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The Stability and D-Stability Analysis of Toeplitz Linear Systems

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Abstract. The stability and D-stability analysis of Toeplitz linear systems is important and interesting problem in applied mathematics. Toeplitz matrices often appear in control theory, as well as across the computational mathematics. In this paper, we study and analyze Toeplitz system of linear equations of the form $Tx = b$, where T is a Toeplitz matrix, and provide some new results on its stability and D-stability. The D-stability is a stronger type of stability and it ensures that a given Toeplitz matrix remains stable under diagonal scaling. We develop and proposed a theoretical approach for the computation of spectrum and the relationship between D-stability and structured singular values for Toeplitz matrices. We make use of EigTool to visualize the pseudo-spectrum of higher dimensional Toeplitz matrices from Toeplitz system of linear equations. The behavior of the spectrum, pseudo-spectrum, singular values, and structured singular values of Hermitian Toeplitz matrices for an optimal mass transportation is demonstrated through various numerical experimentation.

1. Introduction

Toeplitz matrices are the class of matrices having the constant elements along their all diagonals. The Toeplitz matrices does appear in the various research problems across pure and applied mathematics as well as in engineering disciplines. The Toeplitz matrices are invariant under shift to their elements along the diagonal. Fast and super fast algorithms and Toeplitz linear systems can be solved and analyzed using a variety of numerical techniques.

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Toeplitz matrices appear in a number of real-world applications in engineering and applied mathematics, we recommend to see [30] and the references therein. The term Toeplitz solvers is used for algorithms which solves Toeplitz linear systems. In [30, 50], Toeplitz solvers having the complexity $O(n^2)$ and $O(n\, log^2 n)$ operations were developed. In [40], an iterative method and preconditioned conjugate gradient methods were developed to solve and analyze Toeplitz systems.

The Toeplitz-plus-Hankel matrix and their spectral properties were studied in [24], and such matrices are well-known as a displacement-structured matrix class [50]. There exists structured matrices A, B for a class of displacement structure such that the linear operator $\mathscr{L}(M) = AM - MB$ transforms matrices such that their rank must not depend upon the matrix order. The analysis on the spectral properties of Toeplitz matrices is an active research direction. In [51], the asymptotic distribution of the spectrum of Toeplitz matrices was studied and analyzed. In [33, 46–48], the analysis of the extreme values of the spectrum of Toeplitz matrices has been studied.

An extensive amount of literature have been developed on Toeplitz linear systems as well as the least-squares problems, see [31, 40, 50] and the references therein. In [13] authors had given an overview on fast algorithms for computing the singular values of Toeplitz matrices. The fast evaluations on the characteristics polynomials of Toeplitz matrices were developed and analyzed in [14,52,53]. The eigen-decomposition of Toeplitz matrices was derived via the diagonalization of simple matrix which is some how closely related with Toeplitz matrices and possesses the circulant or the Tau structure, see [15, 39, 49].

The new results on analysis of determinant for three different kinds of symmetric Toeplitz matrices were developed and presented by Wang and Sun, see [26]. A symmetric Toeplitz matrix having entries as the Horadam sequence was constructed and studied in [16]. Furthermore, they had proposed novel results on the analysis for determinants, inverse and matrix factorization of Toeplitz matrices. An extensive amount of research work has done to study and analyze structured matrices whose entries satisfies some particular choice of recurrence relations, see [4, 20, 21, 26, 27, 29, 34, 55].

A new algorithm was presented in [1] to solve and study Toeplitz system of linear equations. This was mainly done by exploiting special structure of Toeplitz matrices. The new proposed algorithm was based on the factorization of matrices and the Sherman-Morrison formula. In [7], a fast algorithm for solving Toeplitz linear systems was presented and it results in a circular transformation of the Toeplitz linear systems. A fast Fourier transformation to invert triangular Toeplitz matrices was given by D. A. Bini [5]. The methods based on Gaussian elimination, cyclic reduction, LU decomposition, and Toeplitz decomposition using Sherman-Morrison formula were proposed to solve Toeplitz system of linear equations, see [8, 17, 25, 28, 35, 41, 54].

A well-known mathematical tool in system theory is the approximation to structured singular values (μ-values) and its lower and upper bounds [2]. The μ-value addresses an important and central problem for the analysis and study of linear time-invariant systems. The μ-value and its lower and upper bounds can measure the stability analysis of linear systems which are subject to unstructured perturbations or uncertainties. The computation of the μ-value and its lower and upper bounds are possible with respect to all kind of perturbations. The uncertainties could be of pure real uncertainties, pure complex uncertainties, or a combination of real and complex uncertainties. Regretfully, the exact computation of structured singular values is a NP-hard problem [42].

Arrow and McManus [3], and then Enthoven and Arrow [18] were the first to introduce and study the notion of D-stability analysis, also known as the diagonal stability analysis. They introduced this mathematical tool while studying the dynamics of equilibrium problems. The n-dimensional real-valued matrix A is a D-stable matrix If and only if the matrices DA or AD are stable for any positive diagonal matrix D. In [6, 9, 10, 19] and the reference therein, the characterization of the mathematical problems involving the D-stability of different structured matrices have been studied extensively.

The mathematical formulations of the D-stability and μ-value are interconnected. New findings about the relationship between D-stability of real-valued square matrices and μ-values were developed and examined in [11]. According to these findings, a given n-dimensional real-valued matrix must have μ-values that are strictly less than 1 and more than or equal to 0, in order to be D-stable. Fo the analysis and new results on conditions to strong D-stability in terms of μ-values were given in [32]. The findings on the relationships among real-valued μ-values, strong D-stability, and D-stability were presented in [43]. Some new results on the relationships between H-stable, $D(\alpha)$-stable, semi-stable matrices, and μ-values were developed and reported in [36].

The computation of spectra and pseudo-spectra of structured matrices corresponds to different linear and non-linear mathematical problems helps to analyze their behavior. The spectrum and pseudo-spectrum of D-stable matrices for an economic model was the subject of some recent novel mathematical and computational studies, see [44]. In [37] the spectral characterization of D-stable matrices that surfaced in the transportation problems were analyzed. A thorough and in-depth mathematical analysis on stability, D-stability, and pseudo-spectrum for structured matrices that arise across economic models was given in [38]. The latest mathematical findings on the relationships between μ-values and Schur stability were examined and presented in [45].

2. The Preliminaries and Sufficient Conditions for D-Stability

This section provides a background to D-stable and strong D-stable matrices and the computation of μ-values by reviewing the well-known mathematical results and definitions. We also review some of the basic and existing results on the relationships between structured singular values and D-stable, strong D-stable matrices.

Definition 2.1. *The set of block-diagonal matrices, $\mathbb{B}_1$, is defined as*

$$\mathbb{B}_1 := \{diag\,(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_S I_{r_S}; \Delta_1, \Delta_2, \cdots, \Delta_F) : \delta_i \in \mathbb{K},\ \Delta_j \in \mathbb{K}^{m_j, m_j},\ i = 1 : S,\ j = 1 : F\},$$

where $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$.

Definition 2.2. *[2] The structured singular value* $M \in \mathbb{C}^{n,n}$*, is represented by* $\mu_{\mathbb{B}_1}(M)$ *and is defined by*

$$\mu_{\mathbb{B}_1}(M) := \begin{cases} 0, & if\ det(I - M\Delta) \neq 0,\ \forall \Delta \in \mathbb{B}_1 \\ (min\{\|\Delta\|_2 :\ det(I - M\Delta) = 0,\ \forall \Delta \in \mathbb{B}_1\})\ ,^{-1} & else \end{cases}$$

where ***min*** *is taken over all* $\Delta \in \mathbb{B}_1$.

