Int. J. Anal. Appl. (2026), 24:168

International Journal of Analysis and Applications

Perturbation Analysis of Discrete-Time Wilson Frames

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Abstract. In this paper, we study the effect of perturbations of the generators on discrete-time Wilson frames (DTWFs) for $l^2(\mathbb{Z})$. We find that if the original system forms a DTWF and the perturbed system, obtained by adding another pair of generators in $l^2(\mathbb{Z})$, is a discrete-time Wilson Bessel sequence, then the system generated by the additive component alone is also a discrete-time Wilson Bessel sequence. Moreover, a sufficient condition under which the system corresponding to the additive component constitutes a DTWF is derived. Finally, we study discrete-time Wilson systems generated by finite linear combinations of DTWF generators and establish necessary and sufficient conditions for such systems to form DTWFs.

1. INTRODUCTION

The mathematical framework of frames was first formulated by Duffin and Schaeffer [7] in 1952 as a stable, yet redundant, alternative to orthonormal bases. Although this early contribution laid the foundation, frame theory evolved into a major research area only after the pioneering work of Daubechies, Grossmann, and Meyer [5], which demonstrated its effectiveness in time–frequency methods and signal decomposition.

Let $\mathcal{H}$ denote a separable complex Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A countable family $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $l, u > 0$ satisfying

$$l\|g\|^2 \leq \sum_{k\in\mathbb{N}} |\langle g, g_k \rangle|^2 \leq u\|g\|^2, \qquad \text{for every } g \in \mathcal{H}.$$

The parameters l and u are referred to as frame bounds. Frames provide stable expansions in $\mathcal{H}$ while allowing redundancy, which distinguishes them from bases. A sequence $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ is said to be a Riesz basis if it is complete and if there exist constants $l, u > 0$ such that

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Received: Mar. 20, 2026.

2020 *Mathematics Subject Classification.* 42C15, 46B15.

Key words and phrases. frame perturbation theory; Riesz bases; Wilson frames.

https://doi.org/10.28924/2291-8639-24-2026-168

ISSN: 2291-8639

$$l\sum_{k\in\mathbb{N}}|\gamma_k|^2 \le \left\|\sum_{k\in\mathbb{N}}\gamma_k g_k\right\|^2 \le u\sum_{k\in\mathbb{N}}|\gamma_k|^2, \qquad \forall\,\{\gamma_k\}_{k\in\mathbb{N}}\in\ell^2(\mathbb{N}).$$

A Riesz basis can be characterized as the image of an orthonormal basis under a bounded linear operator that admits a bounded inverse. Thus, while every Riesz basis is a frame, the converse is not true due to the possible presence of redundancy.

Gabor systems constitute a prominent class of frames in $L^2(\mathbb{R})$. However, when such systems are constructed at critical density so as to form Riesz bases, the Balian–Low theorem imposes a severe restriction: the generating function cannot be sharply localized simultaneously in both the temporal and spectral domains. This phenomenon motivated the search for alternative orthonormal constructions that retain strong localization in both variables.

An important contribution in this direction was made by Wilson [14, 15] who proposed a symmetrization procedure in the frequency domain by coupling frequency components of opposite sign. Utilizing this principle, Daubechies, Jaffard, and Journé [6] established the existence of orthonormal systems, now referred to as Wilson bases, for $L^2(\mathbb{R})$. These systems are generated by functions of the form

$$\psi_j^k(x) = \begin{cases} e_k\cos(2k\pi x)\, w\left(x-\dfrac{j}{2}\right), & \text{if } j \text{ is even},\\[2ex] 2\sin(2(k+1)\pi x)\; w\left(x-\dfrac{j+1}{2}\right), & \text{if } j \text{ is odd},\end{cases}$$

where

$$e_k = \begin{cases}\sqrt{2}, & k=0\\ 2, & k\ge 1,\end{cases}$$

and w denotes a smooth window function exhibiting rapid decay. This construction successfully overcomes the localization obstruction described by the Balian–Low theorem, thereby producing orthonormal bases with improved joint concentration in time and frequency.

Around the same period, Malvar [13] introduced a different class of orthonormal systems, known as local trigonometric bases, consisting of functions of the form $w_j\cos\big((k+\frac{1}{2})\pi(\cdot-j)\big)$, $j\in\mathbb{Z}$, $k\in\mathbb{N}_0$. In this setting, each window w_j is compactly supported and arranged so that overlaps occur only between adjacent windows. This localization strategy differs fundamentally from the Wilson construction. Later works [1, 4] extended Malvar's framework, and biorthogonal adaptations allowing greater flexibility were developed by Jawerth and Sweldens [9]. Nevertheless, the compact support constraint imposed on the windows restricts the generality of this approach, which often makes Wilson-type systems more adaptable in applications.

Further analytical properties of Wilson systems have been investigated extensively. Feichtinger, Gröchenig, and Walnut [8] showed that Wilson bases generated by exponentially decaying windows do not yield unconditional bases for a broad class of modulation spaces on $\mathbb{R}$, including classical smoothness spaces such as Bessel potential and Schwartz spaces. They also established that unconditionality fails in $L^p(\mathbb{R})$ whenever $p \neq 2$. Subsequent contributions addressed approximation behavior [3], extensions to general time–frequency lattices [11], and adaptations to non-rectangular lattice structures motivated by engineering applications [14]. Wojdyllo [16, 17] explored variants involving modified constructions and higher redundancy, while Lian et al. [12] studied discrete-time analogues. Moreover, motivated by the asymmetric trigonometric components appearing in the Wilson formulation, Bittner [2, 3] proposed generalized systems with distinct window function.

