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Stability Analysis of Atangana-Baleanu Fractional Delay Differential Equations

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Abstract. This work investigates the Ulam-Hyers and generalized Ulam-Hyers-Rassias stability of fractional delay differential equations involving the Atangana-Baleanu derivative with Mittag-Leffler kernel. Numerical simulations validate the theoretical results, showing close agreement between exact and approximate solutions and confirming bounded error behaviour. These findings demonstrate the effectiveness of the Atangana-Baleanu fractional delay framework for modelling systems with memory and delay, particularly in epidemiological and immune response applications.

1. Introduction and preliminaries

In recent years, fractional calculus has developed rapidly and has attracted considerable attention in the mathematical sciences. Among the many definitions of fractional derivatives are the Riemann-Liouville and Caputo fractional derivatives (with singular kernels), the Caputo-Fabrizio fractional derivative (with a non-singular local kernel) introduced in [10], and the Atangana-Baleanu fractional derivative (with a non-singular, non-local kernel) introduced in [7].

We believe that the main objective of the work by Caputo and Fabrizio was to further improve the modeling of systems with memory effects; however, several limitations of the Caputo-Fabrizio fractional derivative were subsequently identified. In particular, the Caputo-Fabrizio derivative employs a local kernel, and the associated anti-derivative corresponds to an average of the function

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and its integral rather than a true fractional integral. Caputo and Fabrizio addressed the singular kernel problem present in the Riemann-Liouville and Caputo derivatives by introducing a new derivative with an exponential kernel.

This kernel-related issue naturally raises the question: *What is the most accurate kernel for describing the dynamics of memory-effect systems?* This question is motivated by the fact that many physical phenomena appear to lie beyond the descriptive capability of such kernels. By extending the exponential kernel in the Caputo-Fabrizio derivative to the Mittag-Leffler function, Atangana and Baleanu provided a possible resolution to this issue in [7]. Consequently, the Atangana-Baleanu derivative with a Mittag-Leffler kernel is regarded as both non-singular and non-local, offering a more realistic representation of memory effects compared to the exponential kernel.

Delay differential equations, like other classes of differential equations, have also been generalized to fractional order; see, for example, [16, 21, 22].

In this paper, we examine the Ulam stability of Atangana-Baleanu derivative fractional delay differential equations of the form

$$\begin{cases} {}^{ABC}_{t_0}D^{\zeta}_{t}x(t) = f(t, x_t), \qquad t \in [t_0, T], \qquad t \neq t_\tau \\ x_{t_0} = \eta_0. \end{cases} \tag{1.1}$$

Here, ${}^{ABC}_{t_0}D^{\zeta}_{t}x(t)$ denotes the Atangana-Baleanu fractional derivative with a Mittag-Leffler kernel. Let $C = PC([-r, 0], \mathbb{R}^n)$ denote the space of continuous functions defined on $[-r, 0]$. For any element $\eta \in C$, we have $\eta : [-r, 0] \to \mathbb{R}^n$ equipped with the norm

$$\|\eta\| = \sup_{-r \leq s \leq 0} \|\eta(s)\|,$$

where $r > 0$ and $\|\cdot\|$ denotes a norm in $\mathbb{R}^n$. Suppose that $x \in C([t_0 - r, T], \mathbb{R}^n)$. We denote by x_t the translation of the restriction of x to the interval $[t_0 - r, T]$. More precisely, x_t is an element of C defined by

$$x_t(s) = x(t + s), \qquad -r \leq s \leq 0.$$

The set $C([0, T])$ is defined as

$$C([0, T]) = \{\eta \in C : \|\eta\| < \rho, \ \forall \rho > 0\}.$$

Furthermore, $C([t_0 - r, T], \mathbb{R}^n)$ denotes the Banach space of all continuous functions mapping $[t_0 - r, T]$ into $\mathbb{R}^n$, endowed with the supremum norm

$$\|x\|_\infty = \sup_{t \in [t_0 - r, T]} \|x(t)\|.$$

One of the central concerns in the qualitative theory of differential equations is the stability of solutions. Stability theory enables the comparison of solution behaviours corresponding to different initial conditions. As a fundamental aspect of mathematical modelling, the study of stability in differential systems has wide-ranging applications across numerous scientific and engineering disciplines; see, [8, 13–15].

Analysing the qualitative behaviour of these model equations is essential, especially when simulating real-world problems in the domains of biology, medicine, and economics [6].
[17], [11], [20], [9], [21], examine the qualitative features of the existence and uniqueness of the solutions to the initial value problem for the Atangana-Baleanu, Caputo, and Caputo-Fabrizio non-linear impulsive fractional differential equations.
The remainder of this work is organized as follows. In the next section, we present foundational definitions, propositions, lemmas, and theorems that are essential for the development of the main results. Section 3 contains the main results of the work, focusing on Ulam-Hyers stability and Ulam-Hyers-Rassias stability. In Section 4, a realistic infection-immune interaction model is introduced to illustrate the applicability of the obtained theoretical results. Finally, Section 5 concludes the paper and summarizes the main findings.

