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Scaled Partial Component Consensus in Discrete-Time Leader-Following Multi-Agent Systems

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Abstract. This paper investigates the scaled partial component consensus problem for discrete-time leader-following multi-agent systems. The objective is to achieve proportional agreement with a leader while enforcing consensus only on a prescribed subset of state components. By introducing scaled tracking errors and an inner coupling matrix, the coordination task is formulated in a component-selective manner. A permutation-based transformation is employed to rearrange the global error dynamics into independent component-wise subsystems. Based on this decomposition, sufficient conditions ensuring scaled partial component consensus are established using Lyapunov analysis and spectral stability arguments. The results explicitly characterize the interplay among the agent dynamics, network topology, inner coupling structure, and coupling gain. Numerical simulations demonstrate that the scaled states of the followers converge to the scaled leader trajectory for the selected components, while the unselected components remain unconstrained, thereby confirming the effectiveness of the proposed framework.

1. Introduction

Consensus and coordination of networked multi-agent systems (MASs) have become a central topic in distributed control and cooperative autonomy, motivated by applications ranging from formation control and task allocation to distributed sensing and cyber-physical security. Classical consensus aims to achieve agreement of full state vectors (or outputs) among agents through local interactions, but many modern scenarios demand more structured objectives, including coordination on only a subset of components, limited information exchange for privacy, and proportional (scaled) agreement rather than equality. These requirements have stimulated several complementary research directions, notably partial (component-selective) consensus and scaled consensus, which provide richer and more flexible coordination specifications than standard consensus.

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Partial component consensus formalizes the idea that only certain state components are required to synchronize, while the remaining components are allowed to evolve freely. Early studies developed partial-stability-based methodologies to analyze component-selective convergence for linear MASs, establishing systematic transformations that separate the global dynamics into component-wise subsystems and enabling Lyapunov-type arguments for selected components ([1], [2], [14]). Leader-following partial component consensus has also been investigated for nonlinear MASs, where pinning/leader information and component-selection mechanisms are used to drive only prescribed components of followers toward the leader trajectory ([19], [24]). More recently, partial component consensus has been extended to increasingly realistic environments, including higher-order nonlinear dynamics [7], mean-square settings with delays, uncertain Markov switching topologies, and denial-of-service (DoS) attacks [25], as well as prescribed-time, finite-time, and fixed-time partial component coordination under switching topologies and external disturbances ([22], [20], [21]). Beyond control-oriented settings, partial agreement ideas appear in secure plan consensus and task assignment [10], distributed fusion of probabilistic information [11], and crowd/collective computing with partial agreement mechanisms [9], reflecting the broad relevance of component-selective coordination. Parallel to these developments, partial information transmission has been explored for privacy-preserving average consensus, underscoring that practical consensus protocols may intentionally limit what is shared or what must agree [23], and impulsive/partial communication patterns have been used to improve robustness and efficiency in tracking and consensus tasks [6].

Scaled consensus, initiated in a concise and influential form in [15], generalizes agreement by requiring that appropriately scaled agent states converge to a common value, thereby encoding proportional relationships among agents. This paradigm is particularly meaningful when heterogeneous agents, calibration factors, or performance weights naturally induce proportional coordination rather than strict equality. The scaled consensus framework has been extended to time-delay settings [16], switching and switched-network environments ([12], [17]), and nonlinear constraints such as output saturation [18]. Scaled consensus has also been studied in distributed parameter and networked PDE contexts, such as wave-equation networks, highlighting its versatility beyond finite-dimensional MASs [3]. In leader-following scenarios, finite-time scaled consensus has been analyzed to accelerate convergence and enhance transient performance [4]. Recent work has further connected scaled consensus concepts with impulsive designs and edge-dynamic coordination, indicating that scaling ideas can be integrated with more complex network models and protocol mechanisms ([5], [13]).

Despite the significant progress on partial component consensus and scaled consensus as largely separate lines, a notable gap remains in developing a unified discrete-time leader-following framework that simultaneously addresses (i) component-selective objectives and (ii) proportional (scaled) coordination, while providing verifiable convergence conditions and practical gain-design rules. Existing partial component consensus studies typically enforce convergence of unscaled errors for selected components ([8], [24], [19]), whereas scaled consensus studies usually require scaled agreement across all components and do not incorporate component-selection mechanisms ([15], [17], [12]). Moreover, in discrete-time leader-following settings, ensuring that the implemented control law induces the desired closed-loop scaled error recursion is nontrivial when heterogeneous scaling factors are present, which can invalidate the theoretical error model if not handled carefully. This creates a methodological need for protocol formulations and analyses that explicitly preserve the scaled-error dynamics used in proofs, while retaining the component-wise decomposition benefits of partial stability approaches ([1], [2]) and the proportional coordination semantics of scaled consensus [15].

Motivated by these observations, this paper proposes a discrete-time leader-following *scaled partial component consensus* framework that integrates the inner-coupling component-selection mechanism from partial component consensus with the proportional agreement objective of scaled consensus. The key analytical step is to derive a global scaled error system and then employ a commutation/permutation transformation, as in partial-stability-based methodologies ([1], [2], [8]), to rearrange the Kronecker-structured dynamics into independent component-wise subsystems. This decomposition enables a Lyapunov analysis tailored to the selected components only, yielding tractable sufficient conditions for convergence in terms of matrix contraction inequalities. To reduce conservatism, a spectral-radius-based Schur stability characterization is also established, aligning discrete-time convergence with standard linear stability theory and complementing norm-based conditions that are convenient for analysis ([17], [12]). In the practically relevant undirected pinned case, the paper further develops an explicit eigenvalue-based gain selection interval, which connects convergence guarantees to the extreme eigenvalues of the pinned Laplacian and thereby provides a directly implementable tuning rule consistent with gain-design perspectives in scaled consensus studies [18].

