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Classes of close-to-convex Functions Defined using Beta Negative Binomial Distribution

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Abstract. In this paper, we make use of convolution of the power series whose coefficients are the beta negative binomial distribution probabilities and a power series of an analytic function in the unit disk $\mathbb{D}$, to introduce a novel class of Ma-Minda type close-to-convex functions associated with the q-analogue of sine function. In addition, we find bounds for the growth and distortion of functions belonging to our class and some of its various subclasses. Moreover, we obtain the classical Fekete-Szegö inequality of functions belonging to our class and some of its various subclasses.

1. Introduction

The probability distribution of a random variable constitutes a core concept within the domains of statistics and probability theory. Their probability mass functions have been crucial in probability theory and many other mathematical sciences. There has been tremendous studies make use of distribution series in mathematical sciences. Using the probability distributions, many researchers have investigated some important features in the field of geometric function theory such as coefficient estimates and the Fekete and Szegö functional problem, see for example [10], [33] and the related references included therein.

In addition, a random variable X is said to have a beta negative binomial distribution with the parameters $\theta, \alpha, \beta > 0$ if its probability mass function can be represented in the following manner

$$f(\theta;\alpha;\beta) := Prob(X = k) = \binom{\theta + k - 1}{k} \frac{B(\alpha + \theta, \beta + k)}{B(\alpha, \beta)},$$

where $k = 0, 1, 2, 3, \cdots$ and $B(\alpha, \beta)$ is the beta function with the positive parameters α and β.

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In this paper, we make use of convolution of two power series one whose coefficients are the beta negative binomial distribution probabilities and the other one is an analytic function in the unit disk $\mathbb{D}$, to introduce a new class of Ma-Minda-type close-to-convex functions associated with the q-analogue of sine function.

Let $\mathcal{A}$ be the family of all analytic functions f that are defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = 1 - f'(0)$. Any function $f \in \mathcal{A}$ has the following Taylor-Maclaurin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where} \ \ z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. As known univalent functions are injective (one-to-one) functions. Hence, they are invertible and the inverse functions may not be defined on the entire unit disk $\mathbb{D}$. In fact, according to Koebe one-quarter Theorem [15], the image of $\mathbb{D}$ under any function $f \in \mathcal{S}$ contains the disk $D(0, 1/4)$ of center 0 and radius $1/4$. Accordingly, every function $f \in \mathcal{S}$ has an inverse $f^{-1} = g$ which is defined as

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

$$f(g(w)) = w, \quad |w| < r(f); \ \ r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w + c_2 w^2 + c_3 w^3 + c_4 w^4 + \cdots, \tag{1.2}$$

where $c_2 = -a_2$, $c_3 = 2a_2^2 - a_3$, and $c_4 = -(5a_2^3 - 5a_2 a_3 + a_4)$.

A function $f \in \mathcal{A}$ is said to be bi-univalent if both f and f^{-1} are univalent in $\mathbb{D}$. Therefore, let Σ denote the class of all bi-univalent functions in $\mathcal{A}$ which are given by equation (1.1). For more information about univalent and bi-univalent functions we refer the readers to the articles [23], [25], [29], [30] the monograph [15], [18] and the references therein.

The research in the geometric function theory has been very active in recent years, the typical problem in this field is studying a functional made up of combinations of the initial coefficients of the functions $f \in \mathcal{A}$. For a function in the class $\mathcal{S}$, it is well-known that $|a_n|$ is bounded by n. Moreover, the coefficient bounds give information about the geometric properties of those functions. For instance, the bound for the second coefficients of the class $\mathcal{S}$ gives the growth and distortion bounds for the class.

Coefficient related investigations of functions belong to the class Σ began around the 1970. It is worth mentioning that, in the year 1967, Lewin [23] studied the class of bi-univalent functions and derived the bound for $|a_2|$. Later on, in the year 1969, Netanyahu [29] showed that the maximum value of $|a_2|$ is $\frac{4}{3}$ for functions belong to the class Σ. In addition, in the year 1979, Brannan and Clunie [10] proved that $|a_2| \leq \sqrt{2}$ for functions in the class Σ. Since then, many researchers investigated the coefficient bounds for various subclasses of the bi-univalent function class Σ.

However, not much is known about the bounds of the general coefficients $|a_2|$ for $n \geq 4$. In fact, the coefficient estimate problem for the general coefficient $|a_n|$ is still an open problem.