2. Preliminaries

Definition 2.1. *[12] Let $n < \zeta \leq n+1$, $n \in \mathbb{N}$ and f be a real function defined on $t \in [a,b]$ the Riemann-Liouville fractional integral of order β is defined by*

$$({}_aI_t^{\zeta})f(t) = \frac{1}{\Gamma(\zeta)} \int_a^t (t-p)^{\zeta-1} f(p)dp, \tag{2.1}$$

where

$$\Gamma(\zeta) = \int_0^{\infty} t^{\zeta-1} e^{-t} dt \qquad \zeta > 0, \tag{2.2}$$

is the gamma function, and the Riemann-Liouville derivative of order β equivalent of equation 2.1 *is defined as*

$${}^{RL}_{t_0}\mathrm{D}_t^{\zeta} x(t) = \frac{1}{\Gamma(n-\zeta)} D^n \int_{t_0}^t (t-p)^{n-\zeta-1} x(p)dp. \tag{2.3}$$

Definition 2.2. *[11] Let $0 < \zeta \leq n+1$, $n \in \mathbb{N}$. The Caputo fractional derivative of order ζ of a function $x(t)$ is defined as*

$${}^{C}_{t_0}\mathrm{D}_t^{\zeta} x(t) = \frac{1}{\Gamma(n-\zeta)} \int_{t_0}^t (t-p)^{n-\zeta-1} D^n x(p)dp. \tag{2.4}$$

Definition 2.3. *[12] Let $f \in H^{'}(a,b)$, $a < b$, $\zeta \in (0,1)$, then the definition of the new (left Caputo) fractional derivative in the sense of Caputo and Fabrizio becomes*

$$({}^{CFC}_{a}D_t^{\zeta})f(t) = \frac{M(\zeta)}{1-\zeta} \int_a^t f^{'}(p) e^{[-\zeta \frac{(t-p)}{1-\zeta}]} dp, \tag{2.5}$$

where $M(\zeta)$ is a normalization function such that $M(0) = M(1) = 1$.

Definition 2.4. *[1] In the right case we have*

$$({}^{CFC}_{t}D_b^{\zeta})f(t) = \frac{-M(\zeta)}{1-\zeta} \int_t^b f^{'}(p) e^{[-\zeta \frac{(p-t)}{1-\zeta}]} dp. \tag{2.6}$$

Definition 2.5. *[2] Let $f \in H^{'}(a,b)$, $a < b$, $\zeta \in (0,1)$, then the definition of the new (left Riemann-Liouville) fractional derivative in the sense of Caputo and Fabrizio becomes,*

$$({}_{a}^{CFR}\mathrm{D}^{\zeta} f)(t) = \frac{M(\zeta)}{1-\zeta}\frac{d}{dt}\int_{a}^{t} f(p)e^{[-\zeta\frac{(t-p)}{1-\zeta}]}dp. \tag{2.7}$$

The associated fractional integral is

$$({}_{a}{}^{CF}\mathrm{I}^{\zeta} f)(t) = \frac{1-\zeta}{M(\zeta)}f(t) + \frac{\zeta}{M(\zeta)}\int_{a}^{t} f(p)dp, \tag{2.8}$$

and in the right Riemann-Liouville fractional derivative in the sense of Caputo and Fabrizio is

$$({}^{CFR}\mathrm{D}_{b}^{\zeta} f)(t) = \frac{-M(\zeta)}{1-\zeta}\frac{d}{dt}\int_{t}^{b} f(p)e^{[-\zeta\frac{(t-p)}{1-\zeta}]}dp. \tag{2.9}$$

The associated fractional integral is

$$({}_{t}{}^{CF}\mathrm{I}_{b}^{\zeta} f)(t) = \frac{1-\zeta}{M(\zeta)}f(t) + \frac{\zeta}{M(\zeta)}\int_{a}^{t} f(p)dp \tag{2.10}$$

Definition 2.6. *[7] Let $f \in H^{'}(a,b), b > a, \zeta \in (0,1)$, the left Atangana-Baleanu derivative in Caputo sense is given as*

$$({}_{a}^{AB}\mathrm{D}_{t}^{\zeta} f)(t) = \frac{M(\zeta)}{1-\zeta}\int_{a}^{t} f^{'}(p)E_{\zeta}\Big[-\zeta\frac{(t-p)^{\zeta k}}{1-\zeta}\Big]dp, \tag{2.11}$$

where $M(\zeta)$ has the same properties as in Caputo-Fabrizio.

Definition 2.7. *[7] Let $f \in H^{'}(a,b), b > a, \zeta \in (0,1)$, the right Atangana-Baleanu derivative in Caputo sense is given as*

$$({}_{a}^{AB}\mathrm{D}_{t}^{\zeta} f)(t) = \frac{-M(\zeta)}{1-\zeta}\int_{a}^{t} f^{'}(p)E_{\zeta}\Big[-\zeta\frac{(t-p)^{\zeta k}}{1-\zeta}\Big]dp. \tag{2.12}$$

Definition 2.8. *[3] Let $f \in H^{'}(a,b), b > a, \zeta \in (0,1)$, the left Atangana-Baleanu derivative in Riemann-Liouville sense is given as*

$$({}_{a}^{AB}\mathrm{D}^{\zeta} f)(t) = \frac{M(\zeta)}{1-\zeta}\frac{df}{dt}\int_{a}^{t} f(p)E_{\zeta}\Big[-\zeta\frac{(t-p)^{\zeta k}}{1-\zeta}\Big]dp. \tag{2.13}$$

