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On the Existence of Disjoint Infinite Solution Sets for a Critical Hardy-Sobolev-Maz'ya Equation

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Abstract. In this work, we study the following Hardy–Sobolev–Maz'ya equation involving critical growth:

$$\begin{cases} -\Delta u - \lambda \dfrac{u}{|y|^2} = \dfrac{|u|^{2^*(s)-2}u}{|y|^s} + \mu |u|^{q-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain containing a point $x^0 = (0, z^0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, with $2 \le k < N$, $x = (y, z)$, $0 \le s < 2$, and $2^*(s) = \dfrac{2(N-s)}{N-2}$. We assume $0 \le \lambda < \overline{\lambda} := \dfrac{(k-2)^2}{4}$ for $k > 2$, and $\lambda = 0$ when $k = 2$, with parameters satisfying $1 < q < 2$, $\mu > 0$, and $N > \dfrac{s+2+2q}{q-1}$. Using an approximating argument, local Pohozaev-type identities, and variational methods, including the Fountain Theorem and its dual version, we establish the existence of two disjoint and infinite sets of solutions under these assumptions.

1. INTRODUCTION

We are interested in the following Hardy-Sobolev-Maz'ya problem

$$\begin{cases} -\Delta u - \lambda \frac{u}{|y|^2} = \frac{|u|^{2^*(s)-2}u}{|y|^s} + \mu |u|^{q-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ that contains some points $x^0 = \left(0, z^0\right)$, $0 \le s < 2$, $1 < q < 2$, $\mu > 0$ and $2^*(s) := \frac{2(N-s)}{N-2}$ is the Hardy-Sobolev exponent. The energy functional

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corresponding to problem (1.1) is defined as follows

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega \frac{u^2}{|y|^2}dx - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|y|^s}dx - \frac{\mu}{q}\int_\Omega |u|^q dx,$$

for all $u \in H_0^1(\Omega)$.

Prior to presenting our main results, we will briefly review the literature related to problem (1.1). Devillanova and Solimini, [5], proved that if $N \geq 7$, the problem (1.1) when $q = 2$, $s = 0$ and $\lambda = 0$ has infinitely many solutions. Their key approach involves proving the strong convergence of approximate solutions to (1.1). In order to accomplish this objective, the authors obtained estimates for approximating solutions of (1.1) within a carefully defined safe region. Subsequently, they employed a local Pohozaev identity get the result. In [4], Cao and Yan established the existence of infinitely many solutions for (1.1) with $\mu > 0$, $s = 0$ and $q = 2$ if $N \geq 7$ and $0 \leq \lambda < \frac{(N-2)^2}{4} - 4$.

Building upon similar ideas as those [4, 5], the authors in [10] established the existence of infinitely many solutions to the problem (1.1) if $N > 6 + s$ and certain conditions on λ with $\mu > 0$ and $q = 2$. For further related results, interested readers are referred to [3, 7] and the references therein.

However, unlike to [4, 10], our work faces technical challenges due to the absence of a reverse Hölder inequality when $q < 2$, complicating the application of the Moser iteration. The significant result of this work relies on Lemma 2.2, which enabled us to overcome these difficulties. Consequently, this work can be regarded as the extension of the aforementioned results in the case of Hardy-Sobolev-Maz'ya equations. To the best of our knowledge, there are no similar results concerning the problem (1.1). The main result of this paper is the following

Theorem 1.1. *If* $N > \frac{s+2+2q}{q-1}$, $1 < q < 2$ *and* $\lambda \in [0, \overline{\lambda} - 4)$, *then there exists a sequence of solutions* $(v_k)_k$ *of the problem* (1.1) *such that* $I(v_k) < 0$ *and* $I(v_k) \to 0$ *as* $k \to +\infty$.

Under the conditions of Theorem 1.1, we also prove

Theorem 1.2. *There exists a sequence of solutions* $(u_k)_k$ *of the problem* (1.1) *such that* $I(u_k) > 0$ *and* $I(u_k) \to +\infty$ *as* $k \to +\infty$.

This paper is organized as follows. Section 2 is devoted to the strong convergence of approximating solutions in $H_0^1(\Omega)$ of the problem (1.1). In Section 3, we establish the main result by applying the Fountain theorem and its dual form. We would also like to note that the Lusternik-Schnirelman theory used in [1] does not seem applicable to our case.

At the end of this section, we establish some notations that will be used in the sequel. We denote the norm in the space $L^p(\Omega)$ for $1 \leq p < \infty$ by

$$\|u\|_p := \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}},$$

and the norm in the Sobolev space $H_0^1(\Omega)$ by

$$||u|| := \left(\int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

2. Strong convergence of approximating solutions in $H_0^1(\Omega)$.

We consider the following perturbed problem:

$$\begin{cases} -\Delta u - \lambda \frac{u}{|y|^2} = \frac{|u|^{2^*(s)-2-\varepsilon}u}{|y|^s} + \mu |u|^{q-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where ε is a small positive constant. The energy functional corresponding to problem (2.1) is defined by

$$I_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega \frac{u^2}{|y|^2}dx - \frac{1}{2^*(s)-\varepsilon}\int_\Omega \frac{|u|^{2^*(s)-\varepsilon}}{|y|^s}dx - \frac{\mu}{q}\int_\Omega |u|^q dx,$$

for all $u \in H_0^1(\Omega)$. Let's introduce some notations and terminologies that will be used later on. Let u be a solution of (2.1). Define $\tilde{u} := |u|$ (extended by zero out of Ω), by a direct computation we get that

$$\int_{\mathbb{R}^N} \left(\nabla\tilde{u}\nabla\phi - \frac{\lambda}{|y|^2}\tilde{u}\phi \right) dx \leq \int_{\mathbb{R}^N} \frac{\left(2\tilde{u}^{2^*(s)-1} + A\right)}{|y|^s}\phi dx, \quad \forall\phi \in H^1\left(\mathbb{R}^N\right), \phi \geq 0, \tag{2.2}$$

where $A(\mu, N)$ is a positive constant independent of ε. Therefore, From now on we can only consider the estimates of solutions to (2.1) in $H^1\left(\mathbb{R}^N\right)$, which also relieves us from considering the sign of u and the bounded domain Ω.

Definition 2.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a given sequence. We say that $(u_n)_{n\in\mathbb{N}}$ is a controlled sequence if each u_n is a solution to problem (2.2).*

For any $\lambda > 0$ and $x \in \mathbb{R}^N$, we define

$$\rho_{x,\lambda}(u) := \lambda^{\frac{N}{2^*}} u\left(\lambda(. - x)\right), u \in H_0^1(\Omega).$$

We have the following result.