In the year 1933, Fekete and Szegö [17] found the maximum value of $|a_3 - \lambda a_2^2|$, as a function of the real parameter $0 \leq \lambda \leq 1$ for a univalent function f. Since then, maximizing the modulus of the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for $f \in \mathcal{A}$ with any complex λ is called the Fekete-Szegö problem. There are many researchers investigated the Fekete-Szegö functional and the other coefficient estimates problems, for example see the articles [2], [6], [11], [13], [17], [20], [21], [25], [34] and the references therein.

2. Definitions and Lemmas

In this section we present some information that are curial for the main results of this paper. The beta distribution is a continuous probability distribution defined on the interval $[0, 1]$ in terms of two positive parameters α and β. The probability mass function of the beta distribution for $0 \leq p \leq 1$ and the parameters $\alpha, \beta > 0$ is defined as follows

$$f(p; \alpha; \beta) := \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)},$$

where the beta function

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

is a normalization constant to ensure that the total probability is one.

On the other hand, the negative binomial distribution is a discrete probability distribution that models the number of failures needed to get θ successes in a sequence of independent Bernoulli trials, where each trial has two potential outcomes called success and failure. In each trial the probability of success is p and of failure is $1 - p$. We observe this sequence until a predefined number θ of successes occurs. Therefore, the random number of observed failures X follows the negative binomial distribution. Moreover, if k is the number of failures, then the probability mass function of the negative binomial distribution is defined as follows

$$f(k; \theta; p) := Prob(X = k) = \binom{\theta + k - 1}{k}(1-p)^k P^\theta.$$

Now, for a discrete random variable X, the beta negative binomial distribution of X is the number of failures in a sequence of independent and identically distributed Bernoulli trials before a specified number of successes θ occurs. The probability p of success on each trial remains constant within any given experiment but varies a cross different experiments following a beta distribution. Moreover, if $f(p; \alpha; \beta)$ and $f(k; \theta; p)$ are the probability mass functions of beta and negative binomial distributions respectively, we get the mass function $f(\theta; \alpha; \beta)$ of the beta-negative binomial distribution by marginalization

$$\begin{aligned} f(\theta;\alpha;\beta) &= \int_0^1 f(k;\theta;p)f(p;\alpha;\beta)dp \\ &= \binom{\theta+k-1}{k}\frac{B(\alpha+\theta,\beta+k)}{B(\alpha,\beta)}. \end{aligned} \tag{2.1}$$

Note that, the beta geometric distribution is just a special case of the beta negative binomial distribution when either $\theta = 1$ or $\beta = 1$. A shifted form of the beta negative binomial distribution is called the beta Pascal distribution. For more information about the beta distribution and negative binomial distribution and their applications, we encourage the interested readers to consult the monographs [19], [39], the articles [7], [8], [9], [35], [40] and the related references included therein.

Recently, for $z \in \mathbb{D}$, $\alpha > 0$ and $\beta > 0$, Wanas and Al-Ziadi [37] introduced the following power series whose coefficients are the probabilities of the beta-negative binomial distribution

$$\mathcal{B}^{\theta}_{\alpha,\beta} = z + \sum_{n=2}^{\infty}\binom{\theta+n-2}{n-1}\frac{B(\alpha+\theta,\beta+n-1)}{B(\alpha,\beta)}z^n. \tag{2.2}$$

It is clear that, using the ratio test, the radius of convergence of the last series is infinity.

The convolution of specific analytic functions has been of very important in geometric function theory. It can be used to express a wide variety of differential and integral operators. The convolution, also referred to as Hadamard product, of two analytic functions $f(z)$ as described in equation (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is expressed as follows:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Moreover, the convolution operation provides a deeper mathematical exploration and enhances our understanding of the geometric and symmetric properties of functions within the space $\mathcal{H}$. Its significance in operator theory and geometric function theory is well-established and thoroughly discussed in the available literature. For those seeking further insights into convolution within geometric function theory, we recommend consulting the monographs [15] and [18] as well as the articles [4], [12] and the associated references therein.