In 2020, Kohli, Panwar and Kaushik [10] defined discrete-time Wilson (DTW) system $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z}); m = 0, 1 \ldots M-1; k \in \mathbb{Z}; L, M \in \mathbb{N}\right\}$ generated by a pair of sequences $g_0, g_{-1} \in l^2(\mathbb{Z})$ as follows:

$$\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} = \begin{cases} (E_{\frac{m}{M}} T_{\frac{kL}{2}} + E_{\frac{-m}{M}} T_{\frac{kL}{2}})g_0 & \text{if } k \in 2\mathbb{Z}, k \neq 0, \\ \frac{1}{i}(E_{\frac{m+1}{M}} T_{\frac{(k+1)L}{2}} - E_{\frac{-(m+1)}{M}} T_{\frac{(k+1)L}{2}})g_{-1} & \text{if } k \in 2\mathbb{Z}+1, \\ \frac{1}{\sqrt{2}}(E_{\frac{m}{M}} + E_{\frac{-m}{M}})g_0 & \text{if } k = 0, \end{cases} \tag{1.1}$$

where $k \in \mathbb{Z}$, $L, M \in \mathbb{N}$ and $m = 0, 1, 2, \cdots, M-1$.

For any $n \in \mathbb{Z}$, the above DTW system [10] can be rewritten as

$$\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}(n) = \begin{cases} \sqrt{2}\cos(\frac{2\pi mn}{M})g_0(n) & \text{if } k = 0, \\ 2\cos(\frac{2\pi mn}{M})g_0(n - \frac{kL}{2}) & \text{if } k \in 2\mathbb{Z}, k \neq 0, \\ 2\sin(\frac{2\pi(m+1)n}{M})g_{-1}(n - \frac{(k+1)L}{2}) & \text{if } k \in 2\mathbb{Z}+1. \end{cases}$$

For $g_0 = g_{-1} = g$, the DTW system [10] has the form

$$\psi^{g}_{\frac{m}{M},kL} = \begin{cases} (E_{\frac{m}{M}} T_{\frac{kL}{2}} + E_{\frac{-m}{M}} T_{\frac{kL}{2}})g & \text{if } k \in 2\mathbb{Z}, k \neq 0, \\ \frac{1}{i}(E_{\frac{m+1}{M}} T_{\frac{(k+1)L}{2}} - E_{\frac{-(m+1)}{M}} T_{\frac{(k+1)L}{2}})g & \text{if } k \in 2\mathbb{Z}+1, \\ \frac{1}{\sqrt{2}}(E_{\frac{m}{M}} + E_{\frac{-m}{M}})g & \text{if } k = 0, \end{cases}$$

where $k \in \mathbb{Z}$, $L, M \in \mathbb{N}$ and $m = 0, 1, 2, \cdots, M-1$.

The discrete-time Wilson system [10] (generated by a pair of sequences $g_0, g_{-1} \in l^2(\mathbb{Z})$), $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z}); m = 0, 1 \ldots M-1; k \in \mathbb{Z}; L, M \in \mathbb{N}\right\}$ is called a discrete-time Wilson frame (DTWF), if there exist constants $A > 0$ and $B > 0$ satisfying:

$$A\left\|f\right\|^2 \leq \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \left|\left\langle f, \psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} \right\rangle\right|^2 \leq B\left\|f\right\|^2, \text{ for all } f \in l^2(\mathbb{Z}). \tag{1.2}$$

Kohli et al. [10] provided the necessary and sufficient conditions for the discrete-time Wilson system to be DTWF and discussed its properties. Inspired from their work, we investigate the effect of perturbation of the generators on the discrete-time Wilson frame (DTWF). Section 2 discusses that, if a perturbed system generated by $g_0 + h_0$ and $g_{-1} + h_{-1}$ is Bessel, then the system generated solely by the perturbation pair h_0, h_{-1} is also Bessel, provided the system corresponding to g_0, g_{-1} is DTWF. It also investigates the sufficient condition for the system generated by h_0, h_{-1} to be DTWF. In section 3, we study the necessary and sufficient condition for the discrete-time Wilson system generated by a finite linear combination of DTWF generators, to be a DTWF itself. For convenience of notation, the DTW system $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z}); m = 0, 1 \ldots M-1; k \in \mathbb{Z}; L, M \in \mathbb{N}\right\}$ associated with $g_0, g_{-1} \in l^2(\mathbb{Z})$ will be denoted by $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$ throughout this article.

2. Perturbation of DTW Frames

In this section, we study how the discrete-time Wilson frame behaves under the additive changes in the generators. Let $g_0, g_{-1} \in l^2(\mathbb{Z})$ be a pair of sequences used to generate a discrete-time Wilson system $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$. To analyze the perturbations, we consider another pair $h_0, h_{-1} \in l^2(\mathbb{Z})$ and the modified generators $g_0 + h_0$ and $g_{-1} + h_{-1}$. The corresponding system $\left\{\psi^{(g_0+h_0,g_{-1}+h_{-1})}_{\frac{m}{M},kL} : g_0 + h_0, g_{-1} + h_{-1} \in l^2(\mathbb{Z})\right\}$ is viewed as a perturbed version of the original family.

