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A Mixed Extragradient Algorithm with Double Acceleration for Non-Lipschitz Bilevel Split Variational Inequality Problems

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Abstract. In this paper, we study the problem of approximating solutions of the bilevel split variational inequality problem (BSVIP) in real Hilbert spaces. The operator associated with the upper-level problem is assumed to be strongly monotone and uniformly continuous, while the operators in the lower-level problem are quasimonotone and uniformly continuous. To address this problem, we propose a new algorithm obtained by combining the modified subgradient extragradient method with a modified Tseng extragradient method. In contrast to existing modified subgradient extragradient approaches for solving the BSVIP, the proposed method avoids the computation of projections onto two auxiliary half-spaces containing the feasibility sets, thereby reducing the computational complexity of each iteration. Moreover, the step-size rules employed in the algorithm are self-adaptive and do not require prior knowledge of the norm of the bounded linear operator or the Lipschitz constants of the involved mappings. Under mild assumptions on the control parameters, we establish strong convergence of the proposed scheme. The algorithm further incorporates two inertial extrapolation terms, which help accelerate the convergence process. Numerical experiments are provided to illustrate the effectiveness and practical advantages of the proposed method when compared with several existing algorithms. The obtained results extend, unify, and improve a number of previously known results in the literature.

1. Introduction

Let K and Q be nonempty, closed, and convex subsets of two real Hilbert spaces H_1 and H_2, respectively, and let $B : H_1 \to H_2$ be a bounded linear operator. Consider two operators $T : H_1 \to H_1$ and $S : H_2 \to H_2$. In [4], Censor *et al.* introduced the *split variational inequality problem* (SVIP), which consists of finding a point $u^* \in K$ such that

$$\langle Tu^*, u - u^* \rangle \geq 0, \quad \forall u \in K, \tag{1.1}$$

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and whose image $v^* = Bu^* \in Q$ satisfies

$$\langle Sv^*, v - v^* \rangle \geq 0, \quad \forall v \in Q. \tag{1.2}$$

The SVIP unifies the classical variational inequality problem with the split feasibility framework and has attracted considerable attention due to its theoretical significance and practical relevance.

It is worth noting that when $T = S = 0$ in (1.1)–(1.2), the SVIP reduces to the well-known *split feasibility problem* (SFP), which was originally studied by Censor and Elfving [5] in the context of inverse problems in infinite-dimensional spaces. The SFP is formulated as follows: find $u^* \in K$ such that

$$Bu^* \in Q. \tag{1.3}$$

Over the past decades, the SFP has been extensively investigated by many researchers (see, for instance, [6,7,12,29] and the references therein). In addition to its theoretical interest, the SFP has been successfully applied to a variety of practical problems, including signal processing, image reconstruction, and intensity-modulated radiation therapy [7,8,10].

In this work, we focus on the *bilevel split variational inequality problem* (BSVIP), which can be interpreted as solving a variational inequality problem over the solution set of the SVIP. This problem was first introduced by Ahn *et al.* [3] and is formulated as follows: find $u^* \in \Gamma$ such that

$$\langle Mu^*, u - u^* \rangle \geq 0, \quad \forall u \in \Gamma, \tag{1.4}$$

where

$$\Gamma = \{u^* \in VI(K, T) : Bu^* \in VI(Q, S)\},$$

denotes the solution set of (1.1)–(1.2). The operator $M : H_1 \to H_1$ is assumed to be strongly monotone and Lipschitz continuous on H_1.

The BSVIP constitutes a broad framework that encompasses several important problems studied in optimization and variational analysis. In particular, if $M = 0$ in (1.4), the BSVIP reduces to the SVIP. On the other hand, if we take $H_1 = H$, let $M : H \to H$ be strongly monotone, set $T = F : H \to H$, choose $S = 0$, and take $Q = H_2$, then the BSVIP reduces to the *bilevel variational inequality problem* (BVIP) defined by

$$\langle Mu^*, v - u^* \rangle \geq 0, \quad \forall v \in VI(K, F), \tag{1.5}$$

where $VI(K, F)$ denotes the solution set of the classical variational inequality problem

$$\langle Fv^*, w - v^* \rangle \geq 0, \quad \forall w \in K. \tag{1.6}$$

The dual variational inequality of VIP is formulated as follows:

$$\text{Find } v^* \in K \text{ such that } \langle Fw, w - v^* \rangle \geq 0 \ \forall \ w \in K.$$

The solution set of the dual VIP is denoted by S_D. S_D is obviously a closed convex set (possibly). When F is continuous and K is convex, $S_D \subset VI(K, F)$. If F is pseudomonotone and continuous, then $VI(K, F) = S_D$ [45]. The inclusion $VI(K, F) \subset S_D$ is false if F is quasimonotone and continuous [47].

We also use S_T and S_N for the trivial and nontrivial solution sets of VIP; that is,

$$S_T = \{v^* \in K : \langle Fv^*, w - v^* \rangle = 0,\ \forall w \in K\},$$
$$S_N = VI(K, F) \setminus S_T.$$

The BVIP framework includes a wide range of mathematical models such as bilevel convex programming problems [37], bilevel optimization problems [11,26,30,32,46], equilibrium models, and minimum-norm problems over solution sets of variational inequalities [41, 43]. Because of these diverse applications, the study of efficient numerical methods for such problems has received considerable interest.

Among the earliest numerical approaches for solving the classical VIP is the *extragradient method* introduced by Korpelevich [19]. Although this method has been widely used due to its strong theoretical properties, it requires two projections onto the feasible set per iteration, which may significantly increase the computational cost. To overcome this limitation, Censor *et al.* [9] proposed the *subgradient extragradient method*, in which the second projection onto the constraint set is replaced by a projection onto a half-space containing the feasible set. This modification reduces the computational burden and improves the practical efficiency of the algorithm. Later, Tseng [36] introduced another variant, commonly referred to as the *Tseng extragradient method*, which requires only one projection onto the constraint set per iteration.

Recently, several authors have proposed algorithms for solving the BSVIP under various assumptions. For example, Ahn *et al.* [3] and Van *et al.* [38] developed modified subgradient extragradient methods for BSVIP involving Lipschitz continuous pseudomonotone operators in the lower-level problem and a Lipschitz continuous strongly monotone operator in the upper-level problem. Strong convergence results were established under certain restrictive conditions on the control parameters. Subsequently, Huy *et al.* [16] proposed a modified Tseng-type algorithm for solving the BSVIP under similar assumptions.

Despite these advances, the existing methods suffer from several limitations. In particular, most of the available algorithms require prior knowledge of the Lipschitz constants of the involved operators, the modulus of strong monotonicity of the upper-level operator, and the norm of the bounded linear operator. In practice, however, such information may not be readily available or may be difficult to estimate accurately. Moreover, these algorithms are generally not applicable when the underlying operators are non-Lipschitz.

On the other hand, recent studies have investigated extragradient-type methods under weaker assumptions, where the cost operator is assumed to be quasimonotone rather than pseudomonotone [2, 17, 18, 20, 21, 24]. This line of research is particularly important because convergence analyses developed under pseudomonotonicity assumptions cannot be directly extended to the quasimonotone setting.

Another technique that has proved effective in accelerating the convergence of iterative algorithms is the *inertial extrapolation technique*, originally introduced by Polyak [28]. The key idea is to incorporate information from previous iterates in order to improve the convergence speed of the algorithm.

To the best of our knowledge, there is currently no algorithm available in the literature for solving the BSVIP in which the operator in the upper-level problem is strongly monotone but not necessarily Lipschitz continuous. This naturally leads to the following question:

Is it possible to design a non-monotonic self-adaptive algorithm with double inertial extrapolation steps for solving the BSVIP without assuming pseudomonotonicity or Lipschitz continuity of the underlying operators in infinite-dimensional Hilbert spaces?

Motivated by this question and the recent developments in the literature, we propose in this paper a new *mixed extragradient-type method* for solving the BSVIP. The proposed algorithm combines the main ideas of the subgradient extragradient method and Tseng's extragradient method. It is designed to handle uniformly continuous quasimonotone operators in the lower-level problem and a uniformly continuous strongly monotone operator in the upper-level problem.

The main features of the proposed method can be summarized as follows:

- The algorithm incorporates *double inertial extrapolation steps*, which enhance the convergence behavior.
- It employs a *non-monotonic self-adaptive step-size strategy* that is updated dynamically at each iteration.
- The convergence analysis does not require prior knowledge of the Lipschitz constants or strong monotonicity parameters of the involved operators.
- The implementation does not require computation or estimation of the norm of the bounded linear operator.

To demonstrate the practical performance of the proposed algorithm, several numerical experiments are presented and compared with some existing methods in the literature.

The remainder of this paper is organized as follows. In Section 2, we recall some preliminary concepts and auxiliary results that will be used in the subsequent analysis. Section 3 introduces the proposed algorithm and establishes its strong convergence properties. In Section 4, numerical experiments are provided in both finite- and infinite-dimensional settings to illustrate the efficiency of the proposed method. Finally, Section 5 concludes the paper with a summary of the main results.

2. Preliminaries

Let K be a nonempty, closed and convex subset of a real Hilbert space H. We represent the weak and strong convergence of $\{u_n\}$ to u by $u_n \rightharpoonup u$ and $u_n \to u$, respectively, and $w_\omega(u_n)$ represents the set of weak limits of $\{u_k\}$, that is $w_\omega(u_n) = \{u \in H : u_{n_j} \rightharpoonup u$, for some subsequence $\{u_{n_j}\}$ of $u_n\}$. For every point $u \in \mathcal{H}$, the unique nearest point which is denoted by $P_K u$ exists in K such that $\|u - P_K u\| \leq \|u - v\|$, $\forall v \in K$. The mapping P_K is called the metric projection of $\mathcal{H}$ onto K and it is known to be nonexpansive.