The resulting theory clarifies how scaling factors and component selection interact: scaling determines proportional tracking relationships relative to the leader, while the inner coupling matrix specifies which state components must satisfy those relationships. This viewpoint is compatible with modern considerations such as partial information exchange and selective coordination objectives ([23], [6]), and it complements broader partial-agreement themes that arise in multi-agent task allocation, information fusion, and collective computation ([10], [11], [9]). Finally, numerical simulations are provided to verify all standing assumptions and to illustrate the component-selective effect: the scaled states of all followers converge to the scaled leader trajectory for the selected components, while the unselected component may diverge when its intrinsic dynamics are unstable, thereby visually demonstrating the precise objective encoded by scaled partial component consensus.

The remainder of this paper is organized as follows. Section 2 introduces the notation, graph-theoretic preliminaries, and the problem formulation for scaled partial component consensus.

Section 3 presents the main theoretical results, including the error transformation, Lyapunov-based convergence analysis, spectral stability conditions, and eigenvalue-based gain design rules. Section 4 provides numerical simulations that validate the theoretical results and illustrate the component-selective scaled consensus behavior. Finally, concluding remarks and possible directions for future research are discussed in Section 5.

2. Preliminaries

This section introduces the basic graph-theoretic concepts and formulates the scaled partial component consensus problem for discrete-time leader–following multi-agent systems.

2.1. **Graph theory.** Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ of order N, where $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ denotes the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of directed edges. An edge $(v_j, v_i) \in \mathcal{E}$ indicates that node v_i can receive information from node v_j. The weighted adjacency matrix associated with $\mathcal{G}$ is denoted by $A = (a_{ij}) \in \mathbb{R}^{N\times N}$, where $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise.

The set of neighbors of node v_i is defined as $N_i = \{v_j \in \mathcal{V} \,|\, (v_j, v_i) \in \mathcal{E}\}$. A directed graph is said to contain a directed spanning tree if there exists a node, referred to as the root, that has directed paths to all other nodes in the graph. The Laplacian matrix corresponding to $\mathcal{G}$ is defined as $L = (l_{ij}) \in \mathbb{R}^{N\times N}$, where $l_{ij} = -a_{ij}$ for $i \neq j$ and $l_{ii} = \sum_{j=1}^{N} a_{ij}$. It follows that the Laplacian matrix satisfies $\sum_{j=1}^{N} l_{ij} = 0$ for all $i = 1, \ldots, N$.

2.2. **Problem formulation.** Consider a leader–following multi-agent system consisting of N follower agents and one leader agent. Each follower agent is described by an n-dimensional discrete-time state vector $x_i(k) \in \mathbb{R}^n$, while the leader state is denoted by $x_0(k) \in \mathbb{R}^n$. The dynamics of the followers are given by

$$x_i(k+1) = Ax_i(k) + u_i(k), \quad i = 1, \ldots, N, \tag{2.1}$$

where $A = \mathrm{diag}(A_1, \ldots, A_n) \in \mathbb{R}^{n\times n}$ represents the intrinsic system matrix and $u_i(k) \in \mathbb{R}^n$ denotes the distributed control input. The leader evolves according to

$$x_0(k+1) = Ax_0(k). \tag{2.2}$$

To enable component-selective coordination among agents, an inner coupling matrix is introduced in the form $\Gamma = \mathrm{diag}(r_1, \ldots, r_n)$ with $r_j \geq 0$. The nonzero elements of Γ determine which components of the state vectors participate in the coordination process. Without loss of generality, assume that the first ℓ components are selected, that is, $r_j > 0$ for $j = 1, \ldots, \ell$ and $r_j = 0$ for $j = \ell + 1, \ldots, n$. Consequently, the coordination protocol will only enforce agreement on the selected components.

In order to incorporate the concept of scaled consensus, associate with each agent a nonzero scaling factor $\alpha_i \neq 0$ for $i = 0, 1, \ldots, N$. The scaled state of agent i is defined as $y_i(k) = \alpha_i x_i(k)$. To measure the deviation from the leader under the scaling transformation, define the scaled tracking error

$$\tilde{e}_i(k) = \alpha_i x_i(k) - \alpha_0 x_0(k), \quad i = 1, \dots, N.$$

Stacking the scaled errors of all followers yields the global error vector

$$\tilde{e}(k) = (\tilde{e}_1^\top(k), \dots, \tilde{e}_N^\top(k))^\top \in \mathbb{R}^{Nn}.$$

For each state component $j \in \{1, \dots, n\}$, define the component-wise scaled error vector as

$$\tilde{\varepsilon}_j(k) = (\tilde{e}_{1j}(k), \dots, \tilde{e}_{Nj}(k))^\top \in \mathbb{R}^N.$$

This representation allows the global error vector to be decomposed into component-wise blocks corresponding to each state dimension.

Motivated by both scaled consensus protocols and component-selective coupling strategies, consider the following distributed control law

$$u_i(k) = c \sum_{j=1}^{N} a_{ij} \Gamma(\alpha_j x_j(k) - \alpha_i x_i(k)) - c d_i \Gamma(\alpha_i x_i(k) - \alpha_0 x_0(k)), \tag{2.3}$$

where $c > 0$ is the coupling gain and $d_i \geq 0$ indicates whether agent i has access to the leader information.

The objective of this work is to investigate the conditions under which the resulting multi-agent system achieves scaled partial component consensus. Specifically, the system is said to achieve scaled partial component consensus if the scaled tracking errors corresponding to the selected components asymptotically vanish, that is,

$$\lim_{k \to \infty} \tilde{\varepsilon}_j(k) = 0, \quad j = 1, \dots, \ell.$$

This condition implies that the scaled states of all agents agree with the scaled leader state on the selected components, while no consensus requirement is imposed on the remaining components.

