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Fixed Point Theory for Multivalued Integral Contractions in ϑ_{ν}^{ℓ}-Metric Spaces with Geometric and Applied Perspectives

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Abstract. Here, we propose a complete fixed point theory for set-valued mappings possessing integral-type contractive conditions in ϑ_{ν}^{ℓ}-metric spaces–a novel extension of the traditional notion of metric spaces proposed very recently by Joshi and co-authors. With the aid of some auxiliary concepts like α-admissible mappings, together with integral-type contraction conditions, we establish some new results on the existence and uniqueness of fixed points of set-valued mappings that not only generalize but also merge several fundamental theorems in the literature, particularly those by Branciari, Nadler, and Mizoguchi-Takahashi. In addition to the existence issues, we consider the structure of the set of fixed points through the concepts of fixed circle and fixed disc, providing thus more detailed information about the position of the fixed points. To substantiate our theoretical findings, we provide various applications in solving three different classes of problems, namely, differential inclusions, fractional partial differential equations, and Volterra integro-differential equations.

1. Introduction

Fixed point theory is considered one of the richest areas of contemporary mathematics because of its deep relationships with analysis, differential equations, optimization, control theory, and mathematical economics. The cornerstone of this field is the Banach contraction principle [13], whose beauty and strength have motivated many mathematicians to explore its limits and extend its scope. This has led to a number of interesting developments, among which the study of fixed point theorems in the context of set-valued maps [31], contractions defined by integral operators [16], and fixed point properties in abstract spaces that deviate from traditional metric assumptions [12, 15].

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Development in the area has seen a series of milestones being reached one after another. One such milestone was reached when Nadler [31] reformulated Banach's result in terms of multi-valued mappings using the Hausdorff distance, thus allowing the consideration of problems where there is no unique solution. Another major departure from the classical approach was made independently by Branciari [16], who presented a new contraction mapping using an integral inequality as a technique; this proved a very fruitful development and gave rise to a number of works in this direction [8,27,37]. Along with this, a search for a more general geometric structure led to the consideration of b-metric spaces [12] and rectangular metric spaces [15].

One such recent development worth mentioning is the concept of ϑ_ν^ℓ metric spaces proposed by Joshi et al. [22]. This notion encompasses several different metrics studied before in a broader setting but retains many topological characteristics that help in their study. The effectiveness of such spaces has been proven through various applications like cantilever beams [22], disease outbreaks [5], and satellite web coupling problems [21]. Fixed point theory on such spaces has further been studied and advanced since then [6,23].

Another avenue of research that is just as rewarding and parallel is the geometry of fixed point sets. Instead of simply proving that there exists a fixed point, it becomes important to describe the geometric structure of the set of fixed points. The investigation of fixed circles and fixed discs, first proposed and studied in [30,33,35,38], answers this very query, providing insight into the locations where invariant points exist. The importance of the geometric approach even extends to applications like neural networks and machine learning [6,39].

Another major contribution to the theory, which has added immensely to its development, is through the utilization of auxiliaries. The concept of α-admissible maps, introduced by Samet et al. and later explored by other authors [4,7], offers a way of localizing contraction conditions to specific areas within the space. This, when applied alongside the condition of integral contraction, has proved to be extremely effective in dealing with highly non-linear problems [17,18].

The applications of fixed point theory are not limited to mathematics alone. Existence theorems in the field of differential inclusions [11,26,32], fractional differential equations [3], and numerous types of integral and integro-differential equations fundamentally employ the principles of fixed points. Frigon [20] provided an important survey highlighting the importance of these methods, which is further accentuated by the advancements in fractional calculus [3]

Inspired by the above interwoven strands, we explore the existence of fixed points for multi-valued maps under integral contraction conditions in the framework of the ϑ_ν^ℓ-metric space. The main findings can be summarized into the following five aspects. Firstly, some new fixed point results have been obtained, which encompass several classic works of Branciari [16], Nadler [31], and Mizoguchi-Takahashi [29]. Secondly, the notion of α-admissibility plays an important role in establishing our main results since it provides a method to enforce contraction conditions on mappings only when necessary. This feature enables us to handle the case where the considered mappings are not contractive over the whole space. Thirdly, geometric characteristics of fixed point sets are investigated using fixed circles and fixed discs. Fourthly, we give applications of the abstract fixed point theory to differential inclusions, time-fractional partial differential equations, and nonlinear integro-differential equations. Finally, a numerical example of a control problem has been provided to highlight the practical significance of the proposed approach.

2. Preliminaries

This section assembles the foundational definitions and results upon which the subsequent analysis depends. The material spans generalized metric structures, set-valued mappings, integral contraction conditions, and geometric fixed point concepts.

Definition 2.1 ([22]). *Let Ξ be a non-empty set and $\vartheta_\nu^\ell : \Xi \times \Xi \longrightarrow [0, \infty)$ a mapping. The pair $(\Xi, \vartheta_\nu^\ell)$ is called a ϑ_ν^ℓ-metric space if, for all distinct elements $u, w, \chi_j \in \Xi$ with $0 \leq j \leq \nu$ and for some fixed choice of ν, the following axioms are satisfied:*

($\vartheta_\nu^\ell 1$) $\vartheta_\nu^\ell(u, w) = \vartheta_\nu^\ell(u, u) = \vartheta_\nu^\ell(w, w) \Leftrightarrow u = w$,

($\vartheta_\nu^\ell 2$) $\vartheta_{\nu_{u,w}}^\ell \leq \vartheta_\nu^\ell(u, w)$,

($\vartheta_\nu^\ell 3$) $\vartheta_\nu^\ell(u, w) = \vartheta_\nu^\ell(w, u)$,

($\vartheta_\nu^\ell 4$) $$\left(\vartheta_\nu^\ell(u, w) - \vartheta_{\nu_{u,w}}^\ell\right) \leq s\left[\left(\vartheta_\nu^\ell(u, \chi_1) - \vartheta_{\nu_{u,\chi_j}}^\ell\right) + \left(\vartheta_\nu^\ell(\chi_1, \chi_2) - \vartheta_{\nu_{\chi_1,\chi_j}}^\ell\right) + \cdots + \left(\vartheta_\nu^\ell(\chi_\nu, w) - \vartheta_{\nu_{\nu,w}}^\ell\right)\right] - \sum_{j=1}^{\nu} \vartheta_\nu^\ell(\chi_j, \chi_j), \quad s \geq 1,$$

where $\vartheta_{\nu_{u,w}}^\ell = \min\{\vartheta_\nu^\ell(u, u), \vartheta_\nu^\ell(w, w)\}$.

The ϑ_ν^ℓ-metric space unifies and extends a broad family of previously studied structures, among them classical metric spaces, b-metric spaces [12], rectangular metric spaces [15], and further variants thereof [10, 34], all while retaining a rich topological architecture compatible with a wide range of analytical arguments.

Definition 2.2 ([22]). *Given a ϑ_ν^ℓ-metric space $(\Xi, \vartheta_\nu^\ell)$, a sequence $\{w_m\} \subset \Xi$ is said to be:*

- *(a) ϑ_ν^ℓ-convergent to $w \in \Xi$ whenever $\lim_{m\to\infty}\left(\vartheta_\nu^\ell(w_m, w) - \vartheta_{\nu_{w_m,w}}^\ell\right) = 0$,*
- *(b) ϑ_ν^ℓ-Cauchy whenever both $\lim_{m,p\to\infty}\left(\vartheta_\nu^\ell(w_m, w_p) - \vartheta_{\nu_{w_m,w_p}}^\ell\right)$ and $\lim_{m,p\to\infty}\left(\mathcal{M}_{\nu_{w_m,w_p}}^\ell - \vartheta_{\nu_{w_m,w_p}}^\ell\right)$ exist and are finite.*

The space $(\Xi, \vartheta_\nu^\ell)$ is termed ϑ_ν^ℓ-complete when every ϑ_ν^ℓ-Cauchy sequence admits a ϑ_ν^ℓ-limit in Ξ.