Remark 2.1. *[2] The set* $\mathbb{B}_1$ *represents a multi-index of integers. This implies that the computation of* μ*-value depends on given matrix and set* $\mathbb{B}_1$*, the set of block-diagonal matrices.*

Remark 2.2. *[2] The full blocks in* $\mathbb{B}_1$ *can be interpreted as rank-1 matrices, or dyads.*

Remark 2.3. *[2] From the definition of* μ*-value, one can easily verify that for any* $\alpha \in \mathbb{C}$*, we have that* $\mu_{\mathbb{B}_1}(\alpha M) = \alpha \mu_{\mathbb{B}_1}(M)$.

The following Lemma 2.1 provides an alternate expression for the calculation of $\mu_{\mathbb{B}_1}(M)$.

Lemma 2.1. *[2] For given* $M \in \mathbb{C}^{n,n}$ *and for all* $\Delta \in \mathbb{B}_1$*, we have* $\mu_{\mathbb{B}_1}(M) = max\ \rho(\Delta M)$*, where* $\rho(\cdot)$ *represents a matrix's spectral radius, and* ***max*** *is taken over all* $\Delta \in \mathbb{B}_1$.

Definition 2.3. *[10] If the matrix product DM or MD for each positive diagonal matrix D has all of its eigenvalues in the left half of the complex plane, then the given n-dimensional matrix M is a D-stable matrix.*

Remark 2.4. *Similar matrices are represented by the matrix products DM and MD whose D-stability remains intact under perturbations affecting both rows and columns.*

The following four observations were given in [10] holds true for *D*-stable matrices.

Observation 2.1. *Sufficient conditions for D-stability of matrices remains preserve under positive diagonal multiplications.*

Observation 2.2. *Let* $M \in \mathbb{C}^{n,n}$ *so that none of the eigenvalue of M is precisely identical to* 0*, and DM is stable for a positive diagonal matrix D, and hence M-1 is invertible,* $\hat{D}^T M \hat{D}$*,* $\hat{D} M D$*,* M^T *are all D-stable matrices with* $\hat{D}$ *having a positive diagonal structure.*

Observation 2.3. *The* $k \times k$ *principal sub-matrix of M belongs to the euclidean closure of* $k \times k$ *D-stable matrices if* $M \in \mathbb{C}^{n,n}$ *such that DM is stable for a positive diagonal matrix D.*

Observation 2.4. *For all positive diagonal matrices D, the given M is a D-stable matrix if and only if* $det(M \pm iD) \neq 0$.

Definition 2.4. *[22] Let* $\gamma > 0$ *such that* $M + \hat{M}$ *is a D-stable matrix for each* $\hat{M} \in \mathbb{R}^{n,n}$ *with* $\sigma_{max}(\hat{M}) < \gamma$*, then* $M \in \mathbb{C}^{n,n}$ *is said to be strongly D-stable.*

Remark 2.5. *In [32] authors has constructed and analyzed the simpler requirements that hold true for strong D-stability and compares them with those stated in [9]. All of the* 13 *necessary conditions to D-stability [12] are extended to strong D-stability.*

2.1. **Conditions that are sufficient for D-stability:** We review C.R. Johnson's [10] 13 sufficient conditions for the D-stability for a given n-dimensional real-valued matrix M.
C_1 : All the eigenvalues $\lambda_i\,(DM + M^tD) > 0,\ \forall i$, D is a diagonal matrix which is positive.
C_2 : The M-matrix means that all of the principal minors are positive and all of the off-diagonal entries are non-positive.
C_3 : There exists a positive diagonal matrix D such that $MD = B = (b_{ij})$ which satisfies the condition that

$$Re(b_{ii}) > \sum_{j=1}^{n} |b_{ij}|; \quad i = 1 : n,\ j \neq i.$$

C_4 : Given $M \in \mathbb{R}^{n,n}$ is a triangular matrix and the real part of all the off-diagonal entries m_{ii} is strictly positive.
C_5 : Given $M \in \mathbb{R}^{n,n}$ is a sign stable matrix.
C_6 : Every principal minor of $M \in \mathbb{R}^{n,n}$ is positive, and M is a tri-diagonal matrix.
C_7 : Given $M \in \mathbb{R}^{n,n}$ is an oscillatory matrix, that is, M is totally non-negative matrix.
C_8 : For each $x \in \mathbb{R}^{n,1}$, $x \neq 0$, a positive diagonal matrix D exists such that the real part of x^tDMx is positive.
C_9 : For given $M \in \mathbb{R}^{n,n}$, and for every positive definite matrix P, the Hadamard product of P and M is a stable matrix.
C_{10} : The given $M \in \mathbb{R}^{n,n}$ is a strictly sign symmetric matrix and its every minor is positive.
C_{11} : Given $M \in \mathbb{R}^{n,n}$ such that $M \in \mathbb{R}^{2,2} \cap P_0^+$.
C_{12} : Given $M \in \mathbb{R}^{n,n}$ such that $M \in \mathbb{R}^{3,3} \cap P_0^+$, and $M = \begin{pmatrix} x & a & b \\ \alpha & y & c \\ \beta & \alpha & z \end{pmatrix}$.
C_{13} : Given $M \in \mathbb{R}^{n,n}$ such that $M \in \mathbb{R}^{n,n} \cap P_0^+$ satisfies GKK condition with $n \leq 4$.

3. New Results

New results on the stability and D-stability of linear systems with coefficient matrices being as Toeplitz matrices are presented. The mathematical constructions and proofs to our results are based on the ideas of the computation of spectrum and the relationships between structured singular values and D-stability theory.

3.1. **The stability analysis:** According to our first results Theorem 3.2 a given Toeplitz linear system is stable if the real part of each of eigenvalue of $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$ is strictly positive.

Theorem 3.1. *The linear system*

$$\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} b \\ \hat{b} \end{pmatrix}$$

is stable if

$$Re\left[\lambda_i \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right] > 0, \ \forall i.$$

Proof. Let $\alpha = \{1, 2, \ldots, n\}$, and let $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}[\alpha]$, be the principal sub-matrix.

Let $\begin{pmatrix} x \\ \hat{x} \end{pmatrix} \in \mathbb{R}^{n,1}$ so that $\begin{pmatrix} x \\ \hat{x} \end{pmatrix}[\alpha] \neq 0$. Further, we consider the computation of

$$\begin{pmatrix} x \\ \hat{x} \end{pmatrix}^t [\alpha] \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}[\alpha]\begin{pmatrix} x \\ \hat{x} \end{pmatrix}[\alpha] = \begin{pmatrix} x \\ \hat{x} \end{pmatrix}^t \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix}.$$

From the above expression, we have

$$\begin{pmatrix} x \\ \hat{x} \end{pmatrix}^t \lambda_i \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix} > 0, \quad \forall i.$$

This further reduces to

$$\lambda_i \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix}^t \begin{pmatrix} x \\ \hat{x} \end{pmatrix} > 0, \quad \forall i.$$

Also,

$$\lambda_i \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix} > 0, \quad \forall i.$$

Thus, finally, this yields

$$Re\left[\lambda_i \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right] > 0, \quad \forall i.$$

□

According to the following Theorem 3.2, the eigenvalue λ is a legitimate choice that falls inside the spectrum of the matrix-product as shown below $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$.

Theorem 3.2. *Let* $\lambda \in \Lambda\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right]$, *where the spectrum of a matrix is shown by* $\Lambda(\cdot)$, *then*

$$Re\left[\lambda_i \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right] > 0, \quad \forall i.$$

Proof. Assume that for $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$, the eigen pair $\left(\lambda, \begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right)$ exists, then

$$\begin{pmatrix} x \\ \hat{x} \end{pmatrix}^t \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} x \\ \hat{x} \end{pmatrix}^t \lambda_i \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

Thus

$$Re\left(\lambda_i(\cdot)\right) = \frac{\begin{pmatrix} x \\ \hat{x} \end{pmatrix}^T \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix}}{\begin{pmatrix} x \\ \hat{x} \end{pmatrix}^T \begin{pmatrix} x \\ \hat{x} \end{pmatrix}} > 0,$$

where $\lambda_i(\cdot) = \lambda_i\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right], \forall i.$ □

The following Theorem 3.3 show that linear system with $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$ as the coefficient matrix is stable for a matrix $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ whose rank is exactly equal to size of the coefficient matrix.