Definition 2.9. *[7] Let $f \in H^{'}(a,b), b > a, \zeta \in (0,1)$, the right Atangana-Baleanu derivative in Riemann-Liouville sense is given as*

$$({}_{t}^{AB}\mathrm{D}_{b}^{\zeta} f)(t) = \frac{-M(\zeta)}{1-\zeta}\frac{df}{dt}\int_{a}^{t} f(p)E_{\zeta}\Big[-\zeta\frac{(t-p)^{\zeta k}}{1-\zeta}\Big]dp, \tag{2.14}$$

where E_{ζ} is the one parametric Mittag-Leffler function given as

$$E_{\zeta}(x) = \sum_{k=o}^{\infty}\frac{x^{k}}{\Gamma(k\zeta+1)}. \tag{2.15}$$

Definition 2.10. *[18] Assume that a function f is piecewise smooth over each finite interval in $[0,\infty]$ and that $|f(t)| \leq B^{at}$ for all $t > T$ if there are constants $B > 0$ and $T > 0$. $F(s) = \mathcal{L}f(t);s = \int_{0}^{\infty} e^{-st}f(t)dt, s \in \mathbb{C}$, then the Laplace transform $\mathcal{L}f(t);s$ of $f(t)$ exists.*

Definition 2.11. *If f and g are piecewise continuous on $[0,\infty)$, then the integral defines a specific product, $f*g$.*

$$f*g=\int_0^t f(p)g(t-p)dp, \tag{2.16}$$

and is referred to as the f and g convolution.

Lemma 2.1. *[12] In the Caputo notion, the Laplace transform of the Caputo-Fabrizio derivative is provided as*

$$\mathcal{L}\{{}_0^{CF}\mathrm{D}_t^{\zeta} f(t)\}=\frac{pF(p)-f(0)}{p+\zeta(1-p)}. \tag{81}$$

Lemma 2.2. *[7] The Laplace transform of the Atangana-Baleanu derivative in the Caputo sense is given as,*

$$\mathcal{L}\{{}_0^{AB}\mathrm{D}_t^{\zeta} f(t)\}=\frac{M(\zeta)}{1-\zeta}\frac{p^{\zeta-1}(pF(p)-f(0))}{p^{\zeta}+\frac{\zeta}{1-\zeta}}.$$

Definition 2.12. *[4] The Atangana-Baleanu-Caputo fractional derivative with non-local kernel's fractional integral is provided as*

$${}_0^{AB}\mathrm{I}_t^{\zeta}(f(t))=\frac{1-\zeta}{M(\zeta)}f(t)+\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_0^t f(p)(t-p)^{\zeta-1}dp. \tag{2.17}$$

When ζ is zero we recover the initial function and if also ζ is 1, we obtain the ordinary integral.

Proposition 2.1. *[5] For $x(t)\in[a,b]$ and $n\le\zeta\le n+1$ we have the following equations*

$$\left({}_a^{ABC}\mathrm{D}^{\zeta}\cdot{}_a^{AB}\mathrm{I}^{\zeta}x\right)(t)=x(t) \tag{2.18}$$

$$\left({}_a^{AB}\mathrm{I}^{\zeta}\;{}_a^{ABC}\mathrm{D}^{\zeta}x\right)(t)=x(t)-x(a). \tag{2.19}$$

Theorem 2.1. *[7] The Atangana-Baleanu in Riemann and Caputo sense possess the Lipschitz condition, that is to say, for a given couple function f and h, the following inequalities can be established as*

$$\|{}_0^{ABC}\mathrm{D}_t^{\zeta} f(t)-{}_0^{ABC}\mathrm{D}_t^{\zeta} h(t)\|\le H\|f(t)-h(t)\|. \tag{2.20}$$

Theorem 2.2. *[19] Let $x\in H^{'}(0,1)$, then x is a solution to the fractional initial value problem*

$$\begin{cases}{}_{t_0}^{ABC}D_t^{\zeta}x(t)=f(t,x(t)), \qquad t\in(0,1), \qquad \beta\in(0,1)\\ x(t_0)=x_0\end{cases} \tag{2.21}$$

if and only if, it is a solution to the integral equation

$$x(t)=x_0+\frac{1-\zeta}{M(\zeta)}f(t,x(t))+\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_0^t f(p,x(p))(t-p)^{\zeta-1}dp \tag{2.22}$$

Theorem 2.3. *[19] Consider the fractional initial value problem given as*

$$T_\zeta(x)(t) = f(t, x(t)) \qquad t \in [0, T], \zeta \in (0, 1) \tag{2.23}$$

$x(0) = u$
if $f(t, x(t))$ *is Lipschitz function with Lipschitz constant*

$$\frac{(1-\zeta)\Gamma(\zeta)+1}{\Gamma(\zeta)M(\zeta)} < 1.$$

Then the fractional initial value problem has a unique solution

Remark 2.1. *[21] Let* $0 < \zeta < 1$ *and* $f : [t_0, T] \times C \to \mathbb{R}^n$ *be continuous, then the initial value problem 1.1 is comparable to the following: Fractional integral Volterra with memory*

$$\begin{cases} x_0 = \eta_0 \\ x(t) = \eta_0(0) + \frac{1-\zeta}{M(\zeta)} f(t, x_t) + \\ \frac{\zeta}{M(\zeta)\Gamma(\zeta)} \int_{t_0}^{t} (t-p)^{\zeta-1} f(p, x_p) dp, \qquad t \in [t_0, T]. \end{cases} \tag{2.24}$$

In other words, all solutions of (1.1) are also solutions of (2.24), and vice versa.