Lemma 2.1. *[14] Let $\mathcal{H}$ be a real Hilbert space and K a nonempty closed convex subset of H. Suppose $u \in \mathcal{H}$ and $v \in K$. Then $v = P_K u \iff \langle u - v, v - w \rangle \geq 0, \forall w \in K$.*

Lemma 2.2. *[14] Let K be a closed convex subset of a real Hilbert space H. If $u \in H$, then*
(i) $\|P_K u - P_K v\|^2 \leq \langle P_K u - P_K v, u - v \rangle, \ \forall v \in H$;
(ii) $\langle (I - P_K)u - (I - P_K)v, u - v \rangle \geq \|(I - P_K)u - (I - P_K)v\|^2, \ \forall v \in K$;
(iii) $\|P_K u - v\|^2 \leq \|u - v\|^2 - \|u - P_K u\|^2, \ \forall v \in \mathcal{H}$.

Lemma 2.3. *Let H be a real Hilbert space. Then for every $u, v \in H$ and $\sigma \in \mathbb{R}$, we have*
(i) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v \rangle$;
(ii) $\|u + v\|^2 = \|u\|^2 + 2\langle u, v \rangle + \|v\|^2$;
(iii) $\|\sigma u + (1 - \sigma)v\|^2 = \sigma\|u\|^2 + (1 - \sigma)\|v\|^2 - \sigma(1 - \sigma)\|u - v\|^2$.

Lemma 2.4. *[42] Let $\{u_n\}$ be a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1 - \nu_n)a_n + \nu_n b_n, \ \forall n \geq 1,$$

where $\{\nu_n\} \subset (0, 1)$ with $\sum_{n=0}^{\infty} \nu_n = \infty$. If $\limsup_{n\to\infty} b_n \leq 0$ for every subsequence $\{a_{n_j}\}$ of $\{a_n\}$, the following inequality hold:

$$\liminf_{k\to\infty}(a_{n_{j+1}} - a_{n_j}) \geq 0,$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *[34] Let $\{\rho_n\}$ and $\{\sigma_n\}$ be two non-negative real sequences satisfying*

$$\rho_{n+1} \leq \rho_n + \sigma_n, \ \forall n \geq 1.$$

Suppose $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} \rho_n$ exists.

Definition 2.1. *Let $M : \mathcal{H} \to \mathcal{H}$ be an operator. Then M is called*

(1) *L_M-Lipschitz continuous if there exists $L_M > 0$ such that*

$$\|Mu - Mv\| \leq L\|u - v\|,$$

for each $u, v \in H$;

(2) *uniformly continuous, if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, such that*

$$\|u - v\| < \delta \implies \|Mu - Mv\| < \epsilon, \ \forall u, v \in H;$$

(3) *α–strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle u - v, Mu - Mv \rangle \geq \alpha\|u - v\|^2, \ \forall u, v \in H;$$

(4) *α–inverse strongly monotone (α-co-coercive), if there exists a constant $\alpha > 0$ such that*

$$\langle u - v, Mu - Mv \rangle \geq \alpha\|Mu - Mv\|^2, \ \forall u, v \in H;$$

(5) *monotone if*

$$\langle Mu - Mv, u - v \rangle \geq 0, \ \forall u, v \in H;$$

(6) *pseudomonotone if*

$$\langle Mu, v-u\rangle \geq 0 \Rightarrow \langle Mv, v-u\rangle \geq 0,\ \forall u, v \in H;$$

(7) *quasimonotone*

$$\langle Mu, u-v\rangle > 0 \Rightarrow \langle Mv, u-v\rangle \geq 0\ \forall\ u, v \in H.$$

Lemma 2.6. *[15, 44] Let H be a real Hilbert space, K be a nonempty, closed and convex subset of H and $S : H \to H$ be a quasimonotone and hemicontinuous operator. Suppose that $v \in K$ and for some $u \in K$, we have $\langle Sv, u-v\rangle \geq 0$. Then at least one of the following hold: $\langle Su, u-v\rangle \geq 0$ or $\langle Sv, w-v\rangle \leq 0$, for all $w \in K$.*

Lemma 2.7. *[25, 39] Let K stand for a nonempty convex subset of a real Hilbert space H. The mapping $M : K \to H$ is uniformly continuous if and only if for every $\epsilon > 0$, a constant $D < +\infty$ exists such that*

$$\|Mu - Mv\| \leq D\|u-v\| + \epsilon,\ \forall u, v \in K.$$

Lemma 2.8. *[47] If either*
1. A is pseudomonotone on K and $VI(K, A) \neq \emptyset$,
2. A is the gradient of G, where G is a differential quasiconvex function on an open set C, $K \subset C$ and attains its global minimum on K,
3. A is quasimonotone on K, $A \neq 0$ on K and K is bounded,
4. A is quasimonotone on K, $A \neq 0$ on K and there exists a positive number r such that, for every $v \in K$ with $\|v\| \geq r$, there exists $y \in K$ such that $\|y\| \leq r$ and $\langle Av, y-v\rangle \leq 0$,
5. A is quasimonotone on K and $S_N \neq \emptyset$,
6. A is quasimonotone on K, int$K$ is nonempty and there exists $v^* \in VI(K, A)$ such that $A(v^*) \neq 0$.
Then, S_D is nonempty.

Lemma 2.9. *[48] Let $M : H \to H$ be an α-strongly monotone and (D, ϵ)-uniformly continuous on a real Hilbert space H with $0 < \beta < 1$, $0 < \epsilon_* < 1$, $0 \leq \gamma \leq 1-\beta$ and $0 < \mu < \frac{2\alpha}{\Omega^2}$, where $\Omega = D + \epsilon_*$. Then,*

$$\|(1-\gamma)u - \beta\mu Mu - [(1-\gamma)v - \beta\mu Mv]\| \leq (1-\gamma-\beta\ell)\|u-v\|,\ \forall u, v \in H$$

where $\ell = 1 - \sqrt{1-\mu(2\alpha - \mu\Omega^2)} \in (0, 1]$.

3. Main Results

In this section, we introduce the proposed algorithm and establish its strong convergence in real Hilbert spaces. We begin by stating the assumptions that form the basis for the subsequent convergence analysis.

Assumption 3.1.
(A_1) *Let H_1 and H_2 be two real Hilbert spaces, and the feasible sets K and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively.*
(A_2) *$M : H_1 \to H_1$ is an α-strongly monotone and (D, ϵ)-uniformly continuous on H_1.*

(A_3) *$S : H_2 \to H_2$ is quasimonotone and (D_2, ϵ_2)-uniformly continuous on H_2.*

(A_4) *$T : H_1 \to H_1$ is quasimonotone and (D_1, ϵ_1)-uniformly continuous on H_1.*

(A_5) *$B : H_1 \to H_2$ is a bounded linear operator such that $B \neq 0$, $B^* : H_2 \to H_1$ is the adjoint operator of B.*

(A_6) *The solution set $\Gamma^* = \{u^* \in S_D^* : Bu^* \in S_D^\dagger\} \neq \emptyset$, where S_D^* is the solution set of the dual VIP corresponding to* (1.1) *and $S_D^\dagger$ is the solution set of the dual VIP corresponding to* (1.2).

(A_7) *$\{\beta_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $0 \leq \gamma_n \leq 1 - \beta_n$, $\forall n \geq 0$. The positive sequences $\{\epsilon_n\}$ and $\{\xi_n\}$ satisfy $\lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0 = \lim_{n\to\infty} \frac{\xi_n}{\beta_n}$.*

(A_8) *Let $q_n \subset [0,\infty)$ with $\sum_{n=0}^{\infty} q_n < \infty$ and $p_n \subset [0,\infty)$ with $\sum_{n=0}^{\infty} p_n < \infty$.*

Algorithm 3.1. *A Mixed Extragradient Algorithm with Double Acceleration*

Initialization: *Choose $\phi > 0$, $\psi \geq 3$, $\tau_1 > 0$, $\lambda_1 > 0$, $\tau, \lambda, c \in (0,1)$ and $0 < \mu < \frac{2\alpha}{\Omega^2}$, where $\Omega = D + \epsilon_*$. Let $u_0, u_1 \in \mathcal{H}_1$, $u_0 = h_0$ and set $n = 1$.*

Iterative Steps: *Calculate the next iteration point u_{n+1} as follows:*

Step 1: *Given the iterates u_{n-1} and u_n $(n \geq 1)$, we choose ϕ_n such that $\phi_n \in [0, \bar{\phi}_n]$, where*

$$\bar{\phi}_n = \begin{cases} \min\left\{\phi, \frac{\epsilon_n}{\max\{n^2\|u_n - u_{n-1}\|, n\|u_n - u_{n-1}\|\}}\right\}, & \text{if } u_n \neq u_{n-1}, \\ \phi, & \textit{otherwise}. \end{cases} \tag{3.1}$$

Step 2: *Set*

$$r_n = u_n + \phi_n(u_n - u_{n-1}) \tag{3.2}$$

and compute

$$v_n = P_Q(Br_n - \lambda_n SBr_n), \tag{3.3}$$

$$z_n = P_{Q_n}(Br_n - \lambda_n Sv_n), \tag{3.4}$$

where

$$Q_n = \{g_2 \in H_2 : \langle Br_n - \lambda_n SBr_n - v_n, g_2 - v_n\rangle \leq 0\}, \tag{3.5}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\lambda(\|Br_n - v_n\|^2 + \|z_n - v_n\|^2)}{2\langle SBr_n - Sv_n, z_n - v_n\rangle}, \lambda_n + p_n\right\}, & \text{if } \langle SBr_n - Sv_n, z_n - v_n\rangle > 0, \\ \lambda_n + p_n, & \textit{otherwise}. \end{cases} \tag{3.6}$$

Step 3: *Compute*

$$h_n = r_n + \delta_n B^*(z_n - Br_n), \tag{3.7}$$

where the stepsize δ_n is choosen in such a way that

$$\delta_n = \begin{cases} \frac{c\|z_n - Br_n\|^2}{\|B^*(z_n - Br_n)\|^2}, & \text{if } \|B^*(z_n - Br_n)\| \neq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.8}$$

Step 4: *Choose ψ_n such that $\psi_n \in [0, \bar{\psi}_n]$, where*

$$\bar{\psi}_n = \begin{cases} \min\left\{\frac{n-1}{n+\psi-1}, \frac{\xi_n}{\|h_n - h_{n-1}\|}\right\}, & \text{if } h_n \neq h_{n-1}, \\ \frac{n-1}{n+\psi-1}, & \text{otherwise}. \end{cases} \tag{3.9}$$