Theorem 3.3. *The linear system* $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} b \\ \hat{b} \end{pmatrix}$ *is stable if matrix* $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ *exits, such that* $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}^t \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ *is a positive definite matrix with rank* $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = n$*, where* n *denotes the size of*

$$\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}.$$

Proof. Let $\lambda_i\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix} \neq 0, \forall i$. Consider that

$$\operatorname{rank}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$
$$=$$
$$\operatorname{rank}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}^t \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

As we know that

$$\prod_{i=1}^{m} \lambda_i \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix} \neq 0, \forall i.$$

This further implies that

$$\operatorname{rank}\left[\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}^t \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\right] = n,$$

and this is exactly equal to rank$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$. □

The following Theorem 3.4 show that linear system having $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$ as coefficient matrix is stable if the real part of all the eigenvalues of this matrix are strictly positive and their absolute values are exactly equal to the spectral radius this matrix as well.

Theorem 3.4. *Consider the linear system with coefficient matrix* $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$ *such that*

$$Re\left[\lambda_i\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right] > 0, \forall i.$$

Let $\left(\lambda, \begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right)$ *be an eigen-pair, and* $\left|Re(\lambda_i(\cdot))\right| = \rho\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right]$. *Then,* $\begin{pmatrix} x \\ \hat{x} \end{pmatrix} > 0$, *and* $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\left|\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right| = \rho\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right]\left|\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right|$, *where* $\lambda(\cdot)$ *denotes the spectrum of* $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$, *and* $\rho(\cdot)$ *denotes the spectral radius of a matrix.*

Proof. Let $v = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\left|\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right|$, then

$$v = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\left|\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right| \geq \left|\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right| = Re(\lambda_i(\cdot))\left|\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right|.$$

Further,

$$v = \left|Re(\lambda_i(\cdot))\right|\left|\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right| = \rho\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right]\left|\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right|.$$

Consider that

$$\tilde{v} = v - \rho\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right]\left|\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right| \geq 0.$$

Let $\tilde{v} \geq 0$, this yields

$$\rho\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right]\left|\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right| = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\left|\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right| > 0.$$

This further implies that

$$\rho\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right] > 0, \text{ and } \left|\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\right| > 0.$$

On the other hand, let $\tilde{x} \neq 0$, then we have

$$0 < \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\tilde{v}$$

$$=$$

$$\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\tilde{v} - \rho\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\tilde{v}.$$

This further implies that

$$\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\tilde{v} > \rho\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}.$$

This is not true, and hence $\tilde{v} \neq 0$. □

3.2. **The *D*-stability analysis:** Next we present some new results on the D-stability of the Toeplitz linear systems. The notions of the relationship between structured singular values and D-stability are used in the construction of these results. The subsequent Theorem 3.5 demonstrates that the linear system with $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$ is D-stable if the structural singular value of the coefficient matrix's reciprocal is strictly less than one and bigger than or equal to zero.

Theorem 3.5. *Let* $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$ *be the coefficient matrix for a given system of linear equations. Then* $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$ *is D-stable iff this matrix product is stable, and*

$$0 \leq \mu_{\mathbb{B}_1}\left[\frac{1}{\left(\left(\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right)\right)^2}\right] < 1.$$

Proof. The matrix product $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$ is D-stable iff it is stable and

$$\prod_i \lambda_i\left[\left(\left(\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right)\right)^2 + P^2\right] \neq 0,$$

for a diagonal matrix P that is positive. Furthermore, we have that

$$\prod_i \lambda_i\left[\left(\left(\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right)\right)^2 + P^2\right] \neq 0$$

implies $\prod_i \lambda_i\left(\left(\left(\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right)\right)^2 - P\left(\left(\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right)\right)\right)^{-1}$
$P\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix} \neq 0$ Also,

$$\prod_i \lambda_i\left(\left(\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}\right)\right)^2 - P\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}^{-1} P\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$$

implies that

$$\prod_i \lambda_i \left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}^{-2} \tilde{P} \right] \neq 0, \text{ with } \tilde{P} = P,$$

a positive diagonal matrix.

Finally,

$$\prod_i \lambda_i \left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}^{-2} \tilde{P} \right] \neq 0 \quad \text{implies that}$$

$$0 \leq \mu_{\mathbb{B}_1} \left[\frac{1}{\left(\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} \right)^2} \right] < 1.$$

□

The D-stability of the Toeplitz linear system is guaranteed by the following Theorem 3.6 if the structured singular values of $\tilde{A}$ (as defined in Theorem 3.6) are strictly less than one and greater than or equal to zero, and if all of the real part of the matrix eigenvalues obtained by the coefficient matrix and a positive diagonal matrix are strictly positive.

Theorem 3.6. *Let* $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$ *be a system of linear equations' coefficient matrix. If and only if the real part of the eigenvalues is D-stable, that is,*

$$\lambda_i \left[\begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} + \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}^t \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}^t \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \right] > 0,$$

for a positive diagonal matrix $P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}$*, and* $0 \leq \mu_{\mathbb{B}_1}(\tilde{A}) \leq 1$*, where the matrix* $\tilde{A}$ *is given as:*

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + P \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}^t \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}^t P.$$

Proof. Let $\Delta \in \mathbb{B}_1$ is characterized by its block-diagonal structure and defined as

$$\Delta := \begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix} = \left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \right] \left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \right]^{-1}.$$

Since all the eigenvalues of

$$\begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} + \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}^t \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}^t \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}$$

are nonzero for $\begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}$. This implies that

$$\lambda_i \left[\begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} + \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}^t \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}^t \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \right] \neq 0$$

$$\Longleftrightarrow$$

$$\lambda_i \left[P \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} + \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}^t \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}^t P + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \Delta^{-1} \right] \neq 0,$$

where $\Delta = \begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix}$. Also, we have that all the eigenvalues of

$$\left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + P \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} \right] - \left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - P \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} \right] \Delta \neq 0.$$

Thus, finally we have that all the eigenvalue λ_i of the following matrix

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + P \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} \right) - \left(\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}^t \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}^t P \right)^{-1}$$

are not equal to zero. □

Theorem 3.7 provides the necessary condition for the Toeplitz linear system's D-stability, which has $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$ as coefficient matrix.

Theorem 3.7. *Let* $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$ *be the matrix of coefficients for a system of linear equations. Let* $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ *be a Hermitian matrix. The necessary condition for*

$$\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

to be a D-stable matrix is that the matrix $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} = e^{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}$ *is a stable matrix and*

$$0 \leq \mu_{\mathbb{B}_1} \left[\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + e^{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}} \right)^{-1} \left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - e^{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}} \right) \right] < 1.$$

Proof. For the necessary condition of the matrix $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$ to be D-stable, we aim to show that

$$\lambda_i\left[\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}+e^{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}\right)\begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}\right]\neq 0, \quad \forall i.$$

Let $\Delta=\begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix}\in\mathbb{B}_1$ such that $P=\left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}+\begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix}\right]^{-1}\left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}-\begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix}\right]$. This further allows us to have that

$$\lambda_i\left[\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}+e^{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}\right)\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}+\begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix}\right)^{-1}\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}-\begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix}\right)\right]\neq 0.$$

Thus,

$$\lambda_i\left[\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}+e^{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}\right)^{-1}\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}-e^{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}\right)\begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix}\right]\neq 0, \quad \forall\Delta\in\mathbb{B}_1.$$

This yields the required result by using the definition of structured singular values, that is,

$$0\leq\mu_{\mathbb{B}_1}\left[\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}+e^{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}\right)^{-1}\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}-e^{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}\right)\right]\neq 0.$$

□

4. Numerical Testing

The approximation and visualization of eigenvalues, singular values, structured singular values, and pseudo-spectra for optimal transportation matrices and structured matrices corresponding to Toeplitz dynamical systems are covered in this part through numerical experimentation. We show the level sets which corresponds to the resolvent norm for the pseudo-spectrum in the complex plane and mathematically the resolvent norm is computed by $\|(M-zI_n)^{-1}\|$.