Proposition 2.1. *[10, Proposition C.1] Suppose that $N \geq 3$. Let u_n be a solution of (2.1) with $\varepsilon = \varepsilon_n \to 0$, satisfying $||u_n|| \leq C$ for some constant C. Then when $s = 0$ and $\lambda > 0$,*

(i) u_n can be decomposed as

$$u_n = u_0 + \sum_{j=1}^{m} \rho_{x_{n,j},\Lambda_{n,j}}\left(U_j\right) + \sum_{j=m+1}^{h} \rho_{x_{n,j},\Lambda_{n,j}}\left(U_j\right) + \omega_n, \tag{2.3}$$

where $\omega_n \to 0$ in $H^1(\Omega)$, u_0 is a solution for (1.1). *For $j = 1, 2, \ldots, m$, $\Lambda_{n,j} \to \infty$ as $n \to \infty$, and U_j is a solution of*

$$-\Delta u - \lambda \frac{u}{|y|^2} = b_j \frac{|u|^{2^*(s)-2}u}{|y|^s}, \quad u \in D^{1,2}\left(\mathbb{R}^N\right),$$

for some $b_j \in (0,1]$. *For* $j = m+1, m+2, \dots, h, x_{n,j} \in \Omega, \Lambda_{n,j} d\left(x_{n,j}, \partial\Omega\right) \to \infty, \Lambda_{n,j}|x_{n,j}| \to \infty$ *as* $n \to \infty$, *and* U_j *is a solution of*

$$-\Delta u = b_j |u|^{2^*-2} u, \quad u \in D^{1,2}\left(\mathbb{R}^N\right),$$

for some $b_j \in (0,1]$.

(ii) Set $x_{n,i} = (0, z_{n,i})$ *for* $i = 1, 2, \dots, m$. *For* $i, j = 1, 2, \dots, h$, *if* $i \neq j$, *then as* $n \to \infty$

$$\frac{\Lambda_{n,j}}{\Lambda_{n,i}} + \frac{\Lambda_{n,i}}{\Lambda_{n,j}} + \Lambda_{n,j}\Lambda_{n,i}|x_{n,i} - x_{n,j}|^2 \to \infty,$$

when $t > 0$ *or* $\lambda = 0$, *the same conclusion holds with* $\rho_{x_{n,j},\Lambda_{n,j}} = 0$ *for all* $j = m+1, \dots, h$.

We call $(u_n)_{n\in\mathbb{N}}$ a concentrating sequence if the limit in (2.3) holds in the $H_0^1(\Omega)$ -strong topology. Assume $\|u_n\| \le M$, then from (2.3), we get $k \le C(M)$. Among all the bubbles in (2.3), we can choose a slowest concentration rate, denoted by Λ_n, which concentrates in $x_n = x_{n,i}$ in the slowest way. We may always choose a constant $\bar{C} > 0$ such that the region

$$\mathcal{A}_n^1 := \left(B_{(\bar{C}+5)\Lambda_n^{-\frac{1}{2}}}(x_n) \setminus B_{\bar{C}\Lambda_n^{-\frac{1}{2}}}(x_n)\right) \cap \Omega$$

does not contain any concentration point of u_n for every n. We call this region a safe region for u_n. We consider two thinner subsets as follows

$$\mathcal{A}_n^2 := \left(B_{(\bar{C}+4)\Lambda_n^{-\frac{1}{2}}}(x_n) \setminus B_{(\bar{C}+1)\Lambda_n^{-\frac{1}{2}}}(x_n)\right) \cap \Omega,$$

and

$$\mathcal{A}_n^3 := \left(B_{(\bar{C}+3)\Lambda_n^{-\frac{1}{2}}}(x_n) \setminus B_{(\bar{C}+2)\Lambda_n^{-\frac{1}{2}}}(x_n)\right) \cap \Omega.$$

Lemma 2.1. *[10, Lemma 3.2] Let* (u_n) *be a controlled sequence. Then there is a constant* $C > 0$ *independent of* n, *such that*

$$\left(r^{s-N} \int_{B_r(\hat{x})\cap\Omega} \frac{|u_n|^\tau}{|y|^s} dx\right)^{\frac{1}{\tau}} \le C\Lambda^{\frac{N-s}{2p_1}}, \quad \forall \hat{x} \in \mathbb{R}^N,$$

for all $r \in \left[\bar{C}\Lambda_n^{-\frac{1}{2}}, (\bar{C}+5)\Lambda_n^{-\frac{1}{2}}\right]$, *where* $\tau = \frac{2(N-s)}{2N-s-2}$ *and* $p_1 > 2^*(s)$ *is any constant satisfying* $p_1 < \frac{2(k-s)}{k-2-2\sqrt{\bar{\lambda}-\lambda}}$.

Lemma 2.2. *Let* $(u_n)_{n\in\mathbb{N}}$ *be a controlled sequence. Then there is a positive constant* C *independent of* n *such that*

$$\left(\int_{\mathcal{A}_n^2} \frac{|u_n|^{2\beta^2}}{|y|^s} dx\right)^{\frac{1}{2\beta^2}} \le C\Lambda_n^{\frac{N-s}{2p_1} - \frac{N-s}{4\beta^2}},$$

where $\beta := \frac{2^*(s)}{2}$.

Proof. Define

$$v_n(x) := |u_n|\left(\Lambda_n^{-1/2} x\right), x \in \Omega_n,$$

where $\Omega_n := \left\{x : \Lambda_n^{-1/2} x \in \Omega\right\}$. Using the inequality (2.2), it is easy to check that v_n (extended by zero out of Ω) satisfies

$$\int_{\mathbb{R}^N}\left(\nabla v_n \nabla \phi - \frac{\lambda}{|y|^2} v_n \phi\right) dx \le \Lambda_n^{\frac{s-2}{2}} \int_{\mathbb{R}^N} \frac{\left(2 v_n^{2^*(s)-1} + A\right)}{|y|^s} \phi dx.$$

For fixed $y \in \mathcal{A}_n^2$ and $0 < r < R \le 1$, we set $z_n := \Lambda_n^{1/2} y$ and let $\chi \in C_0^\infty\left(B\left(z_n, R\right)\right)$ be a cut-off function with $0 \le \chi \le 1$, $\chi = 1$ on $B(z_n, r)$ and $|\nabla \chi| \le \frac{1}{R-r}$. For every $t > 1$ and $M > 0$, we define

$$\varphi = \chi^2 v_n \widetilde{v}_n^{2(t-1)},$$

where $\widetilde{v}_n = \min\{v_n, M\}$. In what follows C denotes several positive constants independent of n, r, R, t and M.

Note that $\nabla v_n \nabla \widetilde{v}_n = |\nabla \widetilde{v}_n|^2$ and $\nabla v_n \nabla \widetilde{v}_n v_n \widetilde{v}_n^{2(t-1)-1} = |\nabla \widetilde{v}_n|^2 \widetilde{v}_n^{2(t-1)}$.

We have

$$\begin{aligned}
\int_{\mathbb{R}^N} \nabla v_n \nabla \varphi dx &= \int_{\mathbb{R}^N} \nabla v_n \cdot \left[2\chi \nabla \chi v_n \widetilde{v}_n^{2(t-1)} + \chi^2 \widetilde{v}_n^{2(t-1)} \nabla v_n + 2(t-1)\chi^2 v_n \widetilde{v}_n^{2(t-1)-1} \nabla \widetilde{v}_n\right] dx \\
&= \int_{\mathbb{R}^N} \chi^2 |\nabla v_n|^2 \widetilde{v}_n^{2(t-1)} dx + 2(t-1) \int_{\mathbb{R}^N} \chi^2 |\nabla \widetilde{v}_n|^2 \widetilde{v}_n^{2(t-1)} dx + 2 \int_{\mathbb{R}^N} \chi v_n \widetilde{v}_n^{2(t-1)} \nabla \chi \nabla v_n dx \\
&\le \int_{\mathbb{R}^N} \left(\frac{\lambda v_n}{|y|^2} + \Lambda_n^{\frac{s}{2}-1} \frac{2 v_n^{2^*(s)-1} + A}{|y|^s}\right) \chi^2 v_n \widetilde{v}_n^{2(t-1)} dx.
\end{aligned}$$