Step 5: *Set*

$$w_n = h_n + \psi_n(h_n - h_{n-1}) \tag{3.10}$$

and compute

$$e_n = P_K(w_n - \tau_n T w_n), \tag{3.11}$$

$$d_n = e_n - \tau_n(Te_n - Tw_n), \tag{3.12}$$

where

$$\tau_{n+1} = \begin{cases} \min\left\{\frac{\tau\|w_n - e_n\|}{\|Tw_n - Te_n\|}, \tau_n + q_n\right\}, & \text{if } Tw_n \neq Te_n, \\ \tau_n + q_n, & \text{otherwise}. \end{cases} \tag{3.13}$$

Step 6: *Compute*

$$u_{k+1} = \gamma_n u_n + (1 - \gamma_n) d_n - \beta_n \mu M d_n. \tag{3.14}$$

*Put $n := n + 1$ and return to **Step 1**.*

Remark 3.1. (1) *Algorithm 3.1 is developed by integrating the modified subgradient extragradient method with a modified version of Tseng's extragradient scheme for solving the BSVIP. In contrast to the modified subgradient extragradient methods studied in [3, 38], the proposed algorithm eliminates the need for computing projections onto two auxiliary half-spaces that contain the feasible sets K and Q, thereby reducing computational complexity.*

(2) *To the best of our knowledge, there are no existing results addressing the BVIP and BSVIP in the setting where the upper-level operator M is non-Lipschitz. The results obtained in this work fill this gap and, at the same time, extend, generalize, and improve several related results reported in [3, 16, 27, 34, 39].*

(3) *The operators S and T in the lower-level problem are assumed to be quasimonotone and uniformly continuous. It is important to emphasize that these assumptions are weaker than the commonly imposed pseudomonotonicity and Lipschitz continuity conditions used in [3,16,38], thereby broadening the applicability of the proposed method.*

(4) *The proposed algorithm incorporates two inertial extrapolation terms, which contribute to accelerating its convergence behavior. Furthermore, the inertial update rules* (3.2) *and* (3.10) *are computationally efficient, since the quantities* $\|u_n - u_{n-1}\|$ *and* $\|h_n - h_{n-1}\|$ *are readily available prior to updating the parameters* ϕ_n *and* ψ_n*.*

(5) *Strong convergence of the proposed method is established under mild conditions on the control parameters. Notably, the analysis does not require the restrictive condition* $\lim_{n\to\infty} \gamma_n = \gamma < 1$*, which is commonly assumed in the convergence results of [3, 39].*

(6) *The step-size selection in Algorithm 3.1 is fully adaptive and does not depend on prior knowledge of the norm of the bounded linear operator* $\|B\|$*. This is particularly advantageous, as computing or estimating* $\|B\|$ *is often difficult and, in some cases, infeasible in practice.*

Lemma 3.1. *Suppose Assumption 3.1 holds. Then, the sequences* $\{\lambda_n\}$ *and* $\{\tau_n\}$ *of the step sizes formulated by Algorithm 3.1 are bounded and well defined.*

Proof. Due to the uniform continuity of S, Lemma 2.9, and (3.6), if $\langle SBr_n - Sv_n, z_n - v_n\rangle > 0$ for all $n \geq 1$, we obtain

$$\frac{\lambda(\|Br_n - v_n\|^2 + \|z_n - v_n\|^2)}{2\langle SBr_k - Sv_k, z_k - v_k\rangle} \geq \frac{\lambda\|Br_n - v_n\|\|z_n - v_n\|}{\|SBr_n - Sv_n\|\|z_n - v_n\|} \geq \frac{\lambda\|Br_n - v_n\|}{D_2\|Br_n - v_n\| + \epsilon_2} = \frac{\lambda\|Br_n - v_n\|}{(D_2 + \epsilon_3)\|Br_n - v_n\|} = \frac{\lambda}{E},$$

where $\epsilon_2 = \epsilon_3\|Br_n - v_n\|$ for some $\epsilon_3 \in (0, 1)$ and $E = D_2 + \epsilon_3$. We infer from the definition of λ_{n+1} that the sequence $\{\lambda_n\}$ has lower bound $\min\{\frac{\lambda}{E}, \lambda_1\}$ and has upper bound $\lambda_1 + P$. From Lemma 2.5, we know that $\lim_{n\to\infty} \lambda_n$ exists and $\lim_{n\to\infty} \lambda_n = \lambda^*$. Apparently, we have that $\lambda^* \in [\min\{\frac{\lambda}{E}, \lambda_1\}, \lambda_1 + P]$, where $P = \sum_{n=1}^{\infty} p_n$.

Similarly, following the same approach above, we obtain $\tau^* \in [\min\{\frac{\tau}{F}, \tau_1\}, \tau_1 + U]$, for some $F > 0$, where $U = \sum_{n=1}^{\infty} q_n$. □

Lemma 3.2. *Assume* $\|B^*(z_n - Br_n)\| \neq 0$*, then the sequence* $\{\delta_n\}$ *defined by* (3.8) *has a positive lower bound.*

Proof. If $\|B^*(z_n - Br_n)\| \neq 0$, it implies that

$$\delta_n = \frac{c\|z_n - Br_n\|^2}{\|B^*(z_n - Br_n)\|^2}.$$

Now, since $c \in (0, 1)$, and B is a bounded linear operator, we get

$$\frac{c\|z_n - Br_n\|^2}{\|B^*(z_n - Br_n)\|^2} \geq \frac{c\|z_n - Br_n\|^2}{\|B\|^2\|z_n - Br_n\|^2} = \frac{c}{\|B\|^2}.$$

Hence, $\frac{c}{\|B\|^2}$ is a lower bound of $\{\delta_n\}$. □

Lemma 3.3. *Suppose that Assumption 3.1 holds and the solution set* $\Gamma^* \neq \emptyset$*. If* $\{u_n\}$ *is the sequence generated by Algorithm 3.1. Then, for each* $u^* \in \Gamma^*$ *and* $k \geq 0$*, the following inequality holds:*

$$\|z_n - Bu^*\|^2 \leq \|Br_n - Bu^*\|^2 - \left(1 - \frac{\lambda_n\lambda}{\lambda_{n+1}}\right)\|Br_n - r_n\|^2 - \left(1 - \frac{\lambda_n\lambda}{\lambda_{n+1}}\right)\|v_n - r_n\|^2. \tag{3.15}$$

Proof. Let $u^* \in \Gamma^*$, we have $Bu^* \in S_D^\dagger$. Since $Bu^* \in S_D^\dagger \subset VI(Q,S) \subset Q \subset Q_n$, then using (3.4), Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} \|z_n - Bu^*\|^2 &= \|P_{Q_n}(Br_n - \lambda_n Sv_n) - P_{Q_n}Bu^*\|^2 \\ &\leq \|Br_k - Bu^*\|^2 - 2\lambda_k\langle Br_n - Bu^*, Sv_n\rangle + \lambda_n^2\|Sv_n\|^2 \\ &\quad -\|Br_n - z_n\|^2 + 2\lambda_k\langle Bu^* - z_n, Sv_n\rangle - \lambda_k^2\|Sv_n\|^2 \\ &= \|Br_n - Bu^*\|^2 - \|Br_n - z_n\|^2 - 2\lambda_n\langle Sv_n, z_n - Bu^*\rangle \\ &= \|Br_n - Bu^*\|^2 - \|Br_n - z_n\|^2 - 2\lambda_n\langle Sv_n, z_n - v_n\rangle - 2\lambda_n\langle Sv_n, v_n - Bu^*\rangle. \end{aligned} \tag{3.16}$$

Since $Bu^* \in S_D^\dagger$ and $v_n \in Q$, we have that $\langle Sv_n, v_n - Bu^*\rangle \geq 0$. Thus, from (3.16), we have

$$\begin{aligned} \|z_n - Bu^*\|^2 &\leq \|Br_n - Bu^*\|^2 - \|Br_k - z_k\|^2 - 2\lambda_k\langle Sv_n, z_n - v_n\rangle \\ &= \|Br_n - Bu^*\|^2 - \|Br_n - v_n\|^2 - \|v_n - z_n\|^2 - 2\langle Br_n - v_n, v_n - z_n\rangle - 2\lambda_n\langle Sv_n, z_n - v_n\rangle \\ &= \|Br_n - Bu^*\|^2 - \|Br_n - v_n\|^2 - \|v_n - z_n\|^2 + 2\langle Br_n - \lambda_n Sv_n - v_n, z_n - v_n\rangle. \end{aligned} \tag{3.17}$$

Observe that

$$\begin{aligned} 2\langle Br_n - \lambda_k Sv_n - v_n, z_n - v_n\rangle &= 2\langle Br_n - \lambda_k SBr_k - v_n, z_n - v_n\rangle \\ &\quad +2\lambda_k\langle SBr_k - Sv_k, z_k - v_k\rangle. \end{aligned} \tag{3.18}$$

Since $v_n = P_Q(Br_n - \lambda_n SBr_n)$, then from Lemma 2.1, we have

$$\langle Br_n - \lambda_n SBr_n - v_n, z - v_n\rangle \leq 0, \ \forall z \in Q \implies Q \subset Q_n.$$

This implies from $z_n \in Q_n$ that

$$\langle Br_n - \lambda_n SBr_n - v_n, z_n - v_n\rangle \leq 0. \tag{3.19}$$

Using (3.18) and (3.19), we have

$$2\langle Br_n - \lambda_n Sv_n - v_n, z_n - v_n\rangle \leq 2\lambda_n\langle SBr_n - Sv_n, z_n - v_n\rangle. \tag{3.20}$$

From (3.6), we claim that

$$2\langle SBr_n - Sv_n, z_n - v_n\rangle \leq \frac{\lambda}{\lambda_{n+1}}\|Br_n - v_n\|^2 + \frac{\lambda}{\lambda_{n+1}}\|z_n - v_n\|^2, \ \forall n \geq 1. \tag{3.21}$$

Indeed, if $\langle SBr_n - Sv_n, z_n - v_n\rangle \leq 0$, then (3.21) holds. Putting (3.21) into (3.20), we have

$$2\langle Br_n - \lambda_n Sv_n - v_n, z_n - v_n\rangle \leq \frac{\lambda_n\lambda}{\lambda_{n+1}}\|Br_n - v_n\|^2 + \frac{\lambda_n\lambda}{\lambda_{n+1}}\|z_n - v_n\|^2. \tag{3.22}$$