3. Main Results

This section presents the main theoretical results of the paper. A key step in the analysis is to transform the global scaled error dynamics into a component-wise representation. This transformation reveals that the overall system can be decomposed into independent subsystems associated with individual state components. Based on this structure, a Lyapunov analysis is developed to derive sufficient conditions ensuring scaled partial component consensus.

3.1. **Error transformation.** Let

$$\tilde{e}(k) = (\tilde{e}_1^\top(k), \dots, \tilde{e}_N^\top(k))^\top \in \mathbb{R}^{Nn}$$

denote the stacked scaled error vector defined in Section 2. Under protocol (2.3), the closed-loop error dynamics can be written compactly as

$$\tilde{e}(k+1) = [I_N \otimes A - c(L + D) \otimes \Gamma]\, \tilde{e}(k), \tag{3.1}$$

where $A = \operatorname{diag}(A_1, \dots, A_n)$ and $\Gamma = \operatorname{diag}(r_1, \dots, r_n)$.

The following lemma allows the Kronecker factors in (3.1) to be rearranged so that the global dynamics can be analyzed component-wise.

Lemma 3.1. *There exists a permutation (commutation) matrix $P \in \mathbb{R}^{Nn \times Nn}$ such that*

$$P[(L+D) \otimes \Gamma]P^{-1} = \Gamma \otimes (L+D).$$

Proof. The result follows from the commutation property of the Kronecker product. Specifically, there exists a permutation matrix, often referred to as a commutation or perfect shuffle matrix, that rearranges the stacking order of block vectors. Applying this permutation to $(L+D) \otimes \Gamma$ interchanges the Kronecker factors and yields $\Gamma \otimes (L+D)$. □

Define the transformed error vector

$$\hat{e}(k) = P\tilde{e}(k).$$

Applying Lemma 3.1 to (3.1) gives

$$\hat{e}(k+1) = [A \otimes I_N - c\,\Gamma \otimes (L+D)]\,\hat{e}(k). \tag{3.2}$$

Let $\hat{e}(k) = (\tilde{\varepsilon}_1^\top(k), \dots, \tilde{\varepsilon}_n^\top(k))^\top$, where $\tilde{\varepsilon}_j(k) = (\tilde{e}_{1j}(k), \dots, \tilde{e}_{Nj}(k))^\top$ and denotes the scaled error associated with the j-th state component across all agents. Substituting this block structure into (3.2) shows that each component evolves independently according to

$$\tilde{\varepsilon}_j(k+1) = M_j\,\tilde{\varepsilon}_j(k), \tag{3.3}$$

where

$$M_j = A_j I_N - c r_j (L+D).$$

Therefore, the global system decomposes into n independent subsystems, each corresponding to a particular state component.

3.2. **Scaled partial component consensus.** We now derive sufficient conditions guaranteeing that the scaled errors associated with the selected components converge to zero.

Theorem 3.1. *Suppose that the communication graph contains a directed spanning tree and that the scaling factors $\alpha_i \neq 0$ for all agents. If there exists $\beta \in (0,1)$ such that*

$$M_j^\top M_j < (1-\beta) I_N, \qquad j = 1, \dots, \ell,$$

where

$$M_j = A_j I_N - c r_j (L+D),$$

then the discrete-time multi-agent system (2.1)–(2.2) *under protocol* (2.3) *achieves scaled partial component consensus with respect to the first ℓ components.*

Proof. Consider the Lyapunov function

$$V(k) = \sum_{j=1}^{\ell} \|\tilde{\varepsilon}_j(k)\|^2.$$

From (3.3), we obtain

$$V(k+1) - V(k) = \sum_{j=1}^{\ell} \tilde{\varepsilon}_j^\top(k)(M_j^\top M_j - I_N)\tilde{\varepsilon}_j(k).$$

Using the condition of the theorem, $M_j^\top M_j < (1-\beta)I_N$, it follows that

$$V(k+1) - V(k) \leq -\beta V(k).$$

Therefore

$$V(k+1) \leq (1-\beta)V(k).$$

By recursion, $V(k) \leq (1-\beta)^k V(0)$. Since $0 < 1-\beta < 1$, we have

$$\lim_{k\to\infty} V(k) = 0,$$

which implies

$$\lim_{k\to\infty} \tilde{\varepsilon}_j(k) = 0, \quad j = 1, \ldots, \ell.$$

Recalling that

$$\tilde{\varepsilon}_j(k) = (\alpha_1 x_{1j}(k) - \alpha_0 x_{0j}(k), \ldots, \alpha_N x_{Nj}(k) - \alpha_0 x_{0j}(k))^\top,$$

we conclude that

$$\lim_{k\to\infty} (\alpha_i x_{ij}(k) - \alpha_0 x_{0j}(k)) = 0$$

for all $i = 1, \ldots, N$ and $j = 1, \ldots, \ell$. Hence scaled partial component consensus is achieved. □

Corollary 3.1. *Under the assumptions of Theorem 3.1, if*

$$\|M_j\|_2 < 1, \qquad j = 1, \ldots, \ell,$$

where $M_j = A_j I_N - cr_j(L+D)$, then the multi-agent system achieves scaled partial component consensus with respect to the first ℓ components.