Now, using the convolution, we introduce the following linear operator

$$\mathcal{N}^{\theta}_{\alpha,\beta} : \mathcal{A} \to \mathcal{A}$$

which defined as:

$$\begin{aligned} \mathcal{N}^{\theta}_{\alpha,\beta} f(z) &= \mathcal{N}^{\theta}_{\alpha,\beta}(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \mathcal{N}_n a_n z^n, \end{aligned} \tag{2.3}$$

where

$$\mathcal{N}_n = \binom{\theta+n-2}{n-1}\frac{B(\alpha+\theta,\beta+n-1)}{B(\alpha,\beta)}.$$

Let f and g be analytic functions in the unit disk $\mathbb{D}$. We say the function f is subordinate to the function g , denoted by $f(z) \prec g(z)$ for all $z \in \mathbb{D}$, if there exists a Schwartz function h, with $h(0) = 0$ and $|h(z)| < |z|$ for all $z \in \mathbb{D}$, such that $f(z) = g(h(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent then $f(z) \prec g(z)$ equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D}$. For more information about the Subordination Principle we refer the readers to the monographs [16], [27] and [28].

Now, a region $G \subset \mathbb{C}$ is called convex if the line segment joining any two points in G lies completely inside the region G. The region G is called starlike with respect to $z_0 \in G$ if the line segment joining any point $z \in G$ to the point z_0 lies entirely inside G.

An analytic function $f \in \mathcal{A}$ is called starlike if $f(\mathbb{D})$ is starlike region with respect to the origin. While, f is called convex function if if $f(\mathbb{D})$ is convex region. Analytically, the function $f \in \mathcal{A}$ is starlike if an only if $f \in \mathcal{S}^*$ where

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\},$$

whereas the function f is called convex if and only if $f \in \mathcal{K}$ where

$$\mathcal{K} := \left\{ f \in \mathcal{K} : 1 + Re\left(\frac{zf''(z)}{f'(z)}\right) > 0 \right\}.$$

The concept of subordination is used to unify these classes. For this purpose, let ϕ be an analytic function with positive real part in the unit disk $\mathbb{D}$, $\phi(0) = 1$, $\phi'(0)$ is positive, and ϕ maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions, denoted by $\mathcal{S}^*(\phi)$, is defined as follows

$$\mathcal{S}^*(\phi) := \left\{ f : f \in \mathcal{A} \text{ and } \frac{zf'(z)}{f(z)} \prec \phi(z) \right\},$$

whereas the class of Ma-Minda convex functions, denoted by $\mathcal{K}(\phi)$, is defined as follows

$$\mathcal{K}(\phi) := \left\{ f : f \in \mathcal{A} \text{ and } 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\}.$$

Moreover, a function $f \in \mathcal{A}$ is called bi-starlike Ma-minda type (or bi-convex Ma-Minda type) if both f and f^{-1} are Ma-Minda starlike (or Ma-Minda convex). For more information about the coefficients estimates and Fekete-Szegö inequalities for functions belong to classes of Ma-Minda starlike and convex functions, we refer the interested readers to see, for example, the articles [1], [3], [5], [6], [22], [24], [26], [31], [32], [36], [38] and the related references included therein.

The goal of this article is to implore the concept of convolution of a power series whose coefficients are the beta negative binomial distribution probabilities and a power series of an analytic function in the unit disk $\mathbb{D}$, to introduce a new class of close-to-convex functions of Ma-Minda type that is subordinate to the q-analogue of sine function, which we denote as

$\mathscr{M}_q(\theta, \alpha, \beta, \lambda; Sin)$, which we define as follows.

Definition 2.1. *A function $f \in \Sigma$ is said to be in the class $\mathscr{M}_q(\theta, \alpha, \beta, \lambda; Sin)$ if it satisfies the following subordinations:*

$$1 + \frac{2}{1+2\lambda}\left(\frac{z\left(\mathcal{N}^{\theta}_{\alpha,\beta} f(z)\right)''}{\left(\mathcal{N}^{\theta}_{\alpha,\beta} f(z)\right)'}\right) \prec 1 + \sin(qz), \tag{2.4}$$

and

$$1 + \frac{2}{1+2\lambda}\left(\frac{w\left(\mathcal{N}^{\theta}_{\alpha,\beta} g(w)\right)''}{\left(\mathcal{N}^{\theta}_{\alpha,\beta} g(w)\right)'}\right) \prec 1 + \sin(qw), \tag{2.5}$$

where $\lambda \in [1/2, 1]$, q, θ, α, β are positive real numbers and the function $g(w) = f^{-1}(w)$ is given by the equation (1.2).

In our analysis, the parameters λ and θ plays a crucial role in categorizing our class $\mathscr{M}_q(\theta, \alpha, \beta, \lambda; Sin)$. The choice of λ can significantly influence the properties and behaviors of this class, leading us to identify distinct subclasses based on its values. For example, taking the value $\lambda = 1/2$ gives us the subclass of bi-univalent convex functions defined using beta negative binomial distribution associated with q-analogue sine function, which we denote as $\mathcal{M}_q(\theta, \alpha, \beta; Sin)$. Moreover, taking the value $\theta = 1$ we get the subclass of close-to-convex functions defined using beta geometric distribution related to q-analogue sine function, which we denote as $\mathcal{G}_q(\alpha, \beta, \lambda; Sin)$.