Substituting (3.22) into (3.17), we have

$$\begin{aligned} \|z_n - Bu^*\|^2 &\leq \|Br_n - Bu^*\|^2 - \|Br_n - v_n\| - \|v_n - z_n\|^2 + \frac{\lambda_n\lambda}{\lambda_{n+1}}\|Br_n - v_n\|^2 + \frac{\lambda_n\lambda}{\lambda_{n+1}}\|z_n - v_n\|^2 \\ &= \|Br_n - Bu^*\|^2 - \left(1 - \frac{\lambda_n\lambda}{\lambda_{n+1}}\right)\|Br_n - v_n\| - \left(1 - \frac{\lambda_n\lambda}{\lambda_{n+1}}\right)\|z_n - v_n\|^2. \end{aligned} \tag{3.23}$$

□

Lemma 3.4. *Let $\{u_n\}$ be the sequence generated by Algorithm 3.1. Then, $\{u_n\}$ is bounded.*

Proof. Let $u^* \in \Gamma^*$. From (3.1), we have $\phi_n\|u_n - u_{n-1}\| \leq \epsilon_n$, $\forall n \in \mathbb{N}$. Since by Assumption 3.1 (A_7) we have that $\lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0$. This implies that

$$\frac{\phi_n}{\beta_n}\|u_n - u_{n-1}\| \leq \frac{\epsilon_n}{\beta_n} \to 0, \text{ as } n \to \infty. \tag{3.24}$$

Thus, there exists $C_1 > 0$ such that

$$\frac{\phi_n}{\beta_n}\|u_n - u_{n-1}\| \leq C_1, \ \forall n \in \mathbb{N}. \tag{3.25}$$

Using (3.2) and (3.25), we have

$$\begin{aligned} \|r_n - u^*\| &= \|u_n + \phi_n(u_n - u_{n-1}) - u^*\| \\ &\leq \|u_n - u^*\| + \phi_n\|u_n - u_{n-1}\| \\ &\leq \|u_n - u^*\| + \beta_n\frac{\phi_n}{\beta_n}\|u_n - u_{n-1}\| \\ &\leq \|u_n - u^*\| + \beta_n C_1. \end{aligned} \tag{3.26}$$

From Lemma 3.1, we have that $\lim_{n\to\infty} \lambda_n = \lambda^*$. This implies that $\lim_{n\to\infty}\left(1 - \frac{\lambda_n\lambda}{\lambda_{n+1}}\right) = 1 - \lambda$. Thus, there exists $n_0 \in \mathbb{N}$ such that $1 - \frac{\lambda_n\lambda}{\lambda_{n+1}} > 0$, for all $n \geq n_0$. Therefore, from (3.23), we have

$$\|z_n - Bu^*\| \leq \|Br_n - Bu^*\|. \tag{3.27}$$

By Lemma 2.3 and (3.27), we have

$$\begin{aligned} \|h_n - u^*\|^2 &= \|r_n + \delta_n B^*(z_n - Br_n) - u^*\|^2 \\ &= \|r_n - u^*\|^2 + \delta_n^2\|B^*(z_n - Br_n)\|^2 + 2\delta_n\langle r_n - u^*, B^*(z_n - Br_n)\rangle \\ &= \|r_n - u^*\|^2 + \delta_n^2\|B^*(z_n - Br_n)\|^2 + 2\delta_n\langle Br_n - Bu^*, z_n - Br_n\rangle \\ &= \|r_n - u^*\|^2 + \delta_n^2\|B^*(z_n - Br_n)\|^2 + \delta_n[\|z_n - Bu^*\|^2 - \|Br_n - Bu^*\|^2 - \|z_n - Br_n\|^2] \\ &\leq \|r_n - u^*\|^2 + \delta_n^2\|B^*(z_n - Br_n)\|^2 - \delta_n\|z_n - Br_n\|^2 \\ &= \|r_n - u^*\|^2 - \delta_n[\|z_n - Br_n\|^2 - \delta_n\|B^*(z_n - Br_n)\|^2]. \end{aligned} \tag{3.28}$$

If $\|B^*(z_n - Br_n)\| \neq 0$, then from (3.8), we have that

$$\|z_n - Br_n\|^2 - \delta_n\|B^*(z_n - Br_n)\|^2 = (1 - c)\|Br_n - z_n\|^2 \geq 0. \tag{3.29}$$

Combining (3.26), (3.28) and (3.29), we have

$$\|h_n - u^*\| \leq \|r_n - u^*\| \leq \|u_n - u^*\| + \beta_n C_1, \ \forall n \geq n_0. \tag{3.30}$$

Next, from (3.12) and Lemma 2.3, we have

$$\begin{aligned} \|d_n - u^*\|^2 &= \|e_n - \tau_n(Te_n - Tw_n) - u^*\|^2 \\ &= \|e_n - u^*\|^2 + \tau_n^2\|Te_n - Tw_n\|^2 - 2\tau_n\langle e_n - u^*, Te_n - Tw_n\rangle \\ &= \|w_n - u^*\|^2 + \|w_n - e_n\|^2 - 2\langle e_n - w_n, e_n - u^*\rangle \end{aligned}$$

$$
\begin{aligned}
&+\tau_n^2\|Te_n - Tw_n\|^2 - 2\tau_n\langle e_n - u^*, Te_n - Tw_n\rangle \\
= \; & \|w_n - u^*\|^2 + \|w_n - e_n\|^2 - 2\langle e_n - w_n, e_n - w_n\rangle + 2\langle e_n - w_n, e_n - u^*\rangle \\
&+\tau_n^2\|Te_n - Tw_n\|^2 - 2\tau_n\langle e_n - u^*, Te_n - Tw_n\rangle \\
= \; & \|w_n - u^*\|^2 - \|w_n - e_n\|^2 + 2\langle e_n - w_n, e_n - u^*\rangle \\
&+\tau_n^2\|Te_n - Tw_n\|^2 - 2\tau_n\langle e_n - u^*, Te_n - Tw_n\rangle. \qquad (3.31)
\end{aligned}
$$

Since $e_k = P_K(w_n - \tau_n Tw_n)$ and $u^* \in S_D^* \subset VI(K,T) \subset K$, then by Lemma 2.1, we have

$$\langle e_n - w_n + \tau_n Tw_n, e_n - u^*\rangle \leq 0$$

or, equivalently,

$$\langle e_n - w_n, e_n - u^*\rangle \leq -\tau_n\langle Tw_n, e_n - u^*\rangle. \qquad (3.32)$$

From(3.31) and (3.32), we have that

$$
\begin{aligned}
\|d_n - u^*\|^2 \leq \; & \|w_n - u^*\|^2 - \|w_n - e_n\|^2 - 2\tau_n\langle Tw_n, e_n - u^*\rangle \\
&+\mu_n^2\|Te_n - Tw_n\|^2 - 2\tau_n\langle e_n - u^*, Te_n - Tw_n\rangle \\
= \; & \|w_n - u^*\|^2 - \|w_n - e_n\|^2 + \tau_n^2\|Te_n - Tw_n\|^2 - 2\tau_n\langle e_n - u^*, Te_n\rangle. \qquad (3.33)
\end{aligned}
$$

Since Since $u^* \in S_D^*$ and $e_n \in K$, we have that $\langle Te_n, e_n - u^*\rangle \geq 0$. Thus, from (3.33), we have

$$\|d_n - u^*\|^2 \leq \|w_n - u^*\|^2 - \|w_n - e_n\|^2 + \tau_n^2\|Te_n - Tw_n\|^2. \qquad (3.34)$$

Furthermore, it is not hard to see from (3.13) that

$$\|Te_n - Tw_n\| \leq \frac{\tau}{\tau_{n+1}}\|w_n - e_n\|, \ \forall n \in \mathbb{N}. \qquad (3.35)$$

Using (3.34) and (3.35), we have

$$\|d_n - u^*\|^2 \leq \|w_n - u^*\|^2 - \left(1 - \frac{\tau^2\tau_n^2}{\tau_{n+1}^2}\right)\|w_n - e_n\|^2. \qquad (3.36)$$

From Lemma 3.1, we have that $\lim_{n\to\infty}\tau_n = \tau^*$. This implies that $\lim_{n\to\infty}\left(1 - \frac{\tau^2\tau_n^2}{\tau_{n+1}^2}\right) = 1 - \tau^2$. Thus, there exists $n_0 \in \mathbb{N}$ such that $\left(1 - \frac{\tau^2\tau_n^2}{\tau_{n+1}^2}\right) > 0$, for all $n \geq n_0$. Therefore, from (3.36), we have

$$\|d_n - u^*\| \leq \|w_n - u^*\|. \qquad (3.37)$$

From (3.9), we have $\psi_n\|h_n - h_{n-1}\| \leq \xi_n$, $\forall n \in \mathbb{N}$. Since by Assumption 3.1 (A_7) we have that $\lim_{n\to\infty}\frac{\xi_n}{\beta_n} = 0$. This implies that

$$\frac{\psi_n}{\beta_n}\|h_n - h_{n-1}\| \leq \frac{\xi_n}{\beta_n} \to 0, \ \text{as}\, n \to \infty. \qquad (3.38)$$

Thus, there exists $C_2 > 0$ such that

$$\frac{\psi_n}{\beta_n}\|h_n - h_{n-1}\| \leq C_2, \ \forall n \in \mathbb{N}. \qquad (3.39)$$

Using (3.10) and (3.39), we have

$$\begin{aligned}\|w_n - u^*\| &= \|h_n + \phi_n(h_n - h_{n-1}) - u^*\| \\ &\leq \|h_n - u^*\| + \psi_n\|h_n - h_{n-1}\| \\ &\leq \|h_n - u^*\| + \beta_n\frac{\phi_n}{\beta_n}\|h_n - h_{n-1}\| \\ &\leq \|h_n - u^*\| + \beta_n C_2. \end{aligned} \tag{3.40}$$