Theorem 2.4. *[23] (Banach Contraction Mapping Principle) Let N be a non-empty complete metric space and let* $T : N \to N$ *be a contraction map, then T has a unique fixed point in N. That is, there exists* $x_0 \in N$ *such that* $Tx_0 = x_0$.

Definition 2.13. *Equation (1.1) is said to be Ulam-Hyers stable if there exists a constant* $C_{f,m} > 0$ *such that for some* $\epsilon > 0$ *and for every solution* $\gamma \in C([t_0, T], \mathbf{R}^n)$ *of the inequality*

$$\| {}_{t_0}^{C}\mathrm{D}_t^{\zeta} \gamma(t) - f(t, \gamma(t)) \| \leq \epsilon \tag{2.25}$$

there exists a unique solution $x(t) \in C([t_0, T], \mathbf{R}^n)$ *of equation (1.1) with*

$$\|\gamma(t) - x(t)\| \leq C_{f,m}\epsilon \tag{2.26}$$

Definition 2.14. *Equation (1.1) is said to be generalized Ulam-Hyers stable if there exists* $\kappa_{f,m} \in C([t_0, T].R^+)$ *with* $\kappa_{f,m}(0) = 0$ *such that for every solution* $\gamma \in C([t_0, T], \mathbf{R}^n)$ *of the inequality (2.25), there exists a unique solution* $x \in C([0, T], \mathbf{R}^n)$ *of equation (1.1) which satisfies*

$$\|\gamma(t) - x(t)\| \leq \kappa_{f,m}(\epsilon) \tag{2.27}$$

Definition 2.15. *Equation (1.1) is said to be Ulam-Hyers-Rassias stable with respect to* $\phi, \phi \in C([t_0, T], \mathbb{R})$ *if there exists a real number* $\kappa_{f,\phi} > 0$ *such that for each* $\epsilon > 0$ *and for every solution* $\gamma \in C([t_0, T], \mathbf{R}^n)$ *of the inequality*

$$\| {}_{t_0}^{C}\mathrm{D}_t^{\zeta} \gamma(t) - f(t, \gamma(t)) \| \leq \epsilon\phi(t) \tag{2.28}$$

there exists a unique solution $x \in C([t_0, T], \mathbf{R}^n)$ *of equation (1.1) with*

$$\|\gamma(t) - x(t)\| \leq \epsilon\kappa_{f,\phi}\phi(t) \tag{2.29}$$

Definition 2.16. *Equation (1.1) is said to be generalized Ulam-Hyers-Rassias stable with respect to θ if there exists a real number $\kappa_{f,\phi} > 0$ such that for every solution $\gamma \in C([t_0,T],\mathbf{R}^n)$ of inequality*

$$\|{}^{C}_{t_0}\mathrm{D}^{\zeta}_{t}\gamma(t) - f(t,\gamma(t))\| \leq \phi(t) \tag{2.30}$$

there exists a unique solution $x \in C([t_0,T],\mathbf{R}^n)$ of equation (1.1) which satisfies

$$\|\gamma(t) - x(t)\| \leq \kappa_{f,\phi}\phi(t) \tag{2.31}$$

Remark 2.2. *We say that $\gamma \in C([t_0,T],\mathbf{R}^n)$ is a solution of the inequality (2.25) if there exists a function $\gamma(t) \in PC([t_0,T],\mathbf{R}^n)$ such that*

(1) $|g_t| \leq \epsilon$
(2) ${}^{ABC}_{t_0}D^{\zeta}_{t}\gamma(t) = f(t,\gamma_t) + g_t$

Remark 2.3. *We say that $\gamma \in C([t_0,T],\mathbf{R}^n)$ is a solution of the inequality (2.24) if there exists a function $\gamma(t) \in C([t_0,T],\mathbf{R}^n)$ such that*

(1) $|g_t| \leq \phi(t)$
(2) ${}^{ABC}_{t_0}D^{\zeta}_{t}\gamma(t) = f(t,\gamma_t) + g_t$

Theorem 2.5. *[23] Assume the following conditions are satisfied*
(D_1) The function $f : [t_0,T] \times C \to \mathbb{R}^n$ is continuous.
(D_2) There exist positive constants μ such that $\|f(t,x)\| \leq \mu$, for each $t \in [t_0,T]$ and $x \in \mathbb{R}$.
Then the IVP (1.1) has a solution on $t \in [t_0,T]$.

Theorem 2.6. *[23] Assume the conditions of theorem 2.5 are satisfied with the additional conditions*
(D_4) There exist positive constants ν such that

$$\|f(t,x_t) - f(t,y_t)\| \leq K\|x-y\|$$

Then the IVP (1.1) has a unique solution on $t \in [t_0,T]$ provided

$$0 < \Big[\frac{1-\zeta}{M(\zeta)}K + \frac{K}{M(\zeta)\Gamma(\zeta)}T^{\zeta}\Big] < 1. \tag{2.32}$$

3. Results

Lemma 3.1. *Let $\gamma \in C([t_0,T],\mathbf{R}^n)$ be a solution of the inequality (2.25) then γ is a solution of the following integral inequality*

$$\Big|\gamma(t) - \omega_0(0) - \frac{1-\zeta}{M(\zeta)}f(t,\gamma_t) - \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}f(p,\gamma_p)dp\Big|$$
$$\leq \Big(\frac{1-\zeta}{M(\zeta)} + \frac{T^{\zeta}}{M(\zeta)\Gamma(\zeta)}\Big)\epsilon$$