By the Young's inequality, we have

$$|(\chi \nabla v_n) \cdot (v_n \nabla \chi)| \le |\nabla \chi|^2 v_n^2 + \frac{1}{4} \chi^2 |\nabla v_n|^2.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \chi^2 |\nabla v_n|^2 \widetilde{v}_n^{2(t-1)} dx + 2(t-1) \int_{\mathbb{R}^N} \chi^2 |\nabla \widetilde{v}_n|^2 \widetilde{v}_n^{2(t-1)} dx \\
&\le \int_{\mathbb{R}^N} \left(\frac{\lambda v_n}{|y|^2} + \Lambda_n^{\frac{s}{2}-1} \frac{2 v_n^{2^*(s)-1} + A}{|y|^s}\right) \chi^2 v_n \widetilde{v}_n^{2(t-1)} dx + 2 \int_{\mathbb{R}^N} \widetilde{v}_n^{2(t-1)} \left[|\nabla \chi|^2 v_n^2 + \frac{1}{4} \chi^2 |\nabla v_n|^2\right] dx,
\end{aligned}$$

and it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \chi^2 |\nabla v_n|^2 \widetilde{v}_n^{2(t-1)} dx + 4(t-1) \int_{\mathbb{R}^N} \chi^2 |\nabla \widetilde{v}_n|^2 \widetilde{v}_n^{2(t-1)} dx \\
&\quad \le 2 \int_{\mathbb{R}^N} \left(\frac{\lambda v_n}{|y|^2} + \Lambda_n^{\frac{s}{2}-1} \frac{2 v_n^{2^*(s)-1} + A}{|y|^s}\right) \chi^2 v_n \widetilde{v}_n^{2(t-1)} dx + 4 \int_{\mathbb{R}^N} \widetilde{v}_n^{2(t-1)} |\nabla \chi|^2 v_n^2 dx.
\end{aligned} \tag{2.4}$$

Now we take $t = \frac{2^*(s)}{2} > 1$. Consider $\Psi_M := \chi v_n \widetilde{v}_n^{t-1}$, we have

$$\begin{aligned}
\nabla \Psi_M &= \nabla \chi v_n \widetilde{v}_n^{(t-1)} + \chi \nabla v_n \widetilde{v}_n^{(t-1)} + (t-1) \chi v_n \widetilde{v}_n^{(t-2)} \nabla \widetilde{v}_n \\
&= \chi \nabla v_n \widetilde{v}_n^{(t-1)} + (t-1) \chi \widetilde{v}_n^{(t-1)} \nabla \widetilde{v}_n + \nabla \chi v_n \widetilde{v}_n^{(t-1)},
\end{aligned}$$

since $t > 1$,

$$\begin{aligned}|\nabla\Psi_M|^2 &\le C\left[\chi^2|\nabla v_n|^2\widetilde{v}_n^{2(t-1)} + (t-1)^2\chi^2\widetilde{v}_n^{2(t-1)}|\nabla\widetilde{v}_n|^2 + \widetilde{v}_n^{2(t-1)}|\nabla\chi|^2{v_n}^2\right]\\ &\le Ct\left[\chi^2|\nabla v_n|^2\widetilde{v}_n^{2(t-1)} + 4(t-1)\chi^2\widetilde{v}_n^{2(t-1)}|\nabla\widetilde{v}_n|^2 + \widetilde{v}_n^{2(t-1)}|\nabla\chi|^2{v_n}^2\right].\end{aligned} \tag{2.5}$$

Formula (2.4) and (2.5) yields that

$$\int_{\mathbb{R}^N}|\nabla\Psi_M|^2dx \le\Lambda_n^{\frac{s}{2}-1}\int_{\mathbb{R}^N}\frac{v_n^{2^*(s)}\chi^2\widetilde{v}_n^{2(t-1)}}{|y|^s}dx + A\Lambda_n^{\frac{s}{2}-1}\int_{\mathbb{R}^N}\frac{\chi^2 v_n\widetilde{v}_n^{2(t-1)}}{|y|^s} + \int_{\mathbb{R}^N}\widetilde{v}_n^{2(t-1)}|\nabla\chi|^2v_n^2dx].$$

By the Hardy embedding theorem, we have

$$\left(\int_{\mathbb{R}^N}\frac{(\Psi_M)^{2^*(s)}}{|y|^s}dx\right)^{\frac{2}{2^*(s)}} \le\Lambda_n^{\frac{s}{2}-1}\int_{\mathbb{R}^N}\frac{v_n^{2^*(s)}\chi^2\widetilde{v}_n^{2(t-1)}}{|y|^s}dx + A\Lambda_n^{\frac{s}{2}-1}\int_{\mathbb{R}^N}\frac{\chi^2 v_n\widetilde{v}_n^{2(t-1)}}{|y|^s} + \int_{\mathbb{R}^N}\widetilde{v}_n^{2(t-1)}|\nabla\chi|^2v_n^2dx].$$

By Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{v_n^{2^*(s)}\chi^2\widetilde{v}_n^{2(t-1)}}{|y|^s}dx &= \int_{\mathbb{R}^N}\frac{v_n^{2^*(s)-2}\chi^2v_n^2\widetilde{v}_n^{2(t-1)}}{|y|^s}dx\\ &\le\left[\int_{\mathcal{B}(z_n,R)}\frac{v_n^{2^*(s)}}{|y|^s}dx\right]^{\frac{2^*(s)-2}{2^*(s)}}\left[\int_{\mathbb{R}^N}\frac{\left(\chi v_n\widetilde{v}_n^{t-1}\right)^{2^*(s)}}{|y|^s}dx\right]^{\frac{2}{2^*(s)}}\\ &=\left[\int_{\mathcal{B}(z_n,R)}\frac{v_n^{2^*(s)}}{|y|^s}dx\right]^{\frac{2^*(s)-2}{2^*(s)}}\left[\int_{\mathbb{R}^N}\frac{(\Psi_M)^{2^*(s)}}{|y|^s}dx\right]^{\frac{2}{2^*(s)}}.\end{aligned}$$

Since $\widetilde{v}_n \le v_n$, we have

$$\int_{\mathbb{R}^N}\widetilde{v}_n^{2(t-1)}|\nabla\chi|^2v_n^2dx \le \int_{\mathbb{R}^N}v_n^{2^*}|\nabla\chi|^2dx,$$

and

$$\int_{\mathbb{R}^N}\chi^2v_n\widetilde{v}_n^{2(t-1)}dx \le \int_{\mathbb{R}^N}\chi^2v_n^{2^*-1} \le \int_{\mathcal{B}(z_n,R)}v_n^{2^*-1}.$$

So

$$\begin{aligned}\left(\int_{\mathbb{R}^N}\frac{(\Psi_M)^{2^*(s)}}{|y|^s}dx\right)^{\frac{2}{2^*(s)}} \le& C\Lambda_n^{\frac{s}{2}-1}\left[\int_{\mathcal{B}(z_n,R)}\frac{v_n^{2^*(s)}}{|y|^s}dx\right]^{\frac{2-s}{N-s}}\left(\int_{\mathbb{R}^N}\frac{(\Psi_M)^{2^*(s)}}{|y|^s}dx\right)^{\frac{2}{2^*}}\\ &+ C\Lambda_n^{\frac{s}{2}-1}\int_{\mathbb{R}^N}\frac{\chi^2v_n^{2^*(s)-1}}{|y|^s}dx + C\int_{\mathbb{R}^N}v_n^{2^*(s)}|\nabla\chi|^2dx.\end{aligned}$$