Combining (3.14), (3.30), (3.37), (3.40), and recalling Lemma 2.9, we have

$$\begin{aligned}\|u_{n+1} - u^*\| &= \|(1-\gamma_n)d_n - \beta_n\mu M d_n - [(1-\gamma_n)u^* - \beta_n\mu M d_n] + \gamma_n(u_n - u^*) - \beta_n\mu M u^*\| \\ &\leq \|(1-\gamma_n)d_n - \beta_n\mu M d_n - [(1-\gamma_n)u^* - \beta_n\mu M d_n]\| + \gamma_n\|u_n - u^*\| + \beta_n\mu\|Mu^*\| \\ &\leq (1-\gamma_n-\beta_n\ell)\|d_n - u^*\| + \gamma_n\|u_n - u^*\| + \beta_n\mu\|Mu^*\| \\ &\leq (1-\gamma_n-\beta_n\ell)\|w_n - u^*\| + \gamma_n\|u_n - u^*\| + \beta_n\mu\|Mu^*\| \\ &\leq (1-\gamma_n-\beta_n\ell)\|h_n - u^*\| + \beta_n C_2 + \gamma_n\|u_n - u^*\| + \beta_n\mu\|Mu^*\| \\ &\leq (1-\gamma_n-\beta_n\ell)\|u_n - u^*\| + \beta_n C_1 + \beta_n C_2 + \gamma_n\|u_n - u^*\| + \beta_n\mu\|Mu^*\| \\ &= (1-\beta_n\ell)\|u_n - u^*\| + \beta_n C_1 + \beta_n C_2 + \beta_n\mu\|Mu^*\| \\ &= (1-\beta_n\ell)\|u_n - u^*\| + \beta_n\ell\frac{\mu\|Mu^*\| + C_3}{\ell} \\ &\leq \max\left\{\|u_n - u^*\|, \frac{\mu\|Mu^*\| + C_3}{\ell}\right\},\end{aligned}$$

where $C_3 = C_1 + C_2$ and $\ell = 1 - \sqrt{1-\mu(2\alpha - \mu\Omega^2)} \in (0,1]$. Thus, by induction, for every $n \geq 0$, we obtain

$$\|u_k - u^*\| \leq \max\left\{\|u_0 - u^*\|, \frac{\mu\|Mu^*\| + C_3}{\ell}\right\}.$$

Hence, the sequence $\{u_n\}$ is bounded and so are the sequences $\{d_n\}$, $\{e_n\}$, $\{w_n\}$, $\{h_n\}$, $\{z_n\}$, $\{v_n\}$, $\{r_n\}$ and Md_n. □

Lemma 3.5. *Suppose Assumption 3.1 holds and $\{u_n\}$ is the sequence generated by Algorithm 3.1. If there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightharpoonup u^\dagger \in H_1$ and $\lim_{j\to\infty}\|r_{n_j} - h_{n_j}\| = 0 = \lim_{j\to\infty}\|w_{n_j} - d_{k_j}\|$. Then, $u^\dagger \in \Gamma^*$.*

Proof. Let $u^* \in \Gamma$. Now, from (3.28) and (3.29), we have

$$\|h_{n_j} - u^*\|^2 \leq \|r_{n_j} - u^*\|^2 - \delta_{n_j}(1-c)\|Br_{n_j} - z_{n_j}\|^2. \tag{3.41}$$

It follows that

$$\begin{aligned}\delta_{n_j}(1-c)\|Br_{n_j} - z_{n_j}\|^2 &\leq \|r_{n_j} - u^*\|^2 - \|h_{n_j} - u^*\|^2 \\ &\leq (\|r_{n_j} - h_{n_j}\| + \|h_{n_j} - u^*\|)^2 - \|h_{n_j} - u^*\|^2 \\ &= \|r_{n_j} - h_{n_j}\|^2 + 2\|r_{n_j} - h_{n_j}\|\|h_{n_j} - u^*\|.\end{aligned} \tag{3.42}$$

Since $\{h_{n_j}\}$ is bounded and by hypothesis we have that $\lim_{j\to\infty}\|r_{n_j}-h_{n_j}\|=0$, then by (3.42) and the assumptions on the control parameters, we have

$$\lim_{j\to\infty}\|Br_{n_j}-z_{n_j}\|=0. \tag{3.43}$$

Now, from (3.23), we get

$$\begin{aligned}
&\left(1-\frac{\lambda_{n_j}\lambda}{\lambda_{n_j+1}}\right)\|Br_{n_j}-v_{n_j}\|^2+\left(1-\frac{\lambda_{n_j}\lambda}{\lambda_{n_j+1}}\right)\|z_{n_j}-v_{n_j}\|^2\\
&\leq \|Br_{n_j}-Bu^*\|^2-\|z_{n_j}-Bu^*\|^2\\
&=\|Br_{n_j}-z_{n_j}\|^2+\|z_{n_j}-Bu^*\|^2+2\langle Br_{n_j}-z_{n_j},z_{n_j}-Bu^*\rangle-\|z_{n_j}-Bu^*\|^2\\
&\leq \|Br_{n_j}-z_{n_j}\|^2+2\|Br_{n_j}-z_{n_j}\|\|z_{n_j}-Bu^*\|.
\end{aligned} \tag{3.44}$$

Since $\lim_{j\to\infty}\lambda_{n_j+1}=\lambda$ and together with (3.43), we have

$$\lim_{j\to\infty}\|Br_{n_j}-v_{n_j}\|=0=\lim_{j\to\infty}\|z_{n_j}-v_{n_j}\|. \tag{3.45}$$

Now, from (3.4), we have $v_{n_j}=P_Q(Br_{n_j}-\lambda_{n_j}SBr_{n_j})$. By Lemma 2.1, we have

$$\langle Br_{n_j}-\lambda_{n_j}SBr_{n_j}-v_{n_j},u-v_{n_j}\rangle\leq 0,\ \forall u\in Q. \tag{3.46}$$

It follows that

$$\langle Br_{n_j}-v_{n_j},u-v_{n_j}\rangle-\lambda_{n_j}\langle SBr_{n_j},u-v_{n_j}\rangle\leq 0. \tag{3.47}$$

This means that

$$\begin{aligned}
\langle Br_{n_j}-v_{n_j},u-v_{n_j}\rangle &\leq \lambda_{n_j}\langle SBr_{n_j},u-v_{n_j}\rangle\\
&= \lambda_{n_j}\langle SBr_{n_j},Br_{n_j}-v_{n_j}\rangle+\lambda_{k_j}\langle SBr_{n_j},u-Br_{n_j}\rangle.
\end{aligned} \tag{3.48}$$

Thus, from (3.48) we have

$$\lambda_{n_j}^{-1}\langle Br_{n_j}-v_{n_j},u-v_{n_j}\rangle+\langle SBr_{n_j},v_{n_j}-Br_{k_j}\rangle\leq\langle SBr_{n_j},u-Br_{n_j}\rangle. \tag{3.49}$$

Since from Lemma 3.38, we have $\lim_{j\to\infty}\lambda_{n_j}=\lambda>0$. Then applying (3.45) in (3.49), we have

$$\liminf_{j\to\infty}\langle SBr_{n_j},u-Br_{n_j}\rangle\geq 0,\ \forall u\in U. \tag{3.50}$$

Notice that

$$\begin{aligned}
\langle Sv_{n_j},u-v_{n_j}\rangle &\leq \langle Sv_{n_j},u-Br_{n_j}\rangle+\langle Sv_{n_j},Br_{n_j}-v_{n_j}\rangle\\
&\leq \langle Sv_{n_j}-SBr_{n_j},u-Br_{n_j}\rangle+\langle SBr_{n_j},u-Br_{n_j}\rangle+\langle Sv_{n_j},Br_{n_j}-v_{n_j}\rangle
\end{aligned} \tag{3.51}$$

By the uniform continuity of S on H_2 and (3.45), we get

$$\lim_{j\to\infty}\|Sv_{n_j}-SBr_{n_j}\|=0. \tag{3.52}$$

Combining (3.45), (3.51) and (3.52), we have

$$\liminf_{j\to\infty}\langle Sv_{n_j},u-v_{n_j}\rangle\geq 0. \tag{3.53}$$

Next, we show that $Bu^\dagger \in S_D^\dagger$. To show this, we first consider the case where $\limsup_{j\to\infty}\langle Sv_{n_j}, u - v_{n_j}\rangle > 0$, for all $u \in Q$. Without loss of generality, there exists a subsequence $v_{n_{j_i}}$ of v_{n_j} such that

$$\lim_{i\to\infty}\langle Sv_{n_{j_i}}, u - v_{n_{j_i}}\rangle > 0, \ \forall u \in Q. \tag{3.54}$$

It implies that one can find $i_0 \in \mathbb{N}$ such that

$$\langle Sv_{n_{j_i}}, u - v_{n_{j_i}}\rangle > 0, \ \forall i > i_0. \tag{3.55}$$

By the quasimonotonicity of S, we have

$$\langle Su, u - v_{n_{j_i}}\rangle \geq 0, \ \forall i > i_0. \tag{3.56}$$

From (3.2), we have

$$\|r_{n_{j_i}} - u_{n_{j_i}}\| = \beta_{n_{j_i}}\frac{\phi_{n_{j_i}}}{\beta_{n_{j_i}}}\|u_{n_{j_i}} - u_{n_{j_i}-1}\| \to 0 \,\text{as}\, i \to \infty. \tag{3.57}$$

Since $\{u_{n_{j_i}}\}$ is a subsequence of $\{u_{n_j}\}$ and $u_{n_{j_i}} \rightharpoonup u^\dagger \in H_1$. Furthermore, since B is a bounded linear operator we know that $Br_{n_{j_i}} \rightharpoonup Bu^\dagger$ and by (3.45), we have $\lim_{j\to\infty}\|Br_{n_{j_i}} - v_{n_{j_i}}\| = 0$. Thus, it implies that $v_{k_{j_i}} \rightharpoonup Bu^\dagger$.

Now, passing the limit as $i \to \infty$ in (3.56), we have

$$\lim_{i\to\infty}\langle Su, u - v_{n_{j_i}}\rangle = \langle Su, u - Bu^\dagger\rangle \geq 0. \tag{3.58}$$

Hence, $Bu^\dagger \in S_D^\dagger$.