Proof. By remark 2.2 we have

$$\gamma(t) = \omega_0(0) + \frac{1-\zeta}{M(\zeta)}f(t,\gamma_t) + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}f(p,\gamma_p)dp +$$

$$\frac{1-\zeta}{M(\zeta)}g(t)+\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}g(p)dp$$

from this we have

$$\begin{aligned}
&\Big|\gamma(t)-\omega_0(0)-\frac{1-\zeta}{M(\zeta)}f(t,\gamma_t)-\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}f(p,\gamma_p)dp\Big|\\
&=\Big|\frac{1-\zeta}{M(\zeta)}g(t)+\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}g(p)dp\Big|\\
&\leq\frac{1-\zeta}{M(\zeta)}\|g(t)\|+\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}\|g(p)\|dp\|\\
&\leq\frac{1-\zeta}{M(\zeta)}\epsilon+\frac{\epsilon\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}dp\\
&\leq\frac{1-\zeta}{M(\zeta)}\epsilon+\frac{\epsilon T^{\zeta}}{M(\zeta)\Gamma(\zeta)}\\
&=\Big(\frac{1-\zeta}{M(\zeta)}+\frac{T^{\zeta}}{M(\zeta)\Gamma(\zeta)}\Big)\epsilon
\end{aligned}$$

□

Theorem 3.1. *Assume that there exists a unique solution $x\in C([t_0,T],\mathbf{R}^n)$ of equation (1.1) and for each $\epsilon>0$ and a real number $\kappa_{f,n}>0$ such that for every solution $\gamma\in C([t_0,T],\mathbf{R}^n)$ of the inequality (2.24) satisfying*

$$|\gamma-x|\leq\kappa_{f,n}\epsilon,$$

then the unique solution of equation(1.1) is Ulam-Hyers stable.

Proof.

$$\begin{aligned}
|\gamma(t)-x(t)|&=\Big\|\frac{1-\zeta}{M(\zeta)}[f(t,\omega)-f(t,\eta)]\Big\|+\Big\|\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}[f(p,\omega)-f(p,\eta)]dp\Big\|\\
&\quad+\Big\|\frac{1-\zeta}{M(\zeta)}g(t)+\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}g(p)dp\|\\
&\leq\frac{1-\zeta}{M(\zeta)}\Big\|f(t,\omega)-f(t,\eta)\Big\|+\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}\Big\|f(p,\omega)-f(p,\eta)\Big\|dp+\frac{1-\zeta}{M(\zeta)}\Big\|g(t)\Big\|\\
&\quad+\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}\Big\|g(p)\Big\|dp\\
&\leq\frac{1-\zeta}{M(\zeta)}K\Big\|\omega-\eta\Big\|+\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}K\Big\|\omega-\eta\Big\|dp+\frac{1-\zeta}{M(\zeta)}\Big\|g(t)\Big\|+\\
&\quad\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}\Big\|g(p)\Big\|dp\\
&\leq\frac{1-\zeta}{M(\zeta)}K\|\gamma-x\|+\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}K\Big\|\gamma-x\Big\|_\infty\\
&\quad+\frac{1-\zeta}{M(\zeta)}\epsilon+\frac{T^{\zeta}}{M(\zeta)\Gamma(\zeta)}\epsilon
\end{aligned}$$

$$\leq \Big(\frac{1-\zeta}{M(\zeta)}K + \frac{K}{M(\zeta)\Gamma(\zeta)}T^{\zeta}\Big)\|\gamma - x\|_{\infty}$$

$$+\Big(\frac{1-\zeta}{M(\zeta)}\epsilon + \frac{KT^{\zeta}}{M(\zeta)\Gamma(\zeta)}\epsilon\Big)$$

$$|\gamma - x| \leq \frac{1}{1-c}\Big(\frac{1-\zeta}{M(\zeta)} + \frac{T^{\zeta}}{M(\zeta)\Gamma(\zeta)}\Big)\epsilon$$

$$|\gamma - x| \leq \kappa_{f,\phi}\epsilon$$

where

$$c = 0 < \Big(\frac{1-\zeta}{M(\zeta)}K + \frac{KT^{\zeta}}{M(\zeta)\Gamma(\zeta)}\Big) < 1.$$

Therefore, the solution of equation (1.1) is Ulam-Hyers stable. □

Lemma 3.2. *Let $\gamma \in C([t_0, T], \mathbf{R}^n)$ be a solution of the inequality* (2.30) *then γ is a solution of the following integral inequality*

$$\Big|\gamma(t) - \omega_0(0) - \frac{1-\zeta}{M(\zeta)}f(t,\gamma_t) - \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}f(p,\gamma_p)dp\Big|$$

$$\leq \Big(\frac{1-\zeta}{M(\zeta)} + \frac{T^{\zeta}}{M(\zeta)\Gamma(\zeta)}\Big)\phi(t)$$

Proof. By remark 2.3 we have

$$\gamma(t) = \omega_0(0) + \frac{1-\zeta}{M(\zeta)}f(t,\gamma_t) + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}f(p,\gamma_p)dp +$$

$$\frac{1-\zeta}{M(\zeta)}g(t) + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}g(p)dp$$

from this we have

$$\Big|\gamma(t) - \omega_0(0) - \frac{1-\zeta}{M(\zeta)}f(t,\gamma_t) - \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}f(p,\gamma_p)dp\Big|$$

$$= \Big|\frac{1-\zeta}{M(\zeta)}g(t) + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}g(p)dp\Big|$$

$$\leq \frac{1-\zeta}{M(\zeta)}||g(t)|| + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}||g(p)||dp||$$

$$\leq \frac{1-\zeta}{M(\zeta)}\phi(t) + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}\phi(p)dp$$

$$\leq \frac{1-\zeta}{M(\zeta)}\phi(t) + \frac{T^{\zeta}}{M(\zeta)\Gamma(\zeta)}\phi(t)$$

$$= \Big(\frac{1-\zeta}{M(\zeta)} + \frac{T^{\zeta}}{M(\zeta)\Gamma(\zeta)}\Big)\phi(t)$$