It is observed that if the original system, $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$ is a DTWF for $l^2(\mathbb{Z})$ and the perturbed system $\left\{\psi^{(g_0+h_0,g_{-1}+h_{-1})}_{\frac{m}{M},kL} : g_0 + h_0, g_{-1} + h_{-1} \in l^2(\mathbb{Z})\right\}$ is a DTW Bessel sequence for $l^2(\mathbb{Z})$, then the system corresponding to the the additive component $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ may not be a DTWF for $l^2(\mathbb{Z})$. The following example supports the above statement.

Example 2.1. *Consider* $L = 2, M = 4, g_0(n) = e_0, g_{-1}(n) = e_1, h_0(n) = e_1$ *and* $h_{-1}(n) = e_3$. *Then the system* $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$ *is a DTWF for* $l^2(\mathbb{Z})$ *and* $\left\{\psi^{(g_0+h_0,g_{-1}+h_{-1})}_{\frac{m}{M},kL} : g_0 + h_0, g_{-1} + h_{-1} \in l^2(\mathbb{Z})\right\}$ *forms a DTW Bessel sequence for* $l^2(\mathbb{Z})$ *with Bessel bound 48. But the system* $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ *does not form a DTWF for* $l^2(\mathbb{Z})$.

Under the assumption that the original system is a frame, the next result shows that any additive component that keeps the perturbed system Bessel must itself generate a Bessel system.

Theorem 2.1. *Let* $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$ *be a DTWF for* $l^2(\mathbb{Z})$ *associated with* g_0, g_{-1}. *Assume that* $\left\{\psi^{(g_0+h_0,g_{-1}+h_{-1})}_{\frac{m}{M},kL} : g_0 + h_0, g_{-1} + h_{-1} \in l^2(\mathbb{Z})\right\}$ *is a DTW Bessel sequence for* $h_0, h_{-1} \in l^2(\mathbb{Z})$, *then the system* $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ *forms a DTW Bessel sequence for* $l^2(\mathbb{Z})$.

Proof. Since the system $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$ is a DTWF for $l^2(\mathbb{Z})$, using (1.2), we obtain positive real numbers A, B satisfying:

$$A\left\|f\right\|^2 \leq \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 \leq B\left\|f\right\|^2, \text{ for all } f \in l^2(\mathbb{Z}).$$

Again as $\left\{\psi^{(g_0+h_0,g_{-1}+h_{-1})}_{\frac{m}{M},kL} : g_0+h_0, g_{-1}+h_{-1} \in l^2(\mathbb{Z})\right\}$ is given to be a DTW Bessel sequence for $l^2(\mathbb{Z})$, there exists $C > 0$ such that

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0+h_0,g_{-1}+h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 \le C\left\|f\right\|^2, \text{ for all } f \in l^2(\mathbb{Z}). \tag{2.1}$$

For $f \in l^2(\mathbb{Z})$, we write

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(h_0,h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 &= 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}h_0(\cdot)\right\rangle\right|^2 \\
&\quad + 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \sin\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}h_{-1}(\cdot)\right\rangle\right|^2 \\
&\quad - 2\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)h_0(\cdot)\right\rangle\right|^2 \\
&\le 8\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}(h_0+g_0)(\cdot)\right\rangle\right|^2 \\
&\quad + 8\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}g_0(\cdot)\right\rangle\right|^2 \\
&\quad + 8\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \sin\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}(h_{-1}+g_{-1})(\cdot)\right\rangle\right|^2 \\
&\quad + 8\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \sin\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}g_{-1}(\cdot)\right\rangle\right|^2 \\
&= 2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0+h_0,g_{-1}+h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 \\
&\quad + 2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 \\
&\quad + 4\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)(h_0+g_0)(\cdot)\right\rangle\right|^2 \\
&\quad + 4\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)g_0(\cdot)\right\rangle\right|^2 \\
&\le 2(C + B + 6M\left\|g_0\right\|^2 + 4M\left\|h_0\right\|^2)\left\|f\right\|^2.
\end{aligned}$$

Thus, $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ is a DTW Bessel sequence for $l^2(\mathbb{Z})$ with the bound $2(C + B + 6M\left\|g_0\right\|^2 + 4M\left\|h_0\right\|^2)$. □

If $g_0 = g_{-1} = g$, then the above theorem has the following form:

Theorem 2.2. *Let* $\left\{\psi^{g}_{\frac{m}{M},kL} : g \in l^2(\mathbb{Z})\right\}$ *be a DTWF for* $l^2(\mathbb{Z})$ *and for* $h \in l^2(\mathbb{Z})$, $\left\{\psi^{g+h}_{\frac{m}{M},kL} : g+h \in l^2(\mathbb{Z})\right\}$ *is a DTW Bessel sequence for* $l^2(\mathbb{Z})$, *then* $\left\{\psi^{h}_{\frac{m}{M},kL} : h \in l^2(\mathbb{Z})\right\}$ *is a DTW Bessel sequence for* $l^2(\mathbb{Z})$.