In the second case, we consider $\limsup_{j\to\infty}\langle Sv_{n_j}, u - v_{n_j}\rangle = 0$, for all $u \in Q$. Let $\{\theta_j\}$ be a non-increasing positive sequence defined by

$$\theta_j = |\langle Sv_{n_j}, u - v_{n_j}\rangle| + \frac{1}{j+1}. \tag{3.59}$$

Thus, we have that

$$\lim_{j\to\infty}\theta_j = \lim_{j\to\infty}|\langle Sv_{n_j}, u - v_{n_j}\rangle| + \lim_{j\to\infty}\frac{1}{j+1} = 0. \tag{3.60}$$

This means that

$$\langle Sv_{n_j}, u - v_{n_j}\rangle + \theta_j > 0, \tag{3.61}$$

for each $j \geq 1$. Since $v_{n_j} \subset Q$, it follows that $\{Sv_{n_j}\}$ is strictly non-zero and $\liminf_{j\to\infty}\|Sv_{n_j}\| = K_0 > 0$. Thus, we deduce that

$$\|Sv_{n_j}\| > \frac{K_0}{2}. \tag{3.62}$$

Furthermore, let the sequence $\{\rho_{n_j}\}$ be defined by $\rho_{n_j} = \frac{Sv_{n_j}}{\|Sv_{n_j}\|^2}$. It follows that

$$\langle Sv_{n_j}, \rho_{n_j}\rangle = 1. \tag{3.63}$$

Now, we can deduce from (3.61) that

$$\langle Sv_{n_j}, u + \theta_j \rho_{n_j} - v_{n_j} \rangle > 0, \tag{3.64}$$

Due to the quasimonotoniclity of S on H_2, we have

$$\langle S(u + \theta_j \rho_{n_j}), u + \theta_j \rho_{n_j} - v_{n_j} \rangle \geq 0. \tag{3.65}$$

Using (3.65), (3.62), the definition of ρ_{n_j} and the Lipschitz continuity of S on H_2, we have

$$\begin{aligned}
\langle Su, u + \theta_j \rho_{n_j} - v_{n_j} \rangle &= \langle Su - S(u + \theta_j \rho_{n_j}) + S(u + \theta_j \rho_{n_j}), u + \theta_j \rho_{n_j} - v_{n_j} \rangle \\
&= \langle Su - S(u + \theta_j \rho_{n_j}), u + \theta_j \rho_{n_j} - v_{n_j} \rangle + \langle S(u + \theta_j \rho_{n_j}), u + \theta_j \rho_{n_j} - v_{n_j} \rangle \\
&\geq \langle Su - S(u + \theta_j \rho_{n_j}), u + \theta_j \rho_{n_j} - v_{n_j} \rangle \\
&\geq -\|Su - S(u + \theta_j \rho_{n_j})\| \|u + \theta_j \rho_{n_j} - v_{n_j}\| \\
&\geq -L_2 \|\theta_j \rho_{n_j}\| \|u + \theta_j \rho_{n_j} - v_{n_j}\| \\
&\geq -\frac{2L_2}{K_0} \theta_j \|u + \theta_j \rho_{n_j} - v_{n_j}\|.
\end{aligned} \tag{3.66}$$

Since $\{u_{n_j}\}$ is a subsequence of $\{u_n\}$ such that $u_{n_j} \rightharpoonup u^\dagger \in H_1$, it follows also from (3.57) that the subsequence $\{r_{n_j}\}$ of $\{r_n\}$ converges weakly to $u^\dagger \in H_1$. Again, since B is a bounded linear operator, it implies that $Br_{n_j} \rightharpoonup Bu^\dagger$. Thus, by (3.45), it follows that $v_{n_j} \rightharpoonup Bu^\dagger$. In (3.66), letting $j \to \infty$ and using the fact that $\lim_{j\to\infty} \theta_j = 0$ and the boundedness of $\{\|u + \theta_j \rho_{n_j} - v_{n_j}\|\}$, we obtain

$$\langle Su, u - Bu^\dagger \rangle \geq 0, \ \forall u \in Q.$$

This implies that $Bu^\dagger \in S_D^\dagger$.

Next, from (3.36), we have

$$\|d_{n_j} - u^*\|^2 \leq \|w_{n_j} - u^*\|^2 - \left(1 - \frac{\tau^2 \tau_{n_j}^2}{\tau_{n_j+1}^2}\right) \|w_{n_j} - e_{n_j}\|^2, \tag{3.67}$$

which implies that

$$\begin{aligned}
\left(1 - \frac{\tau^2 \tau_{n_j}^2}{\tau_{n_j+1}^2}\right) \|w_{n_j} - e_{n_j}\|^2 &\leq \|w_{n_j} - u^*\|^2 - \|d_{n_j} - u^*\|^2 \\
&= \|w_{n_j} - d_{n_j}\|^2 + \|d_{n_j} - u^*\|^2 + 2\langle w_{n_j} - d_{n_j}, d_{n_j} - u^* \rangle - \|d_{n_j} - u^*\|^2 \\
&\leq \|w_{n_j} - d_{n_j}\|^2 + 2\|w_{n_j} - d_{n_j}\| \|d_{n_j} - u^*\|.
\end{aligned}$$

Due to our hypothesis $\lim_{j\to\infty} \|w_{n_j} - d_{k_j}\| = 0$ and the boundedness of $\{d_{n_j}\}$, we obtain

$$\lim_{j\to\infty} \|w_{n_j} - e_{k_j}\| = 0.$$

Also, from (3.10), we have

$$\|w_{n_{j_i}} - h_{n_{j_i}}\| = \beta_{n_{j_i}} \frac{\psi_{n_{j_i}}}{\beta_{n_{j_i}}} \|h_{n_{j_i}} - h_{n_{j_i}-1}\| \to 0 \text{ as } i \to \infty. \tag{3.68}$$

Following similar approach above, we can show that $u^\dagger \in S_D^*$. Hence, we have that $u^\dagger \in \Gamma$. □

Lemma 3.6. *Let $\{u_n\}$ be the sequence generated by Algorithm* (3.1)*; for all $n \in \mathbb{N}$, we have*

$$\begin{aligned} \|u_{n+1} - u^*\|^2 &\leq (1 - \beta_n \ell)\|u_n - u^*\|^2 \\ &\quad + \beta_n \ell \left[\frac{\phi_n}{\beta_n \ell}\|u_n - u_{n-1}\|C_4 + \frac{\psi_n}{\beta_n \ell}\|h_n - h_{n-1}\|C_5 + \frac{2\mu}{\ell}\langle Mu^*, u^* - u_{n+1}\rangle \right]. \end{aligned} \tag{3.69}$$

where u^* is a solution of the BSVIP (1.4).

Proof. Indeed, from (3.2), Lemma 2.3 and (3.25), we have

$$\begin{aligned} \|r_n - u^*\|^2 &= \|u_n + \phi_n(u_n - u_{n-1}) - u^*\|^2 \\ &= \|u_n - u^*\|^2 + 2\phi_n\langle u_n - u^*, u_n - u_{n-1}\rangle + \phi_n^2\|u_n - u_{n-1}\|^2 \\ &\leq \|u_n - u^*\|^2 + 2\phi_n\|u_n - u^*\|\|u_n - u_{n-1}\| + \phi_n^2\|u_n - u_{n-1}\|^2 \\ &= \|u_n - u^*\|^2 + \phi_n\|u_n - u_{n-1}\|(2\|u_n - u^*\| + \phi_n\|u_n - u_{n-1}\|) \\ &\leq \|u_n - u^*\|^2 + \phi_n\|u_n - u_{n-1}\|C_4. \end{aligned} \tag{3.70}$$

where $C_4 = \sup(2\|u_n - u^*\| + \phi_n\|u_n - u_{n-1}\|) < \infty$.

Also, from (3.10), Lemma 2.3 and (3.39), we have

$$\begin{aligned} \|w_n - u^*\|^2 &= \|h_n + \psi_n(h_n - h_{n-1}) - u^*\|^2 \\ &= \|h_n - u^*\|^2 + 2\psi_n\langle h_n - u^*, h_n - u_{n-1}\rangle + \phi_n^2\|h_n - h_{n-1}\|^2 \\ &\leq \|h_n - u^*\|^2 + 2\psi_n\|h_n - u^*\|\|h_n - h_{n-1}\| + \phi_n^2\|h_n - h_{n-1}\|^2 \\ &= \|h_n - u^*\|^2 + \psi_n\|h_n - h_{n-1}\|(2\|h_n - u^*\| + \psi_n\|h_n - h_{n-1}\|) \\ &\leq \|h_n - u^*\|^2 + \psi_n\|h_n - h_{n-1}\|C_5. \end{aligned} \tag{3.71}$$

where $C_5 = \sup(2\|h_n - u^*\| + \psi_n\|h_n - h_{n-1}\|) < \infty$. Now, using (3.14), (3.36), (3.37), (3.71), Lemma 2.9, and Lemma 2.3, we obtain

$$\begin{aligned} \|u_{n+1} - u^*\|^2 &= \|(1-\gamma_n)d_n - \beta_n\mu Md_n - [(1-\gamma_n)u^* - \beta_n\mu Md_n] + \gamma_n(u_n - u^*) - \beta_n\mu Mu^*\|^2 \\ &= \|(1-\gamma_n)d_n - \beta_n\mu Md_n - [(1-\gamma_n)u^* - \beta_n\mu Md_n] + \gamma_n(u_n - u^*)\|^2 \\ &\quad + 2\beta_n\mu\langle Mu^*, u^* - u_{n+1}\rangle \\ &\leq \{\|(1-\gamma_n)d_n - \beta_n\mu Md_n - [(1-\gamma_n)u^* - \beta_n\mu Md_n]\| + \gamma_n\|u_n - u^*\|\}^2 \\ &\quad + 2\beta_n\mu\langle Mu^*, u^* - u_{n+1}\rangle \\ &\leq [(1-\gamma_n - \beta_n\ell)\|d_n - u^*\| + \gamma_n\|u_n - u^*\|]^2 + 2\beta_n\mu\langle Mu^*, u^* - u_{n+1}\rangle \\ &\leq (1-\gamma_n - \beta_n\ell)\|d_n - u^*\|^2 + \gamma_n\|u_n - u^*\|^2 + 2\beta_n\mu\langle Mu^*, u^* - u_{n+1}\rangle \\ &\leq (1-\gamma_n - \beta_n\ell)\|w_n - u^*\|^2 - (1-\gamma_n - \beta_n\ell)\left(1 - \frac{\tau^2\tau_n^2}{\tau_{n+1}^2}\right)\|w_n - e_n\|^2 \\ &\quad + \gamma_n\|u_n - u^*\|^2 + 2\beta_n\mu\langle Mu^*, u^* - u_{n+1}\rangle \\ &\leq (1-\gamma_n - \beta_n\ell)[\|h_n - u^*\|^2 + \psi_n\|h_n - h_{n-1}\|C_5] \end{aligned}$$