Proof. For each $j \in \{1, \ldots, \ell\}$, the component-wise scaled error satisfies

$$\tilde{\varepsilon}_j(k+1) = M_j \tilde{\varepsilon}_j(k).$$

Taking the Euclidean norm gives

$$\|\tilde{\varepsilon}_j(k+1)\| = \|M_j \tilde{\varepsilon}_j(k)\| \leq \|M_j\|_2 \, \|\tilde{\varepsilon}_j(k)\|.$$

Since $\|M_j\|_2 < 1$, by recursion we obtain

$$\|\tilde{\varepsilon}_j(k)\| \leq \|M_j\|_2^k \|\tilde{\varepsilon}_j(0)\|.$$

Hence

$$\lim_{k\to\infty} \tilde{\varepsilon}_j(k) = 0, \qquad j = 1, \ldots, \ell,$$

which implies scaled partial component consensus. □

Remark 3.1. *If $\alpha_i = 1$ for all agents, the scaled tracking error reduces to the standard tracking error $x_i(k) - x_0(k)$. In this case, the proposed framework reduces to the classical partial component consensus problem studied in the literature.*

Remark 3.2. *If all state components participate in the coordination process, that is, $\ell = n$, then the proposed framework reduces to a scaled consensus problem. In this case, Theorem 3.1 ensures that*

$$\alpha_i x_i(k) \to \alpha_0 x_0(k), \qquad i = 1, \dots, N,$$

which means that the scaled states of all agents asymptotically agree with the scaled leader state.

3.3. **A spectral-radius condition.** The contraction condition in Theorem 3.1 is convenient for analysis, yet it can be conservative because it enforces a uniform bound in the Euclidean norm. For discrete-time linear systems, a sharper characterization of asymptotic convergence is provided by Schur stability, equivalently by a spectral-radius condition. We next state an alternative sufficient condition in terms of the spectral radius.

Theorem 3.2 (Spectral condition for scaled partial component consensus)**.** *Assume that the first ℓ components are selected, i.e., $r_j > 0$ for $j = 1, \dots, \ell$ and $r_j = 0$ for $j = \ell + 1, \dots, n$. Let $M_j = A_j I_N - c r_j(L + D)$ be defined in* (3.3)*. If*

$$\rho(M_j) < 1, \qquad j = 1, \dots, \ell, \tag{3.4}$$

then the discrete-time multi-agent system (2.1)–(2.2) *under protocol* (2.3) *achieves scaled partial component consensus with respect to the first ℓ components, i.e.,*

$$\lim_{k\to\infty} \tilde{\varepsilon}_j(k) = 0, \qquad j = 1, \dots, \ell.$$

Proof. From (3.3), each selected component evolves according to the linear recursion

$$\tilde{\varepsilon}_j(k+1) = M_j \tilde{\varepsilon}_j(k), \qquad j = 1, \dots, \ell.$$

Fix an arbitrary $j \in \{1, \dots, \ell\}$. The condition $\rho(M_j) < 1$ implies that M_j is Schur stable. A standard consequence of Schur stability is the existence of a positive definite solution to the discrete-time Lyapunov equation. More precisely, pick any matrix $Q_j \in \mathbb{R}^{N\times N}$ such that $Q_j \succ 0$. Define

$$P_j := \sum_{k=0}^{\infty} (M_j^\top)^k Q_j M_j^k. \tag{3.5}$$

We first show that the series (3.5) converges and that $P_j \succ 0$.

Since $\rho(M_j) < 1$, by Gelfand's formula there exists $\eta \in (0, 1)$ and a constant $C > 0$ such that $\|M_j^k\| \le C\eta^k$ for all $k \ge 0$ in some induced matrix norm. Therefore,

$$\|(M_j^\top)^k Q_j M_j^k\| \le \|M_j^k\|^2 \|Q_j\| \le C^2 \eta^{2k} \|Q_j\|,$$

and hence the series (3.5) is absolutely convergent. Moreover, each summand $(M_j^\top)^k Q_j M_j^k$ is positive semidefinite and the $k = 0$ term equals $Q_j \succ 0$, which implies $P_j \succ 0$.

Next, we verify that P_j satisfies a Lyapunov identity. Using (3.5),

$$\begin{aligned} M_j^\top P_j M_j &= M_j^\top \left(\sum_{k=0}^{\infty} (M_j^\top)^k Q_j M_j^k \right) M_j \\ &= \sum_{k=0}^{\infty} (M_j^\top)^{k+1} Q_j M_j^{k+1} = \sum_{k=1}^{\infty} (M_j^\top)^k Q_j M_j^k. \end{aligned}$$

Subtracting this expression from P_j gives the discrete-time Lyapunov equation

$$P_j - M_j^\top P_j M_j = Q_j \succ 0. \tag{3.6}$$

Now define the Lyapunov function for the j-th subsystem by

$$V_j(k) := \tilde{\varepsilon}_j^\top(k) P_j \tilde{\varepsilon}_j(k).$$

Since $P_j \succ 0$, $V_j(k)$ is positive definite and radially unbounded in $\tilde{\varepsilon}_j(k)$. Along trajectories,

$$\begin{aligned} V_j(k+1) - V_j(k) &= \tilde{\varepsilon}_j^\top(k+1) P_j \tilde{\varepsilon}_j(k+1) - \tilde{\varepsilon}_j^\top(k) P_j \tilde{\varepsilon}_j(k) \\ &= \tilde{\varepsilon}_j^\top(k) M_j^\top P_j M_j \tilde{\varepsilon}_j(k) - \tilde{\varepsilon}_j^\top(k) P_j \tilde{\varepsilon}_j(k) \\ &= -\tilde{\varepsilon}_j^\top(k) \big(P_j - M_j^\top P_j M_j \big) \tilde{\varepsilon}_j(k). \end{aligned}$$

Using (3.6), we obtain

$$V_j(k+1) - V_j(k) = -\tilde{\varepsilon}_j^\top(k) Q_j \tilde{\varepsilon}_j(k) \le -\lambda_{\min}(Q_j) \|\tilde{\varepsilon}_j(k)\|^2.$$

Hence $V_j(k)$ is strictly decreasing unless $\tilde{\varepsilon}_j(k) = 0$. Moreover, since $V_j(k) \ge \lambda_{\min}(P_j)\|\tilde{\varepsilon}_j(k)\|^2$ and $V_j(k+1) \le V_j(k)$, the monotone convergence theorem yields that $V_j(k)$ converges and the summability condition

$$\sum_{k=0}^{\infty} \|\tilde{\varepsilon}_j(k)\|^2 < \infty$$

holds. In particular, $\|\tilde{\varepsilon}_j(k)\| \to 0$ as $k \to \infty$.

