility. Thus, $\phi\theta\psi = 0$. Therefore, $R * M$ is semicommutative. □

The next result demonstrates that the property of being σ-skew strongly M-*nil*-reversible is a local-global invariant between a right Ore ring R and its classical right quotient ring Q. This theorem provides a unified generalization of the local quotient equivalence for strongly right α-reversible rings established by Liang and X. Zhu [29, Proposition 2.19], as well as the quotient transfer conditions for strongly M-reversible rings introduced by A. Bhooshan and et. al [**?**].

Theorem 3.2. *Let R be a ring, M be a monoid and $\sigma : M \to Aut(R)$ be a compatible monoid homomorphism. If R is a right Ore ring and Q is its classical right quotient ring, then R is σ-skew strongly M-nil-reversible if and only if Q is $\bar{\sigma}$-skew strongly M-nil-reversible.*

Proof. We need to prove Q is $\bar{\sigma}$-skew strongly M-*nil*-reversible, while the opposite is easy. Assume R is σ-skew strongly M-*nil*-reversible. Let $\varphi = \Sigma_{i=1}^{n}\mu_i g_i, \psi = \Sigma_{j=1}^{m}\lambda_j h_j \in Q * M$ satisfying $\varphi\psi \in nil(Q) * M$. Then, by Proposition 2.1.16 [19], suppose $\mu_i = a_i u^{-1}$ and $\lambda_j = b_j v^{-1}$ for each $a_i, b_j \in R$ for all i, j there are regular elements u and v in R, and for every index j, there exist a regular element s in R and an element c_j in R satisfying $u^{-1}b_j = c_j s^{-1}$. Also by Proposition 2.1.16 [19]. Suppose $\varphi_1 = \Sigma_{i=1}^{n}a_i g_i, \psi_1 = \Sigma_{j=1}^{m}b_j h_j$ and $\psi_2 = \Sigma_{j=1}^{m}c_j h_j$. From $\varphi\psi \in nil(Q) * M$ we have $\varphi\psi = \Sigma_{i=1}^{n}\Sigma_{j=1}^{m}\mu_i\lambda_j g_i h_j = \Sigma_{i=1}^{n}\Sigma_{j=1}^{m}a_i(u^{-1}b_j)v^{-1}g_i h_j = \Sigma_{i=1}^{n}\Sigma_{j=1}^{m}a_i c_j(vs)^{-1}g_i h_j = \varphi_1\psi_2(vs)^{-1} = 0$, which implies that $\varphi_1\sigma_{g_i}(\psi_2) \in nil(Q) * M$. So, $\sigma_{g_i}(\varphi_1)\psi_2 \in nil(Q) * M$ since $R * M$ is an σ-compatible ring. Since R is σ-skew strongly M-*nil*-reversible and σ is an automorphism of R, it follows that $R * M$ is semicommutative by Lemma 3.1. Therefore, $\sigma(\varphi_1)u\psi_2 = \sigma(\varphi_1)\psi_1 s = 0$, so $\sigma(\varphi_1)\psi_1 = 0$. Again by using Proposition 2.1.16 [19], for all i we have $d_i \in R$ and regular element $t \in R$ satisfying $v^{-1}\sigma(a_i) = d_i t^{-1}$. Since σ is an automorphism and a compatible with the monoid structure of M we have $k_i \in R$ such that $d_i = \sigma(k_i)$ for each i. Let $\varphi_2 = \Sigma_{i=1}^{n}k_i g_i$, then $\varphi_1 t\psi_1 = \Sigma_{i+j=k}a_i t\sigma(b_j)g_i h_j = \Sigma_{i+j=k}vd_i\sigma(b_j)g_i h_j = v\varphi_2\sigma(\psi_1) = 0$. This shows that $\varphi_2\sigma_{g_i}(\psi_1) \in nil(Q) * M$. Since R is σ-skew strongly M-*nil*-reversible, we have $\psi_1\sigma(\varphi_2) = 0$. Therefore, $\psi_1\sigma(\varphi_2)$ is nilpotent, since $R * M$ is an σ-compatibility. Thus,
$\psi\bar{\sigma}(\varphi) = (\Sigma_{j=1}^{m}\lambda_j h_j)(\Sigma_{i=1}^{n}(\sigma(\mu_i)g_i) = (\Sigma_{j=1}^{m}b_j v^{-1}h_j)(\Sigma_{i=1}^{n}\sigma(a_i)\sigma(u)^{-1}g_i) =$
$\Sigma_{i+j}b_j(v^{-1}\sigma(a_i))\alpha(u)^{-1}g_i h_j = \Sigma_{i+j}b_j d_i(\sigma(u)t)^{-1}g_i h_j = \psi_1\sigma(\varphi_2)(\sigma(u)t)^{-1} = 0$. Therefore, Q is $\bar{\sigma}$-skew strongly M-*nil*-reversible. □

Corollary 3.3. *[7, Theorem 2]. If M be a monoid and R be a right Ore ring with classical right quotient ring Q of a ring R, then the ring R is strongly M-reversible if and only if its quotient ring Q is strongly M-reversible.*

Proposition 3.4. *For a ring R, M be a monoid, $\sigma : M \to Aut(R)$ a compatible monoid homomorphism and if e is a central idempotent of R such that $\sigma(e) = e$, then the following statements are equivalent:*
(1) R is σ-skew strongly M-nil-reversible.
(2) eR and $(1 - e)R$ are σ-skew strongly M-nil-reversible.