Towards the sufficient condition for the system corresponding to the additive component, $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ to be a DTWF, we have the following result:

Theorem 2.3. *Let* $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$ *be a DTWF associated with* $g_0, g_{-1} \in l^2(\mathbb{Z})$ *with frame bounds A and B, and for* $h_0, h_{-1} \in l^2(\mathbb{Z})$, *the system* $\left\{\psi^{(g_0+h_0,g_{-1}+h_{-1})}_{\frac{m}{M},kL} : g_0+h_0, g_{-1}+h_{-1} \in l^2(\mathbb{Z})\right\}$ *is a DTW Bessel sequence for* $l^2(\mathbb{Z})$ *with Bessel bound C. If* $A > 2C + 6M(\left\|g_0\right\|^2 + 2\left\|h_0\right\|^2)$, *then the system* $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ *forms a DTWF for* $l^2(\mathbb{Z})$.

Proof. Using Theorem 2.1, we get that $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ is a DTW Bessel sequence for $l^2(\mathbb{Z})$. To derive the lower frame inequality, for $f \in l^2(\mathbb{Z})$

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 - 2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0+h_0,g_{-1}+h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 \\
&\qquad\qquad \leq 8\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \cos\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}h_0(\cdot)\right\rangle\right|^2 \\
&\qquad\qquad\quad + 8\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \sin\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}h_{-1}(\cdot)\right\rangle\right|^2 \\
&\qquad\qquad\quad + 6\sum_{m=0}^{M-1}\left|\left\langle f, \cos\left(\frac{2\pi m(\cdot)}{M}\right)g_0(\cdot)\right\rangle\right|^2 \\
&\qquad\qquad\quad + 8\sum_{m=0}^{M-1}\left|\left\langle f, \cos\left(\frac{2\pi m(\cdot)}{M}\right)h_0(\cdot)\right\rangle\right|^2 \\
&\qquad\qquad = 2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(h_0,h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 \\
&\qquad\qquad\quad + 6\sum_{m=0}^{M-1}\left|\left\langle f, \cos\left(\frac{2\pi m(\cdot)}{M}\right)g_0(\cdot)\right\rangle\right|^2 \\
&\qquad\qquad\quad + 12\sum_{m=0}^{M-1}\left|\left\langle f, \cos\left(\frac{2\pi m(\cdot)}{M}\right)h_0(\cdot)\right\rangle\right|^2 \\
&\qquad\qquad \leq 2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(h_0,h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 + 6M(\left\|g_0\right\|^2 + 2\left\|h_0\right\|^2)\left\|f\right\|^2.
\end{aligned}
$$

Using (1.2) and (2.1) in the above inequality, we obtain

$$
\frac{1}{2}(A - 2C - 6M\left\|g_0\right\|^2 - 12M\left\|h_0\right\|^2)\left\|f\right\|^2 \leq \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(h_0,h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2, \text{ for all } f \in l^2(\mathbb{Z}).
$$

Thus, $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ forms a DTWF for $l^2(\mathbb{Z})$. □

Corollary 2.1. *Let* $\left\{\psi^{g}_{\frac{m}{M},kL} : g \in l^2(\mathbb{Z})\right\}$ *be a DTWF for* $l^2(\mathbb{Z})$ *with frame bounds A and B and for* $h \in l^2(\mathbb{Z})$*, the system* $\left\{\psi^{(g+h)}_{\frac{m}{M},kL} : g+h \in l^2(\mathbb{Z})\right\}$ *is a DTW Bessel sequence for* $l^2(\mathbb{Z})$ *with Bessel bound C, then* $\left\{\psi^{h}_{\frac{m}{M},kL} : h \in l^2(\mathbb{Z})\right\}$ *is a DTWF for* $l^2(\mathbb{Z})$ *if* $A > 4C$.

The system $\left\{\psi^{(g_0-h_0,g_{-1}-h_{-1})}_{\frac{m}{M},kL} : g_0-h_0, g_{-1}-h_{-1} \in l^2(\mathbb{Z})\right\}$ is referred to as a backward-perturbed system generated by $(g_0-h_0, g_{-1}-h_{-1})$. If the system $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$ is a DTWF, then next theorem presents the necessary and sufficient condition for the DTW system $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ to be discrete Wilson Bessel sequence under some condition satisfied by backward-perturbed system $\left\{\psi^{(g_0-h_0,g_{-1}-h_{-1})}_{\frac{m}{M},kL} : g_0-h_0, g_{-1}-h_{-1} \in l^2(\mathbb{Z})\right\}$. Further, it provides a sufficient condition for the system $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ to be a DTWF for $l^2(\mathbb{Z})$.

Theorem 2.4. *Let* $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$ *be a DTWF for* $l^2(\mathbb{Z})$ *with frame bounds A and B, then* $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ *is a DTW Bessel sequence for* $l^2(\mathbb{Z})$ *if and only if there exists a constant* $C > 0$ *such that*

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0-h_0,g_{-1}-h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 \leq C\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 \text{, for all } f \in l^2(\mathbb{Z}). \tag{2.2}$$

In addition, if

$$A(1-2C) > 4M(2\left\|g_0\right\|^2 + 3\left\|h_0\right\|^2),$$

then $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ *is a DTWF for* $l^2(\mathbb{Z})$.