Since $\mathcal{B}\left(y,\Lambda_n^{-1/2}\right)\subset\mathcal{A}_n^1$, and $\mathcal{A}_n^1$ does not contain any concentration point of v_n, we can deduce that

$$\begin{aligned}\Lambda_n^{\frac{s}{2}-1}\left[\int_{\mathcal{B}(z_n,R)}\frac{v_n^{2^*(s)}}{|y|^s}dx\right]^{\frac{2-s}{N-s}} &\le \Lambda_n^{\frac{s}{2}-1}\left[\int_{\mathcal{B}(z_n,1)}\frac{v_n^{2^*(s)}}{|y|^s}dx\right]^{\frac{2-s}{N-s}}\\ &= \left[\int_{\mathcal{B}(y,\Lambda_n^{\frac{-1}{2}})}\frac{|u_n|^{2^*(s)}}{|y|^s}dx\right]^{\frac{2-s}{N-s}} \to 0,\end{aligned}$$

as $n \to +\infty$. It follows that

$$\left(\int_{\mathbb{R}^N} \frac{(\Psi_M)^{2^*(s)}}{|y|^s} dx\right)^{\frac{2}{2^*(s)}} \leq C\int_{\mathbb{R}^N} \frac{\chi^2 v_n^{2^*(s)-1}}{|y|^s} dx + C\int_{\mathbb{R}^N} v_n^{2^*(s)}|\nabla\chi|^2 dx.$$

Let M go to infinity, we obtain that

$$\left(\int_{\mathcal{B}(z_n,r)} \frac{v_n^{2^*(s)t}}{|y|^s} dx\right)^{\frac{2}{2^*(s)}} \leq C\int_{\mathbb{R}^N} \frac{\chi^2 v_n^{2^*(s)-1}}{|y|^s} dx + C\int_{\mathbb{R}^N} v_n^{2^*(s)}|\nabla\chi|^2 dx. \tag{2.6}$$

It then follows from (2.6) and Young inequality that

$$\begin{aligned}\left(\int_{\mathcal{B}(z_n,r)} \frac{v_n^{t2^*(s)}}{|y|^s} dx\right)^{\frac{1}{t2^*(s)}} &\leq \frac{C}{(R-r)^{\frac{1}{t}}}\left[\int_{\mathcal{B}(z_n,R)} v_n^{2^*(s)} dx\right]^{\frac{1}{2t}} + C\left[\int_{\mathcal{B}(z_n,R)} \frac{v_n^{2^*(s)-1}}{|y|^s} dx\right]^{\frac{1}{2t}}\\ &\leq C\left[\frac{1}{(R-r)^{\frac{1}{t}}}+1\right]\left[\int_{\mathcal{B}(z_n,R)} \frac{v_n^{2^*(s)}}{|y|^s} dx\right]^{\frac{1}{2^*(s)}} + C\\ &< \infty.\end{aligned}$$

By applying the interpolation inequality and Young's inequality, we obtain

$$\begin{aligned}\left(\int_{\mathcal{B}(z_n,r)} \frac{v_n^{t2^*(s)}}{|y|^s} dx\right)^{\frac{1}{2^*t}} &\leq C\left[\frac{1}{(R-r)^{\frac{1}{t}}}+1\right]\|v_n\|^k_{L_s^\tau(\mathcal{B}(z_n,R))} \times \|v_n\|^{1-k}_{L_s^{2^*(s)t}(\mathcal{B}(z_n,R))} + C\\ &\leq \frac{1}{2}\|v_n\|_{L_s^{2^*(s)t}(\mathcal{B}(z_n,R))} + C\left[\frac{1}{(R-r)^{\frac{1}{t}}}+1\right]\|v_n\|_{L_s^\tau(\mathcal{B}(z_n,R))} + C,\end{aligned}$$

where the number k is given by $\frac{1}{2^*} = \frac{k}{\tau} + \frac{1-k}{t2^*}$. Using iteration argument, we set $r_i = R - \frac{R-r}{i+1}$, $i \in \mathbb{N}$, we obtain

$$\begin{aligned}\left(\int_{\mathcal{B}(z_n,r_0)} \frac{v_n^{t2^*(s)}}{|y|^s} dx\right)^{\frac{1}{2^*t}} \leq& \frac{1}{2^i}\|v_n\|_{L_s^{2^*(s)ts}(\mathcal{B}(z_n,r_i))} + C\sum_{j=1}^{i}\frac{1}{2^{j-1}}\left[\frac{1}{\left(r_j - r_{j-1}\right)^{\frac{1}{t}}}+1\right]\|v_n\|_{L_s^\tau\left(\mathcal{B}(z_n,r_j)\right)}\\ &+ C\sum_{j=1}^{i}\frac{1}{2^{j-1}}.\end{aligned} \tag{2.7}$$

It is easy to see that

$$\sum_{i=1}^{+\infty}\frac{1}{2^{i-1}}\left[\frac{1}{\left(r_i - r_{i-1}\right)^{\frac{1}{t}}}+1\right] < \infty \quad \textit{and} \quad \sum_{j=1}^{+\infty}\frac{1}{2^{j-1}} < \infty.$$

Letting i go to infinity in (2.7), we obtain then that

$$\left(\int_{\mathcal{B}(z_n,r)} \frac{v_n^{t2^*(s)}}{|y|^s} dx\right)^{\frac{1}{2^*t}} \leq C\|v_n\|_{L_s^\tau(\mathcal{B}(z_n,1))} + C. \tag{2.8}$$

Letting $r \to \frac{1}{2}$ in (2.8), we infer that

$$\left(\int_{\mathcal{B}(z_n,\frac{1}{2})} \frac{v_n^{t2^*(s)}}{|y|^s} dx\right)^{\frac{1}{2^*s}} \leq C \|v_n\|_{L_s^\tau(\mathcal{B}(z_n,1))} + C. \tag{2.9}$$

On the other hand, we have from $B_{\Lambda_n^{-\frac{1}{2}}}(y) \subset \mathcal{A}_n^1$ and Lemma 2.1

$$\|v_n\|_{L_s^\tau(\mathcal{B}(z_n,1))} \leq C\Lambda_n^{\frac{N-s}{2p_1}}.$$

Together with (2.9), this implies that

$$\left(\int_{\mathcal{B}(y,\frac{1}{2}\Lambda_n^{\frac{-1}{2}})} \frac{|u_n|^{2\beta^2}}{|y|^s} dx\right)^{\frac{1}{2\beta^2}} \leq C\Lambda_n^{\frac{N-s}{2p_1}-\frac{N-s}{4\beta^2}}.$$

□

As a consequence of the previous Lemma we have the following estimates which play a crucial role in the proof of the main results.

Proposition 2.2. *Let u_n be a weak solution of (2.1) with $\varepsilon = \varepsilon_n \to 0$. Then there is a constant $C > 0$ independent of n, such that*

$$\left(\int_{\mathcal{A}_n^2} \frac{|u_n|^p}{|y|^s} dx\right)^{\frac{1}{p}} \leq C\Lambda_n^{\frac{N-s}{2p_1}-\frac{N-s}{2p}},$$

where $p_1 > 2^*(s)$ is any constants, satisfying $p_1 < \frac{2(k-s)}{k-2-2\sqrt{\bar{\lambda}-\lambda}}$, and $1 \leq p \leq 2^*(s)$.