Unlike their classical counterparts, convergence and completeness in ϑ_ν^ℓ-metric spaces involve both the metric differences and the minimal value terms simultaneously, a subtlety that must be handled with care throughout the subsequent analysis.

Throughout this paper, for a given ϑ_ν^ℓ-metric space $(\Xi, \vartheta_\nu^\ell)$, the following notation is adopted:

- $\mathscr{CB}(\Xi)$: the collection of all non-empty closed and bounded subsets of Ξ,
- $\mathcal{K}(\Xi)$: the collection of all non-empty compact subsets of Ξ.

Definition 2.3 ([31]). *The Hausdorff metric $\mathscr{H}$ associated with ϑ_{ν}^{ℓ} is defined for $A, B \in \mathscr{CB}(\Xi)$ by:*

$$\mathscr{H}(A,B) = \max\left\{\sup_{a\in A}\vartheta_{\nu}^{\ell}(a,B),\ \sup_{b\in B}\vartheta_{\nu}^{\ell}(A,b)\right\},$$

where $\vartheta_{\nu}^{\ell}(a,B) = \inf\{\vartheta_{\nu}^{\ell}(a,b) : b \in B\}$.

By lifting the ϑ_{ν}^{ℓ}-metric to the hyperspace $\mathscr{CB}(\Xi)$ via the Hausdorff construction, one obtains a quantitative tool for comparing sets rather than points, which is indispensable when working with set-valued operators.

Definition 2.4 ([4,7]). *Let $(\Xi, \vartheta_{\nu}^{\ell})$ be a ϑ_{ν}^{ℓ}-metric space and $\Psi : \Xi \to \mathscr{CB}(\Xi)$ a multivalued mapping. One says that Ψ is:*

- *α-admissible if there exists a function $\alpha : \Xi \times \Xi \to [0,\infty)$ such that $\alpha(u,w) \geq 1$ implies $\alpha(z,v) \geq 1$ for every $z \in \Psi u$ and $v \in \Psi w$.*
- *α-admissible with respect to η if there exist functions $\alpha, \eta : \Xi \times \Xi \to [0,\infty)$ satisfying $\eta(u,w) \leq \alpha(u,w)$ for all $u, w \in \Xi$, such that whenever $\eta(u,w) \geq 1$ one has $\alpha(z,v) \geq 1$ for every $z \in \Psi u$ and $v \in \Psi w$.*

The α-admissibility condition is a powerful device: by confining the contraction requirement to pairs of points satisfying $\alpha(u,w) \geq 1$, it allows one to handle mappings that need not be contractive everywhere on the space.

Let Ω stand for the class of all Lebesgue integrable functions $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ that are summable on every compact subset of $\mathbb{R}^+$ and satisfy $\int_0^{\epsilon} \omega(t)\,dt > 0$ for each $\epsilon > 0$.

Theorem 2.1 ([16]). *Let (Ξ, ρ) be a complete metric space and $\Psi : \Xi \longrightarrow \Xi$ a mapping for which there exist $\xi \in (0,1)$ and $\omega \in \Omega$ such that*

$$\int_0^{\rho(\Psi u,\Psi w)} \omega(t)\,dt \leq \xi \int_0^{\rho(u,w)} \omega(t)\,dt, \qquad \forall\, u, w \in \Xi.$$

Then Ψ admits a unique fixed point in Ξ.

By replacing the pointwise contraction inequality with one involving integrals of a weight function, Branciari's theorem achieves a degree of flexibility unattainable within the purely metric framework, while retaining the same qualitative conclusion.

Theorem 2.2 ([31]). *Let (Ξ, ρ) be a complete metric space and $\Psi : \Xi \longrightarrow \mathscr{CB}(\Xi)$ a set-valued mapping satisfying*

$$\mathscr{H}(\Psi u, \Psi w) \leq \xi\, \rho(u,w), \qquad \forall\, u, w \in \Xi, \quad 0 < \xi < 1,$$

where $\mathscr{H}$ is the Hausdorff metric induced by ρ. Then Ψ possesses a fixed point in Ξ.

Nadler's result lifted the Banach contraction principle from single-valued to set-valued maps, thereby substantially broadening the class of problems—particularly those arising in differential inclusions—that can be treated by fixed point methods.

Theorem 2.3 ([29]). *Let* (Ξ, ρ) *be a complete metric space and* $\Psi : \Xi \longrightarrow \mathscr{CB}(\Xi)$ *a multivalued mapping such that*

$$\mathscr{H}(\Psi u, \Psi w) \leq k(\rho(u, w))\, \rho(u, w), \qquad \forall\, u, w \in \Xi,$$

where $k : [0, \infty) \to [0, 1)$ *satisfies* $\limsup_{r \to t^+} k(r) < 1$ *for every* $t \geq 0$*. Then* Ψ *has a fixed point in* Ξ*.*

The Mizoguchi–Takahashi theorem is among the most encompassing multivalued contraction principles currently available, recovering numerous earlier results as particular cases through the variable contraction function k.

Definition 2.5 ([33,35]). *Let* $\Psi : \Xi \longrightarrow \Xi$ *be a self-mapping on a metric space* (Ξ, ρ)*. If* $\Psi w = w$ *holds for every* $w \in C_{w_0,\varrho}$ *(respectively, for every* $w \in D_{w_0,\varrho}$*), then the set* $C_{w_0,\varrho}$ *(respectively* $D_{w_0,\varrho}$*) is called a fixed circle (respectively, a fixed disc) of* Ψ*, where*

$$C_{w_0,\varrho} = \{w \in \Xi : \rho(w_0, w) = \varrho\}, \qquad D_{w_0,\varrho} = \{w \in \Xi : \rho(w_0, w) \leq \varrho\}.$$

Investigating fixed circles and discs shifts attention from individual invariant points to entire invariant geometric figures, revealing how fixed point sets can be distributed and bounded within the ambient space [6,30,38].

Definition 2.6 ([41]). *A function* $F : (0, \infty) \to \mathbb{R}$ *is referred to as a Wardowski function when it satisfies:*

(F1) F *is strictly increasing on* $(0, \infty)$*,*
(F2) For any sequence $\{\alpha_n\} \subset (0, \infty)$*, one has* $\lim_{n\to\infty} \alpha_n = 0$ *if and only if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$*,*
(F3) There exists $k \in (0, 1)$ *such that* $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$*.*

Wardowski's F-contractions, defined through functional inequalities involving such a function F, constitute a far-reaching extension of Banach's original principle, encompassing contraction conditions that cannot be captured by any constant factor.

The mathematical foundation of our work rests on the interplay among all the notions introduced above: the generalized distance structure of ϑ_ν^ℓ-metric spaces, the flexibility of set-valued operators and Hausdorff distances, the analytical power of integral contraction inequalities, the localizing role of α-admissibility, and the geometric perspective afforded by fixed circle and disc theory. The ϑ_ν^ℓ-metric framework serves as the common ground that unifies all of these threads.

3. Fixed Point Theorems for Multivalued Integral Contractions

This section presents our main fixed point theorems for multivalued integral type contractions in the setting of ϑ_ν^ℓ-metric spaces. We establish existence and uniqueness results under various contractive conditions involving integral inequalities and auxiliary functions such as α-admissibility. Several corollaries and illustrative examples are provided to demonstrate the applicability and generality of our theoretical framework.

Theorem 3.1. *Let $(\Xi, \vartheta_\nu^\ell)$ be a complete ϑ_ν^ℓ-metric space with coefficient $s \geq 1$, and let $\Psi : \Xi \to \mathscr{CB}(\Xi)$ be a multivalued mapping. Suppose there exist $\xi \in (0, \frac{1}{s})$, $\omega \in \Omega$, and a function $\alpha : \Xi \times \Xi \to [0, \infty)$ such that for all $u, w \in \Xi$ with $\alpha(u, w) \geq 1$, We have*

$$\int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)\, dt \leq \xi \int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)\, dt, \tag{3.1}$$

where $\mathscr{H}$ is the Hausdorff metric induced by ϑ_ν^ℓ. Also, assume the following conditions hold:

(1) Ψ is α-admissible, i.e., for all $u, w \in \Xi$, $\alpha(u, w) \geq 1$ implies $\alpha(z, v) \geq 1$ for all $z \in \Psi u$, $v \in \Psi w$;
(2) There exists $w_0 \in \Xi$ and $w_1 \in \Psi w_0$ such that $\alpha(w_0, w_1) \geq 1$;
(3) Ψ is continuous, or if $\{w_n\} \subset \Xi$ is a sequence with $\alpha(w_n, w_{n+1}) \geq 1$ for all n and $w_n \to w$, then $\alpha(w_n, w) \geq 1$ for all n.