Example 2.1. *Let f be a bi-univalent function. Then f is said to be in the class $\mathcal{M}_q(\theta, \alpha, \beta; Sin)$ if it satisfies the following subordinations:*

$$1 + \frac{z\left(\mathcal{N}^{\theta}_{\alpha,\beta} f(z)\right)''}{\left(\mathcal{N}^{\theta}_{\alpha,\beta} f(z)\right)'} \prec 1 + \sin(qz), \tag{2.6}$$

and

$$1 + \frac{w\left(\mathcal{N}^{\theta}_{\alpha,\beta} g(w)\right)''}{\left(\mathcal{N}^{\theta}_{\alpha,\beta} g(w)\right)'} \prec 1 + \sin(qw), \tag{2.7}$$

where the parameters q, θ, α, β are positive real numbers and the function $g(w) = f^{-1}(w)$ is given by the equation (1.2).

Example 2.2. *Let f be a bi-univalent function. Then f is said to be in the class $\mathcal{G}_q(\alpha, \beta, \lambda; Sin)$ if it satisfies the following subordinations:*

$$1 + \frac{2}{1+2\lambda}\left(\frac{z\left(\mathcal{G}_{\alpha,\beta} f(z)\right)''}{\left(\mathcal{G}_{\alpha,\beta} f(z)\right)'}\right) \prec 1 + \sin(qz), \tag{2.8}$$

and

$$1+\frac{2}{1+2\lambda}\left(\frac{w\left(\mathcal{G}_{\alpha,\beta}g(w)\right)''}{\left(\mathcal{G}_{\alpha,\beta}g(w)\right)'}\right)\prec 1+\sin(qw), \tag{2.9}$$

where $\lambda\in[1/2,1]$, q,α,β *are positive real numbers and the function* $g(w)=f^{-1}(w)$ *is given by the equation (1.2).*

The following lemma, extensively elaborated upon in existing literature, represents well-established principles that hold significant importance for the research we are currently presenting.

Lemma 2.1. *[21] If L belongs to the Caratheodory class* $\mathcal{P}$, *then for* $z\in\mathbb{D}$ *the function* Ω *can be written as*

$$L(z)=1+p_1z+p_2z^2+p_3z^3+\cdots$$

In addition, $|p_n|\leq 2$ *for* $n\geq 1$. *Moreover, for any* $\zeta\in\mathbb{C}$, *we have*

$$|p_2-\zeta p_1^2|\leq 2\max\{1,|1-2\zeta|\}.$$

In particular, if ζ *is a real number, then*

$$|p_2-\zeta p_1^2|\leq\begin{cases}-4\zeta+2, & \text{if } \zeta\leq 0,\\ 2, & \text{if } 0\leq\zeta\leq 1,\\ 4\zeta-2, & \text{if } \zeta\geq 1.\end{cases}$$

The primary goal of this study is to establish the bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to the class $\mathscr{M}_q(\theta,\alpha,\beta,\lambda;Sin)$, which we introduce using the convolution and the beta negative binomial distribution. In addition, we examine the corresponding Fekete-Szegö problem for functions belong to the presenting class and some of its various subclasses. We also provide relevant connections of our main results with those considered in earlier research papers.

3. Growth and Distortion Bounds

In this section, we provide estimates for the initial Taylor-Maclaurin coefficients for the functions belong to the class $\mathscr{M}_q(\theta,\alpha,\beta,\lambda;Sin)$ which are given by equation (1.1) as well as some of its various subclasses. Moreover, we present the Fekete-Szegö inequalities for the functions belong to our class and some of its various special cases.