Proof. Let $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$ is a DTWF for $l^2(\mathbb{Z})$ with frame bounds A and B. Suppose that $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ be a DTW Bessel sequence for $l^2(\mathbb{Z})$. Thus, there exists a positive constant P such that

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(h_0,h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 \leq P\left\|f\right\|^2 \text{, for all } f \in l^2(\mathbb{Z}). \tag{2.3}$$

For any $f \in l^2(\mathbb{Z})$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0-h_0,g_{-1}-h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 &= 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}(g_0-h_0)(\cdot)\right\rangle\right|^2\\ &\quad + 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \sin\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}(g_{-1}-h_{-1})(\cdot)\right\rangle\right|^2\\ &\quad - 2\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)(g_0-h_0)(\cdot)\right\rangle\right|^2\end{aligned}$$

$$
\begin{aligned}
&\leq 8\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}h_0(\cdot)\right\rangle\right|^2\\
&\quad+8\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}g_0(\cdot)\right\rangle\right|^2\\
&\quad+8\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}h_{-1}(\cdot)\right\rangle\right|^2\\
&\quad+8\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}g_{-1}(\cdot)\right\rangle\right|^2
\end{aligned}
$$

$$
\begin{aligned}
&=2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2+2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2\\
&\quad+4\sum_{m=0}^{M-1}\left|\left\langle f,\cos\Big(\frac{2\pi m(\cdot)}{M}\Big)g_0(\cdot)\right\rangle\right|^2\\
&\quad+4\sum_{m=0}^{M-1}\left|\left\langle f,\cos\Big(\frac{2\pi m(\cdot)}{M}\Big)h_0(\cdot)\right\rangle\right|^2\\
&\leq 2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2\\
&\quad+2(P+2M\left\|g_0\right\|^2+2M\|h_0\|^2)\left\|f\right\|^2\\
&\leq C\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2,
\end{aligned}
$$

where $C=\dfrac{2}{A}(A+P+2M\left\|g_0\right\|^2+2M\|h_0\|^2)>0$. Therefore, the inequality (2.2) holds. Conversely, suppose that the inequality (2.2) holds. Then, proceeding as in Theorem 2.3, we get

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 &\leq 2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi^{(h_0-g_0,h_{-1}-g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2\\
&\quad+2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2\\
&\quad+4\sum_{m=0}^{M-1}\left|\left\langle f,\cos\Big(\frac{2\pi m(\cdot)}{M}\Big)(h_0-g_0)(\cdot)\right\rangle\right|^2\\
&\quad+4\sum_{m=0}^{M-1}\left|\left\langle f,\cos\Big(\frac{2\pi m(\cdot)}{M}\Big)g_0(\cdot)\right\rangle\right|^2\\
&\leq 2((C+1)B+4M\|h_0\|^2+6M\left\|g_0\right\|^2)\left\|f\right\|^2,
\end{aligned}
$$

for all $f \in l^2(\mathbb{Z})$. For lower frame inequality, using the similar argument as in Theorem 2.3, one can prove that for any $f \in l^2(\mathbb{Z})$

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 - 2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0-h_0,g_{-1}-h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2$$
$$\leq 2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(h_0,h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 + 4M(2\left\|g_0\right\|^2 + 3\left\|h_0\right\|^2)\left\|f\right\|^2.$$

As A is given to be lower frame bound for $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$, using (2.2) we derive that

$$\frac{1}{2}(A(1-2C) - 6M\left\|h_0\right\|^2 - 4M\left\|g_0\right\|^2)\left\|f\right\|^2 \leq \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(h_0,h_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2,$$

for all $f \in l^2(\mathbb{Z})$. Since $A(1-2C) > 4M(2\left\|g_0\right\|^2 + 3\left\|h_0\right\|^2)$, we have that the system $\left\{\psi^{(h_0,h_{-1})}_{\frac{m}{M},kL} : h_0, h_{-1} \in l^2(\mathbb{Z})\right\}$ is a DTWF for $l^2(\mathbb{Z})$. □

Corollary 2.2. *Let* $\left\{\psi^{g}_{\frac{m}{M},kL} : g \in l^2(\mathbb{Z})\right\}$ *be a DTWF for* $l^2(\mathbb{Z})$ *with frame bounds* A *and* B *and let* $h \in l^2(\mathbb{Z})$ *be any function. Then,* $\left\{\psi^{h}_{\frac{m}{M},kL} : h \in l^2(\mathbb{Z})\right\}$ *is a DTW Bessel sequence for* $l^2(\mathbb{Z})$ *if and only if there exists a constant* $C > 0$ *such that*

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g-h)}_{\frac{m}{M},kL}\right\rangle\right|^2 \leq C\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{g}_{\frac{m}{M},kL}\right\rangle\right|^2, \text{ for all } f \in l^2(\mathbb{Z}).$$

Further, if $0 < C < \frac{1}{4}$, *then* $\left\{\psi^{h}_{\frac{m}{M},kL} : h \in l^2(\mathbb{Z})\right\}$ *is a DTWF for* $l^2(\mathbb{Z})$.

3. System Generated by Finite Linear Family

In this section, we study the behaviour of the discrete-time Wilson frames generated by finite linear combinations of generators. In particular, we investigate conditions under which finite linear combinations of the given generating functions preserve the DTWF structure for $l^2(\mathbb{Z})$.

Proposition 3.1. *Let* $g_0^i = g$ *and* $g_{-1}^i = h$ *for all* $i \in S_r = \{0, 1, 2, \cdots, r\}$ *where* $g, h \in l^2(\mathbb{Z})$. *The system* $\left\{\psi^{(g,h)}_{\frac{m}{M},kL} : g, h \in l^2(\mathbb{Z})\right\}$ *is a DTWF for* $l^2(\mathbb{Z})$ *iff* $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0 = \sum_{i=1}^r g_0^i = rg, g_{-1} = \sum_{i=1}^r g_{-1}^i = rh\right\}$ *is a DTWF for* $l^2(\mathbb{Z})$.