Proof. (1)⇔(2). It is easy to see that subrings and finite direct products of σ-skew strongly M-*nil*-reversible rings also exhibit the property of being σ-skew strongly M-*nil*-reversible. □

According to Liu [26], a ring R is called M-Armendariz if whenever elements $\alpha = \sum_{i=1}^{n} a_i g_i$ and $\beta = \sum_{j=1}^{m} b_j h_j$ in the monoid ring $R[M]$ satisfy $\alpha\beta = 0$, then $a_i b_j = 0$ for all i and j. Building upon this notion, the following theorem provides a comprehensive generalization by establishing a strong sufficient condition under which M-Armendarizness guarantees the preservation of σ-skew strong M-*nil*-reversibility.

Theorem 3.5. *Let R be a ring, M be a monoid with a strict ordering relation and $\sigma : M \to Aut(R)$ is a homomorphism that is a compatible monoid of M. Assume that R is an M-Armendariz, then R is σ-skew strongly M-nil-reversible.*

Proof. Let $\phi = \Sigma_{i=1}^{n} b_i g_i$ and $\psi = \Sigma_{j=1}^{m} a_j h_j \in R * M$ satisfying $\phi\psi \in nil(R) * M$ implies that $b_i \sigma_{g_i}(\sigma_s(a_j)) \in nil(R)$ for any $s \in M$ and any i, j. We write

$$(b_1 g_1 + b_2 g_2 + \cdots + b_n g_n)(s)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0. \qquad (2.1)$$

With $g_1 < g_2 < \ldots < g_n, h_1 < h_2 < \ldots < h_m$. We will use transfinite induction on a strictly totally ordered set $\leq$ to show that $\psi\phi \in nil(R) * M$. If we take $m = 1$ in Eq. (2.1), then we have $(b_1 g_1 + b_2 g_2 + \cdots + b_n g_n)s(a_1 h_1) = 0$. Therefore, we obtain $b_i \sigma_{g_i}(\sigma_s(a_1)) \in nil(R)$ for each $1 \leq i \leq n$. Since M is a strictly totally ordered monoid, we have $g_1 h_1 < g_i h_1 \leq g_i h_j = g_1 h_1$ for $i \neq 1$ or $j \neq 1$. Thus, Eq. (2.1) becomes

$$(b_1 g_1 + b_2 g_2 + \cdots + b_n g_n)s(a_2 h_2 + a_3 h_3 + \cdots + a_m h_m) = 0. \qquad (2.2)$$

The case $n = 1$ is proved by similar argument. By applying the induction hypothesis for all $2 \leq i \leq n$ and $2 \leq j \leq m$. Suppose that $b_i \sigma_{g_i}(\sigma_s(a_j)) = 0$ for all $1 \leq i \leq n, 1 \leq j \leq m$, with $\Omega \in M$ is such that for any g_i and h_j, $g_i h_j < \Omega$. We will show that $b_i \sigma_{g_i}(\sigma_s(a_j)) \in nil(R)$ for any g_i and h_j with $g_i h_j = \Omega$. Set $X = \{(g_i, h_j) | g_i h_j = \Omega\}$. Then X is a finite set. We write X as $\{(g_{i_\lambda}, h_{j_\lambda}) | \lambda = 1, 2, \ldots, d\}$ such that $g_{i_1} < g_{i_2} < \ldots < g_{i_d}$. Since M is cancellative, $g_{i_1} = g_{i_2}$ and $g_{i_1} h_{j_1} = g_{i_2} h_{j_2} = \Omega$ imply $h_{j_1} = h_{j_2}$ since $\leq$ is a strict order, $g_{i_1} < g_{i_2}$ and $g_{i_1} h_{j_1} = g_{i_2} h_{j_2} = \Omega$ imply $h_{j_2} < h_{j_1}$. Thus we have $h_{j_\lambda} < h_{j_{\lambda-1}} < \ldots < h_{j_2} < h_{j_1}$. Now $\sum_{(g_i, h_j) \in X} b_i \sigma_{g_i}(\sigma_s(a_j)) = \sum_{\lambda=1}^{d} b_{i_\lambda} \sigma_{g_{i_\lambda}}(\sigma_s(a_{j_\lambda})) = 0$. For any $\lambda \geq 2, g_{i_1} h_{j_\lambda} < g_{i_\lambda} h_{j_\lambda} = \lambda$, and so $b_{i_1} \sigma_{g_{i_\lambda}}(\sigma_s(a_{j_\lambda})) \in nil(R)$ by induction hypothesis. Thus, $b_{i_1} \sigma_{g_{i_\lambda}}(a_{j_\lambda}) \in nil(R)$ because R is an M-Armendariz and σ is automorphism.