Theorem 3.1. *Let the function f be a bi-univalent function given by equation (1.1). If f belongs to the class* $\mathscr{M}_q(\theta,\alpha,\beta,\lambda;Sin)$, *then*

$$|a_2|\leq\frac{q(1+2\lambda)}{\sqrt{|6q\theta(\theta+1)(1+2\lambda)A_2-8\theta^2(2q\lambda+q-2)A_1^2|}}, \tag{3.1}$$

and

$$|a_3|\leq\frac{q(1+2\lambda)}{6\theta(\theta+1)A_2}+\frac{q^2(1+2\lambda)^2}{16\theta^2A_1^2}, \tag{3.2}$$

where

$$A_1 = \frac{\Gamma(\alpha+\theta)\Gamma(\beta+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\theta+\beta+1)\Gamma(\alpha)\Gamma(\beta)},$$

$$A_2 = \frac{\Gamma(\alpha+\theta)\Gamma(\beta+2)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\theta+\beta+2)\Gamma(\alpha)\Gamma(\beta)}.$$

Proof. Let f be a bi-univalent function in the class $\mathscr{M}_q(\theta,\alpha,\beta,\lambda;Sin)$. Then, using Definition 2.1, we can find two Schwarz functions $u(z)$ and $v(w)$ on the unit disk $\mathbb{D}$ such that

$$1+\frac{2}{1+2\lambda}\left(\frac{z\left(\mathcal{N}^{\theta}_{\alpha,\beta}f(z)\right)''}{\left(\mathcal{N}^{\theta}_{\alpha,\beta}f(z)\right)'}\right) = 1+\sin(qu(z)), \tag{3.3}$$

and

$$1+\frac{2}{1+2\lambda}\left(\frac{w\left(\mathcal{N}^{\theta}_{\alpha,\beta}g(w)\right)''}{\left(\mathcal{N}^{\theta}_{\alpha,\beta}g(w)\right)'}\right) = 1+\sin(qv(w)). \tag{3.4}$$

Now, we define the following analytic functions

$$\delta(z) = \frac{1+u(z)}{1-u(z)} = 1+\delta_1 z+\delta_2 z^2+\delta_3 z^3+\cdots \tag{3.5}$$

and

$$\gamma(w) = \frac{1+v(w)}{1-v(w)} = 1+\gamma_1 w+\gamma_2 w^2+\gamma_3 w^3+\cdots \tag{3.6}$$

It is clear that, these functions $\delta(z)$ and $\gamma(w)$ are analytic in the open unit disk $\mathbb{D}$ and belong to the Caratheodory class. In addition, $\delta(0)=1=\gamma(0)$, they have positive real parts, $|\delta_j|\le 2$ and $|\gamma_j|\le 2$ for all $j\in\mathbb{N}$.

Therefore, we can rewrite equation (3.5) and equation (3.6) in the following manner

$$\begin{aligned} u(z) &= \frac{\delta(z)-1}{\delta(z)+1} \\ &= \frac{1}{2}\left(\delta_1 z+\left(\delta_2-\frac{\delta_1^2}{2}\right)z^2+\left(\frac{\delta_1^3}{4}-\delta_1\delta_2+\delta_3\right)z^3+\cdots\right) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} v(w) &= \frac{\gamma(w)-1}{\gamma(w)+1} \\ &= \frac{1}{2}\left(\gamma_1 w+\left(\gamma_2-\frac{\gamma_1^2}{2}\right)w^2+\left(\frac{\gamma_1^3}{4}-\gamma_1\gamma_2+\gamma_3\right)w^3+\cdots\right). \end{aligned} \tag{3.8}$$

Now, by consulting equation (3.7), the right-hand side of equation (3.3) can be written as follows

$$\begin{aligned} &1+\sin\left(q\frac{\delta(z)-1}{\delta(z)+1}\right) \\ &=1+\frac{q\delta_1}{2}z+q\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\cdots \end{aligned} \tag{3.9}$$

Similarly, by equation (3.8), the right-hand side of equation (3.4) can be written as follows

$$\begin{aligned} &1+\sin\left(q\frac{\gamma(z)-1}{\gamma(z)+1}\right) \\ &=1+\frac{q\gamma_1}{2}w+q\left(\frac{\gamma_2}{2}-\frac{\gamma_1^2}{4}\right)w^2+\cdots \end{aligned} \tag{3.10}$$

On one hand, considering equation (3.9), then upon comparing coefficients of both-sides of equation (3.3), we get the following two equations

$$\frac{4\theta A_1}{1+2\lambda}a_2=\frac{q}{2}\delta_1, \tag{3.11}$$

and

$$\frac{6\theta(\theta+1)A_2}{1+2\lambda}a_3-\frac{8\theta^2A_1^2}{1+2\lambda}a_2^2=q\left(\frac{\delta_2}{2}-\frac{\delta_1^2}{4}\right), \tag{3.12}$$

where

$$A_1=\frac{\Gamma(\alpha+\theta)\Gamma(\beta+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\theta+\beta+1)\Gamma(\alpha)\Gamma(\beta)},$$

$$A_2=\frac{\Gamma(\alpha+\theta)\Gamma(\beta+2)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\theta+\beta+2)\Gamma(\alpha)\Gamma(\beta)}.$$