Proof. Using Hölder's inequality we have

$$\left(\int_{\mathcal{A}_n^2} \frac{|u_n|^p}{|y|^s}\right)^{\frac{1}{p}} \leq \left(\int_{\mathcal{A}_n^2} \frac{|u_n|^{2\beta^2}}{|y|^s}\right)^{\frac{1}{2\beta^2}} \Lambda_n^{\left(-\frac{N-s}{2p}\right)\left(1-\frac{p}{2\beta^2}\right)},$$

using Lemma 2.2, we get

$$\left(\int_{\mathcal{A}_n^2} \frac{|u_n|^p}{|y|^s}\right)^{\frac{1}{p}} \leq C\Lambda_n^{\left(-\frac{N-s}{2p}\right)\left(1-\frac{p}{2\beta^2}\right)} \Lambda_n^{\frac{N-s}{2p_1}-\frac{N-s}{4\beta^2}},$$

we conclude that

$$\left(\int_{\mathcal{A}_n^2} \frac{|u_n|^p}{|y|^s} dx\right)^{\frac{1}{p}} \leq C\Lambda_n^{\frac{N-s}{2p_1}-\frac{N-s}{2p}}.$$

□

Proposition 2.3. *We have*

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^2 dx, \quad \int_{\mathcal{A}_n^3} \frac{\lambda |u_n|^2}{|y|^2} dx \leq C \int_{\mathcal{A}_n^2} \frac{\left(|u_n|^{2^*(s)} + 1\right)}{|y|^s} dx + C\Lambda_n \int_{\mathcal{A}_n^2} \frac{|u_n|^2}{|y|^s} dx. \tag{2.10}$$

In particular,

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^2 dx, \quad \int_{\mathcal{A}_n^3} \frac{\lambda |u_n|^2}{|y|^2} dx \leq C\Lambda_n^{\frac{2+s-N}{2}+\frac{(N-s)}{p_1}}. \tag{2.11}$$

Proof. Let $\phi_n \in C_0^\infty\left(\mathcal{A}_n^2\right)$ be a function with $\phi_n = 1$ in $\mathcal{A}_n^3$, $0 \leq \phi_n \leq 1$ and $|\nabla\phi_n| \leq C\Lambda_n^{\frac{1}{2}}$. From

$$\int_\Omega \left(\nabla u_n \nabla\left(\phi_n^2 u_n\right) - \frac{\lambda\phi_n^2 u_n^2}{|y|^s}\right) dx \leq \int_\Omega \frac{\left(2|u_n|^{2^*(s)-1} + A\right)}{|y|^s}\phi_n^2|u_n| dx.$$

we get

$$\int_\Omega \nabla|u_n\phi_n u_n|^2 - \frac{\lambda\left(\phi_n u_n\right)^2}{|y|^2} dx \leq C\int_{\mathcal{A}_n^2} \frac{\left(|u_n|^{2^*(s)} + 1\right)}{|y|^s} dx + C\Lambda_n \int_{\mathcal{A}_n^2} \frac{|u_n|^2}{|y|^s} dx.$$

Using the fact that the norms are equivalent and Hardy inequality we can prove (2.10).

On the other hand, from (2.10) and Proposition 2.2, we have

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^2 dx, \quad \int_{\mathcal{A}_n^3} \frac{\lambda|u_n|^2}{|y|^2} dx \leq C\Lambda_n^{\frac{2^*(t)}{2p_1}(N-t)-\frac{N-t}{2}} + C\Lambda_n^{\frac{N-t}{p_1}-\frac{N-t}{2}+1}$$

Since $p_1 > 2^*(s)$, we see

$$\frac{2^*(s)}{2p_1}(N-s) - \frac{N-s}{2} = \frac{N-s}{p_1} + \frac{2^*(s)(2-s)}{2p_1} < \frac{N-s}{p_1} - \frac{N-s}{2} + 1,$$

we get

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^2 dx, \quad \int_{\mathcal{A}_n^3} \frac{\lambda|u_n|^2}{|y|^s} dx \leq C\Lambda_n^{\frac{N-s}{p_1}-\frac{N-s}{2}+1}.$$

□

Proposition 2.4. *For any u_n which is a solution of (2.1) with $\varepsilon = \varepsilon_n \to 0$ as $n \to +\infty$, satisfying $\|u_n\| \leq C$ for some constant independent of n, the sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly in $H_0^1(\Omega)$.*

Proof. Take a $t_n \in [\bar{C}+2, \bar{C}+3]$, satisfying

$$\begin{aligned}
&\int_{\partial B_{t_n\Lambda_n^{-\frac{1}{2}}}(x_n)} \left(\Lambda_n^{-1}\frac{|u_n|^{2^*(s)-\varepsilon_n}}{|y|^s} + |u_n|^q + \Lambda_n^{-1}|\nabla u_n|^2 + \Lambda_n^{-1}\frac{\lambda u_n^2}{|y|^2}\right) d\sigma \\
&\leq C\Lambda_n^{\frac{1}{2}} \int_{\mathcal{A}_n^3} \left(\Lambda_n^{-1}\frac{|u_n|^{2^*(s)-\varepsilon_n}}{|y|^s} + |u_n|^q + \Lambda_n^{-1}|\nabla u_n|^2 + \Lambda_n^{-1}\frac{\lambda u_n^2}{|y|^2}\right) dx.
\end{aligned} \tag{2.12}$$

Applying Proposition 2.2, (2.12) and (2.11), we get

$$\begin{aligned}
&\int_{\partial B_{t_n\Lambda_n^{-\frac{1}{2}}}(x_n)} \left(\Lambda_n^{-1}\frac{u_n^{2^*(s)-\varepsilon_n}}{|y|^s} + |u_n|^q + \Lambda_n^{-1}|\nabla u_n|^2 + \Lambda_n^{-1}\frac{\lambda u_n^2}{|y|^2}\right) d\sigma \\
&\leq C\Lambda_n^{\frac{1}{2}} \left(C\Lambda_n^{-1}\Lambda_n^{\left(\frac{(N-t)}{2p_1}-\frac{N-t}{2(2^*(s)-\varepsilon_n)}\right)2^*(s)-\varepsilon_n} + C\Lambda_n^{\frac{Nq}{2p_1}-\frac{N}{2}} + C\Lambda_n^{-1}\Lambda_n^{\frac{(N-s)}{p_1}+\frac{2+s-N}{2}}\right) \\
&\leq C\Lambda_n^{\frac{1}{2}-\frac{N-s}{2}+\frac{(N-s)}{p_1}},
\end{aligned} \tag{2.13}$$

since

$$-1 + \left(\frac{(N-s)}{2p_1} - \frac{N-s}{2\left(2^*(s)-\varepsilon_n\right)}\right)\left(2^*(s)-\varepsilon_n\right) < -\frac{N-s}{2} + \frac{(N-s)}{p_1},$$

and

$$\frac{Nq}{2p_1}-\frac{N}{2}<-\frac{N-s}{2}+\frac{(N-s)}{p_1}.$$

We have three different cases:

(i) $B_{t_n\Lambda_n^{-\frac{1}{2}}}(x_n)\cap\left(\mathbb{R}^N\backslash\Omega\right)\neq\emptyset$;