Because the above argument holds for each $j = 1, \ldots, \ell$, we conclude that

$$\lim_{k \to \infty} \tilde{\varepsilon}_j(k) = 0, \qquad j = 1, \ldots, \ell,$$

which is precisely scaled partial component consensus. □

Remark 3.3. *The contraction condition in Theorem 3.1 implies the spectral condition in Theorem 3.2. Indeed, if $M_j^\top M_j \prec (1-\beta) I_N$ for some $\beta \in (0,1)$, then $\|M_j\|_2^2 = \lambda_{\max}(M_j^\top M_j) < 1 - \beta < 1$, and hence $\rho(M_j) \le \|M_j\|_2 < 1$. Therefore, Theorem 3.2 provides a less conservative, spectral-radius-based sufficient condition.*

3.4. **Eigenvalue-based gain selection under undirected pinned graphs.** While Theorem 3.2 provides a less conservative Schur-stability condition, it remains implicit in the coupling gain c for general directed graphs. In the practically relevant undirected pinned case, one can further derive an explicit interval for c in terms of the extreme eigenvalues of $L+D$.

Assumption 3.1. *The communication graph among the followers is undirected and connected, and at least one follower is pinned to the leader, i.e., $D \neq 0$. Consequently, $L+D$ is symmetric positive definite with eigenvalues*

$$0 < \lambda_{\min}(L+D) \leq \lambda_i(L+D) \leq \lambda_{\max}(L+D), \quad i = 1, \ldots, N.$$

Theorem 3.3 (Eigenvalue-based gain design for DT scaled partial component consensus)**.** *Under Assumption 3.1, consider the component-wise recursion*

$$\tilde{\varepsilon}_j(k+1) = M_j \tilde{\varepsilon}_j(k), \qquad M_j = A_j I_N - c r_j (L+D),$$

for each selected component $j = 1, \ldots, \ell$. If the gain $c > 0$ is chosen such that

$$\frac{A_j - 1}{r_j\, \lambda_{\min}(L+D)} \;<\; c \;<\; \frac{A_j + 1}{r_j\, \lambda_{\max}(L+D)}, \qquad j = 1, \ldots, \ell, \tag{3.7}$$

then $\rho(M_j) < 1$ for all $j = 1, \ldots, \ell$, and hence scaled partial component consensus is achieved.

Proof. Fix any $j \in \{1, \ldots, \ell\}$. Since $L+D$ is symmetric positive definite, it admits an orthogonal eigen-decomposition

$$L + D = U \Lambda U^\top,$$

where $U^\top U = I_N$ and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ with $\lambda_i \in [\lambda_{\min}(L+D), \lambda_{\max}(L+D)]$. Then

$$M_j = A_j I_N - c r_j (L+D) = U\Big(A_j I_N - c r_j \Lambda\Big)U^\top,$$

so the eigenvalues of M_j are $\mu_i(M_j) = A_j - c r_j \lambda_i$. Hence

$$\rho(M_j) = \max_{1 \leq i \leq N} |A_j - c r_j \lambda_i| = \max\Big\{ |A_j - c r_j \lambda_{\min}(L+D)|,\ |A_j - c r_j \lambda_{\max}(L+D)| \Big\}.$$

Therefore, $\rho(M_j) < 1$ is guaranteed if

$$|A_j - c r_j \lambda_{\min}(L+D)| < 1 \quad \text{and} \quad |A_j - c r_j \lambda_{\max}(L+D)| < 1.$$

These inequalities are equivalent to

$$\frac{A_j - 1}{r_j\, \lambda_{\min}(L+D)} < c < \frac{A_j + 1}{r_j\, \lambda_{\max}(L+D)},$$

which is exactly (3.7). The conclusion follows for each selected $j = 1, \ldots, \ell$. □

Corollary 3.2 (Simplified gain range for marginally stable component dynamics)**.** *Under Assumption 3.1, suppose that $0 < A_j \leq 1$ for all selected components $j = 1, \ldots, \ell$. If*

$$0 < c < \min_{1 \leq j \leq \ell} \frac{A_j + 1}{r_j\, \lambda_{\max}(L+D)},$$

then scaled partial component consensus is achieved.

Proof. When $0 < A_j \leq 1$, the lower bound $(A_j - 1)/(r_j \lambda_{\min}(L+D))$ is nonpositive, hence any $c > 0$ satisfies it. The stated upper bound ensures the right inequality in (3.7) for every $j = 1, \dots, \ell$. Thus (3.7) holds and Theorem 3.3 applies. □

Remark 3.4. *Theorem 3.3 provides an explicit and verifiable tuning rule: the gain c can be selected from an interval determined by $\lambda_{\min}(L+D)$ and $\lambda_{\max}(L+D)$, together with the component parameters A_j and r_j. This eigenvalue-based characterization is particularly convenient for implementation and robustness studies, since the extreme eigenvalues quantify the pinned connectivity strength of the network.*