On the other hand, considering equation (3.10), then upon comparing coefficients of both-sides of equation (3.4), we get the following two equations

$$\frac{-4\theta A_1}{1+2\lambda}a_2=\frac{q}{2}\gamma_1, \tag{3.13}$$

and

$$\frac{6\theta(\theta+1)A_2}{1+2\lambda}(2a_2^2-a_3)-\frac{8\theta^2A_1^2}{1+2\lambda}a_2^2=q\left(\frac{\gamma_2}{2}-\frac{\gamma_1^2}{4}\right), \tag{3.14}$$

Now, using equation (3.11) and equation (3.13), we get the following two equations

$$\delta_1=-\gamma_1, \tag{3.15}$$

and

$$\frac{128\theta^2A_1^2}{(1+2\lambda)^2}a_2^2=q^2(\delta_1^2+\gamma_1^2). \tag{3.16}$$

Moreover, adding equation (3.12) and equation (3.14) gives the following equation

$$\left[48\theta(\theta+1)A_2-64\theta^2A_1^2\right]a_2^2=q(1+2\lambda)[2(\delta_2+\gamma_2)-(\delta_1^2+\gamma_1^2)].$$

Thus, using equation (3.16), the last equation can written as follows

$$\left[48\theta(\theta+1)A_2 - 64\theta^2 A_1^2\right] a_2^2 = 2q(1+2\lambda)(\delta_2+\gamma_2) - \frac{128\theta^2 A_1^2}{q(1+2\lambda)} a_2^2.$$

Therefore, solving for a_2^2 in the last equation, we get the following equation

$$a_2^2 = \frac{q^2(1+2\lambda)^2(\delta_2+\gamma_2)}{24q\theta(\theta+1)(1+2\lambda)A_2 - 32\theta^2[q(1+2\lambda)-2]A_1^2}. \tag{3.17}$$

Therefore, using the facts $|\delta_2| \leq 2$ and $|\gamma_2| \leq 2$, last equation gives the desired estimation of $|a_2|$ as presented in inequality (3.1).

Secondly, we are looking for the estimation of $|a_3|$. Subtracting equation (3.14) from equation (3.12) then using equation (3.15), we get the following equation

$$a_3 = \frac{q(1+2\lambda)(\delta_2-\gamma_2)}{24\theta(\theta+1)A_2} + a_2^2. \tag{3.18}$$

Therefore, using equation (3.16), the last equation can be written as follows

$$a_3 = \frac{q(1+2\lambda)(\delta_2-\gamma_2)}{24\theta(\theta+1)A_2} + \frac{q^2(1+2\lambda)^2(\delta_1^2+\gamma_1^2)}{128\theta^2 A_1^2}. \tag{3.19}$$

Hence, applying the initial conditions $|\delta_i| \leq 2$ and $|\gamma_i| \leq 2$, for $i = 1, 2$, on the last equation (3.19) we get the desired estimation of $|a_3|$ that presented in inequality (3.2). This completes the proof of Theorem 3.1. □

The following corollary gives the initial coefficient bounds of functions belonging to the subclass $\mathcal{M}_q(\theta, \alpha, \beta; Sin)$ which consists of close-to-convex functions that subordinate to the $q-$analogue sine function. For more information about the coefficients estimates for functions belong to classes closed-to-convex functions, we refer the interested readers to see, for example, the articles [3], [5], [6], [31], [36], [38] and the related references included therein.

Corollary 3.1. *Let the function f be a bi-univalent function given by equation (1.1). If f belongs to the class $\mathcal{M}_q(\theta, \alpha, \beta; Sin)$, then*

$$|a_2| \leq \frac{q}{\sqrt{|3q\theta(\theta+1)A_2 - 4\theta^2(q-1)A_1^2|}}, \tag{3.20}$$

and

$$|a_3| \leq \frac{q}{3\theta(\theta+1)A_2} + \frac{q^2}{4\theta^2 A_1^2}, \tag{3.21}$$

The following corollary give the initial coefficient bounds of functions belonging to the subclass $\mathcal{G}_q(\alpha, \beta, \lambda; Sin)$ which consists of Ma-Minda-type of close-to-convex functions that are subordinate to the $q-$analogue sine function.