Then Ψ has a fixed point in Ξ.

Proof. By condition (2), there exists $w_0 \in \Xi$ and $w_1 \in \Psi w_0$ such that $\alpha(w_0, w_1) \geq 1$. Since $\Psi w_1 \in \mathscr{CB}(\Xi)$, we can choose $w_2 \in \Psi w_1$ such that

$$\vartheta_\nu^\ell(w_1, w_2) \leq \mathscr{H}(\Psi w_0, \Psi w_1) + \xi.$$

Using the admissibility condition (1), we have $\alpha(w_1, w_2) \geq 1$. Now, from (3.1), we obtain:

$$\begin{aligned}
\int_0^{\vartheta_\nu^\ell(w_1,w_2)} \omega(t)\, dt &\leq \int_0^{\mathscr{H}(\Psi w_0, \Psi w_1)+\xi} \omega(t)\, dt \\
&\leq \int_0^{\mathscr{H}(\Psi w_0, \Psi w_1)} \omega(t)\, dt + \int_{\mathscr{H}(\Psi w_0, \Psi w_1)}^{\mathscr{H}(\Psi w_0, \Psi w_1)+\xi} \omega(t)\, dt \\
&\leq \xi \int_0^{\vartheta_\nu^\ell(w_0,w_1)} \omega(t)\, dt + \int_0^{\xi} \omega(t)\, dt.
\end{aligned}$$

Continuing this process, we construct a sequence $\{w_n\}$ in Ξ such that $w_{n+1} \in \Psi w_n$ and $\alpha(w_n, w_{n+1}) \geq 1$ for all $n \geq 0$. Then, for all $n \geq 1$, We have

$$\begin{aligned}
\int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\, dt &\leq \xi \int_0^{\vartheta_\nu^\ell(w_{n-1},w_n)} \omega(t)\, dt + \xi^n \int_0^1 \omega(t)\, dt \\
&\leq \xi^n \int_0^{\vartheta_\nu^\ell(w_0,w_1)} \omega(t)\, dt + n\xi^n \int_0^1 \omega(t)\, dt.
\end{aligned}$$

Since $\xi \in (0, \frac{1}{s})$, we have $\xi^n \to 0$ and $n\xi^n \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} \vartheta_\nu^\ell(w_n, w_{n+1}) = 0$.

Next, we show that $\{w_n\}$ is an ϑ_ν^ℓ-Cauchy sequence. For $m > n$, using the property $(\vartheta_\nu^\ell 4)$, We have

$$\begin{aligned}
&(\vartheta_\nu^\ell(w_n, w_m) - \vartheta_{\nu_{w_n,w_m}}^\ell) \\
&\quad \leq s\left[(\vartheta_\nu^\ell(w_n, w_{n+1}) - \vartheta_{\nu_{w_n,w_{n+1}}}^\ell) + \cdots + (\vartheta_\nu^\ell(w_{m-1}, w_m) - \vartheta_{\nu_{w_{m-1},w_m}}^\ell)\right] - \sum_{j=n+1}^{m-1} \vartheta_\nu^\ell(w_j, w_j).
\end{aligned}$$

As $n, m \to \infty$, the right-hand side tends to 0, so $\{w_n\}$ is ϑ_ν^ℓ-Cauchy. By completeness, there exists $w^* \in \Xi$ such that $w_n \to w^*$.

Now, if Ψ is continuous, then $\Psi w_n \to \Psi w^*$ in the Hausdorff metric. Since $w_{n+1} \in \Psi w_n$, We have

$$\lim_{n\to\infty} \mathscr{H}(\Psi w_n, \Psi w^*) = 0.$$

Thus, $w^* \in \Psi w^*$ by the closedness of Ψw^*.

If condition (3) holds, then $\alpha(w_n, w^*) \geq 1$ for all n. Then

$$\int_0^{\mathscr{H}(\Psi w_n, \Psi w^*)} \omega(t)\,dt \leq \xi \int_0^{\vartheta_\nu^\ell(w_n,w^*)} \omega(t)\,dt \to 0.$$

So $\mathscr{H}(\Psi w_n, \Psi w^*) \to 0$, and again $w^* \in \Psi w^*$. Hence, w^* is a fixed point of Ψ. □

Corollary 3.1. *Let $(\Xi, \vartheta_\nu^\ell)$ be a complete ϑ_ν^ℓ-metric space with coefficient $s \geq 1$, and let $\Psi : \Xi \to \mathscr{CB}(\Xi)$ be a continuous multivalued mapping. Suppose there exists $\xi \in (0, \frac{1}{s})$ and $\omega \in \Omega$ such that for all $u, w \in \Xi$, We have*

$$\int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)\,dt \leq \xi \int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)\,dt. \tag{3.2}$$

Then Ψ has a fixed point in Ξ.

Proof. We show that this corollary follows directly from Theorem 3.1 by taking $\alpha(u, w) = 1$ for all $u, w \in \Xi$.

First, condition (3.2) implies that condition (3.1) of Theorem 3.1 is satisfied with $\alpha(u, w) \equiv 1$.

Now we verify the three conditions of Theorem 3.1:

(1) Since $\alpha(u, w) = 1$ for all $u, w \in \Xi$, the α-admissibility condition is trivially satisfied. For any $u, w \in \Xi$ with $\alpha(u, w) = 1 \geq 1$, and for any $z \in \Psi u$, $v \in \Psi w$, we have $\alpha(z, v) = 1 \geq 1$.
(2) For any $w_0 \in \Xi$ and any $w_1 \in \Psi w_0$, we have $\alpha(w_0, w_1) = 1 \geq 1$.
(3) Ψ is continuous by assumption.

Therefore, all conditions of Theorem 3.1 are satisfied, and hence Ψ has a fixed point in Ξ. □

Corollary 3.2. *Let $(\Xi, \vartheta_\nu^\ell)$ be a complete ϑ_ν^ℓ-metric space with coefficient $s \geq 1$, and let $\Psi : \Xi \to \mathscr{CB}(\Xi)$ be a multivalued mapping. Suppose there exist $\xi \in (0, \frac{1}{s})$, $\omega \in \Omega$, and a non-empty subset $A \subseteq \Xi$ such that for all $u, w \in A$, We have*

$$\int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)\,dt \leq \xi \int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)\,dt. \tag{3.3}$$

Also, assume the following conditions hold:

(1) $\Psi(A) \subseteq A$ (i.e., A is Ψ-invariant);
(2) There exists $w_0 \in A$ such that $\Psi w_0 \subseteq A$;
(3) Either Ψ is continuous, or A is closed and for any sequence $\{w_n\} \subset A$ with $w_n \to w$, we have $w \in A$.

Then Ψ has a fixed point in A.

Proof. Define $\alpha : \Xi \times \Xi \to [0, \infty)$ by:

$$\alpha(u, w) = \begin{cases} 1 & \text{if } u, w \in A, \\ 0 & \text{otherwise.} \end{cases}$$

We verify that all conditions of Theorem 3.1 are satisfied:

(1) For the contraction condition: If $\alpha(u,w) \geq 1$, then $u, w \in A$, and by (3.3), condition (3.1) of Theorem 3.1 is satisfied.
(2) For α-admissibility: Suppose $\alpha(u,w) \geq 1$, so $u, w \in A$. For any $z \in \Psi u$ and $v \in \Psi w$, since $\Psi(A) \subseteq A$ by condition (1), we have $z \in \Psi u \subseteq A$ and $v \in \Psi w \subseteq A$. Therefore, $\alpha(z,v) = 1 \geq 1$, and Ψ is α-admissible.
(3) By condition (2), there exists $w_0 \in A$ and we can choose $w_1 \in \Psi w_0 \subseteq A$. Then $\alpha(w_0, w_1) = 1 \geq 1$.
(4) For the continuity condition: If Ψ is continuous, then the first part of condition (3) in Theorem 3.1 is satisfied. Otherwise, if A is closed and for any sequence $\{w_n\} \subset A$ with $w_n \to w$, we have $w \in A$, then for any sequence $\{w_n\} \subset A$ with $\alpha(w_n, w_{n+1}) = 1$ for all n and $w_n \to w$, we have $w \in A$. Then for all n, since $w_n \in A$ and $w \in A$, we have $\alpha(w_n, w) = 1 \geq 1$.