Example 1. Optimal Mass Transportation: We consider following 4-dimensional complex valued Hermitian Toeplitz matrix T_0 [23] given as:

$$T_o = \begin{pmatrix} 10 & 5+2i & 4+3i & 2+i \\ 5-2i & 10 & 5+2i & 4+3i \\ 4-3i & 5-2i & 10 & 5+2i \\ 2-i & 4-3i & 5-2i & 10 \end{pmatrix}.$$

Figure 1 displays the spectral features of the Toeplitz matrix T_1, including the computation of spectrum, singular values, structured singular values, and pseudo-spectrum. In Figure 2, we plot the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further it shows that how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations.

In Figure 2, we plot the value of inverse of the resolvent norm. We show real part of pseudomode in magenta. The pseudomode displays the right singular vector that corresponds to the least singular value in the matrix $(zI_4 - T_0)$.

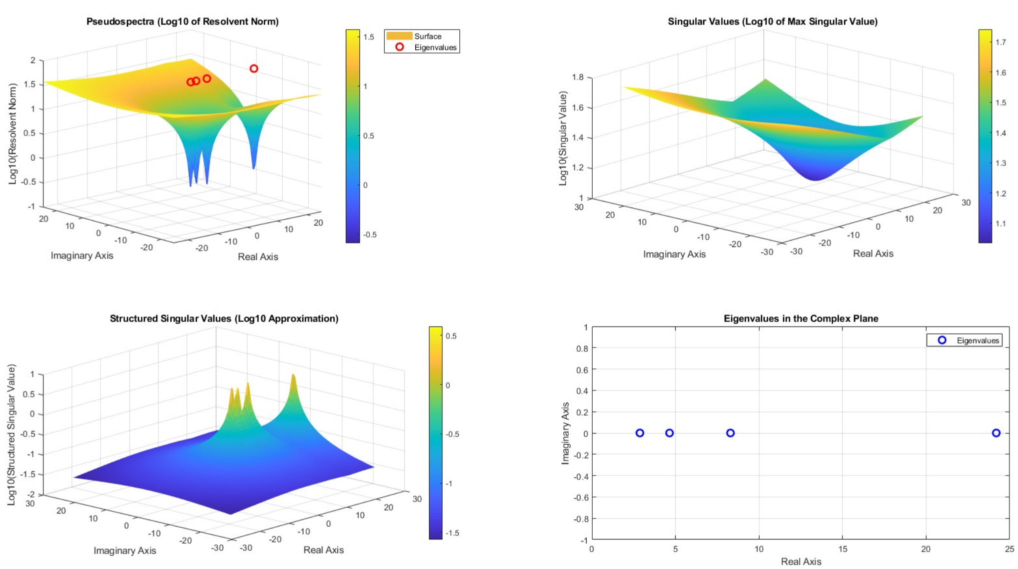


FIGURE 1. Spectral properties of matrix T_0 in Example-1.

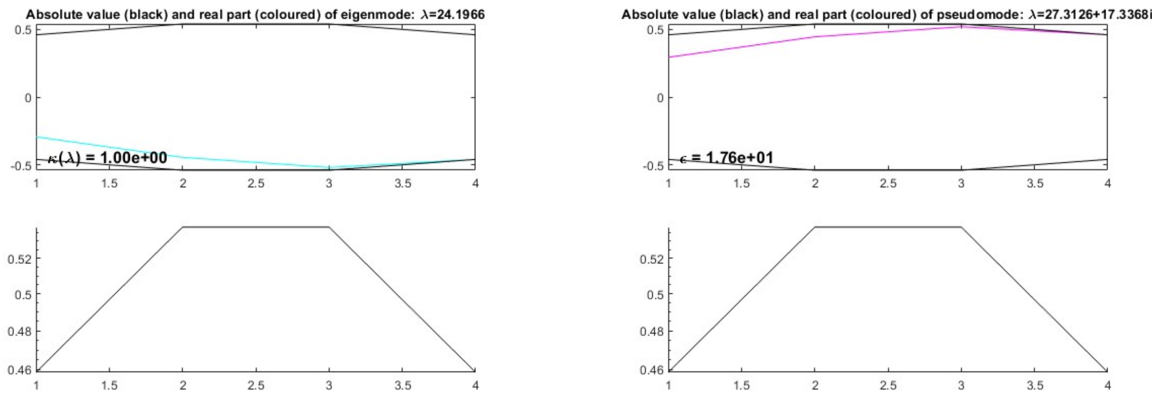


FIGURE 2. Eigenmode (left) and inverse of resolvent norm (right) of matrix T_0 in Example-1

Example 2. We consider a system of linear equations with coefficient matrix being a Toeplitz matrix [1]

$$\underbrace{\begin{pmatrix} -1 & -1 & 2 & 0 & 1 & 1 \\ -1 & -1 & -1 & 2 & 0 & 1 \\ 2 & -1 & -1 & -1 & 2 & 0 \\ 0 & 2 & -1 & -1 & -1 & 2 \\ 1 & 0 & 2 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2 & -1 & -1 \end{pmatrix}}_{T_1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -3 \\ 1 \end{pmatrix}.$$

Figure 3 displays the spectral features of the Toeplitz matrix T_1, including the computation of spectrum, singular values, structured singular values, and pseudo-spectrum. In Figure 4, we plot the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further it shows that how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations.

In Figure 4, we plot the value of inverse of the resolvent norm. We show real part of pseudomode in magenta. The pseudomode displays the right singular vector that corresponds to the least singular value in the matrix $(zI_6 - T_1)$.

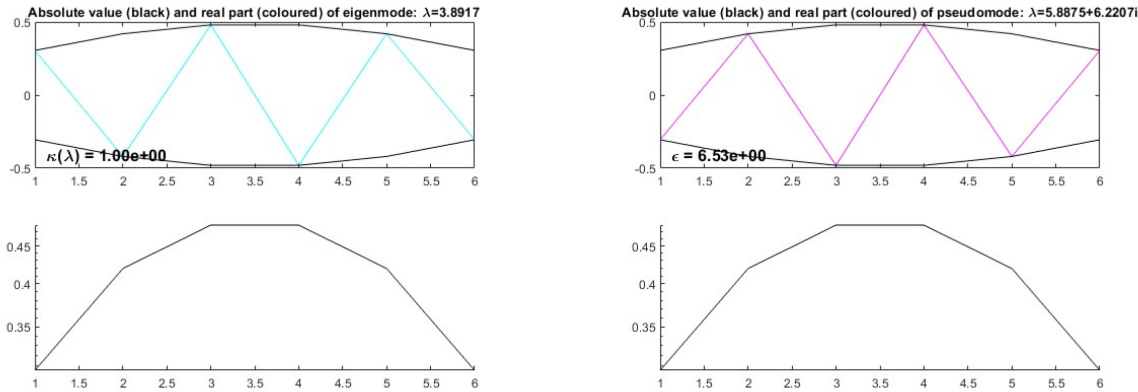


Figure 4. Eigenmode (left) and inverse of resolvent norm (right) of matrix T_1 in Example-2