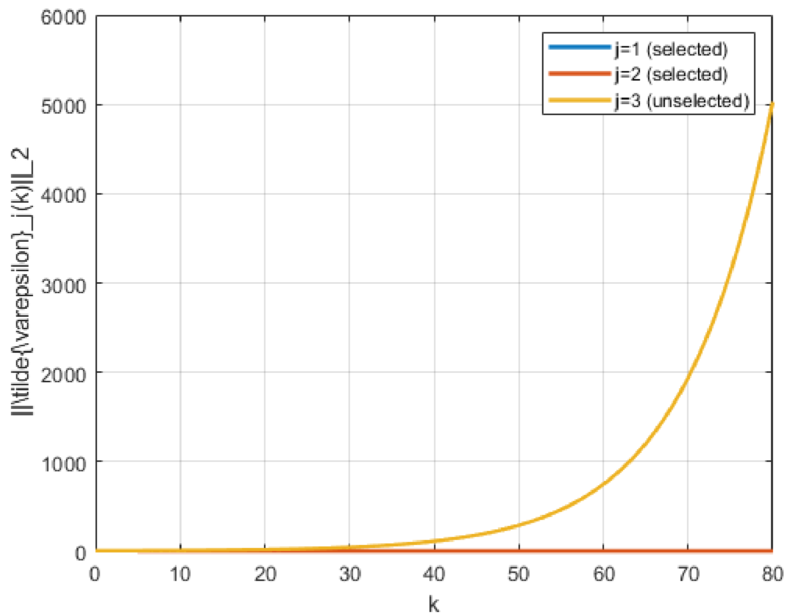

FIGURE 1. Discrete-time scaled partial component consensus: norms of component-wise scaled errors $E_j(k) = \|\tilde{\varepsilon}_j(k)\|_2$. Selected components ($j = 1, 2$) converge to zero, whereas the unselected component ($j = 3$) is not regulated ($r_3 = 0$) and may grow.

4. NUMERICAL EXAMPLE

This section validates the discrete-time scaled partial component consensus results established in Section 3. The simulation is designed to satisfy all assumptions and to explicitly verify the convergence statements in Theorem 3.1, Corollary 3.1, and Theorem 3.2, together with the gain-design condition in Theorem 3.3 under Assumption 3.1. In particular, we show that the scaled states of all followers synchronize with the scaled leader state for the selected components, while the unselected component may remain divergent because it is excluded from the inner coupling.

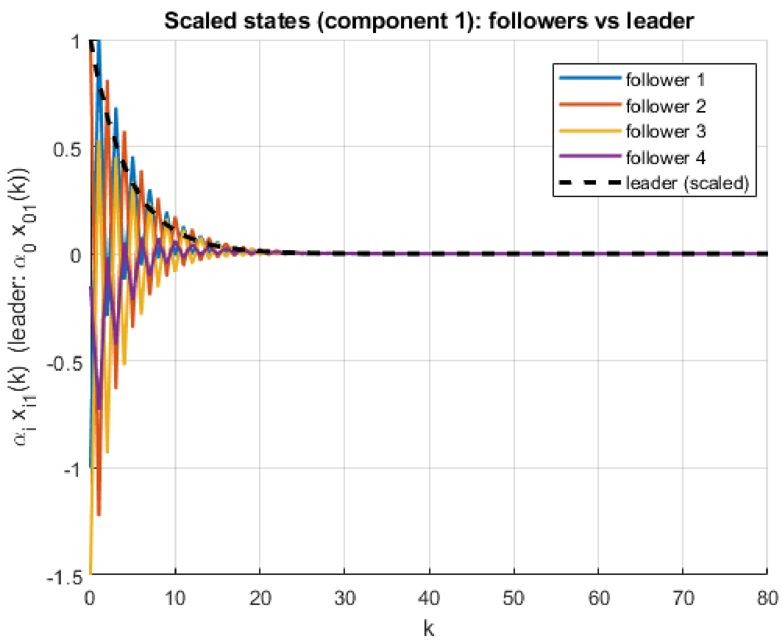

FIGURE 2. Scaled states for component $j = 1$: follower trajectories $\alpha_i x_{i1}(k)$ converge to the scaled leader trajectory $\alpha_0 x_{01}(k)$, confirming scaled tracking on the selected component.

4.1. **System model, scaling, and protocol implementation.** Consider $N = 4$ followers and one leader with $n = 3$ state components. The follower dynamics and leader dynamics are

$$x_i(k+1) = Ax_i(k) + u_i(k), \qquad i = 1, \dots, N, \qquad x_0(k+1) = Ax_0(k),$$

where

$$A = \mathrm{diag}(A_1, A_2, A_3) = \mathrm{diag}(0.8,\, 0.9,\, 1.1).$$

The inner coupling matrix is chosen as

$$\Gamma = \mathrm{diag}(r_1, r_2, r_3) = \mathrm{diag}(1, 1, 0),$$

so the selected set is $\ell = 2$ (components $j = 1, 2$) and the third component is unselected ($r_3 = 0$). The scaling factors are

$$\alpha_0 = 1, \qquad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1,\, 2,\, -1,\, 0.5),$$

which satisfy $\alpha_i \neq 0$ for all followers as required in Theorem 3.1. Define the scaled tracking error $\tilde{e}_i(k) = \alpha_i x_i(k) - \alpha_0 x_0(k)$ and the component-wise stacked errors $\tilde{\varepsilon}_j(k) \in \mathbb{R}^N$ as in Section 2. To ensure that the simulated closed-loop dynamics match the theoretical scaled-error model (3.1)–(3.3), the control input is implemented in the scaled form so that the induced error recursion satisfies

$$\tilde{e}(k+1) = \Big[I_N \otimes A - c(L + D) \otimes \Gamma \Big] \tilde{e}(k).$$

Equivalently, each follower applies

$$u_i(k) = \frac{c}{\alpha_i}\left(\sum_{j=1}^{N} a_{ij}\,\Gamma\Big(\alpha_j x_j(k) - \alpha_i x_i(k)\Big) - d_i\,\Gamma\Big(\alpha_i x_i(k) - \alpha_0 x_0(k)\Big)\right),$$

which yields the component-wise subsystems (3.3) with $M_j = A_j I_N - c r_j (L+D)$ for $j = 1, \ldots, n$.