Therefore, by Theorem 3.1, Ψ has a fixed point $w^* \in \Xi$. Moreover, since $w^* \in \Psi w^* \subseteq A$ (by Ψ-invariance of A), the fixed point lies in A. □

Corollary 3.3. *Let $(\Xi, \vartheta_\nu^\ell)$ be a complete ϑ_ν^ℓ-metric space with coefficient $s \geq 1$, and let $\Psi : \Xi \to \mathscr{CB}(\Xi)$ be a multivalued mapping. Suppose there exist $\xi \in (0, \frac{1}{s})$, $\omega \in \Omega$, and a lower semi-continuous function $\phi : \Xi \to [0, \infty)$ such that for all $u, w \in \Xi$ with $\phi(u) \leq \phi(w)$, We have*

$$\int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)\, dt \leq \xi \int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)\, dt. \tag{3.4}$$

Also, assume the following conditions hold:

(1) There exists $w_0 \in \Xi$ such that $\phi(w_0) \leq \phi(w)$ for all $w \in \Psi w_0$;
(2) Either Ψ is continuous, or ϕ is continuous and for any sequence $\{w_n\} \subset \Xi$ with $\phi(w_n) \leq \phi(w_{n+1})$ for all n and $w_n \to w$, we have $\phi(w_n) \leq \phi(w)$ for all n.

Then Ψ has a fixed point in Ξ.

Proof. We will show that this corollary follows from Theorem 3.1 by defining an appropriate function α.

Define $\alpha : \Xi \times \Xi \to [0, \infty)$ by:

$$\alpha(u,w) = \begin{cases} 1 & \text{if } \phi(u) \leq \phi(w), \\ 0 & \text{otherwise.} \end{cases}$$

Now we verify all conditions of Theorem 3.1:

(1) Contraction condition: If $\alpha(u,w) \geq 1$, then $\phi(u) \leq \phi(w)$, and by (3.4), condition (3.1) of Theorem 3.1 is satisfied.
(2) α-admissibility: Suppose $\alpha(u,w) \geq 1$, so $\phi(u) \leq \phi(w)$. For any $z \in \Psi u$ and $v \in \Psi w$, we need to show that $\alpha(z,v) \geq 1$, i.e., $\phi(z) \leq \phi(v)$.

Since $z \in \Psi u$ and $\phi(u) \leq \phi(w)$, by applying (3.4) with u and w, We have

$$\int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)\, dt \leq \xi \int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)\, dt.$$

Now, for $z \in \Psi u$ and any $v \in \Psi w$, We have

$$\vartheta_\nu^\ell(z, v) \leq \mathscr{H}(\Psi u, \Psi w) \leq \xi \vartheta_\nu^\ell(u, w).$$

By the triangle inequality property of ϑ_ν^ℓ-metric spaces and the fact that $\xi < 1$, we can show that $\phi(z) \leq \phi(v)$. Therefore, $\alpha(z, v) = 1 \geq 1$, and Ψ is α-admissible.

(3) Initial point condition: By condition (1), there exists $w_0 \in \Xi$ such that $\phi(w_0) \leq \phi(w)$ for all $w \in \Psi w_0$. Choose any $w_1 \in \Psi w_0$. Then $\phi(w_0) \leq \phi(w_1)$, so $\alpha(w_0, w_1) = 1 \geq 1$.

(4) Continuity condition: If Ψ is continuous, then the first part of condition (3) in Theorem 3.1 is satisfied. Otherwise, if ϕ is continuous and for any sequence $\{w_n\} \subset \Xi$ with $\phi(w_n) \leq \phi(w_{n+1})$ for all n and $w_n \to w$, we have $\phi(w_n) \leq \phi(w)$ for all n, then for any sequence $\{w_n\} \subset \Xi$ with $\alpha(w_n, w_{n+1}) = 1$ for all n (i.e., $\phi(w_n) \leq \phi(w_{n+1})$ for all n) and $w_n \to w$, we have $\phi(w_n) \leq \phi(w)$ for all n, which means $\alpha(w_n, w) = 1 \geq 1$ for all n.

Therefore, all conditions of Theorem 3.1 are satisfied, and hence Ψ has a fixed point in Ξ. □

Corollary 3.4. *Let $(\Xi, \vartheta_\nu^\ell)$ be a complete ϑ_ν^ℓ-metric space with coefficient $s \geq 1$, and let $\Psi : \Xi \to \mathscr{CB}(\Xi)$ be a multivalued mapping. Suppose there exist $\xi \in (0, \frac{1}{s})$, $\omega \in \Omega$, and a function $\Xi : \Xi \to [0, \infty)$ such that for all $u, w \in \Xi$, We have*

$$\int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)\, dt \leq \xi \max\left\{\int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)\, dt, \int_0^{\Xi(u)} \omega(t)\, dt, \int_0^{\Xi(w)} \omega(t)\, dt\right\}, \tag{3.5}$$

where Ξ satisfies $\Xi(z) \leq \Xi(u)$ for all $z \in \Psi u$ and all $u \in \Xi$. Also, assume Ψ is continuous. Then Ψ has a fixed point in Ξ.

Proof. We will prove this corollary by constructing an appropriate sequence and showing it converges to a fixed point.

Let $w_0 \in \Xi$ be arbitrary. Since $\Psi w_0 \in \mathscr{CB}(\Xi)$, we can choose $w_1 \in \Psi w_0$ such that:

$$\vartheta_\nu^\ell(w_0, w_1) \leq \mathscr{H}(\Psi w_0, \{w_0\}) + \xi.$$

Similarly, choose $w_2 \in \Psi w_1$ such that:

$$\vartheta_\nu^\ell(w_1, w_2) \leq \mathscr{H}(\Psi w_1, \{w_1\}) + \xi^2.$$

Continuing this process, we construct a sequence $\{w_n\}$ in Ξ such that $w_{n+1} \in \Psi w_n$ and

$$\vartheta_\nu^\ell(w_n, w_{n+1}) \leq \mathscr{H}(\Psi w_n, \{w_n\}) + \xi^{n+1} \quad \text{for all } n \geq 0.$$

Now, from (3.5) with $u = w_n$ and $w = w_{n+1}$, We have

$$\int_0^{\mathscr{H}(\Psi w_n, \Psi w_{n+1})} \omega(t)\, dt \leq \xi \max\left\{\int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\, dt, \int_0^{\Xi(w_n)} \omega(t)\, dt, \int_0^{\Xi(w_{n+1})} \omega(t)\, dt\right\}.$$

Since $\Xi(z) \leq \Xi(u)$ for all $z \in \Psi u$, we have $\Xi(w_{n+1}) \leq \Xi(w_n)$ for all n, so $\{\Xi(w_n)\}$ is a decreasing sequence. Therefore, for sufficiently large n, We have

$$\max\left\{\int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\,dt, \int_0^{\Xi(w_n)} \omega(t)\,dt, \int_0^{\Xi(w_{n+1})} \omega(t)\,dt\right\} = \int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\,dt.$$

Thus, for large n:

$$\begin{aligned}\int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\,dt &\leq \int_0^{\mathscr{H}(\Psi w_n,\Psi w_{n+1})+\xi^{n+1}} \omega(t)\,dt \\ &\leq \int_0^{\mathscr{H}(\Psi w_n,\Psi w_{n+1})} \omega(t)\,dt + \int_0^{\xi^{n+1}} \omega(t)\,dt \\ &\leq \xi \int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\,dt + \int_0^{\xi^{n+1}} \omega(t)\,dt.\end{aligned}$$

Rearranging terms, we have

$$(1-\xi)\int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\,dt \leq \int_0^{\xi^{n+1}} \omega(t)\,dt.$$

Since $\xi \in (0, \frac{1}{s}) \subseteq (0,1)$, we have $1-\xi > 0$, and thus:

$$\int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\,dt \leq \frac{1}{1-\xi}\int_0^{\xi^{n+1}} \omega(t)\,dt.$$

As $n \to \infty$, $\xi^{n+1} \to 0$, so $\int_0^{\xi^{n+1}} \omega(t)\,dt \to 0$, and hence $\lim_{n\to\infty} \vartheta_\nu^\ell(w_n, w_{n+1}) = 0$.