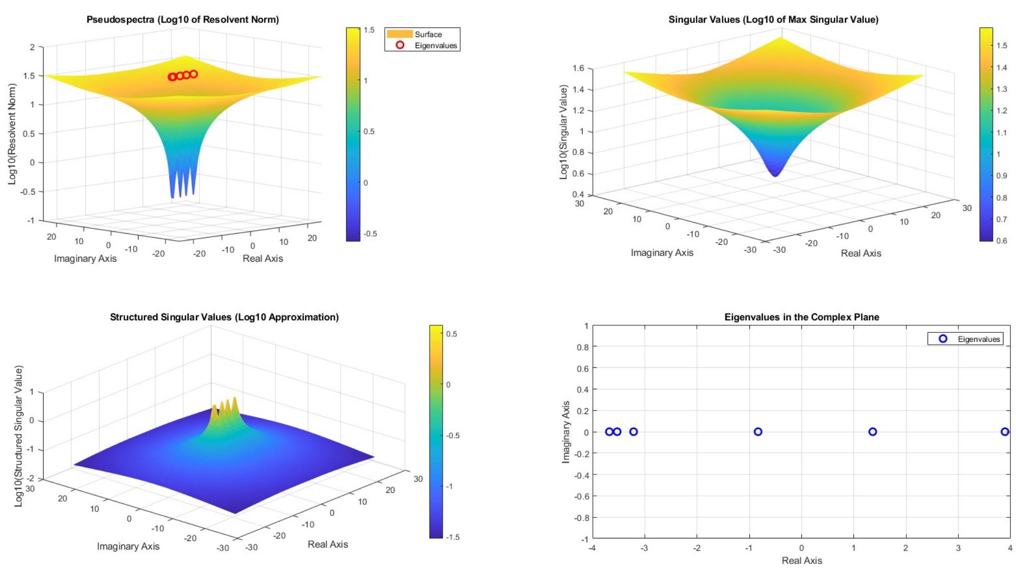


Figure 3. Spectral properties of matrix T_1 in Example-2.

Example 3. We consider a banded Toeplitz matrix T_2 of size 1000×1000. The pseudo-spectrum of T_2 is presented in Figure 5. In Figure 6, we plot the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode and minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further it shows that how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations.

In Figure 6, we plot the value of inverse of the resolvent norm. We show real part of pseudomode in magenta. The right singular vector corresponding to the smallest singular value to matrix $(zI_{1000} - T_2)$, is shown in pseudomode.

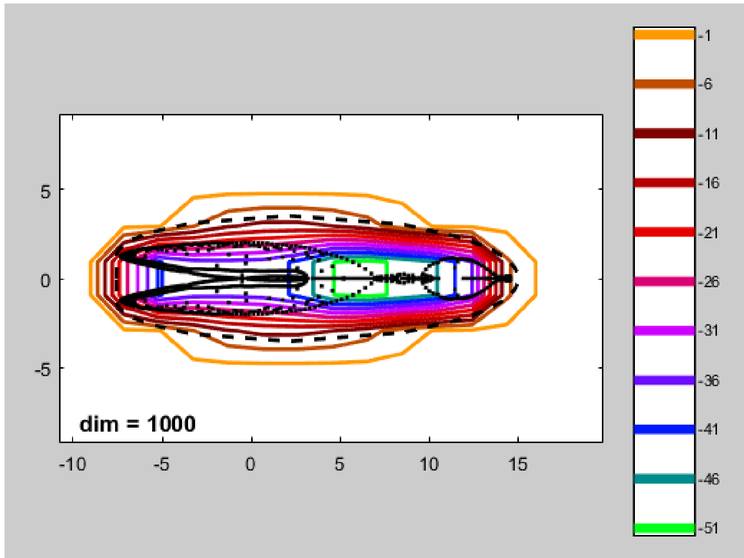


FIGURE 5. Pseudo-spectrum of banded Toeplitz matrix T_2 in Example-3.

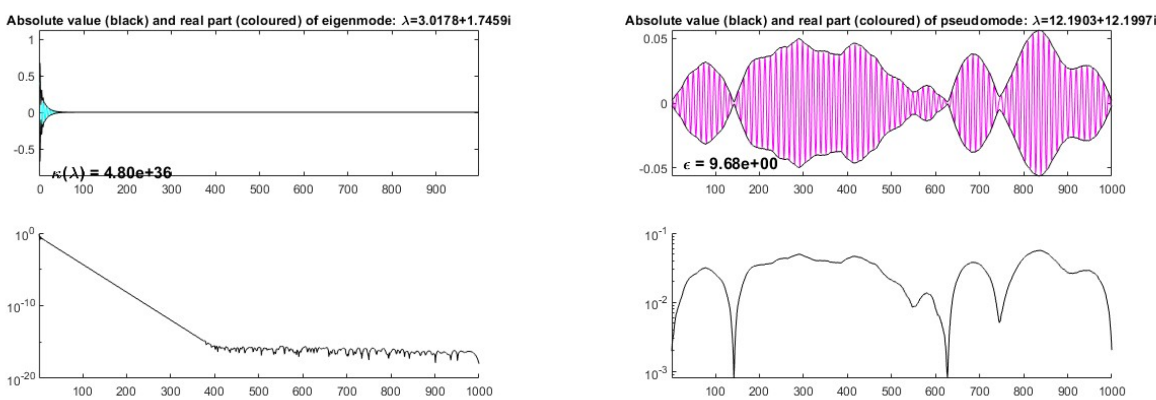


FIGURE 6. Eigenmode (left) and inverse of resolvent norm (right) of matrix T_2 in Example-3

Example 4. We consider a Grcar matrix T_3 of size 1000×1000. The pseudo-spectrum of T_3 is presented in Figure 7. In Figure 8, we plot the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode and minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further it shows that how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations.

In Figure 8, we plot the value of inverse of the resolvent norm. We show real part of pseudomode in magenta. The right singular vector corresponding to the smallest singular value to matrix $(zI_{1000} - T_3)$, is shown in pseudomode.

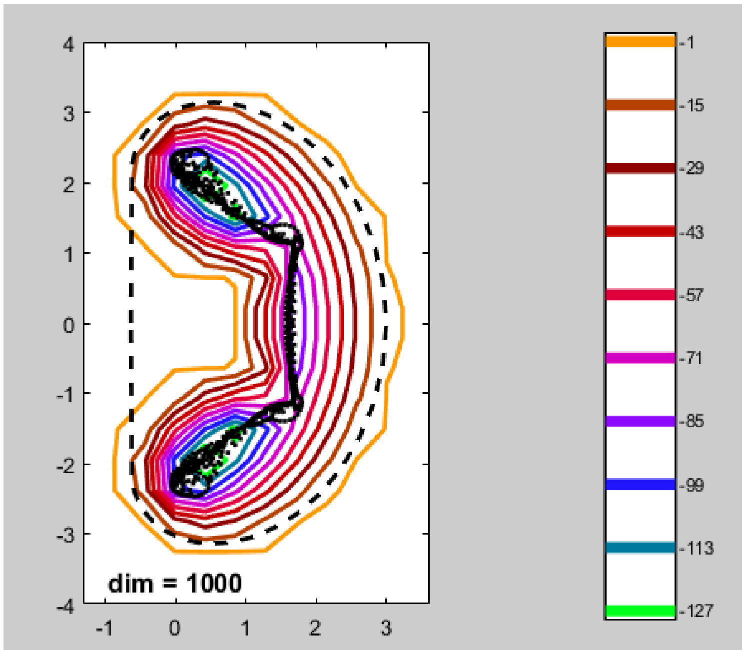


FIGURE 7. Pseudo-spectrum of Grcar matrix T_3 in Example-4.