(ii) $B_{t_n\Lambda_n^{-\frac{1}{2}}}(x_n)\subset\Omega$ and $x^0\notin\overline{B_{t_n\Lambda_n^{-\frac{1}{2}}}(x_n)}$;

(iii) $B_{t_n\Lambda_n^{-\frac{1}{2}}}(x_n)\subset\Omega$ and $x^0\in\overline{B_{t_n\Lambda_n^{-\frac{1}{2}}}(x_n)}$;

Let $p_n(s)=2^*(s)-\varepsilon_n$, we have the following local Pohozaev identity for u_n on $B_n=B_{t_n\Lambda_n^{-\frac{1}{2}}}(x_n)\cap\Omega$:

$$\begin{aligned}&\left[\frac{N-s}{p_n(s)}-\frac{N-2}{2}\right]\int_{B_n}\frac{|u_n|^{p_n(s)}}{|y|^s}dx+\mu\int_{B_n}|u_n|^q dx+\frac{\lambda}{2}\int_{B_n}\frac{u_n^2}{|y|^2}\left(y_0\cdot y\right)dx\\&=\frac{N-2}{2}\int_{\partial B_n}\left(\nabla u_n\cdot\nu\right)u_n d\sigma+\frac{1}{2}\int_{\partial B_n}|\nabla u_n|^2\left(x-x_0\right)\cdot\nu d\sigma\\&\quad+\frac{\lambda}{2}\int_{\partial B_n}\frac{u_n^2}{|y|^2}\left(x-x_0\right)\cdot\nu d\sigma+\frac{1}{p_n(s)}\int_{\partial B_n}\frac{|u_n|^{p_n(s)}\left(x-x_0\right)\cdot\nu}{|y|^s}d\sigma\\&\quad+\frac{\mu}{q}\int_{\partial B_n}|u_n|^q\left(x-x_0\right)\cdot\nu d\sigma,\end{aligned}\tag{2.14}$$

where ν is the outward normal to ∂B_n. The point $x_0=(y_0,0)$ in (2.14) is chosen as follows.

- In case (i), we take $x_0\in\mathbb{R}^N\backslash\Omega$ with $|x_0-x_n|\leq 2t_n\Lambda_n^{-\frac{1}{2}}$ and $\nu\cdot(x-x_0)\leq 0$ in $\partial\Omega\cap B_n$. With this x_0, we can check that $y_0\cdot y\geq 0$ in B_n.
- In case (ii), we take a point $x_0=x_n$. Then $y_0\cdot y\geq 0$ in B_n.
- In case (iii), we take $y_0=0$. Thus, in any case $y_0\cdot y>0$ in B_n.

Since $p_n(s)<2^*(s)$ and $\frac{N-s}{p_n(s)}-\frac{N-2}{2}>0$, hence the first term in the left-hand side of (2.14) is non-negative and by the choice of x_0, the third term in the left-hand side of (2.14) is also non-negative. Hence (2.14) can be rewritten as

$$\begin{aligned}\mu\int_{B_n}|u_n|^q dx\leq&\frac{N-2}{2}\int_{\partial B_n}\left(\nabla u_n\cdot\nu\right)u_n d\sigma+\frac{1}{2}\int_{\partial B_n}|\nabla u_n|^2\left(x-x_0\right)\cdot\nu d\sigma+\frac{\lambda}{2}\int_{\partial B_n}\frac{u_n^2}{|y|^2}\left(x-x_0\right)\cdot\nu d\sigma\\&+\frac{1}{p_n(t)}\int_{\partial B_n}\frac{|u_n|^{p_n(t)}\left(x-x_0\right)\cdot\nu}{|y|^s}d\sigma+\frac{\mu}{q}\int_{\partial B_n}|u_n|^q\left(x-x_0\right)\cdot\nu d\sigma.\end{aligned}\tag{2.15}$$

Now we decompose ∂B_n into $\partial B_n=\partial_i B_n\cup\partial_e B_n$, where $\partial_i B_n=\partial B_n\cap\Omega$ and $\partial_e B_n=\partial B_n\cap\partial\Omega$. Observing that $u_n=0$ on $\partial\Omega$, we have

$$\begin{aligned}&\frac{N-2}{2}\int_{\partial_e B_n}\left(\nabla u_n\cdot\nu\right)u_n d\sigma+\frac{1}{2}\int_{\partial_e B_n}|\nabla u_n|^2\left(x-x_0\right)\cdot\nu d\sigma+\frac{\lambda}{2}\int_{\partial_e B_n}\frac{u_n^2}{|y|^2}\left(x-x_0\right)\cdot\nu d\sigma\\&+\frac{1}{p_n(s)}\int_{\partial_e B_n}\frac{|u_n|^{p_n(s)}\left(x-x_0\right)\cdot\nu}{|y|^s}d\sigma+\frac{\mu}{q}\int_{\partial_e B_n}|u_n|^q\left(x-x_0\right)\cdot\nu d\sigma\\&=\frac{1}{2}\int_{\partial_e B_n}|\nabla u_n|^2\left(x-x_0\right)\cdot\nu d\sigma\leq 0.\end{aligned}$$

Hence, we can rewrite (2.15) as

$$\mu\int_{B_n}|u_n|^q dx \leq \frac{N-2}{2}\int_{\partial_i B_n}(\nabla u_n\cdot\nu)\,u_n d\sigma+\frac{1}{2}\int_{\partial_i B_n}|\nabla u_n|^2\,(x-x_0)\cdot\nu d\sigma+\frac{\lambda}{2}\int_{\partial_i B_n}\frac{u_n^2}{|y|^2}\,(x-x_0)\cdot\nu d\sigma$$
$$+\frac{1}{p_n(t)}\int_{\partial_i B_n}\frac{|u_n|^{p_n(s)}\,(x-x_0)\cdot\nu}{|y|^s}d\sigma+\frac{\mu}{q}\int_{\partial_i B_n}|u_n|^q\,(x-x_0)\cdot\nu d\sigma. \tag{2.16}$$

By (2.13), noting that $|x-x_0|\leq C\Lambda_n^{-\frac{1}{2}}$ for $x\in\partial_i B_n$, we have

$$\begin{aligned}\text{RHS of (2.16)} \leq & C\Lambda_n^{-\frac{1}{2}}\int_{\partial_i B_n}\left(|\nabla u_n|^2+\frac{|u_n|^{p_n(s)}}{|y|^s}+|u_n|^q+\lambda\frac{u_n^2}{|y|^2}\right)d\sigma+C\int_{\partial_i B_n}|\nabla u_n|u_n|d\sigma\\ \leq & C\Lambda_n^{-\frac{1}{2}}\int_{\partial_i B_n}\left(|\nabla u_n|^2+\frac{|u_n|^{p_n(t)}}{|y|^t}+|u_n|^q+\lambda\frac{u_n^2}{|y|^2}\right)d\sigma+\int_{\partial_i B_n}\left(\Lambda_n^{-1}|\nabla u_n|^2+C|u_n|^q\right)d\sigma\\ \leq & C\Lambda_n^{1-\frac{N-s}{2}+\frac{(N-s)}{p_1}}+C\Lambda_n^{\frac{1}{2}-\frac{N-s}{2}+\frac{(N-s)}{p_1}}\\ \leq & C\Lambda_n^{1-\frac{N-s}{2}+\frac{(N-s)}{p_1}}.\end{aligned}$$