Corollary 3.2. *Let the function f be a bi-univalent function given by equation (1.1). If f belongs to the class $\mathcal{G}_q(\alpha,\beta,\lambda;Sin)$, then*

$$|a_2| \leq \frac{q(1+2\lambda)}{\sqrt{|12q(1+2\lambda)A_2^* - 8(2q\lambda+q-2)(A_1^*)^2|}}, \tag{3.22}$$

and

$$|a_3| \leq \frac{q(1+2\lambda)}{12A_2^*} + \frac{q^2(1+2\lambda)^2}{16(A_1^*)^2}, \tag{3.23}$$

where

$$A_1^* = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)\Gamma(\alpha)\Gamma(\beta)},$$

$$A_2^* = \frac{\Gamma(\alpha+1)\Gamma(\beta+2)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+3)\Gamma(\alpha)\Gamma(\beta)}.$$

4. Fekete-Szegö Functional Inequalities

In this section, we consider the classical Fekete-Szegö problem for functions belong to our class $\mathscr{M}_q(\theta,\alpha,\beta,\lambda;Sin)$. The following theorem gives the Fekete-Szegö inequality for functions belonging to our aforementioned class.

Theorem 4.1. *Let f be a bi-univalent function given by equation (1.1). If f belongs to the class $\mathscr{M}_q(\theta,\alpha,\beta,\lambda;Sin)$, then for some $\lambda\in\mathbb{R}$ the following inequality holds*

$$|a_3-\zeta a_2^2| \leq \frac{q(1+2\lambda)}{6\theta(\theta+1)A_2}\min\left\{\max\{1,\mathcal{F}_1\},\max\{1,\mathcal{F}_2\}\right\}, \tag{4.1}$$

where

$$\mathcal{F}_1 = \frac{q(1+2\lambda)[4\theta A_1^2+3\zeta(\theta+1)A_2]}{8\theta A_1^2},$$

$$\mathcal{F}_2 = \frac{32q\theta^3A_1^2[(6-3\zeta)(\theta+1)A_2-4\theta A_1^2]}{(1+2\lambda)^3}.$$

Proof. Firstly, consulting equation (3.12), we get the following equation

$$a_3 = \frac{q(1+2\lambda)}{6\theta(\theta+1)A_2}\left(\frac{\delta_2}{2}-\frac{\delta_1^2}{4}\right) - \frac{4\theta^2A_1^2}{3\theta(\theta+1)A_2}a_2^2. \tag{4.2}$$

Hence, for any complex number ζ, the last equation gives us the following equation

$$a_3-\zeta a_2^2 = \frac{q(1+2\lambda)}{12\theta(\theta+1)A_2}\left(\delta_2-\frac{\delta_1^2}{2}\right) - \left(\frac{4\theta^2A_1^2+3\zeta\theta(\theta+1)A_2}{3\theta(\theta+1)A_2}\right)a_2^2. \tag{4.3}$$

Therefore, using equation (3.11), the last equation can be written as follows

$$a_3-\zeta a_2^2 = \frac{q(1+2\lambda)}{12\theta(\theta+1)A_2}\left\{\delta_2-K_1\delta_1^2\right\}, \tag{4.4}$$

where

$$K_1 = \frac{8\theta A_1^2 + q(1+2\lambda)[4\theta A_1^2 + 3\zeta(\theta+1)A_2]}{16\theta A_1^2}.$$

Now, applying Lemma 2.1 on the last equation, we obtain the following inequality

$$\begin{aligned} |a_3 - \zeta a_2^2| &\leq \frac{q(1+2\lambda)}{6\theta(\theta+1)A_2} \max\{1, |2K_1 - 1|\} \\ &= \frac{q(1+2\lambda)}{6\theta(\theta+1)A_2} \max\{1, |\mathcal{F}_1|\}, \end{aligned} \tag{4.5}$$

where

$$\mathcal{F}_1 = \frac{q(1+2\lambda)[4\theta A_1^2 + 3\zeta(\theta+1)A_2]}{8\theta A_1^2}.$$

Secondly, consulting equation (3.14), we get the following equation

$$a_3 = \frac{q}{2B}\left(-\gamma_2 + \frac{\gamma_1^2}{2}\right) + \left(\frac{4B - (1+2\lambda)A^2}{2B}\right)a_2^2, \tag{4.6}$$

where

$$A = \frac{4\theta A_1}{1+2\lambda} \text{ and } B = \frac{6\theta(\theta+1)A_2}{1+2\lambda}.$$