$$
\begin{aligned}
& -(1-\gamma_n-\beta_n\ell)\left(1-\frac{\tau^2\tau_n^2}{\tau_{n+1}^2}\right)\|w_n-e_n\|^2+\gamma_n\|u_n-u^*\|^2+2\beta_n\mu\langle Mu^*,u^*-u_{n+1}\rangle \\
\leq \quad & (1-\gamma_n-\beta_n\ell)\|h_n-u^*\|^2+\psi_n\|h_n-h_{n-1}\|C_5-(1-\gamma_n-\beta_n\ell)\left(1-\frac{\tau^2\tau_n^2}{\tau_{n+1}^2}\right)\|w_n-e_n\|^2 \\
& +\gamma_n\|u_n-u^*\|^2+2\beta_n\mu\langle Mu^*,u^*-u_{n+1}\rangle \qquad (3.72)
\end{aligned}
$$

Combining (3.72), (3.41) and (3.70), we have

$$
\begin{aligned}
& \|u_{n+1}-u^*\|^2 \\
\leq \quad & (1-\gamma_n-\beta_n\ell)\|r_n-u^*\|^2-(1-\gamma_n-\beta_n\ell)\delta_n(1-c)\|Br_n-z_n\|^2 \\
& +\psi_n\|h_n-h_{n-1}\|C_5-(1-\gamma_n-\beta_n\ell)\left(1-\frac{\tau^2\tau_n^2}{\tau_{n+1}^2}\right)\|w_n-e_n\|^2 \\
& +\gamma_n\|u_n-u^*\|^2+2\beta_n\mu\langle Mu^*,u^*-u_{n+1}\rangle \\
\leq \quad & (1-\gamma_n-\beta_n\ell)\|u_n-u^*\|^2+\phi_n\|u_n-u_{n-1}\|C_4-(1-\gamma_n-\beta_n\ell)\delta_n(1-c)\|Br_n-z_n\|^2 \\
& +\psi_n\|h_n-h_{n-1}\|C_5-(1-\gamma_n-\beta_n\ell)\left(1-\frac{\tau^2\tau_n^2}{\tau_{n+1}^2}\right)\|w_n-e_n\|^2 \\
& +\gamma_n\|u_n-u^*\|^2+2\beta_n\mu\langle Mu^*,u^*-u_{n+1}\rangle \\
= \quad & (1-\beta_n\ell)\|u_n-u^*\|^2-(1-\gamma_n-\beta_n\ell)\delta_{n_j}(1-c)\|Br_n-z_n\|^2-(1-\gamma_n-\beta_n\ell)\left(1-\frac{\tau^2\tau_n^2}{\tau_{n+1}^2}\right)\|w_n-e_n\|^2 \\
& +\beta_n\ell\left[\frac{\phi_n}{\beta_n\ell}\|u_n-u_{n-1}\|C_4+\frac{\psi_n}{\beta_n\ell}\|h_n-h_{n-1}\|C_5+\frac{2\mu}{\ell}\langle Mu^*,u^*-u_{n+1}\rangle\right] \qquad (3.73) \\
\leq \quad & (1-\beta_n\ell)\|u_n-u^*\|^2 \\
& +\beta_n\ell\left[\frac{\phi_n}{\beta_n\ell}\|u_n-u_{n-1}\|C_4+\frac{\psi_n}{\beta_n\ell}\|h_n-h_{n-1}\|C_5+\frac{2\mu}{\ell}\langle Mu^*,u^*-u_{n+1}\rangle\right].
\end{aligned}
$$

□

Theorem 3.1. *Suppose $\{u_n\}$ is the sequence generated by Algorithm 3.1 under Assumption 3.1. Then, $\{u_n\}$ strongly converges to a unique solution of the BSVIP* (1.4).

Proof. Let $u^*\in\Gamma^*$. We, claim that $\{\|u_n-u^*\|^2\}$ converges to zero. Indeed, from Lemma 2.4, (3.24) and (3.38), it suffices to prove that $\limsup_{j\to\infty}\langle Mu^*,u^*-u_{n_j+1}\rangle\leq 0$, for every subsequence $\{\|u_{n_j}-u^*\|\}$ of $\{\|u_n-u^*\|\}$ fulfilling

$$\liminf_{j\to\infty}(\|u_{n_j+1}-u^*\|-\|u_{n_j}-u^*\|)\geq 0. \qquad (3.74)$$

In what follows, suppose $\{|u_{n_j}-u^*\|\}$ is a subsequence of $\{\|u_k-u^*\|\}$ such that (3.74) holds. Then, from (3.73), $\lim_{j\to\infty}\beta_{n_j}=0$, Lemma 3.1 and Assumption 3.1 (A_7), we have

$$\limsup_{j\to\infty}\left[(1-\gamma_{n_j}-\beta_{n_j}\ell)\delta_{n_j}(1-c)\|Br_{n_j}-z_{n_j}\|^2+(1-\gamma_{n_j}-\beta_n\ell)\left(1-\frac{\tau^2\tau_{n_j}^2}{\tau_{n_j+1}^2}\right)\|w_{n_j}-e_{n_j}\|^2\right]$$

$$\leq \limsup_{j\to\infty}\Big((1-\beta_{n_j}\ell)\|u_{n_j}-u^*\|^2-\|u_{n_j+1}-u^*\|^2\Big)$$
$$+\limsup_{j\to\infty}\left(\beta_{n_j}\ell\left[\frac{\phi_{n_j}}{\beta_{n_j}\ell}\|u_{n_j}-u_{n_j-1}\|C_4+\frac{\psi_{n_j}}{\beta_{n_j}\ell}\|h_{n_j}-h_{n_j-1}\|C_5+\frac{2\mu}{\ell}\|Mu^*\|\|u^*-u_{n_j+1}\|\right]\right)$$
$$=-\liminf_{j\to\infty}\Big(\|u_{n_j+1}-u^*\|^2-\|u_{n_j}-u^*\|^2\Big)$$
$$\leq 0.$$

This implies that

$$\lim_{j\to\infty}\|Br_{n_j}-z_{n_j}\|=0,\ \text{ and }\ \lim_{j\to\infty}\|w_{n_j}-e_{n_j}\|=0. \tag{3.75}$$

Now, from (3.7), we have

$$\|h_{n_j}-r_{n_j}\|=\delta_{n_j}\|B^*(z_{n_j}-Br_{n_j})\|=c\|z_{n_j}-Br_{n_j}\|^2\to 0 \text{ as } n\to\infty. \tag{3.76}$$

Clearly, if $\|B^*(z_{n_j}Br_{n_j})\|=0$, we still have that $\lim_{j\to\infty}\|h_{n_j}-r_{n_j}\|=0$.

Next, combining (3.75), (3.12) and (3.35), we have

$$\|e_{n_j}-d_{n_j}\|\leq\frac{\tau_{n_j}\tau}{\tau_{n_j+1}}\|w_{n_j}-e_{n_j}\|\to 0 \text{ as } j\to\infty. \tag{3.77}$$

From (3.77) and (3.75), we have

$$\|d_{n_j}-w_{n_j}\|\leq\|d_{n_j}-e_{n_j}\|+\|e_{n_j}-w_{n_j}\|\to 0 \text{ as } j\to\infty. \tag{3.78}$$

Also, using (3.78), (3.68), (3.76), and (3.57), we have

$$\|d_{n_j}-u_{n_j}\|\leq\|d_{n_j}-w_{n_j}\|+\|w_{n_j}-h_{n_j}\|+\|h_{n_j}-r_{n_j}\|+\|r_{n_j}-u_{n_j}\|\to 0 \text{ as } j\to\infty. \tag{3.79}$$

Since $\lim_{j\to\infty}\beta_{n_j}=0$ and Md_{n_j} is bounded, using (3.14) and (3.79) together with the boundedness of $\{\gamma_{n_j}\}$, we have

$$\|u_{n_j+1}-d_{n_j}\|\leq\gamma_n\|u_{n_j}-d_{n_j}\|+\beta_{n_j}\mu\|Md_{n_j}\|\to 0 \text{ as } j\to\infty. \tag{3.80}$$

Now, combining (3.79) and (3.80), we have

$$\|u_{n_j+1}-u_{n_j}\|\leq\|u_{n_j+1}-d_{n_j}\|+\|d_{n_j}-u_{n_j}\|\to 0 \text{ as } j\to\infty. \tag{3.81}$$

Since u_{n_j} is bounded, then $w_\omega(u_n)$ is nonempty. If we arbitrary take $\bar{u}\in w_\omega(u_n)$, then, a subsequence $u_{k_{j_i}}$ of $\{u_{n_j}\}$ exists such that $u_{n_{j_i}}\rightharpoonup\bar{u}$ as $i\to\infty$. By (3.57), we know that $r_{k_{j_i}}\rightharpoonup\bar{u}$ as $j\to\infty$. Now, applying Lemma 3.5, (3.76) and (3.78), we get $\bar{u}\in\Gamma^*$.