□

Theorem 3.2. *Assume that there exists a unique solution $x \in C([t_0, T], \mathbf{R}^n)$ of equation (1.1) and there exist a nondecreasing continuous function $\phi(t) \in C([t_0, T], \mathbf{R}^n)$ and a real number $\kappa_{f,\phi} > 0$ such that for every solution $\gamma \in C([t_0, T], \mathbf{R}^n)$ of the inequality (2.30) satisfying*

$$|\gamma - x| \leq \kappa_{f,\phi}\phi(t),$$

then the unique solution of equation(1.1) is stable via modified Ulam-Hyers-Rassias stability concepts.

Proof.

$$|\gamma(t) - x(t)| = \Big\|\frac{1-\zeta}{M(\zeta)}[f(t,\omega) - f(t,\eta)]\Big\| + \Big\|\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}[f(p,\omega) - f(p,\eta)]dp\Big\|$$

$$+\Big\|\frac{1-\zeta}{M(\zeta)}g(t) + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}g(p)dp\|$$

$$\leq \frac{1-\zeta}{M(\zeta)}\Big\|f(t,\omega) - f(t,\eta)\Big\| + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}\Big\|f(p,\omega) - f(p,\eta)\Big\|dp + \frac{1-\zeta}{M(\zeta)}\Big\|g(t)\Big\|$$

$$+\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}\Big\|g(p)\Big\|dp$$

$$\leq \frac{1-\zeta}{M(\zeta)}K\Big\|\omega - \eta\Big\| + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}K\Big\|\omega - \eta\Big\|dp + \frac{1-\zeta}{M(\zeta)}\Big\|g(t)\Big\| +$$

$$\frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}\Big\|g(p)\Big\|dp$$

$$\leq \frac{1-\zeta}{M(\zeta)}K\|\gamma - x\| + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{t_0}^{t}(t-p)^{\zeta-1}K\Big\|\gamma - x\Big\|_\infty$$

$$+\frac{1-\zeta}{M(\zeta)}\phi(t) + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\theta\phi(t)$$

$$\leq \Big(\frac{1-\zeta}{M(\zeta)}K + \frac{K}{M(\zeta)\Gamma(\zeta)}T^\zeta\Big)\Big\|\gamma - x\Big\|_\infty$$

$$+\Big(\frac{1-\zeta}{M(\zeta)}\phi(t) + \frac{T^\zeta}{M(\zeta)\Gamma(\zeta)}\phi(t)\Big)$$

$$|\gamma - x| \leq \frac{1}{1-c}\Big(\frac{1-\zeta}{M(\zeta)}\phi(t) + \frac{T^\zeta}{M(\zeta)\Gamma(\zeta)}\phi(t)\Big)$$

$$|\gamma - x| \leq \kappa_{f,\phi}\phi(t)$$

where

$$c = 0 < \Big(\frac{1-\zeta}{M(\zeta)}K + \frac{KT^\zeta}{M(\zeta)\Gamma(\zeta)}\Big) < 1.$$

Therefore, the solution of equation (1.1) is generalized Ulam-Hyers-Rassias stable. □

4. Illustration

Consider the following Atangana–Baleanu fractional delay single-population infection–immune interaction model with memory and delay, which represents the dynamics of an infectious disease incorporating immune memory effects

$$\begin{cases} {}_{0}^{ABC}D_t^{\zeta}x(t) &= -\alpha x(t) + \beta x(t-\tau) + u(t), \quad t \in [0,T], \\ x_0(\theta) &= \eta_0(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{4.1}$$

where

- $x(t)$ represents the infected cell concentration,
- $\alpha > 0$ is the natural clearance rate,
- $\beta \geq 0$ models delayed immune feedback,
- $\tau > 0$ is the immune-response delay,
- $u(t)$ is an external stimulus (treatment or exposure),
- $0 < \zeta < 1$.

Let the right hand side of system (4.1) be represented as

$$f(t, x_t) = -\alpha x(t) + \beta x(t-\tau) + u(t)$$

Since $f(t, x_t)$ is globally Lipschitz continuous with respect to x_t, it follows immediately that system (4.1) admits a unique solution. Consequently, by applying fractional Grönwall inequalities with Mittag–Leffler kernels, we deduce that approximate solutions satisfies

$$\left| {}^{ABC}D_t^{\zeta}\gamma(t) - f(t, \gamma_t) \right| \leq \epsilon.$$

This concludes the Ulam–Hyers and modified Ulam–Hyers–Rassias stability of system (4.1).