Therefore, for the complex number ζ, the last equation gives us the following equation

$$a_3 - \zeta a_2^2 = \frac{q}{2B}\left(-\gamma_2 + \frac{\gamma_1^2}{2}\right) + \left(\frac{(4-2\zeta)B - (1+2\lambda)A^2}{2B}\right)a_2^2. \tag{4.7}$$

Thus, using equation (3.13), the last equation can be written as follows

$$a_3 - \zeta a_2^2 = \frac{q}{2B}\left\{-\gamma_2 + K_2\gamma_1^2\right\}, \tag{4.8}$$

where

$$K_2 = \frac{2 + qA^2[(4-2\zeta)B - (1+2\lambda)A^2]}{4}.$$

Now, applying Lemma 2.1 on the last equation, we get the following inequality

$$\begin{aligned} |a_3 - \zeta a_2^2| &\leq \frac{q}{B} \max\{1, |2K_2 - 1|\} \\ &= \frac{q}{B} \max\{1, |\mathcal{F}_2|\}, \end{aligned} \tag{4.9}$$

where

$$\mathcal{F}_2 = \frac{32q\theta^3 A_1^2[(6-3\zeta)(\theta+1)A_2 - 4\theta A_1^2]}{(1+2\lambda)^3}.$$

Finally, Using inequality (4.5) and inequality (4.9), we get the desired Fekete-Szegö inequality that is presented in inequality (4.10). This completes the proof. □

The following corollary presents the Fekete-Zsegö inequality of functions belonging to the subclass $\mathcal{M}_q(\theta, \alpha, \beta; Sin)$ which consists of close-to-convex functions defined using beta negative binomial distribution that subordinate to the $q-$analogue sine function. For more information about the Fekete-Szegö inequalities for functions belong to close-to-convex functions, we refer the interested readers to see, for example, the articles [1], [3], [5], [6], [31], [32], [36], [38] and the related references included therein.

Corollary 4.1. *Let f be a bi-univalent function given by equation (1.1). If f belongs to the class $\mathcal{M}_q(\theta, \alpha, \beta; Sin)$, then for some complex number ζ the following inequality holds*

$$|a_3 - \zeta a_2^2| \leq \frac{q}{3\theta(\theta+1)A_2} \min\left\{\max\{1, \mathcal{F}_1^*\}, \max\{1, \mathcal{F}_2^*\}\right\}, \tag{4.10}$$

where

$$\mathcal{F}_1^* = \frac{4q\theta A_1^2 + 3q\zeta(\theta+1)A_2}{4\theta A_1^2},$$
$$\mathcal{F}_2^* = 4q\theta^3 A_1^2[(6-3\zeta)(\theta+1)A_2 - 4\theta A_1^2].$$

The following corollary presents the Fekete-Zsegö inequality of functions belonging to the subclass $\mathcal{G}_q(\alpha, \beta, \lambda; Sin)$ which consists of Ma-Minda-type close-to-convex functions defined using the beta geometric distribution that subordinated to the $q-$analogue sine function.

Corollary 4.2. *Let f be a bi-univalent function given by equation (1.1). If f belongs to the class $\mathcal{G}_q(\alpha, \beta, \lambda; Sin)$, then for some complex number ζ the following inequality holds*

$$|a_3 - \zeta a_2^2| \leq \frac{q(1+2\lambda)}{12A_2} \min\left\{\max\{1, \mathcal{F}_3\}, \max\{1, \mathcal{F}_4\}\right\}, \tag{4.11}$$

where

$$\mathcal{F}_3 = \frac{q(1+2\lambda)[2A_1^2 + 3\zeta A_2]}{4A_1^2},$$
$$\mathcal{F}_4 = \frac{64qA_1^2[(6-3\zeta)A_2 - 2A_1^2]}{(1+2\lambda)^3}.$$

5. Conclusion

This research paper has studied a novel class of close-to-convex functions defined using the beta negative binomial distribution that is subordinate to $q-$analogue sine function. The author successfully obtained the growth and distortion bounds for functions belonging to our class and some of its various subclasses. Moreover, we derived the classical Fekete-Szegö inequality of functions belonging to our class and some of its various subclasses. Furthermore, we found estimates for the initial coefficients and the classical Fekete-Szegö functional problem associated with the logarithmic function for functions belonging to our class. The findings of this research are expected to inspire the researchers to connect our class with orthogonal polynomials. Also, the insights provided in this paper are anticipated to motivate researchers to broaden these concepts to encompass harmonic functions and symmetric q-calculus.

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