Again, since $u_{k_{j_i}}\rightharpoonup\bar{u}$ as $i\to\infty$, it implies that

$$\limsup_{j\to\infty}\langle Mu^*,u^*-u_{n_j}\rangle=\lim_{i\to\infty}\langle Mu^*,u^*-u_{k_{j_i}}\rangle=\langle Mu^*,u^*-\bar{u}\rangle.$$

Since u^* is a solution of the BSVIP (1.4), it implies that

$$\limsup_{j\to\infty}\langle Mu^*,u^*-u_{n_j}\rangle=\langle Mu^*,u^*-\bar{u}\rangle\leq 0. \tag{3.82}$$

Using (3.81) and (3.82), we have

$$\limsup_{j\to\infty}\langle Mu^*, u^* - u_{n_j+1}\rangle = \limsup_{j\to\infty}\langle Mu^*, u^* - u_{n_j}\rangle = \langle Mu^*, u^* - \bar{u}\rangle \le 0. \tag{3.83}$$

Combining Lemma 2.4, Lemma 3.6, (3.83), (3.24), and (3.38), we have that $u_n \to u^*$ as $n \to \infty$. Hence, we know that $\{u_n\}$ strongly converges to a unique solution of the the BSVIP (1.4). □

4. Numerical Examples

In this section, we present several numerical experiments to demonstrate the effectiveness of the proposed method in comparison with some well-established strongly convergent algorithms available in the literature. In particular, we compare our approach with Algorithm 1 of Van *et al.* [38] (denoted by VTA Alg 1), Algorithm 3.1 of Huy *et al.* [16] (denoted by HVHA Alg 3.1), and Algorithm 3.1 of Peng *et al.* [3] (denoted by PLZL Alg 3.1). All numerical computations are carried out using MATLAB 2018a on a standard personal computer.

For Algorithm 3.1 (referred to as OMM Alg 3.2), the parameters are selected as follows:

$$\phi = 0.35, \quad \psi = 4, \quad \tau_1 = 0.75, \quad \lambda_1 = 0.65, \quad \tau = 0.25, \quad \lambda = 0.10,$$
$$\epsilon_n = \xi_n = \frac{1}{(n+1)^2}, \quad \beta_n = \frac{1}{(n+2)}, \quad p_n = q_n = \frac{1}{(n+1)^{1.05}}, \quad \gamma_n = \frac{1}{n+3}, \quad \mu = \frac{0.6\alpha}{L_M^2}.$$

For VTA Alg 1, we choose

$$\lambda_1 = 0.75, \quad \mu_1 = 0.65, \quad \lambda = 0.10, \quad \mu = 0.25, \quad \delta_n = 0.85,$$
$$\alpha_n = \frac{1}{(n+2)}, \quad \gamma_n = \frac{1}{n+3}, \quad \alpha = \frac{0.6\alpha}{L_M^2}.$$

For HVHA Alg 3.1, the parameters are taken as

$$\lambda_1 = 0.75, \quad \mu_1 = 0.65, \quad \lambda = 0.10, \quad \mu = 0.25, \quad \delta_n = 0.85,$$
$$\alpha_n = \frac{1}{(n+2)}, \quad \gamma = \frac{0.6\alpha}{L_M^2}.$$

For PLZL Alg 3.1, we select

$$\theta = 0.35, \quad \mu = 0.10, \quad \beta_1 = 0.65, \quad \tau = 1.3, \quad \gamma = 2.5,$$
$$\epsilon_n = \frac{1}{(n+1)^2}, \quad \sigma_n = \frac{1}{(n+2)}, \quad \xi_n = \frac{1}{(n+1)^{1.05}}, \quad \lambda = \frac{0.6\alpha}{L_M^2}.$$

Example 4.1. *Next, let the operators $S, T : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ be defined by*

$$Su = Tu = \left(\frac{1}{\|u\| + s} + \|u\|\right)u, \ \forall u \in \ell_2(\mathbb{R}), s > 0.$$

It is not hard to verify that S and T are uniformly continuous and pseudomonotone operators (hence, quasimonotone). Furthermore, we define the mapping $M : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ by

$$Mu = \frac{u}{2}, \ \forall u \in \ell_2(\mathbb{R}).$$

For this numerical experiment, we consider the stopping criterion $E_n = \|u_{n+1} - u_n\| < 10^{-5}$ *and the following cases for starting points:*

Case I: $u_0 = \left(2, \frac{1}{4}, \frac{1}{6}, \cdots\right)$ *and* $u_1 = \left(1, \frac{1}{6}, \frac{1}{8}, \cdots\right)$.

Case II: $u_0 = \left(-4, \frac{1}{3}, \frac{1}{7}, \cdots\right)$ *and* $u_1 = \left(-2, \frac{1}{5}, \frac{1}{6}, \cdots\right)$.

Case III: $u_0 = \left(4, \frac{1}{6}, \frac{1}{5}, \cdots\right)$ *and* $u_1 = \left(\frac{1}{5}, \frac{1}{3}, \frac{1}{4}, \cdots\right)$.

Case IV: $u_0 = \left(-\frac{1}{3}, \frac{1}{7}, \frac{1}{10}, \cdots\right)$ *and* $u_1 = \left(2, \frac{1}{6}, \frac{1}{5}, \cdots\right)$.

In each of the cases, we plot the graphs of errors against the number of iterations as shown in Table 1 and Figure 1.

Table 1. Comparison of various methods for four Cases

Case 1 Results		
Method	***Iterations***	***CPU time (s)***
VTA Alg 1	*52*	*0.01431*
PLZL Alg 3.1	*61*	*0.01489*
HVHA Alg 3.1	*35*	*0.02392*
OMM Alg 3.2	*25*	*0.00543*
Case 2 Results		
Method	***Iterations***	***CPU time (s)***
VTA Alg 1	*56*	*0.00498*
PLZL Alg 3.1	*65*	*0.00466*
HVHA Alg 3.1	*37*	*0.00387*
OMM Alg 3.2	*27*	*0.00501*
Case 3 Results		
Method	***Iterations***	***CPU time (s)***
VTA Alg 1	*56*	*0.00385*
PLZL Alg 3.1	*65*	*0.00322*
HVHA Alg 3.1	*37*	*0.00215*
OMM Alg 3.2	*27*	*0.00255*
Case 4 Results		
Method	***Iterations***	***CPU time (s)***
VTA Alg 1	*45*	*0.00233*
PLZL Alg 3.1	*52*	*0.00265*
HVHA Alg 3.1	*30*	*0.00160*
OMM Alg 3.2	*22*	*0.00206*

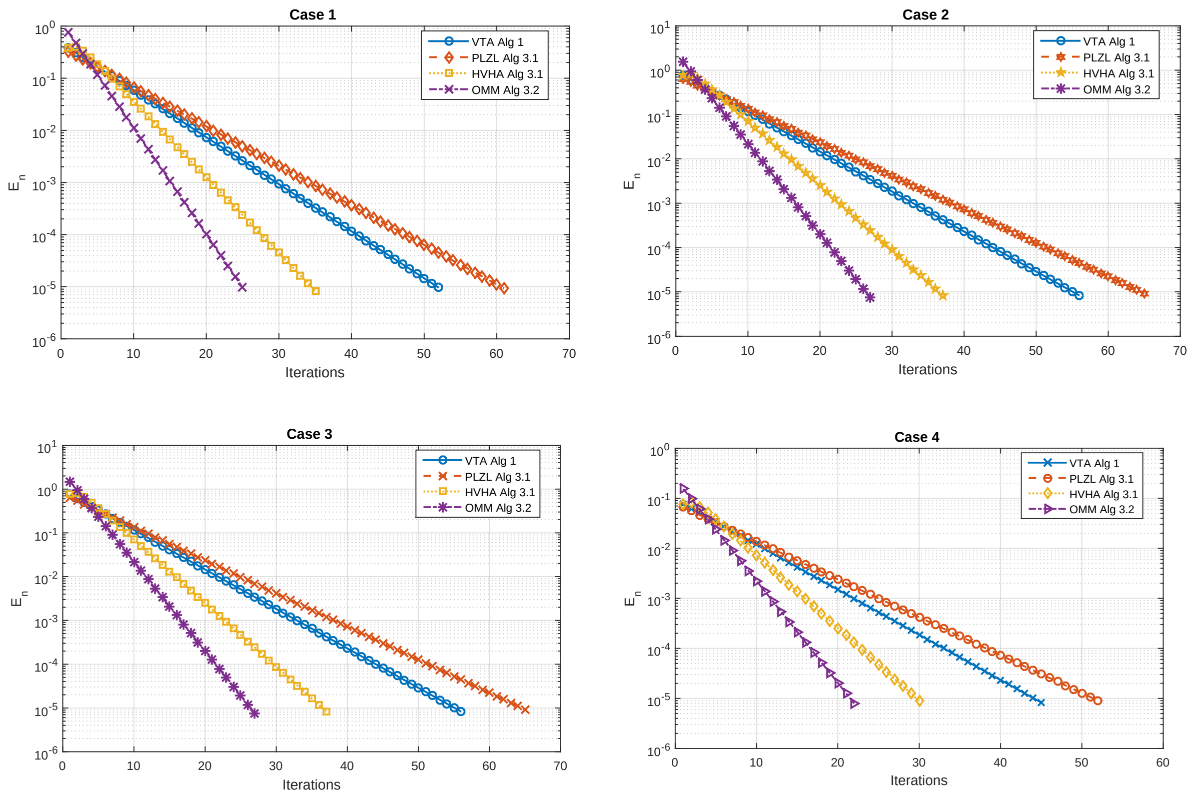

Figure 1. Plot of convergence results for various methods.

5. Conclusion

In this paper, we have developed a robust extragradient-type algorithm for solving the bilevel split variational inequality problem (BSVIP) in real Hilbert spaces. The proposed scheme integrates key features of the subgradient extragradient method and Tseng's extragradient method, resulting in an efficient hybrid approach. Furthermore, the algorithm incorporates two inertial extrapolation steps, which significantly enhance its convergence behavior.

Strong convergence of the method is established under mild conditions imposed on the control parameters. Notably, the operators involved in both the upper- and lower-level problems are assumed to be uniformly continuous, which represents a weaker requirement than the Lipschitz continuity condition commonly adopted in the literature. Consequently, the results presented here extend, generalize, and unify several existing works.

An additional advantage of the proposed method is that its implementation does not rely on prior knowledge of the norm of the bounded linear operator or the Lipschitz constants of the underlying mappings. Numerical results reported in Section 4 clearly demonstrate the superior performance of the proposed algorithm when compared with several well-known methods from the literature.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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