We present numerical simulations illustrating the Ulam–Hyers–type stability behavior of system (4.1) as follows:

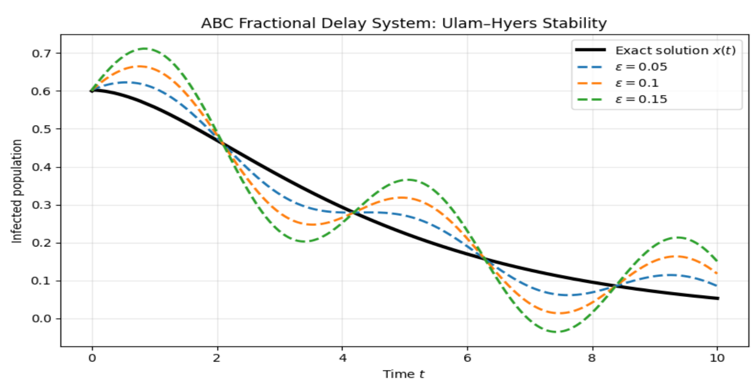

Figure 1. Time evolution of the exact solution $x(t)$ and ϵ-approximate solutions $\gamma(t)$ of the Atangana-Baleanu fractional delay differential system (4.1). The close agreement between the trajectories demonstrates the robustness of the system with respect to small perturbations, illustrating the Ulam–Hyers stability of the unique solution.

Figure 1 illustrates the temporal evolution of the state variable $x(t)$ governed by the Atangana-Baleanu fractional delay differential equation. The solid curve represents the exact solution of the system, while the dashed curves correspond to ϵ-approximate solutions for different perturbation levels. Owing to the non-singular kernel of the Atangana-Baleanu derivative, the solution exhibits a smooth and memory-dependent decay profile, which is consistent with the biological interpretation of gradual immune response and long–term memory effects. The influence of the delay term is evident in the moderate persistence of the state variable, as the delayed feedback counteracts rapid decay and induces a smoother transition over time. The approximate solutions remain uniformly close to the exact solution throughout the entire time interval. Although small oscillations are introduced due to the perturbation term, their amplitudes remain bounded and do not grow unrestrained as time progresses. This behavior confirms that the system is robust with respect to small deviations in the fractional dynamics, a fundamental requirement for Ulam–Hyers stability. The increasing values of ϵ produce proportionally larger deviations, yet the overall qualitative behavior of the system is preserved, demonstrating the structural stability of the fractional delay model.

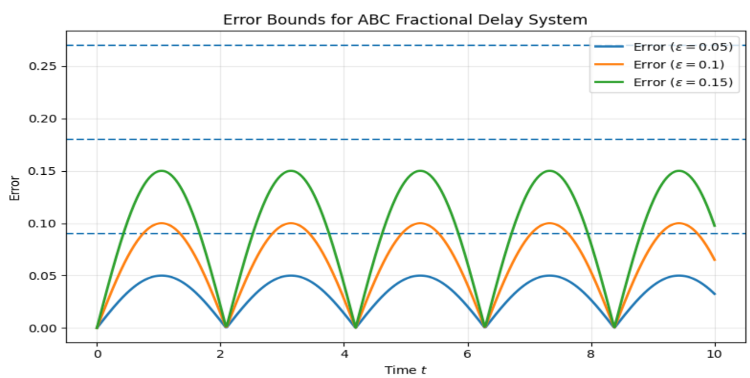

FIGURE 2. Comparison between the exact solution $x(t)$ and $\phi(t)$-bounded approximate solutions of the Atangana-Baleanu fractional delay differential equation. The time-dependent perturbation governed by the nondecreasing function $\phi(t)$ illustrates the generalized Ulam-Hyers-Rassias stability behaviour of the system (4.1).

Figure 2 depicts the absolute error between the approximate solutions and the exact solution as a function of time. The error curves clearly remain below their respective theoretical upper bounds, represented by horizontal dashed lines. This observation validates the existence of a constant $\kappa_{f,n} > 0$ such that

$$|\gamma(t) - x(t)| \leq \kappa_{f,n}\epsilon,$$

for all t in the considered interval. The boundedness of the error curves confirms that perturbations do not accumulate due to the memory effects of the Atangana–Baleanu derivative, which effectively dampens excessive deviations caused by delays or external fluctuations. Furthermore, the error dynamics demonstrate that the fractional delay system possesses a stabilizing memory mechanism.

Unlike classical integer–order delay systems, where errors may amplify due to delayed feedback, the fractional formulation ensures that past states are incorporated in a weighted and decaying manner. As a result, the system exhibits a strong resistance to perturbations, reinforcing the theoretical prediction of Ulam-Hyers stability for the proposed model.

5. Conclusion

This work investigates the Ulam-Hyers stability and the generalized Ulam-Hyers-Rassias stability of solutions to fractional delay differential equations. The numerical simulations are in strong agreement with the analytical stability results. In particular, the close correspondence between the exact and approximate solutions, together with the bounded error behavior, confirms that the Atangana-Baleanu fractional delay differential equation satisfies the conditions of Theorems 11 and 15. These results underscore the suitability of the Atangana-Baleanu fractional framework for modeling real-world systems with memory and delay, especially in applications related to epidemiology and immune response dynamics.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] T. Abdeljawad, Fractional Operators with Exponential Kernels and a Lyapunov-type inequality, Adv. in Diff. Eq. 313 (2017). https://doi.org/10.1186/s13662-017-1285-

[2] T. Abdeljawa, D. Baleanu, On Fractional Derivatives with exponential Kernel and their discrete versions, Reports on Mathematics Physics, 80 (2017). https://doi.org/10.1016/S0034-4877(17)30059-9