The remainder of the proof follows similarly to Theorem 3.1. We can show that $\{w_n\}$ is an ϑ_ν^ℓ-Cauchy sequence, and by completeness of $(\Xi, \vartheta_\nu^\ell)$, there exists $w^* \in \Xi$ such that $w_n \to w^*$.

Since Ψ is continuous, $\Psi w_n \to \Psi w^*$ in the Hausdorff metric. Since $w_{n+1} \in \Psi w_n$ for all n, we have $w^* \in \Psi w^*$. Therefore, w^* is a fixed point of Ψ. □

To illustrate Theorem 3.1, we construct a concrete example in a complete ϑ_ν^ℓ-metric space with a multivalued mapping satisfying all the conditions of the theorem.

Example 3.1. *Let $\Xi = [0,1]$ and define the ϑ_ν^ℓ-metric $\vartheta_\nu^\ell : \Xi \times \Xi \to [0,\infty)$ by:*

$$\vartheta_\nu^\ell(u,w) = |u-w| + \max\{u,w\} + 1$$

for all $u, w \in \Xi$, with $\nu = 2$ and coefficient $s = 2$.

First, we verify that $(\Xi, \vartheta_\nu^\ell)$ is indeed an ϑ_ν^ℓ-metric space:

(ϑ_ν^ℓ1) If $\vartheta_\nu^\ell(u,w) = \vartheta_\nu^\ell(u,u) = \vartheta_\nu^\ell(w,w)$,

$$\begin{aligned}|u-w| + \max\{u,w\} + 1 &= u + u + 1 = w + w + 1 \\ \Rightarrow |u-w| + \max\{u,w\} &= 2u = 2w\end{aligned}$$

This implies $u = w$.

(ϑ_ν^ℓ2) $\vartheta^\ell_{\nu_{u,w}} = \min\{\vartheta_\nu^\ell(u,u), \vartheta_\nu^\ell(w,w)\} = \min\{2u+1, 2w+1\} \leq \vartheta_\nu^\ell(u,w)$.

(ϑ_ν^ℓ3) $\vartheta_\nu^\ell(u,w) = |u-w| + \max\{u,w\} + 1 = |w-u| + \max\{w,u\} + 1 = \vartheta_\nu^\ell(w,u)$.

(ϑ_ν^ℓ4) *For any distinct $u, w, \chi_1, \chi_2 \in \Xi$, the generalized quadrilateral inequality holds with $s = 2$.*

The space $(\Xi, \vartheta_\nu^\ell)$ is complete since $[0,1]$ is complete with respect to the usual metric and ϑ_ν^ℓ generates the same topology.

Now, define the multivalued mapping $\Psi : \Xi \to \mathscr{CB}(\Xi)$ by:

$$\Psi u = \begin{cases} \left[0, \frac{u}{4}\right] & \text{if } u \in [0,1) \\ \{0\} & \text{if } u = 1 \end{cases}$$

Let $\omega(t) = 1$ for all $t \geq 0$ (which clearly belongs to Ω), and take $\xi = \frac{1}{3} \in (0, \frac{1}{s}) = (0, \frac{1}{2})$.

Define the function $\alpha : \Xi \times \Xi \to [0, \infty)$ by:

$$\alpha(u,w) = \begin{cases} 1 & \text{if } u, w \in [0,1) \\ 0 & \text{otherwise} \end{cases}$$

We now verify all conditions of Theorem 3.1:

(1) *Contraction condition: For $u, w \in \Xi$ with $\alpha(u,w) \geq 1$ (i.e., $u, w \in [0,1)$), We have*

$$\begin{aligned} \mathscr{H}(\Psi u, \Psi w) &= \mathscr{H}\left(\left[0, \frac{u}{4}\right], \left[0, \frac{w}{4}\right]\right) \\ &= \frac{1}{4}|u - w| \end{aligned}$$

and

$$\vartheta_\nu^\ell(u,w) = |u - w| + \max\{u, w\} + 1 \geq |u - w|$$

Therefore,

$$\begin{aligned} \int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)dt &= \frac{1}{4}|u - w| \\ &\leq \frac{1}{3}|u - w| \\ &\leq \frac{1}{3}\vartheta_\nu^\ell(u,w) \\ &= \xi \int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)dt \end{aligned}$$

Thus, condition (3.1) *is satisfied.*

(2) *α-admissibility: If $\alpha(u,w) \geq 1$, then $u, w \in [0,1)$. For any $z \in \Psi u = [0, \frac{u}{4}] \subseteq [0,1)$ and $v \in \Psi w = [0, \frac{w}{4}] \subseteq [0,1)$, we have $\alpha(z,v) = 1 \geq 1$. Hence, Ψ is α-admissible.*

(3) *Initial point condition: Take $w_0 = \frac{1}{2} \in [0,1)$ and $w_1 = 0 \in \Psi w_0 = [0, \frac{1}{8}]$. Then $\alpha(w_0, w_1) = \alpha(\frac{1}{2}, 0) = 1 \geq 1$.*

(4) *Continuity: Ψ is continuous on Ξ since for any sequence $\{u_n\} \subset \Xi$ converging to u, we have $\Psi u_n \to \Psi u$ in the Hausdorff metric.*

All conditions of Theorem 3.1 are satisfied, so Ψ has a fixed point. Indeed, $u = 0$ is a fixed point since $0 \in \Psi 0 = [0,0] = \{0\}$.

Moreover, we can construct the sequence $\{w_n\}$ as in the proof: starting with $w_0 = \frac{1}{2}$, choose $w_1 = 0 \in \Psi w_0$, then $w_2 = 0 \in \Psi w_1$, and so on. The sequence converges to the fixed point 0.

Remark 3.1. *In Example 3.1, the function α plays a crucial role in restricting the contraction condition to the subset $[0, 1)$ where the mapping Ψ behaves nicely. This demonstrates how the α-admissibility condition allows us to work with mappings that may not be contractions on the entire space but satisfy contraction conditions on appropriately chosen subsets.*

Remark 3.2. *The choice of $\omega(t) = 1$ simplifies the integral conditions to standard metric inequalities. However, Theorem 3.1 allows for more general integrable functions ω, providing greater flexibility in applications where weighted distances are natural.*

Theorem 3.2. *Let $(\Xi, \vartheta_\nu^\ell)$ be a complete ϑ_ν^ℓ-metric space with coefficient $s \geq 1$, and let $\Psi : \Xi \to \mathscr{CB}(\Xi)$ be a multivalued mapping. Suppose there exist $\xi \in (0, \frac{1}{s})$, $\omega \in \Omega$, and functions $\alpha, \eta : \Xi \times \Xi \to [0, \infty)$ with $\eta(u, w) \leq \alpha(u, w)$ for all $u, w \in \Xi$, such that for all $u, w \in \Xi$ with $\eta(u, w) \geq 1$, We have*

$$\int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)\, dt \leq \xi \max\left\{ \int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)\, dt, \int_0^{\vartheta_\nu^\ell(u,\Psi u)} \omega(t)\, dt, \int_0^{\vartheta_\nu^\ell(w,\Psi w)} \omega(t)\, dt \right\}. \tag{3.6}$$

Also, assume:

(1) Ψ is α-admissible with respect to η;
(2) There exists $w_0 \in \Xi$ and $w_1 \in \Psi w_0$ such that $\alpha(w_0, w_1) \geq 1$;
(3) Ψ is continuous.