For the case where $n \geq 2$ for M is cancellative. We can repeat this process to show that $b_i \sigma_{g_i}(\sigma_s(a_j)) \in nil(R)$ for all $s \in M$ and all i, j. Consequently, we can see that $a_j \sigma_{h_j}(\sigma_s(b_i)) \in nil(R)$ for all $s \in M, 1 \leq j \leq m, 1 \leq i \leq n$, by Armendariz and σ is automorphism. Thus, $\psi\phi \in nil(R) * M$. Therefore, R is σ-skew strongly M-*nil*-reversible. □

The following theorem provides a unified characterization that substantially extends the result of [3, Proposition 2.4] (as well as related equivalence properties studied in [28, 29]). Under the assumption of the σ-skew Armendariz condition, we establish that the classical concepts of reversibility and σ-compatibility are collectively equivalent to the broader property of σ-skew strong M-*nil*-reversibility.

Theorem 3.6. *Let R be a ring, M be a monoid, $\sigma : M \to Aut(R)$ a compatible monoid homomorphism. If R is an σ-skew Armendariz, then R is both reversible and σ-compatible if and only if R is σ-skew strongly M-nil-reversible.*

Proof. We prove to necessity. Let R is both reversible and σ-compatible. Let $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ be nonzero elements in $R * M$ satisfy $\varphi\psi \in nil(R) * M$. Since R is σ-skew Armendariz, $b_i\sigma_{g_i}(a_j) = 0$ and so we have

$$b_ia_j = 0 \Rightarrow a_jb_i = 0 \Leftrightarrow a_j\sigma_{h_j}(b_i) = 0$$

by hypothesis, this holds for all i and j. Consequently, $\psi\varphi$ is nilpotent, which implies that R is σ-skew strongly M-*nil*-reversible. □

According to [11], the categories of semiprime rings and strongly α-skew reversible rings are found to be independent, here we have.

Proposition 3.7. *Let R be a ring, M be a monoid, $\sigma : M \to Aut(R)$ a monoid homomorphism. A ring R is σ-rigid if and only if R is σ-skew strongly M-nil-reversible and semiprime.*

Proof. Note that any σ-rigid ring is reduced (and σ is a homomorphism) by [27, p. 218]. Assume $\varphi\psi \in nil(R) * M$ for $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ be nonzero elements in $R * M$. Then $b_i\sigma_{g_i}(a_j)\sigma_{g_i}(b_i\sigma_{g_i}(a_j)) = b_i\sigma_{g_i}(a_jb_i)\sigma_{g_i}(a_j) = 0$. Since R is σ-rigid, $b_i\sigma_{g_i}(a_j) \in nil(R)$ and thus R is σ-skew strongly M-*nil*-reversible.
Conversely, assume $b_i\sigma_{g_i}(b_i) = 0$ for $b_i \in R$. For any $r \in R, 0 = b_i\sigma_{g_i}(b_i)\sigma_{g_i}(r) = b_i\sigma_{g_i}(rb_i)$. Since R is σ-skew strongly M-*nil*-reversible, $\sigma_{g_i}(b_irb_i) = 0$ for any $r \in R$ and thus $b_i = 0$ since σ is a homomorphism of a semiprime ring R. Therefore, R is σ-rigid. □

The next result establishes a comprehensive localization characterization, demonstrating that the property of being σ-skew strongly M-*nil*-reversible is strictly invariant under central localization. Specifically, this theorem unifies and generalizes classical localization transfer theorems for reversible and *nil*-reversible rings (such as those established in [28] and [29]) by extending them to skew monoid actions over central regular sets Δ.

Theorem 3.8. *Let R be a ring, M be a monoid, $\sigma : M \to Aut(R)$ a compatible monoid homomorphism and R has a multiplicative monoid Δ, which consists of central regular elements, then R is σ-skew strongly M-nil-reversible if and only if $\Delta^{-1}R$ is σ-skew strongly M-nil-reversible.*

Proof. Suppose R is σ-skew strongly M-*nil*-reversible and $p_i, d_j, u, v \in R$. Let $u^{-1}p_i, v^{-1}d_j \in \Delta^{-1}R * M$ for all i, j satisfy $u^{-1}p_iv^{-1}d_j \in nil(\Delta^{-1}R) * M$. Then $(u^{-1}p_iv^{-1}d_j)^{\ell} = 0$ for some positive integer ℓ. This implies $(p_id_j)^{\ell} = 0$, by compatibility so $p_i\sigma_i(d_j) \in nil(R)$. For any $u^{-1}p_i, v^{-1}d_j \in \Delta^{-1}R * M$ having the property that $(u^{-1}p_i)(v^{-1}d_j) = 0$ for all i, j. Since R is σ-skew strongly M-*nil*-reversible, $d_j\sigma_j(p_i) \in nil(R)$, so $(v^{-1}u^{-1})d_jp_i = 0$ which further yields $(v^{-1}d_j)(u^{-1}p_i) \in nil(\Delta^{-1}R) * M$. Hence $\Delta^{-1}R$ is $\bar{\sigma}$-skew strongly M-*nil*-reversible. The converse part is trivial. □

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