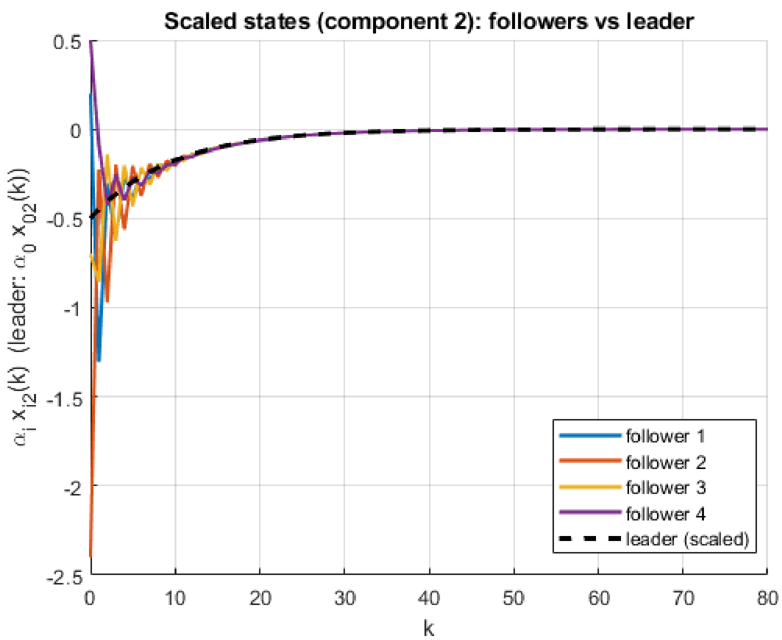

FIGURE 3. Scaled states for component $j = 2$: follower trajectories $\alpha_i x_{i2}(k)$ converge to the scaled leader trajectory $\alpha_0 x_{02}(k)$, confirming scaled tracking on the selected component.

4.2. **Communication graph, pinning, and eigenvalues.** The follower communication topology is the undirected path graph 1–2–3–4 with unit weights, whose adjacency matrix is

$$A_{\mathrm{adj}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The Laplacian is $L = \Delta - A_{\mathrm{adj}}$ with $\Delta = \mathrm{diag}(1, 2, 2, 1)$. The leader is pinned to the network via

$$D = \mathrm{diag}(1, 0, 0, 0),$$

so follower 1 receives leader information. Hence the follower graph is connected and $D \neq 0$, which satisfies Assumption 3.1; consequently, $L + D$ is symmetric positive definite. The eigenvalues are computed as

$$\lambda(L+D) = \{0.1206,\ 1.0000,\ 2.3473,\ 3.5321\},$$

so $\lambda_{\min}(L+D) = 0.1206$ and $\lambda_{\max}(L+D) = 3.5321$.

4.3. **Gain selection and verification of theoretical conditions.** The coupling gain is selected using the explicit interval in Theorem 3.3. For $j = 1, 2$ with $r_1 = r_2 = 1$, the admissible gain interval (3.7) becomes

$$\frac{A_j - 1}{\lambda_{\min}(L + D)} < c < \frac{A_j + 1}{\lambda_{\max}(L + D)}, \qquad j = 1, 2.$$

Since $A_1 = 0.8$ and $A_2 = 0.9$ are less than one, the lower bounds are negative and any $c > 0$ satisfies them. The upper bounds are

$$c < \frac{0.8 + 1}{3.5321} = 0.5096, \qquad c < \frac{0.9 + 1}{3.5321} = 0.5379.$$

We therefore choose $c = 0.45$, which satisfies both component-wise inequalities. With this choice, the matrices M_1 and M_2 satisfy the norm condition in Corollary 3.1:

$$\|M_1\|_2 = 0.7894 < 1, \qquad \|M_2\|_2 = 0.8457 < 1,$$

and thus the spectral-radius condition $\rho(M_j) < 1$ in Theorem 3.2 also holds. Accordingly, the selected component-wise scaled errors $\tilde{\varepsilon}_1(k)$ and $\tilde{\varepsilon}_2(k)$ are guaranteed to converge to zero.

4.4. **Initial conditions and performance measures.** The leader initial state is chosen as

$$x_0(0) = \begin{bmatrix} 1 \\ -0.5 \\ 0.2 \end{bmatrix},$$

and the follower initial states are selected as

$$x_1(0) = \begin{bmatrix} -1.0 \\ 0.2 \\ 1.5 \end{bmatrix}, \quad x_2(0) = \begin{bmatrix} 0.5 \\ -1.2 \\ -0.7 \end{bmatrix}, \quad x_3(0) = \begin{bmatrix} 1.5 \\ 0.7 \\ 0.9 \end{bmatrix}, \quad x_4(0) = \begin{bmatrix} -0.3 \\ 1.0 \\ -1.1 \end{bmatrix}.$$

To evaluate scaled partial component consensus, we track the component-wise scaled error norms

$$E_j(k) = \|\tilde{\varepsilon}_j(k)\|_2, \qquad j = 1, 2, 3,$$

and also plot the scaled states $\alpha_i x_{ij}(k)$ of all followers against the scaled leader trajectory $\alpha_0 x_{0j}(k)$ for each component.