It is standard to show that there is a constant $c_0>0$, see [3], such that

$$\int_{B_n}|u_n|^q\geq c_0\Lambda_n^{-q-N+\frac{Nq}{2}},$$

where $p_1>2^*(s)$ is any constant, satisfying $p_1<\frac{2(k-s)}{k-2-2\sqrt{\bar{\lambda}-\lambda}}$ and $\lambda<\bar{\lambda}-4$. Choose $p_1=\frac{2(N-s)}{(N-2)q-(N+s+2)}+\delta$ with $p_1+\delta<\frac{2(k-s)}{k-2-2\sqrt{\bar{\lambda}-\lambda}}$, where $\delta>0$ is a small constant. This can be achieved if $\lambda<\bar{\lambda}-4$ with this p_1, we know

$$-q-N+\frac{Nq}{2}>1-\frac{N-s}{2}+\frac{(N-s)}{p_1}.$$

So, we obtain a contradiction if $1<q<2$ and $N>\frac{s+2+2q}{q-1}$. □

3. The existence of infinitely many solutions

In this section, we will prove our main results following the ideas in [6,8,9]. We fix an orthonormal basis $\left(e_j\right)_j$ of $H_0^1(\Omega)$ and we define $X_j:=\mathbb{R}e_j$. On $H_0^1(\Omega)$ we consider the antipodal action of $\mathbb{Z}/2$. Let

$$Y_k:=\bigoplus_{j=1}^{k}X_j,\ Z_k:=\overline{\bigoplus_{j=k}^{\infty}X_j}.$$

Proof of Theorem 1.1. By employing arguments analogous to those in [9, Theorem 3.20], it can be easily verified that for all $k\geq k_0$, there exist $\rho_k>r_k>0$ such that $\rho_k\to 0$ as $k\to+\infty$ and for all $n\in\mathbb{N}$

$$a_k^n:=\inf_{\substack{u\in Z_k\\ \|u\|=\rho_k}}I_\mu^{\varepsilon_n}(u)\geq 0,\ b_k^n:=\max_{\substack{u\in Y_k\\ \|u\|=r_k}}I_\mu^{\varepsilon_n}(u)<0,\ b_k:=\max_{\substack{u\in Y_k\\ \|u\|=r_k}}I_\mu(u)<0,$$

and

$$d_k^n := \inf_{\substack{u \in \mathbb{Z}_k \\ \|u\| \leq \rho_k}} I_\mu^{\varepsilon_n}(u) \to 0 \text{ as } k \to +\infty,$$

where

$$B_k := \{u \in Y_k : \|u\| \leq \rho_k\}, \; N_k := \{u \in Z_k : \|u\| = r_k\},$$

and $(\varepsilon_n)_n$ is a decrcasing sequence with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. Then, From [9, Theorem 3.18], the functional I_{ε_n} possesses a sequence of critical points, denoted by $\left(v_k^n\right)_n$, furthermore we use the fact that $I_{\varepsilon_n}\left(v_k^n\right) = c_k^n$ and $(I_{\varepsilon_n})'(v_k^n).v_k^n = 0$, we deduce that

$$\left(\frac{1}{2} - \frac{1}{2^* - \varepsilon_n}\right)\left[\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega \frac{u^2}{|y|^2}\right] - \mu\left(\frac{1}{q} - \frac{1}{2^* - \varepsilon_n}\right)\int_\Omega |v_k^n|^q dx = c_k^n.$$

Since c_k^n is negatif, $2^*(s) < 2^*$ and by using Sobolev's embedding, we conclude that $\left(v_k^n\right)_n$ is bounded in $H_0^1(\Omega)$. Using Proposition 2.4, we can identify a subsequence of $\left(v_k^n\right)_n$ such that

$$\left(v_k^n\right)_n \to v_k,$$

where v_k is the solution of the problem (1.1) at level c_k with $c_k := \lim\limits_{n+\infty} c_k^n$.

Firstly, we want to prove that for any $k \geq k_0,\; c_k < 0$. Indeed, because ∂B_k is compact and the functionals $(I_{\varepsilon_n})_n$ are equicontinuous, then

$$b_k^n \to b_k.$$

Thus

$$c_k \leq b_k < 0.$$

Secondly, we want to prove that $\lim\limits_{k \to +\infty} c_k = 0$. Indeed, for any $k \geq k_0$ there exists $n_k > k$ such that

$$|c_k^{n_k} - c_k| < \frac{1}{k}.$$

Let δ be a fixed number such that $\delta \in (0, \delta_0)$, where

$$\delta_0 := \inf_{u \in H_0^1(\Omega), \|u\|_2 = 1} \int_\Omega |\nabla u|^2 dx > 0.$$

Set

$$\alpha_k := \inf_{u \in Z_k, \|u\|_{p_{n_k}, s} = 1} \int_\Omega \left(|\nabla u|^2 - \delta |u|^2\right) dx, \tag{3.1}$$

where $p_{n_k} := 2^*(s) - \varepsilon_{n_k} \in (2, 2^*)$. We will prove that, up to a subsequence, $\alpha_k \to +\infty$ as $k \to \infty$. While $p_{n_k} < 2^*$, thus the scalar α_k can be achieved by a function $v_k \in z_k$, which satisfies

$$-\Delta v_k = \alpha_k |v_k|^{p_{n_k} - 2} v_k + \delta_0 v_k.$$

Using Young inequality, we get that

$$\alpha_k = \int_\Omega \left(|\nabla v_k|^2 - \delta |v_k|^2\right) dx \geq \left(1 - \frac{\delta}{\delta_0}\right) \int_\Omega |\nabla v_k|^2 dx - C. \tag{3.2}$$

From (3.1) and if $\alpha_k \nrightarrow \infty$ as $k \to \infty$, we find $\int_\Omega |\nabla v_k|^2 dx \leq C$. By Proposition 2.4, we deduce that the sequence $(v_k)_k$ converges strongly in $H_0^1(\Omega)$. Since $v_k \in Z_k$, up to a subsequence, we may assume that

$$v_k \to 0 \text{ in } H_0^1(\Omega).$$

From Hölder inequality, thus $\lim\limits_{k\to\infty} \int_\Omega |v_k|^{p_{n_k}} dx = 0$, which is a contradiction due to $\int_\Omega |v_k|^{p_{n_k}} dx = 1$. Then $\alpha_k \to \infty$ as $k \to \infty$.