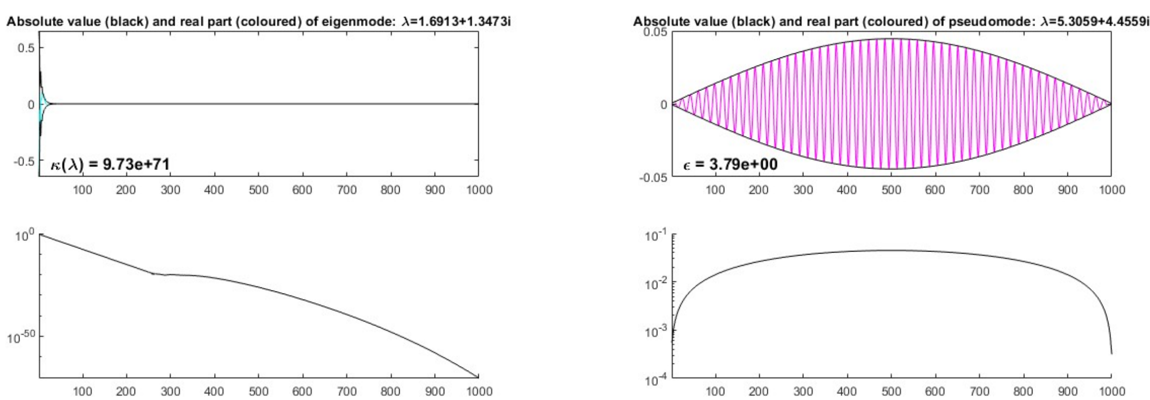
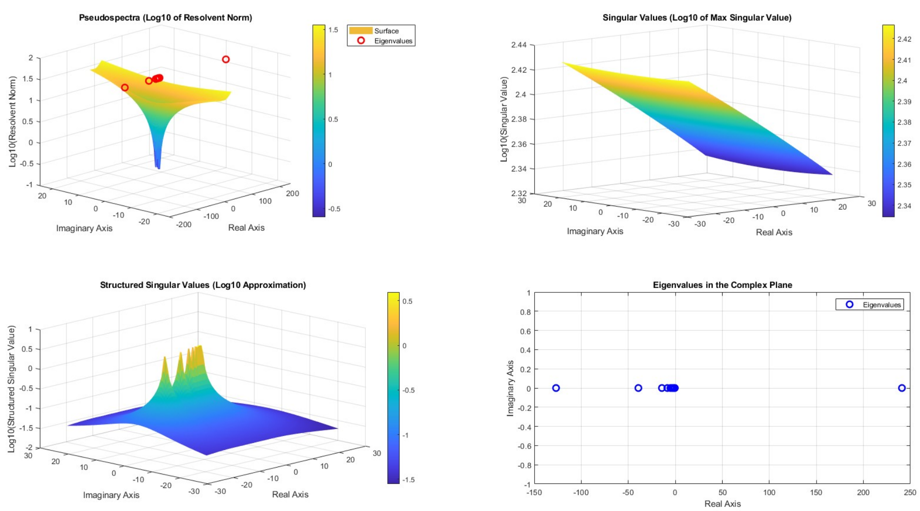
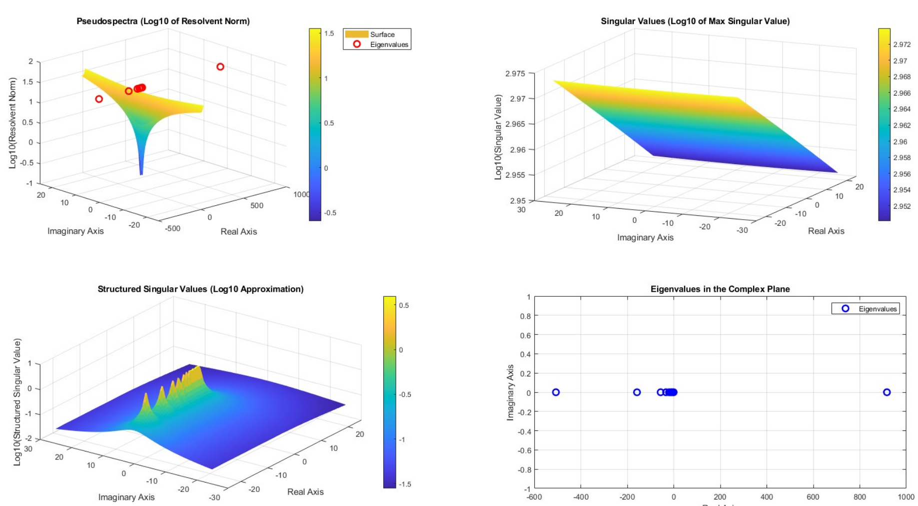


FIGURE 8. Eigenmode (left) and inverse of resolvent norm (right) of Grcar matrix T_3 in Example-4

Example 5. We consider 25, 100 and 250 dimensional Toeplitz matrices. The spectral properties like the computation of spectrum, singular values, structured singular values, and pseudospectrum of 25, 100 and 250 dimensional Toeplitz matrices are presented in Figure 9.

In Figures 10-12, we plot the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode and minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further it shows that how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations. Further, we present the plot of the value of inverse of the resolvent norm. We show real part of pseudomode in magenta. The right singular vector corresponding to the smallest singular value corresponding to matrix $zI - M$, is shown in pseudomode.





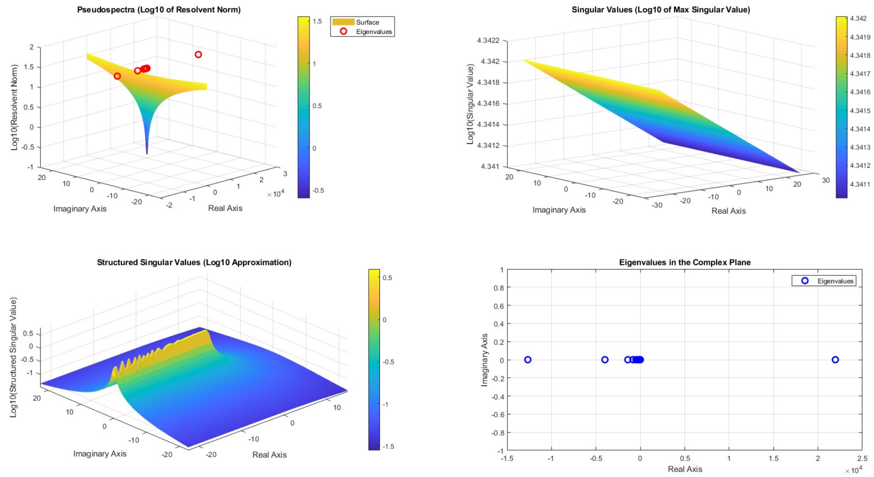


Figure 9. Spectral properties of 25, 100 and 250 dimensional Toeplitz matrices in Example-5.

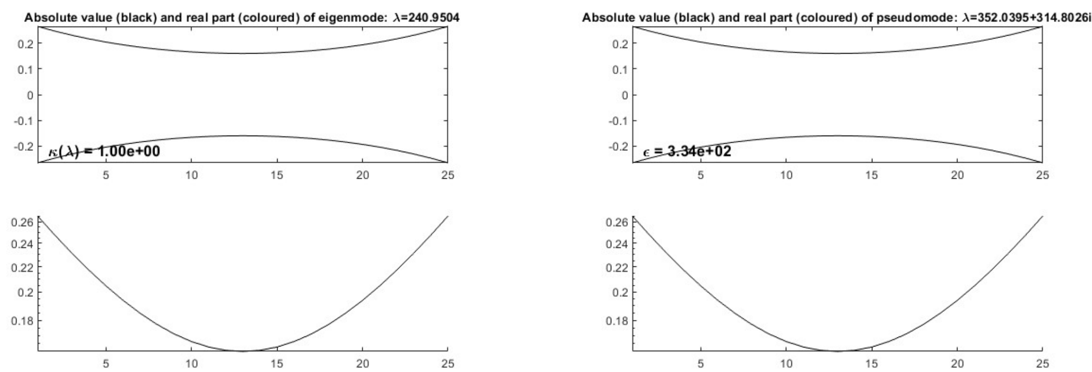


Figure 10. Eigenmode (left) and inverse of resolvent norm (right) of 25 dimensional Toeplitz matrix in Example-5.