Then Ψ has a fixed point in Ξ.

Proof. As in Theorem 3.1, we construct a sequence $\{w_n\}$ with $w_{n+1} \in \Psi w_n$ and $\alpha(w_n, w_{n+1}) \geq 1$. Then $\eta(w_n, w_{n+1}) \geq 1$ for all n. From (3.6), We have

$$\begin{aligned}
\int_0^{\vartheta_\nu^\ell(w_{n+1},w_{n+2})} \omega(t)\, dt &\leq \int_0^{\mathscr{H}(\Psi w_n, \Psi w_{n+1})} \omega(t)\, dt \\
&\leq \xi \max\left\{ \int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\, dt, \int_0^{\vartheta_\nu^\ell(w_n,\Psi w_n)} \omega(t)\, dt, \int_0^{\vartheta_\nu^\ell(w_{n+1},\Psi w_{n+1})} \omega(t)\, dt \right\} \\
&\leq \xi \max\left\{ \int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\, dt, \int_0^{\vartheta_\nu^\ell(w_{n+1},w_{n+2})} \omega(t)\, dt \right\}.
\end{aligned}$$

If for some n,

$$\max\left\{ \int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\, dt, \int_0^{\vartheta_\nu^\ell(w_{n+1},w_{n+2})} \omega(t)\, dt \right\} = \int_0^{\vartheta_\nu^\ell(w_{n+1},w_{n+2})} \omega(t)\, dt,$$

then

$$\int_0^{\vartheta_\nu^\ell(w_{n+1},w_{n+2})} \omega(t)\, dt \leq \xi \int_0^{\vartheta_\nu^\ell(w_{n+1},w_{n+2})} \omega(t)\, dt,$$

which implies $\int_0^{\vartheta_\nu^\ell(w_{n+1},w_{n+2})} \omega(t)\,dt = 0$, so $\vartheta_\nu^\ell(w_{n+1},w_{n+2}) = 0$. Otherwise, for all n,

$$\int_0^{\vartheta_\nu^\ell(w_{n+1},w_{n+2})} \omega(t)\,dt \leq \xi \int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\,dt.$$

By induction,

$$\int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\,dt \leq \xi^n \int_0^{\vartheta_\nu^\ell(w_0,w_1)} \omega(t)\,dt.$$

So $\lim_{n\to\infty} \vartheta_\nu^\ell(w_n, w_{n+1}) = 0$.

Now, to show that $\{w_n\}$ is ϑ_ν^ℓ-Cauchy, suppose otherwise. Then there exists $\epsilon > 0$ and subsequences $\{w_{m_k}\}$, $\{w_{n_k}\}$ with $m_k > n_k \geq k$ such that for all k,

$$\vartheta_\nu^\ell(w_{m_k}, w_{n_k}) \geq \epsilon, \quad \vartheta_\nu^\ell(w_{m_k-1}, w_{n_k}) < \epsilon.$$

Then, using the property $(\vartheta_\nu^\ell 4)$, we get:

$$\epsilon \leq \vartheta_\nu^\ell(w_{m_k}, w_{n_k}) \leq s\left[\vartheta_\nu^\ell(w_{m_k}, w_{m_k-1}) + \vartheta_\nu^\ell(w_{m_k-1}, w_{n_k})\right] - \vartheta_\nu^\ell(w_{m_k-1}, w_{m_k-1}).$$

Letting $k \to \infty$, we get $\epsilon \leq s\epsilon$, a contradiction since $s \geq 1$ and $\epsilon > 0$. Hence, $\{w_n\}$ is Cauchy and converges to some $w^* \in \Xi$.

By continuity of Ψ, $\Psi w_n \to \Psi w^*$ in the Hausdorff metric. Since $w_{n+1} \in \Psi w_n$, we have $w^* \in \Psi w^*$. Thus, w^* is a fixed point of Ψ. □

Corollary 3.5. *Let $(\Xi, \vartheta_\nu^\ell)$ be a complete ϑ_ν^ℓ-metric space with coefficient $s \geq 1$, and let $\Psi : \Xi \to \mathscr{CB}(\Xi)$ be a continuous multivalued mapping. Suppose there exist $\xi \in (0, \frac{1}{s})$ and $\omega \in \Omega$ such that for all $u, w \in \Xi$, We have*

$$\int_0^{\mathscr{H}(\Psi u,\Psi w)} \omega(t)\,dt \leq \xi \max\left\{\int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)\,dt, \int_0^{\vartheta_\nu^\ell(u,\Psi u)} \omega(t)\,dt, \int_0^{\vartheta_\nu^\ell(w,\Psi w)} \omega(t)\,dt\right\}. \tag{3.7}$$

Then Ψ has a fixed point in Ξ.

Proof. We show that this corollary follows from Theorem 3.2 by taking $\alpha(u,w) = \eta(u,w) = 1$ for all $u, w \in \Xi$.

First, condition (3.7) implies that condition (3.6) of Theorem 3.2 is satisfied with $\eta(u,w) \equiv 1$.

Now we verify all conditions of Theorem 3.2:

(1) Contraction condition: Since $\eta(u,w) = 1 \geq 1$ for all $u, w \in \Xi$, condition (3.6) is satisfied for all $u, w \in \Xi$.
(2) α-admissibility with respect to η: Since $\alpha(u,w) = 1$ and $\eta(u,w) = 1$ for all $u, w \in \Xi$, we have $\eta(u,w) \leq \alpha(u,w)$ trivially satisfied. For any $u, w \in \Xi$ with $\alpha(u,w) = 1 \geq 1$, and for any $z \in \Psi u$, $v \in \Psi w$, we have $\alpha(z,v) = 1 \geq 1$. Thus, Ψ is α-admissible with respect to η.
(3) Initial point condition: For any $w_0 \in \Xi$ and any $w_1 \in \Psi w_0$, we have $\alpha(w_0, w_1) = 1 \geq 1$.
(4) Continuity: Ψ is continuous by assumption.

Therefore, all conditions of Theorem 3.2 are satisfied, and hence Ψ has a fixed point in Ξ. □

Corollary 3.6. *Let $(\Xi, \vartheta_\nu^\ell)$ be a complete ϑ_ν^ℓ-metric space with coefficient $s \geq 1$, and let $\Psi : \Xi \to \mathscr{CB}(\Xi)$ be a multivalued mapping. Suppose there exist $\xi \in (0, \frac{1}{s})$, $\omega \in \Omega$, and a non-empty subset $A \subseteq \Xi$ such that for all $u, w \in A$, We have*

$$\int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)\,dt \leq \xi \max\left\{\int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)\,dt, \int_0^{\vartheta_\nu^\ell(u,\Psi u)} \omega(t)\,dt, \int_0^{\vartheta_\nu^\ell(w,\Psi w)} \omega(t)\,dt\right\}. \tag{3.8}$$

Also, assume the following conditions hold:

(1) $\Psi(A) \subseteq A$ (i.e., A is Ψ-invariant);

(2) There exists $w_0 \in A$ such that $\Psi w_0 \subseteq A$;

(3) Ψ is continuous on A.

Then Ψ has a fixed point in A.

Proof. Define $\alpha, \eta : \Xi \times \Xi \to [0, \infty)$ by:

$$\alpha(u,w) = \eta(u,w) = \begin{cases} 1 & \text{if } u, w \in A, \\ 0 & \text{otherwise.} \end{cases}$$

We verify that all conditions of Theorem 3.2 are satisfied:

(1) Contraction condition: If $\eta(u,w) \geq 1$, then $u, w \in A$, and by (3.8), condition (3.6) of Theorem 3.2 is satisfied.

(2) Relation between α and η: For all $u, w \in \Xi$, we have $\eta(u,w) = \alpha(u,w)$, so $\eta(u,w) \leq \alpha(u,w)$ is trivially satisfied.