Now, if we suppose $u \in Z_k$, $\|u\| \leq \rho_k$ and k large enough, then we deduce that

$$\begin{aligned} I_{\varepsilon_n}(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \frac{u^2}{|y|^2} dx - \frac{1}{2^*(s) - \varepsilon_n}\int_\Omega \frac{|u|^{2^*(s)-\varepsilon_n}}{|y|^s} dx - \frac{\mu}{q}\int_\Omega u^q dx \\ &\geq C\frac{1}{2}\|u\|^2 - C\|u\|^{2^*(s)-\varepsilon_n} - C_3\|u\|^q. \end{aligned}$$

While the function $t \to C\frac{t^2}{2} - Ct^{2^*(s)-\varepsilon_{n_k}} - Ct^q$ is decreasing on $[0, \rho_k]$ for k large enough, consequently

$$I_{\varepsilon_{n_k}}(v_k^{n_k}) \geq C\frac{\rho_k^2}{2} - C\rho_k^{2^*(s)-\varepsilon_{n_k}} - C\rho_k^q.$$

Hence we get

$$c_k^{n_k} \geq d_k^{n_k} \geq \frac{\rho_k^2}{2} - C\rho_k^{2^*(s)-\varepsilon_{n_k}} - C\alpha_k^{-\frac{p_{n_k}}{2}}\rho_k^{p_{n_k}} - C\rho_k^q.$$

Since $\rho_k \to 0$, $\alpha_k \to +\infty$ as $k \to \infty$, we can deduce that

$$\lim_{k\to+\infty} c_k = \lim_{k\to+\infty} c_k^{n_k} = 0.$$

Then, the problem (1.1) possesses infinitely many solutions $(v_k)_k$ such that $I(v_k) < 0$ and $I(v_k) \to 0$ as $k \to +\infty$. □

***Proof of Theorem 1.2*.** By employing the same arguments as [9, Theorem 3.7], we can demonstrate that for every k, there exist $\varrho_k > \tau_k > 0$ such that $\varrho_k \to +\infty$ as $k \to +\infty$ and

$$\mathrm{a}_k^n := \max_{\substack{n\in Y_k\\ \|u\|=\varrho_k}} I_\mu^{\varepsilon_n}(u) \leq 0, \ \mathrm{b}_k^n := \inf_{\substack{u\in Z_k\\ \|u\|=r_k}} I_\mu^{\varepsilon_n}(u) \to \infty \text{ as } k \to +\infty,$$

where

$$\mathrm{B}_k := \{u \in Y_k : \|u\| \leq \varrho_k\}, \ \mathrm{N}_k := \{u \in Z_k : \|u\| = \tau_k\},$$

and $(\varepsilon_n)_n$ is a decreasing sequence with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$.

Using [9, Theorem 3.6], we deduce that I_{ε_n} possesses a sequence of critical points, denoted by $\left(u_k^n\right)_n$. Furthermore, $\mathfrak{c}_k^n = I_{\varepsilon_n}\left(u_k^n\right)$, where

$$\mathfrak{c}_k^n := \inf_{\gamma\in\Gamma_k}\max_{u\in \mathrm{B}_k} I_{\varepsilon_n}(\gamma(u)),$$

and

$$\Gamma_k := \left\{\gamma \in C\left(\mathrm{B}_k, H_0^1(\Omega)\right) : \gamma_{|\partial \mathrm{B}_k} = id\right\}.$$

We claim that for any $k \in \mathbb{N}$,

$$\mathfrak{c}_k^n \to \mathfrak{c}_k := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I(\gamma(u)) \text{ as } n \to +\infty.$$

In fact, for any $u \in H_0^1(\Omega)$

$$I(u) = I_{\varepsilon_n}(u) + F_n(u),$$

where

$$F_n(u) = \frac{1}{2^*(s) - \varepsilon_n} \int_\Omega \frac{|u|^{2^*(s)-\varepsilon_n}}{|y|^s} dx - \frac{1}{2^*(s)} \int_\Omega \frac{|u|^{2^*(s)}}{|y|^s} dx.$$

A simple calculation show that for $s > 0$ the function
$h(s) = \frac{1}{2^*(s)-\varepsilon_n} s^{2^*(s)-\varepsilon_n} - \frac{1}{2^*(s)} s^{2^*(s)}$ have a maximum in $s = 1$, so for $\gamma \in \Gamma_k$

$$I(\gamma(u)) \leq I_{\varepsilon_n}(\gamma(u)) + C\left(\frac{1}{2^*(s) - \varepsilon_n} - \frac{1}{2^*(s)}\right).$$

Then

$$c_k \leq \underline{\lim} c_k^n. \tag{3.3}$$

On the other hand, the functionals $I_{\varepsilon_n}(\gamma)$ are equicontinuous on the compact set B_k, then $\lim\limits_{n \to \infty} \sup\limits_{u \in B_k} I_{\varepsilon_n}(\gamma(u)) \to I(\gamma(u))$. Passing to the limit as $n \to +\infty$, we deduce that for any $k \in \mathbb{N}$

$$\overline{\lim_{n \to \infty}}\, \mathfrak{c}_k^n \leq \overline{\lim_{n \to \infty}} \lim_{u \in B_k} I_{\varepsilon_n}(\gamma(u)) = \sup_{u \in B_k} I(\gamma(u)).$$

Since γ is arbitrary, then

$$\overline{\lim_{n \to \infty}}\, \mathfrak{c}_k^n \leq \mathfrak{c}_k. \tag{3.4}$$

Our claim follows from (3.3) and (3.4). By employing arguments similar to those in the proof of Theorem 1.1, we may see that $\left(u_k^n\right)_n$ is bounded in $H_0^1(\Omega)$. Consequently, we can find a subsequence of $\left(u_k^n\right)_n$ that strongly converges to a solution v_k of (1.1) at level $\mathfrak{c}_k$. Then, for every $k \in \mathbb{N}$, there exists $n_k > k$ such that

$$|\mathfrak{c}_k^{n_k} - \mathfrak{c}_k| < \frac{1}{k}. \tag{3.5}$$

By applying Young's inequality and an argument similar to that used in the proof of Theorem 1.1, we get

$$\begin{aligned} I_{\varepsilon_n}(u) &= \frac{1}{2} \int_\Omega |\nabla u|^2 - \lambda \frac{u^2}{|y|^2} dx - \frac{1}{2^*(s) - \varepsilon_n} \int_\Omega \frac{|u|^{2^*(s)-\varepsilon_n}}{|y|^s} dx - \frac{\mu}{q} \int_\Omega u^q dx \\ &\geq \frac{1}{2} \|u\|^2 - \frac{\lambda}{2\bar{\lambda}} \|u\|^2 - C\alpha_k^{-\frac{p_{n_k}}{2}} |u|^{p_{n_k}} - C_4 \\ &\geq \frac{1}{4}\left(1 - \frac{\lambda}{\bar{\lambda}}\right) \|u\|^2 - C\alpha_k^{-\frac{p_{n_k}}{2}} \|u\|^{p_{n_k}} - C_4. \end{aligned}$$

Choosing $r_k = \left(\frac{\alpha_k^{\frac{pn_k}{2}} (1-\frac{\lambda}{\bar{\lambda}})}{6C} \right)^{\frac{1}{pn_k-2}}$, and $\|u\|_{H_0^1(\Omega)} = r_k$, we obtain that

$$I_{\varepsilon_n}(u) \geq \frac{1}{12} \left(\frac{\alpha_k^{\frac{pn_k}{2}} (1-\frac{\lambda}{\bar{\lambda}})}{6C} \right)^{\frac{2}{pn_k-2}} - C_4. \tag{3.6}$$

From the fact that $\alpha_k \to \infty$ as $k \to \infty$, it follows that $b_k^{n_k} \to \infty$ as $k \to \infty$. Using [9, Theorem 3.5], we get that $\mathrm{c}_k^{n_k} \geq \mathrm{b}_k^{n_k}$ and from (3.5), we have

$$\lim_{k\to\infty} \mathrm{c}_k = \lim_{k\to\infty} \mathrm{c}_k^{n_k} = +\infty.$$

Then, the problem (1.1) possesses infinitely many solutions $(u_k)_k$ such that $I(u_k) > 0$ and $I(u_k) \to +\infty$ as $k \to +\infty$. ☐

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