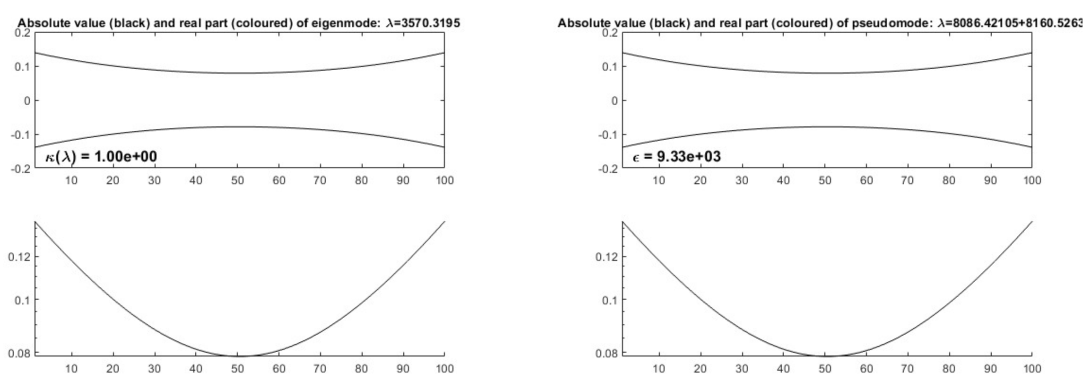


Figure 11. Eigenmode (left) and inverse of resolvent norm (right) of 100 dimensional Toeplitz matrix in Example-5

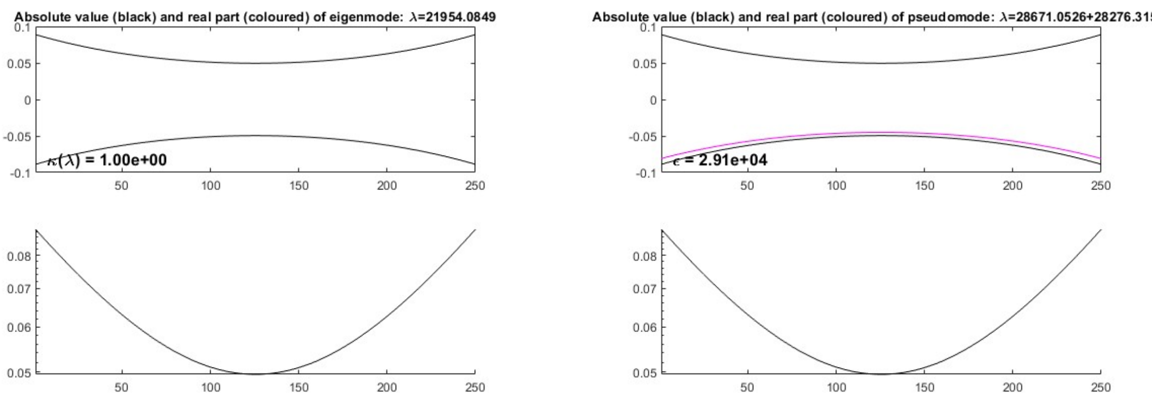


Figure 12. Eigenmode (left) and inverse of resolvent norm (right) of 250 dimensional Toeplitz matrix in Example-5

5. Conclusion

This article introduces new mathematical results on the stability and D-stability analysis of the Toeplitz system of linear equations. Our proposed methodology is to construct the mathematical results on the computation of spectrum and the relationship between D-stability and structured singular values by combining several tools from linear algebra, matrix analysis, and system theory. The behavior of the spectrum and pseudo-spectrum of Toeplitz matrices is demonstrated with various numerical tests. Our future goal is to develop and propose results on strong D-stability, H-stability, $D(\alpha)$-stability, and D-semistability of Toeplitz matrices to further enhance the understanding of D-stability analysis for Toeplitz linear systems. Ismoilova Dildora Erkinovna

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] O.F. Aoulad, C. Tajani, A New Algorithm for Solving Toeplitz Linear Systems, Math. Model. Comput. 10 (2023), 807–815. https://doi.org/10.23939/mmc2023.03.807.

[2] A. Packard, J. Doyle, The Complex Structured Singular Value, Automatica 29 (1993), 71–109. https://doi.org/10.1016/0005-1098(93)90175-s.

[3] K.J. Arrow, M. McManus, A Note on Dynamic Stability, Econometrica 26 (1958), 448–454. https://doi.org/10.2307/1907624.

[4] B. Shi, C. Kızılateş, A New Generalization of the Frank Matrix and Its Some Properties, Comput. Appl. Math. 43 (2023), 19. https://doi.org/10.1007/s40314-023-02524-2.

[5] D. Bini, Parallel Solution of Certain Toeplitz Linear Systems, SIAM J. Comput. 13 (1984), 268–276. https://doi.org/10.1137/0213019.

[6] B.E. Cain, Real, 3×3, D-Stable Matrices, J. Res. Natl. Bur. Stand. Sect. B: Math. Sci. 80B (1976), 75–77. https://doi.org/10.6028/jres.080b.013.

[7] A. Chesnokov, M. Van Barel, A Direct Method to Solve Block Banded Block Toeplitz Systems with Non-Banded Toeplitz Blocks, J. Comput. Appl. Math. 234 (2010), 1485–1491. https://doi.org/10.1016/j.cam.2010.02.029.

[8] M. Chen, On the Solution of Circulant Linear Systems, SIAM J. Numer. Anal. 24 (1987), 668–683. https://doi.org/10.1137/0724044.

[9] C.A. Bahl, B.E. Cain, The Inertia of Diagonal Multiples of 3×3 Real Matrices, Linear Algebr. Appl. 18 (1977), 267–280. https://doi.org/10.1016/0024-3795(77)90056-8.

[10] C.R. Johnson, Sufficient Conditions for D-Stability, J. Econ. Theory 9 (1974), 53–62. https://doi.org/10.1016/0022-0531(74)90074-x.

[11] J. Chen, M.K. Fan, C.C. Yu, On D-Stability and Structured Singular Values, Syst. Control. Lett. 24 (1995), 19–24. https://doi.org/10.1016/0167-6911(94)00036-u.

[12] R.Y. Chang, M.L. Wang, Legendre Polynomials Approximation to Dynamic Linear State Equations with Initial or Boundary Value Conditions, Int. J. Control 40 (1984), 215–232. https://doi.org/10.1080/00207178408933269.

[13] F. Di Benedetto, Computing Eigenvalues and Singular Values of Toeplitz Matrices, CALCOLO 33 (1996), 237–248. https://doi.org/10.1007/bf02576003.

[14] D. Bini, F.D. Benedetto, Solving the Generalized Eigenvalue Problem for Rational Toeplitz Matrices, SIAM J. Matrix Anal. Appl. 11 (1990), 537–552. https://doi.org/10.1137/0611038.

[15] F. Di Benedetto, Generalized Updating Problems and Computation of the Eigenvalues of Rational Toeplitz Matrices, Linear Algebr. Appl. 267 (1997), 187–219. https://doi.org/10.1016/s0024-3795(97)80050-x.

[16] C.M. da Fonseca, C. Kızılateş, N. Terzioğlu, The Determinant and a Factorization of a Toeplitz Matrix with Some Type of Horadam Numbers Entries, Log. J. IGPL 33 (2024), jzae065. https://doi.org/10.1093/jigpal/jzae065.

[17] J.J. Dongarra, C.B. Moler, J.R. Bunch, G.W. Stewart, LINPACK User's Guide, SIAM Press, 1979.

[18] D. Carlson, A Class of Positive Stable Matrices, J. Res. Natl. Bur. Stand. Sect. B: Math. Sci. 78B (1974), 1–2. https://doi.org/10.6028/jres.078b.001.

[19] D. Hershkowitz, Recent Directions in Matrix Stability, Linear Algebr. Appl. 171 (1992), 161–186. https://doi.org/10.1016/0024-3795(92)90257-b.

[20] E. Altınışık, N. Feyza Yalçın, Ş. Büyükköse, Determinants and Inverses of Circulant Matrices with Complex Fibonacci Numbers, Spec. Matrices 3 (2015), 82–90. https://doi.org/10.1515/spma-2015-0008.

[21] E. Kirklar, F. Yilmaz, On the Determinants of Some Kinds of Circulant-Type Matrices with Generalized Number Sequences, Spec. Matrices 3 (2015), 235–243. https://doi.org/10.1515/spma-2015-0023.

[22] E.H. Abed, Strong D-Stability, Syst. Control. Lett. 7 (1986), 207–212. https://doi.org/10.1016/0167-6911(86)90116-7.