Proof. For all $f \in l^2(\mathbb{Z})$, we have

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 = 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \cos\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}\left(\sum_{i=1}^r g_0^i\right)(\cdot)\right\rangle\right|^2$$
$$+ 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \sin\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}\left(\sum_{i=1}^r g_{-1}^i\right)(\cdot)\right\rangle\right|^2$$

$$-2\sum_{m=0}^{M-1}\left|\left\langle f,\cos\Big(\frac{2\pi m(\cdot)}{M}\Big)\Big(\sum_{i=1}^{r} g_0^i\Big)(\cdot)\right\rangle\right|^2$$

$$= r^2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi^{(g,h)}_{\frac{m}{M},kL}\right\rangle\right|^2.$$

Thus, the system $\left\{\psi^{(g,h)}_{\frac{m}{M},kL} : g,h\in l^2(\mathbb{Z})\right\}$ is a DTWF for $l^2(\mathbb{Z})$ iff $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0=\sum_{i=1}^{r} g_0^i = rg, g_{-1}=\sum_{i=1}^{r} g_{-1}^i = rh\right\}$ is a DTWF for $l^2(\mathbb{Z})$. □

The next example illustrates that, in general the above result is not true.

Example 3.1. *If for each i,* $\left\{\psi^{(g_0^i,g_{-1}^i)}_{\frac{m}{M},kL} : g_0^i, g_{-1}^i\in l^2(\mathbb{Z})\right\}$ *is a DTWF for* $l^2(\mathbb{Z})$*, then* $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0=\sum_{i=1}^{r} g_0^i, g_{-1}=\sum_{i=1}^{r} g_{-1}^i, g_0, g_{-1}\in l^2(\mathbb{Z})\right\}$ *need not be a DTWF for* $l^2(\mathbb{Z})$*.*

Assume $r=2, L=2, M=4$. *Let* $g_0^1(n)=e_0, g_{-1}^1(n)=e_1, g_0^2(n)=-g_0^1(n), g_{-1}^2(n)=-g_{-1}^1(n)$. *Then, for each* $i=1,2$, $\left\{\psi^{(g_0^i,g_{-1}^i)}_{\frac{m}{M},kL}\right\}$ *is DTWF for* $l^2(\mathbb{Z})$ *but*

$$\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0=\sum_{i=1}^{2} g_0^i = 0, g_{-1}=\sum_{i=1}^{2} g_{-1}^i = 0\right\} = \{0\},$$

is not a DTWF for $l^2(\mathbb{Z})$*.*

Example 3.2. *If* $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0=\sum_{i=1}^{r} g_0^i, g_{-1}=\sum_{i=1}^{r} g_{-1}^i, g_0, g_{-1}\in l^2(\mathbb{Z})\right\}$ *is a DTWF for* $l^2(\mathbb{Z})$*, then* $\left\{\psi^{(g_0^i,g_{-1}^i)}_{\frac{m}{M},kL} : g_0^i, g_{-1}^i\in l^2(\mathbb{Z})\right\}$ *need not be a DTWF for* $l^2(\mathbb{Z})$*.*

Let $r=2, L=2, M=4, \left\{g_0^1(n)\right\}=e_0, \left\{g_{-1}^1(n)\right\}=e_2, \left\{g_0^2(n)\right\}=e_1$ *and* $\left\{g_{-1}^2(n)\right\}=e_3$.
Then, for $i=1,2$, $\left\{\psi^{(g_0^i,g_{-1}^i)}_{\frac{m}{M},kL} : g_0^i, g_{-1}^i\in l^2(\mathbb{Z})\right\}$ *is not a DTWF for* $l^2(\mathbb{Z})$ *but* $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0=\sum_{i=1}^{2} g_0^i, g_{-1}=\sum_{i=1}^{2} g_{-1}^i\right\}$ *is DTWF for* $l^2(\mathbb{Z})$*.*

The following theorem establishes the necessary and sufficient conditions under which a discrete-time Wilson system, generated by a finite linear combination of discrete-time Wilson frame (DTWF) generators, is itself a discrete-time Wilson frame.

Theorem 3.1. *Let* $\left\{\psi^{(g_0^i,g_{-1}^i)}_{\frac{m}{M},kL} : g_0^i, g_{-1}^i\in l^2(\mathbb{Z})\right\}$, $i\in S_r=\{0,1,2,\cdots,r\}$ *be DTWF for each i for* $l^2(\mathbb{Z})$ *and* $\{\alpha_i\}_{i\in S_r}$ *be any scalars. Then* $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0=\sum_{i=1}^{r}\alpha_i g_0^i, g_{-1}=\sum_{i=1}^{r}\alpha_i g_{-1}^i\right\}$ *is a DTWF for* $l^2(\mathbb{Z})$ *if and only if there exists* $C>0$ *and some* $p\in S_r$ *such that*

$$C\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi^{(g_0^p,g_{-1}^p)}_{\frac{m}{M},kL}\right\rangle\right|^2 \le \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2, \text{ for all } f\in l^2(\mathbb{Z}). \tag{3.1}$$