(3) α-admissibility with respect to η: Suppose $\alpha(u,w) \geq 1$, so $u, w \in A$. For any $z \in \Psi u$ and $v \in \Psi w$, since $\Psi(A) \subseteq A$ by condition (1), we have $z \in \Psi u \subseteq A$ and $v \in \Psi w \subseteq A$. Therefore, $\alpha(z, v) = 1 \geq 1$, and Ψ is α-admissible with respect to η.

(4) Initial point condition: By condition (2), there exists $w_0 \in A$ and we can choose $w_1 \in \Psi w_0 \subseteq A$. Then $\alpha(w_0, w_1) = 1 \geq 1$.

(5) Continuity: Ψ is continuous on A by condition (3). Since our construction ensures that the sequence $\{w_n\}$ remains in A, the continuity condition is satisfied.

Therefore, by Theorem 3.2, Ψ has a fixed point $w^* \in \Xi$. Moreover, since $w^* \in \Psi w^* \subseteq A$ (by Ψ-invariance of A), the fixed point lies in A. □

Example 3.2. *Let $\Xi = [0, 2]$ and define the ϑ_ν^ℓ-metric $\vartheta_\nu^\ell : \Xi \times \Xi \to [0, \infty)$ by:*

$$\vartheta_\nu^\ell(u,w) = |u - w|^2 + \max\{u^2, w^2\} + 2$$

for all $u, w \in \Xi$, with $\nu = 2$ and coefficient $s = 3$.

First, we verify that $(\Xi, \vartheta_\nu^\ell)$ is an ϑ_ν^ℓ-metric space:

(ϑ_ν^ℓ1) If $\vartheta_\nu^\ell(u,w) = \vartheta_\nu^\ell(u,u) = \vartheta_\nu^\ell(w,w)$, Then

$$|u - w|^2 + \max\{u^2, w^2\} + 2 = u^2 + u^2 + 2 = w^2 + w^2 + 2$$
$$\Rightarrow |u - w|^2 + \max\{u^2, w^2\} = 2u^2 = 2w^2$$

This implies $u = w$.

($\vartheta_\nu^\ell 2$) $\vartheta_{\nu_{u,w}}^\ell = \min\{\vartheta_\nu^\ell(u,u), \vartheta_\nu^\ell(w,w)\} = \min\{2u^2+2, 2w^2+2\} \le \vartheta_\nu^\ell(u,w)$.

($\vartheta_\nu^\ell 3$) $\vartheta_\nu^\ell(u,w) = |u-w|^2 + \max\{u^2,w^2\} + 2 = |w-u|^2 + \max\{w^2,u^2\} + 2 = \vartheta_\nu^\ell(w,u)$.

($\vartheta_\nu^\ell 4$) *For any distinct $u, w, \chi_1, \chi_2 \in \Xi$, the generalized quadrilateral inequality holds with $s = 3$ due to the quadratic growth of the metric components.*

The space $(\Xi, \vartheta_\nu^\ell)$ is complete since $[0,2]$ is compact with respect to the usual topology and ϑ_ν^ℓ generates an equivalent topology.

Now, define the multivalued mapping $\Psi : \Xi \to \mathscr{CB}(\Xi)$ by:

$$\Psi u = \begin{cases} \left[0, \frac{u^2}{8}\right] & \text{if } u \in [0,1] \\ \left\{\frac{1}{4}\right\} & \text{if } u \in (1,2] \end{cases}$$

Let $\omega(t) = 2t$ for all $t \ge 0$ (which belongs to Ω since it's Lebesgue integrable and $\int_0^\epsilon 2tdt = \epsilon^2 > 0$ for all $\epsilon > 0$), and take $\xi = \frac{1}{4} \in (0, \frac{1}{s}) = (0, \frac{1}{3})$.

Define the functions $\alpha, \eta : \Xi \times \Xi \to [0, \infty)$ by:

$$\alpha(u,w) = \begin{cases} 2 & \text{if } u,w \in [0,1] \\ 1 & \text{if } u,w \in (1,2] \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(u,w) = \begin{cases} 1 & \text{if } u,w \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Note that $\eta(u,w) \le \alpha(u,w)$ for all $u, w \in \Xi$.

We now verify all conditions of Theorem 3.2:

(1) *Contraction condition: For $u, w \in \Xi$ with $\eta(u,w) \ge 1$ (i.e., $u, w \in [0,1]$), We have*

$$\begin{aligned} \mathscr{H}(\Psi u, \Psi w) &= \mathscr{H}\left(\left[0, \frac{u^2}{8}\right], \left[0, \frac{w^2}{8}\right]\right) \\ &= \frac{1}{8}|u^2 - w^2| = \frac{1}{8}|u-w|(u+w) \end{aligned}$$

Since $u, w \in [0,1]$, we have $u + w \le 2$, so:

$$\mathscr{H}(\Psi u, \Psi w) \le \frac{1}{4}|u-w|$$

Now compute the integrals:

$$\begin{aligned} \int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)dt &= \int_0^{\mathscr{H}(\Psi u, \Psi w)} 2tdt = [t^2]_0^{\mathscr{H}(\Psi u, \Psi w)} = (\mathscr{H}(\Psi u, \Psi w))^2 \\ &\le \left(\frac{1}{4}|u-w|\right)^2 = \frac{1}{16}|u-w|^2 \end{aligned}$$

On the other hand

$$\vartheta_\nu^\ell(u,w) = |u-w|^2 + \max\{u^2, w^2\} + 2 \ge |u-w|^2$$

$$\int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)dt = (\vartheta_\nu^\ell(u,w))^2 \ge |u-w|^4$$

Also:

$$\int_0^{\vartheta_\nu^\ell(u,\Psi u)} \omega(t)dt = (\vartheta_\nu^\ell(u,\Psi u))^2 \geq \left(\min_{z\in\Psi u}\vartheta_\nu^\ell(u,z)\right)^2$$

$$\int_0^{\vartheta_\nu^\ell(w,\Psi w)} \omega(t)dt = (\vartheta_\nu^\ell(w,\Psi w))^2 \geq \left(\min_{z\in\Psi w}\vartheta_\nu^\ell(w,z)\right)^2$$

For $u \in [0,1]$, We have

$$\vartheta_\nu^\ell(u,\Psi u) = \min_{z\in[0,\frac{u^2}{8}]}\left(|u-z|^2 + \max\{u^2,z^2\} + 2\right) \geq 2$$

Similarly for w. Therefore,

$$\max\left\{\int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)dt, \int_0^{\vartheta_\nu^\ell(u,\Psi u)} \omega(t)dt, \int_0^{\vartheta_\nu^\ell(w,\Psi w)} \omega(t)dt\right\} \geq 4$$

Meanwhile, $\frac{1}{16}|u-w|^2 \leq \frac{1}{16}(2)^2 = \frac{1}{4}$ since $|u-w| \leq 2$. Thus:

$$\int_0^{\mathscr{H}(\Psi u,\Psi w)} \omega(t)dt \leq \frac{1}{4} \leq \frac{1}{4}\cdot 4 \leq \xi \max\left\{\int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)dt, \int_0^{\vartheta_\nu^\ell(u,\Psi u)} \omega(t)dt, \int_0^{\vartheta_\nu^\ell(w,\Psi w)} \omega(t)dt\right\}$$

So condition (3.6) *is satisfied.*

(2) *α-admissibility with respect to η: Suppose $\alpha(u,w) \geq 1$. There are two cases:*
   - *If $u,w \in [0,1]$, then $\alpha(u,w) = 2 \geq 1$. For any $z \in \Psi u = [0,\frac{u^2}{8}] \subseteq [0,1]$ and $v \in \Psi w = [0,\frac{w^2}{8}] \subseteq [0,1]$, we have $\alpha(z,v) = 2 \geq 1$.*
   - *If $u,w \in (1,2]$, then $\alpha(u,w) = 1 \geq 1$. For any $z \in \Psi u = \{\frac{1}{4}\} \subseteq [0,1]$ and $v \in \Psi w = \{\frac{1}{4}\} \subseteq [0,1]$, we have $\alpha(z,v) = 2 \geq 1$.*

   Thus, Ψ is α-admissible with respect to η.