[3] T. Abdeljawad, D. Baleanu, Integration by Parts and its Applications of a New Nonlocal Fractional Derivative with Mittag-Leffler Nonsingular kernel, The Journal of Nonlinear Science and Applications, 10 (2017). https://doi.org/10.22436/jnsa.010.03.20

[4] T. Abdeljawad, D. Baleanu, Discrete fractional differences with nonsingular discrete Mittag-Leffler Kernels, Adv. in Dif. Eq. 232 (2016). https://doi.org/10.1186/s13662-016-0949-5

[5] T. Abdeljawad, A Lyapunov type inequality for fractional operators with non singular Mittag-Leffler kernel, J. of ineq. and App. 130 (2017). https://doi.org/10.1186/s13660-017-1400-5

[6] I. Alraddadi, M. P. Ineh, D. K. Igobi, U. Ishtiaq, and I.-L Popa,Stabilization of fractional hybrid systems with applications in biomedical dynamics. J. of Math. and Comp. Sci. 41 (2026), 406-420. https://doi.org/10.22436/jmcs.041.03.07

[7] A. Atangana, and D. Baleanu, New Fractional Derivatives with NonLocal and Non-singular Kernel: Theory and Application to heat transfer model, Therm. Sci. 20 (2016), 763-769. https://doi.org/10.2298/TSCI160111018A

[8] J. Ante, E. Abraham, U. Ebere, W. Udogworen, and C. Akpan, On the Global Existence of solution and Lyapunov Asymptotic Practical Stability for Nonlinear Impulsive Caputo Fractional Derivative via Comparison Principle, Scholars journal of physics, Mathematics and Statistics, 11 (2024), 160-172.

[9] J. Atsu, J. E. Ante, C. M. Orazulume, S. O. Essang, R. E. Francis, S. I. Okeke and W. K. Udogworen, A Lyapunov-based approach to uniform strict stability analysis for nonlinear impulsive Caputo fractional derivatives with weakly singular kernel, International Journal of Mathematical Analysis and Modelling, 8, (2025), 198-215. https://tnsmb.org/journal/index.php/ijmam/article/view/221

[10] M. Caputo and M. Fabrizio, A New Definition of Fractional Derivative without Singular Kernel, Progress in Fractional Differentiation and Applications, 2 (2015),1–7. http://dx.doi.org/10.12785/pfda/010201

[11] F. Chen. A. Chen, and X. Wang, On the Solutions for Impulsive Fractional Functional Differential Equations, Differential Equations and Dynamical Systems, 17 (2009), 379-391. https://doi.org/10.1007/s12591-009-0027-5

[12] D. Igobi and W. Udogworen, Results on Existence and Uniqueness of solutions of Fractional Differential Equations of Caputo-Fabrizio type in the sense of Riemann-Liouville, IAENG International Journal of App. Math. 54 (2024), 1163–1171.

[13] M. P. Ineh, O. O. Marcus, K. N. Inyang, A. Maharaj, and P. A. Nyong, On eventual uniform stability of Caputo fractional hybrid systems with application to library and information systems, Adv. Fixed Point Theory, 16 (2026), Article ID 15. https://doi.org/10.28919/afpt/9826

[14] M. Ineh, U. Ishtiaq, J. Ante, M. Garayev, and I. Popa, A robust uniform practical stability approach for Caputo fractional hybrid systems, AIMS Math. 10 (2025), 7001-7021. https://doi.org/10.3934/math.2025320

[15] M. Ineh and E. Akpan, On Lyapunov stability of Caputo Fractional dynamic equations on time scale using vector Lyapunov functions, Khyyam Journal of Mathematics, 11 (2025), 116-143. https://doi.org/10.22034/kjm.2025.477875.3319

[16] V. Lakshmikantham, Theory of Fractional Functional Differential Equation, Nonlinear Analysis, 69 (2008), 3337-3343.

[17] S. Mohammed, A. Abdeljawad, K. Shah, F. Jarad, Study of Impulsive problems under Mittag-Leffler power law, Heliyon6eo5109, 6 (2020), 1-8.

[18] G. Nchama, R. Mariano, and A. Mecias, Properties of the Caputo-Fabrizio Fractional Derivative, International Journal of Applied Engineering Research, (2021).

[19] M.I. Syam, and Mohammed Al-Refai, Fractional differential equations with Atangana-Baleanu fractional derivative: Analysis analysis and applications, Chaos, Solitons and Fractals: X, 2 (2009).

[20] J. Wang, Y. Zhou, and M. Feckan, Nonlinear Impulsive Problems for Fractional Differential Equations and Ulam Stability, Computers and Mathematics with Application, 64 (2012), 3389-3405.

[21] W. Udogworen, D. Igobi, and U. Jim, Analysis of Atangana-Baleanu-Caputo Impulsive Fractional Delay Differential Equations, IAENG International Journal of Applied Mathematics, 55 (2025), 3883-3891.

[22] W. Udogworen, D. Igobi, A. Hamarsheh, U. Jim, H. A Nabwey and R. George, Impulsive fractional delay differential equations with fixed moments and modified Ulam-Hyers-Rassias stability, Bound Value Probl. (2025). https://doi.org/10.1186/s13661-025-02178-5

[23] W. Udogworen, D. Igobi, A. Hamarsheh, U. Jim, H. A Nabwey and R. George, Analysis of impulsive fractional delay differential equations with Mittag-Leffler kernel and its applications to Hutchinson's model, J. of Math. and Comp. Sci. 42 (2026), 124-142. https://doi.org/10.22436/JMCS.042.01.08