Proof. Since $\left\{\psi^{(g_0^i,g_{-1}^i)}_{\frac{m}{M},kL} : g_0^i, g_{-1}^i\in l^2(\mathbb{Z})\right\}$, $i\in S_r$ is a DTWF for $l^2(\mathbb{Z})$, there exist $A_i, B_i>0$ such that

$$A_i\left\|f\right\|^2 \le \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi^{(g_0^i,g_{-1}^i)}_{\frac{m}{M},kL}\right\rangle\right|^2 \le B_i\left\|f\right\|^2, \text{ for all } f\in l^2(\mathbb{Z}). \tag{3.2}$$

Let us first assume that $\left\{\psi^{(g_0,g_{-1})}_{\frac{m}{M},kL} : g_0 = \sum_{i=1}^r \alpha_i g_0^i, g_{-1} = \sum_{i=1}^r \alpha_i g_{-1}^i\right\}$ is a DTWF sequence for $l^2(\mathbb{Z})$, therefore there exist $A, B > 0$ such that

$$A\left\|f\right\|^2 \leq \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 \leq B\left\|f\right\|^2 \text{ , for all } f \in l^2(\mathbb{Z}). \tag{3.3}$$

Choose $C = \dfrac{A}{B_i} > 0$. Using (3.2) and (3.3), we obtain that

$$C\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0^i,g_{-1}^i)}_{\frac{m}{M},kL}\right\rangle\right|^2 \leq \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2 \text{ , for all } f \in l^2(\mathbb{Z}).$$

Conversely, suppose that there exists $C > 0$ and $p \in S_r$ such that (3.1) holds. Again as $\left\{\psi^{(g_0^i,g_{-1}^i)}_{\frac{m}{M},kL} : g_0^i, g_{-1}^i \in l^2(\mathbb{Z})\right\}$, $i \in S_r = \{0, 1, 2, \cdots, r\}$ is a DTWF for each i, using (3.1) and (3.2) for all $f \in l^2(\mathbb{Z})$, we have

$$\begin{aligned}
CA_p\left\|f\right\|^2 &\leq \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi^{(g_0,g_{-1})}_{\frac{m}{M},kL}\right\rangle\right|^2\\
&= 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}g_0(\cdot)\right\rangle\right|^2\\
&\quad + 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \sin\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}g_{-1}(\cdot)\right\rangle\right|^2\\
&\quad - 2\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)g_0(\cdot)\right\rangle\right|^2\\
&\leq 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}\Big(\sum_{i=1}^r \alpha_i g_0^i\Big)(\cdot)\right\rangle\right|^2\\
&\quad + 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \sin\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}\Big(\sum_{i=1}^r \alpha_i g_{-1}^i\Big)(\cdot)\right\rangle\right|^2
\end{aligned}$$

$$\begin{aligned}
&\leq 4r\sum_{i=1}^r|\alpha_i|^2\Bigg[\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}g_0^i(\cdot)\right\rangle\right|^2\\
&\quad + \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \sin\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}g_{-1}^i(\cdot)\right\rangle\right|^2\Bigg]\\
&\leq 4rP\sum_{i=1}^r\Bigg[\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \cos\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}g_0^i(\cdot)\right\rangle\right|^2\\
&\quad + \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \sin\Big(\frac{2\pi m(\cdot)}{M}\Big)T_{kL}g_{-1}^i(\cdot)\right\rangle\right|^2\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= rP\sum_{i=1}^{r}\Bigg[\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi_{\frac{m}{M},kL}^{(g_0^i,g_{-1}^i)}\right\rangle\right|^2 \\
&\qquad + 2\sum_{m=0}^{M-1}\left|\left\langle f,\cos\Big(\frac{2\pi m(\cdot)}{M}\Big)g_0^i(\cdot)\right\rangle\right|^2\Bigg] \\
&\le rP\sum_{i=1}^{r}\left(B_i+2M\left\|g_0^i\right\|^2\right)\left\|f\right\|^2 ,
\end{aligned}
$$

where $P=\max\limits_{1\le i\le r}|\alpha_i|^2$. Thus, $\left\{\psi_{\frac{m}{M},kL}^{(g_0,g_{-1})}: g_0=\sum_{i=1}^{r}\alpha_i g_0^i, g_{-1}=\sum_{i=1}^{r}\alpha_i g_{-1}^i\right\}$ is a DTWF for $l^2(\mathbb{Z})$. □

In the next theorem, we derive another sufficient condition for the system $\left\{\psi_{\frac{m}{M},kL}^{(g_0,g_{-1})}: g_0, g_{-1}\in l^2(\mathbb{Z})\right\}$ to be a DTWF for $l^2(\mathbb{Z})$.

Theorem 3.2. *Let* $\left\{\psi_{\frac{m}{M},kL}^{(g_0^i,g_{-1}^i)}: g_0^i, g_{-1}^i\in l^2(\mathbb{Z})\right\}, i\in S_r=\{0,1,2,\cdots,r\}$ *be DTWF for* $l^2(\mathbb{Z})$ *with frame bounds* A_i *and* B_i *respectively. Let* $\{\alpha_i\}_{i\in S_r}$ *be any positive scalars. If for some* $p\in S_r$

$$
\alpha_p^2A_p>2\Bigg[\sum_{\substack{i=1\\ i\neq p}}^{r}\alpha_i^2B_i+\sum_{\substack{i,j=1\\ i\neq j;i,j\neq p}}^{r}\alpha_i\alpha_j\sqrt{B_iB_j}\Bigg], \tag{3.4}
$$

then $\left\{\psi_{\frac{m}{M},kL}^{(g_0,g_{-1})}: g_0=\sum_{i=1}^{r}\alpha_i g_0^i, g_{-1}=\sum_{i=1}^{r}\alpha_i g_{-1}^i\right\}$ *is a DTWF for* $l^2(\mathbb{Z})$.