[23] F. Elvander, A. Jakobsson, J. Karlsson, Interpolation and Extrapolation of Toeplitz Matrices via Optimal Mass Transport, IEEE Trans. Signal Process. 66 (2018), 5285–5298. https://doi.org/10.1109/tsp.2018.2866432.

[24] D. Fasino, Spectral Properties of Toeplitz-Plus-Hankel Matrices, Calcolo 33 (1996), 87–98. https://doi.org/10.1007/bf02575710.

[25] D. Fischer, G. Golub, O. Hald, C. Leiva, O. Widlund, On Fourier-Toeplitz Methods for Separable Elliptic Problems, Math. Comput. 28 (1974), 349–368. https://doi.org/10.1090/s0025-5718-1974-0415995-2.

[26] H. Wang, Z.W. Sun, Evaluations of Three Symmetric Toeplitz Determinants, Linear Multilinear Algebr. 72 (2023), 2504–2515. https://doi.org/10.1080/03081087.2023.2267734.

[27] H. Prodinger, Some Combinatorial Matrices and Their Lu-Decomposition, Spec. Matrices 8 (2020), 61–67. https://doi.org/10.1515/spma-2020-0007.

[28] R.W. Hockney, A Fast Direct Solution of Poisson's Equation Using Fourier Analysis, J. ACM 12 (1965), 95–113. https://doi.org/10.1145/321250.321259.

[29] X. Chen, Review of Linear Algebra for Controls, Lecture Notes, University of Washington, 2022. https://faculty.washington.edu/chx/teaching/me547/Linear_Algebra_for_Controls_short.pdf.

[30] J.R. Bunch, Stability of Methods for Solving Toeplitz Systems of Equations, SIAM J. Sci. Stat. Comput. 6 (1985), 349–364. https://doi.org/10.1137/0906025.

[31] J.R. Bunch, Stability of Methods for Solving Toeplitz Systems of Equations, SIAM J. Sci. Stat. Comput. 6 (1985), 349–364. https://doi.org/10.1137/0906025.

[32] J. Lee, T.F. Edgar, Real Structured Singular Value Conditions for the Strong D-Stability, Syst. Control. Lett. 44 (2001), 273–277. https://doi.org/10.1016/s0167-6911(01)00147-5.

[33] M. Kac, W. Murdock, G. Szego, On the Eigen-Values of Certain Hermitian Forms, Ind. Univ. Math. J. 2 (1953), 767–800. https://doi.org/10.1512/iumj.1953.2.52034.

[34] M. Anđelić, Z. Du, C.M. da Fonseca, E. Kılıç, A Matrix Approach to Some Second-Order Difference Equations with Sign-Alternating Coefficients, J. Differ. Equ. Appl. 26 (2020), 149–162. https://doi.org/10.1080/10236198.2019.1709180.

[35] M.A. Malcolm, J. Palmer, A Fast Method for Solving a Class of Tridiagonal Linear Systems, Commun. ACM 17 (1974), 14–17. https://doi.org/10.1145/360767.360777.

[36] M.U. Rehman, A. Shadiyev, Interconnection Between H-Stable, $D(\alpha)$-Stable, D-Semistable Matrices, and μ-Values, Asia Pac. J. Math. 11 (2024), 102. https://doi.org/10.28924/apjm/11-102.

[37] M.U. Rehman, B. Aminov, M.N. Alshehri, M.M. Mohammed, A.O. Mustafa, et al., Spectral Properties of Structured Matrices in Transportation Problems, Eur. J. Pure Appl. Math. 18 (2025), 5637. https://doi.org/10.29020/nybg.ejpam.v18i1.5637.

[38] M.U. Rehman, S.E.M. Hamed, N.E. Taha, A.O. Mustafa, K. Dilmurodov, et al., Analysis of Stability, D-Stability, and Pseudospectra in Economic Modeling, Eur. J. Pure Appl. Math. 18 (2025), 5657. https://doi.org/10.29020/nybg.ejpam.v18i1.5657.

[39] P. Arbenz, Computing Eigenvalues of Banded Symmetric Toeplitz Matrices, SIAM J. Sci. Stat. Comput. 12 (1991), 743–754. https://doi.org/10.1137/0912039.

[40] R.H. Chan, M.K. Ng, Conjugate Gradient Methods for Toeplitz Systems, SIAM Rev. 38 (1996), 427–482. https://doi.org/10.1137/s0036144594276474.

[41] O. Rojo, A New Method for Solving Symmetric Circulant Tridiagonal Systems of Linear Equations, Comput. Math. Appl. 20 (1990), 61–67. https://doi.org/10.1016/0898-1221(90)90165-g.

[42] R. Braatz, P. Young, J. Doyle, M. Morari, Computational Complexity of μ Calculation, IEEE Trans. Autom. Control. 39 (1994), 1000–1002. https://doi.org/10.1109/9.284879.

[43] M.U. Rehman, T.H. Rasulov, F. Amir, D-Stability, Strong D-Stability and μ-Values, Lobachevskii J. Math. 45 (2024), 1227–1233. https://doi.org/10.1134/s1995080224600754.

[44] M.U. Rehman, S.H. Alshabhi, A.O. Mustafa, M.M. Mohammed, S. Aljohani, et al., Spectrum and Pseudspectrum of D-Stable Matrices of Economy Models, J. Math. Comput. Sci. 38 (2024), 298–312. https://doi.org/10.22436/jmcs.038.03.02.

[45] M. Rehman, J. Alzabut, M. Tayyab, F. Amir, Interconnection Between Schur Stability and Structured Singular Values, Contemp. Math. 6 (2025), 63–72. https://doi.org/10.37256/cm.6120255707.

[46] S.V. Parter, On the Extreme Eigenvalues of Truncated Toeplitz Matrices, Bull. Am. Math. Soc. 67 (1961), 191–196. https://doi.org/10.1090/s0002-9904-1961-10563-6.

[47] S. Serra, On the Extreme Spectral Properties of Toeplitz Matrices Generated by L_1 Functions with Several Minima/maxima, BIT Numer. Math. 36 (1996), 135–142. https://doi.org/10.1007/bf01740550.

[48] S. Serra, On the Extreme Eigenvalues of Hermitian (Block) Toeplitz Matrices, Linear Algebr. Appl. 270 (1998), 109–129. https://doi.org/10.1016/s0024-3795(97)00231-0.

[49] S.L. Handy, J.L. Barlow, Numerical Solution of the Eigenproblem for Banded, Symmetric Toeplitz Matrices, SIAM J. Matrix Anal. Appl. 15 (1994), 205–214. https://doi.org/10.1137/s0895479891196673.

[50] T. Kailath, A.H. Sayed, Displacement Structure: Theory and Applications, SIAM Rev. 37 (1995), 297–386. https://doi.org/10.1137/1037082.

[51] U. Grenander, G. Szego, Toeplitz Forms and Their Applications, Chelsea, New York, 1984.

[52] W.F. Trench, On the Eigenvalue Problem for Toeplitz Matrices Generated by Rational Functions, Linear Multilinear Algebra 17 (1985), 337–353. https://doi.org/10.1080/03081088508817665.

[53] F.W. Trench, Numerical Solution of the Eigenvalue Problem for Hermitian Toeplitz Matrices, SIAM J. Matrix Anal. Appl. 10 (1989), 135–146. https://doi.org/10.1137/0610010.

[54] O.B. Widlund, On the Use of Fast Methods for Separable Finite Difference Equations for the Solution of General Elliptic Problems, in: D.J. Rose, R.A. Willoughby, (eds) Sparse Matrices and Their Applications, The IBM Research Symposia Series, Springer, Boston, MA, (1972). https://doi.org/10.1007/978-1-4615-8675-3_11.

[55] Z. Jiang, Y. Zheng, T. Li, Characteristic Polynomial, Determinant and Inverse of a Fibonacci-Sylvester-Kac Matrix, Spec. Matrices 10 (2021), 40–46. https://doi.org/10.1515/spma-2021-0145.