4.5. **Results.**

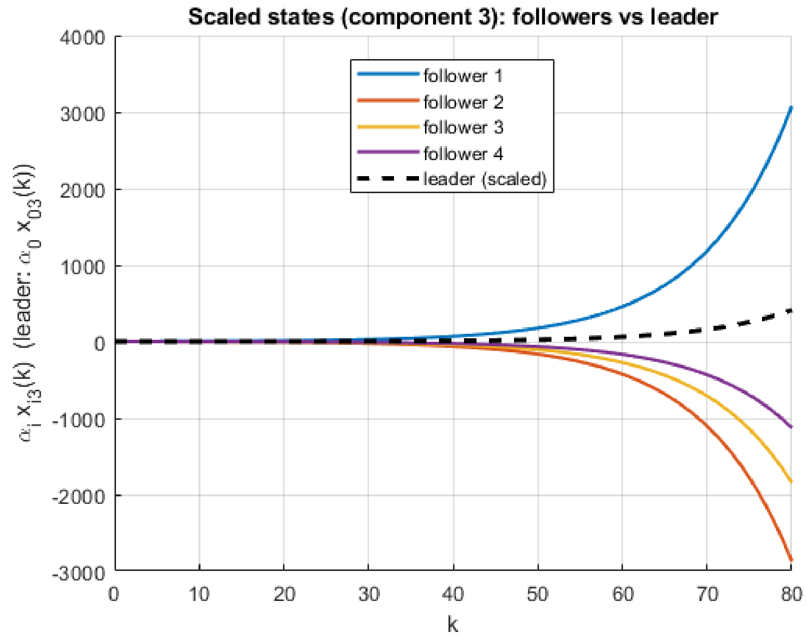

FIGURE 4. Scaled states for component $j = 3$ (unselected): the trajectories $\alpha_i x_{i3}(k)$ do not synchronize with $\alpha_0 x_{03}(k)$ since $r_3 = 0$ and $A_3 > 1$, illustrating the unconstrained component.

Figure 1 reports the norms $E_j(k)$ of the component-wise scaled errors. The selected components ($j = 1, 2$) exhibit exponential decay to zero, consistent with Theorem 3.1 and the sufficient conditions in Corollary 3.1 and Theorem 3.2. In contrast, the third component is unselected because $r_3 = 0$, and therefore it is not regulated by the protocol. Since $A_3 = 1.1 > 1$, this component behaves as an open-loop unstable mode in the scaled error dynamics, so $E_3(k)$ can grow, illustrating the component-selective objective.

Figures 2–4 plot the scaled states $\alpha_i x_{ij}(k)$ of all followers and the scaled leader state $\alpha_0 x_{0j}(k)$ for each component. For $j = 1$ and $j = 2$, all follower trajectories converge to the leader scaled trajectory, i.e., $\alpha_i x_{ij}(k) \to \alpha_0 x_{0j}(k)$, confirming scaled tracking on the selected components. For the unselected component $j = 3$, the follower scaled trajectories do not synchronize with the leader, which matches the model design ($r_3 = 0$) and the absence of convergence guarantees for that component.

Finally, Fig. 5 provides an aggregated visualization of the scaled states of all followers restricted to the selected components $j = 1, 2$. The collapse of all trajectories to a common limit demonstrates the intended scaled partial component consensus behavior.

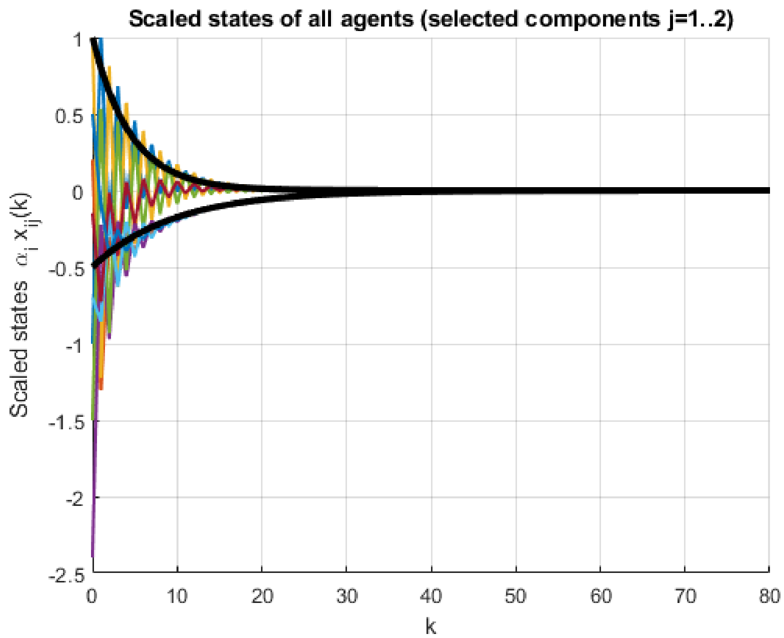

Figure 5. Aggregated plot for selected components ($j = 1, 2$): all follower scaled states collapse to a common trajectory matching the scaled leader, highlighting scaled partial component consensus on the selected set.

5. Conclusion

This paper studied the scaled partial component consensus problem for discrete-time leader–following multi-agent systems. A unified framework was developed to integrate component-selective coordination with scaled consensus objectives. By introducing scaled tracking errors and an inner coupling matrix, the coordination task was formulated so that only a prescribed subset of state components is required to achieve scaled agreement with the leader. Through a permutation-based transformation of the global error dynamics, the closed-loop system was decomposed into independent component-wise subsystems. This decomposition enabled a tractable stability analysis and revealed how the convergence properties of the overall system are determined by the spectral characteristics of the component matrices. Based on this representation, sufficient conditions guaranteeing scaled partial component consensus were established using Lyapunov arguments, together with a spectral stability characterization that provides a practical gain design rule. Numerical simulations verified the theoretical results and illustrated the component-selective nature of the proposed protocol. The scaled states of the followers converge to the scaled leader trajectory for the selected components, while the unselected component remains unconstrained. Future work will focus on extending the proposed framework to more general settings, including switching communication topologies, time-delay systems, and nonlinear multi-agent dynamics, as well as developing distributed and event-triggered coordination strategies.

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