Proof. Since $\left\{\psi_{\frac{m}{M},kL}^{(g_0^i,g_{-1}^i)}: g_0^i, g_{-1}^i\in l^2(\mathbb{Z})\right\}, i\in S_r$ is a DTWF for $l^2(\mathbb{Z})$ with frame bounds A_i and B_i, we have

$$
A_i\left\|f\right\|^2\le\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi_{\frac{m}{M},kL}^{(g_0^i,g_{-1}^i)}\right\rangle\right|^2\le B_i\left\|f\right\|^2 ,\ \text{ for all } f\in l^2(\mathbb{Z}). \tag{3.5}
$$

In view of Theorem 3.1, it suffices to show that $\left\{\psi_{\frac{m}{M},kL}^{(g_0,g_{-1})}: g_0=\sum_{i=1}^{r}\alpha_i g_0^i, g_{-1}=\sum_{i=1}^{r}\alpha_i g_{-1}^i\right\}$ is a DTWF by finding a $C>0$ satisfying (3.1) for some $p\in S_r$.

For all $f\in l^2(\mathbb{Z})$, we write

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi_{\frac{m}{M},kL}^{(g_0,g_{-1})}-\alpha_p\psi_{\frac{m}{M},kL}^{(g_0^p,g_{-1}^p)}\right\rangle\right|^2
&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sum_{i=1}^{r}\alpha_i\psi_{\frac{m}{M},kL}^{(g_0^i,g_{-1}^i)}-\alpha_p\psi_{\frac{m}{M},kL}^{(g_0^p,g_{-1}^p)}\right\rangle\right|^2 \\
&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sum_{\substack{i=1\\ i\neq p}}^{r}\alpha_i\psi_{\frac{m}{M},kL}^{(g_0^i,g_{-1}^i)}\right\rangle\right|^2 \\
&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\sum_{\substack{i=1\\ i\neq p}}^{r}\left\langle f,\alpha_i\psi_{\frac{m}{M},kL}^{(g_0^i,g_{-1}^i)}\right\rangle\right|^2 \\
&\le\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\Bigg[\sum_{\substack{i=1\\ i\neq p}}^{r}\alpha_i^2\left|\left\langle f,\psi_{\frac{m}{M},kL}^{(g_0^i,g_{-1}^i)}\right\rangle\right|^2
\end{aligned}
$$

$$+\sum_{\substack{i,j=1\\ i\neq j;i,j\neq p}}^{r}\alpha_i\alpha_j\left|\left\langle f,\psi_{\frac{m}{M},kL}^{(g_0^i,g_{-1}^i)}\right\rangle\right|\left|\left\langle f,\psi_{\frac{m}{M},kL}{}^{(g_0^j,g_{-1}^j)}\right\rangle\right|\Bigg].$$

Using (3.5) and Cauchy-Schwarz inequality, we obtain that

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi_{\frac{m}{M},kL}^{(g_0,g_{-1})}-\alpha_p\psi_{\frac{m}{M},kL}^{(g_0^p,g_{-1}^p)}\right\rangle\right|^2\leq\left[\sum_{\substack{i=1\\ i\neq p}}^{r}\alpha_i^2B_i+\sum_{\substack{i,j=1\\ i\neq j;i,j\neq p}}^{r}\alpha_i\alpha_j\sqrt{B_iB_j}\right]\|f\|^2\,. \tag{3.6}$$

Now using (3.6)

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi_{\frac{m}{M},kL}^{(g_0^p,g_{-1}^p)}\right\rangle\right|^2&\leq\frac{2}{\alpha_p^2}\left[\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sum_{\substack{i=1\\ i\neq p}}^{r}\alpha_i\psi_{\frac{m}{M},kL}^{(g_0^i,g_{-1}^i)}\right\rangle\right|^2\right.\\&\qquad\left.+\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sum_{i=1}^{r}\alpha_i\psi_{\frac{m}{M},kL}^{(g_0^i,g_{-1}^i)}\right\rangle\right|^2\right]\\&\leq\frac{2}{\alpha_p^2}\left[\sum_{\substack{i=1\\ i\neq p}}^{r}\alpha_i^2B_i+\sum_{\substack{i,j=1\\ i\neq j;i,j\neq p}}^{r}\alpha_i\alpha_j\sqrt{B_iB_j}\right]\|f\|^2\\&\qquad+\frac{2}{\alpha_p^2}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\psi_{\frac{m}{M},kL}^{(g_0,g_{-1})}\right\rangle\right|^2\end{aligned}$$

Choosing $C=\dfrac{\alpha_p^2}{2}-\dfrac{1}{A_p}\left[\sum_{\substack{i=1\\ i\neq p}}^{r}\alpha_i^2B_i+\sum_{\substack{i,j=1\\ i\neq j;i,j\neq p}}^{r}\alpha_i\alpha_j\sqrt{B_iB_j}\right]>0$, we have (3.1) holds. □

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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