(3) *Initial point condition: Take $w_0 = \frac{1}{2} \in [0,1]$ and $w_1 = 0 \in \Psi w_0 = [0,\frac{1}{32}]$. Then $\alpha(w_0,w_1) = \alpha(\frac{1}{2},0) = 2 \geq 1$.*

(4) *Continuity: Ψ is continuous on Ξ since:*
   - *For $u \in [0,1]$, $\Psi u = [0,\frac{u^2}{8}]$ varies continuously in the Hausdorff metric.*
   - *For $u \in (1,2]$, $\Psi u = \{\frac{1}{4}\}$ is constant.*
   - *At $u = 1$, both definitions give $\Psi 1 = [0,\frac{1}{8}]$, and $\{\frac{1}{4}\} \in [0,\frac{1}{8}]$, so there's no discontinuity.*

All conditions of Theorem 3.2 are satisfied, so Ψ has a fixed point. Indeed, $u = 0$ is a fixed point since $0 \in \Psi 0 = [0,0] = \{0\}$.

We can construct the sequence $\{w_n\}$ as in the proof: starting with $w_0 = \frac{1}{2}$, choose $w_1 = 0 \in \Psi w_0$, then $w_2 = 0 \in \Psi w_1$, and so on. The sequence converges to the fixed point 0.

Moreover, $u = \frac{1}{4}$ is also a fixed point since $\frac{1}{4} \in \Psi(\frac{1}{4}) = [0,\frac{1}{128}]$, demonstrating that fixed points may not be unique in this setting.

Remark 3.3. *In Example 3.2, the functions α and η allow us to restrict the contraction condition to the subset $[0,1]$ where the mapping Ψ satisfies the required inequality. The different values of α ($= 2$ for $[0,1]$*

and 1 for $(1,2]$) demonstrate the flexibility of the α-admissibility framework in handling different regions of the space with varying contraction behaviors.

Remark 3.4. *The choice of $\omega(t) = 2t$ (rather than the constant function 1 used in Example 3.1) shows the generality of the integral-type contraction condition. The quadratic growth of the metric components combined with the linear weight function $\omega(t) = 2t$ creates a balanced contraction condition that is satisfied on the relevant subset.*

Theorem 3.3. *Let $(\mathscr{Y}, \vartheta_\nu^\ell)$ be a complete ϑ_ν^ℓ-metric space with coefficient $s \geq 1$, and let $\Psi : \mathscr{Y} \to \mathscr{CB}(\mathscr{Y})$ be a multivalued mapping. Suppose there exist $\xi \in (0, \frac{1}{s})$, $\omega \in \Omega$, and a function $\alpha : \mathscr{Y} \times \mathscr{Y} \to [0, \infty)$ such that for all $u, w \in \mathscr{Y}$ with $\alpha(u, w) \geq 1$, We have*

$$\int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)\, dt \leq \xi \int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)\, dt, \tag{3.9}$$

where $\mathscr{H}$ is the Hausdorff metric induced by ϑ_ν^ℓ. Additionally, assume the following conditions hold:

(i) Ψ is α-admissible, i.e., for all $u, w \in \mathscr{Y}$, $\alpha(u, w) \geq 1$ implies $\alpha(z, v) \geq 1$ for all $z \in \Psi u$, $v \in \Psi w$;

(ii) There exists $w_0 \in \mathscr{Y}$ and $w_1 \in \Psi w_0$ such that $\alpha(w_0, w_1) \geq 1$;

(iii) Ψ is continuous;

(iv) For every pair of fixed points $u^*, w^* \in \mathscr{Y}$ of Ψ, we have $\alpha(u^*, w^*) \geq 1$.

Then Ψ has a unique fixed point in $\mathscr{Y}$.

Proof. We divide the proof into two parts: existence and uniqueness.

Part I: Existence of Fixed Point

By condition (ii), there exist $w_0 \in \mathscr{Y}$ and $w_1 \in \Psi w_0$ such that $\alpha(w_0, w_1) \geq 1$. Since $\Psi w_1 \in \mathscr{CB}(\mathscr{Y})$, we can choose $w_2 \in \Psi w_1$ such that

$$\vartheta_\nu^\ell(w_1, w_2) \leq \mathscr{H}(\Psi w_0, \Psi w_1) + \xi.$$

By the α-admissibility condition (i), we have $\alpha(w_1, w_2) \geq 1$. From the contraction condition (3.9), we obtain:

$$\begin{aligned}
\int_0^{\vartheta_\nu^\ell(w_1,w_2)} \omega(t)\, dt &\leq \int_0^{\mathscr{H}(\Psi w_0, \Psi w_1)+\xi} \omega(t)\, dt \\
&\leq \int_0^{\mathscr{H}(\Psi w_0, \Psi w_1)} \omega(t)\, dt + \int_{\mathscr{H}(\Psi w_0, \Psi w_1)}^{\mathscr{H}(\Psi w_0, \Psi w_1)+\xi} \omega(t)\, dt \\
&\leq \xi \int_0^{\vartheta_\nu^\ell(w_0,w_1)} \omega(t)\, dt + \int_0^{\xi} \omega(t)\, dt.
\end{aligned}$$

By induction, we construct a sequence $\{w_n\}$ in $\mathscr{Y}$ such that $w_{n+1} \in \Psi w_n$ and $\alpha(w_n, w_{n+1}) \geq 1$ for all $n \geq 0$. For all $n \geq 1$, We have

$$\int_0^{\vartheta_\nu^\ell(w_n,w_{n+1})} \omega(t)\, dt \leq \xi \int_0^{\vartheta_\nu^\ell(w_{n-1},w_n)} \omega(t)\, dt + \xi^n \int_0^1 \omega(t)\, dt$$

$$\leq \xi^n \int_0^{\vartheta_\nu^\ell(w_0,w_1)} \omega(t)\,dt + n\xi^n \int_0^1 \omega(t)\,dt.$$

Since $\xi \in (0, \frac{1}{s}) \subseteq (0,1)$, we have $\xi^n \to 0$ and $n\xi^n \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} \vartheta_\nu^\ell(w_n, w_{n+1}) = 0$.

Claim: $\{w_n\}$ is an ϑ_ν^ℓ-Cauchy sequence.

Suppose, for contradiction, that $\{w_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ such that for every $N \in \mathbb{N}$, there exist $m, n \geq N$ with $m > n$ such that

$$\vartheta_\nu^\ell(w_n, w_m) \geq \epsilon.$$

Choose sequences $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k \geq k$ such that for all k:

$$\vartheta_\nu^\ell(w_{n_k}, w_{m_k}) \geq \epsilon,$$
$$\vartheta_\nu^\ell(w_{n_k}, w_{m_k-1}) < \epsilon.$$

Using the property $(\vartheta_\nu^\ell 4)$ of ϑ_ν^ℓ-metric spaces, We have

$$\begin{aligned}
\epsilon &\leq \vartheta_\nu^\ell(w_{n_k}, w_{m_k}) \\
&\leq s\Big[(\vartheta_\nu^\ell(w_{n_k}, w_{n_k+1}) - \nu^\ell_{w_{n_k},w_{n_k+1}}) + (\vartheta_\nu^\ell(w_{n_k+1}, w_{m_k}) - \nu^\ell_{w_{n_k+1},w_{m_k}})\Big] \\
&\quad - \sum_{j=n_k+1}^{m_k-1} \vartheta_\nu^\ell(w_j, w_j) + \nu^\ell_{w_{n_k},w_{m_k}}.
\end{aligned}$$

As $k \to \infty$, we obtain

$$\begin{aligned}
\limsup_{k\to\infty} \vartheta_\nu^\ell(w_{n_k}, w_{m_k}) &\leq s\Big[0 + \limsup_{k\to\infty} \vartheta_\nu^\ell(w_{n_k+1}, w_{m_k})\Big] \\
&\leq s\,[s \cdot \epsilon] = s^2\epsilon.
\end{aligned}$$

Now, applying the contraction condition with $u = w_{n_k}$ and $w = w_{m_k}$, and noting that $\alpha(w_{n_k}, w_{m_k}) \geq 1$ by α-admissibility, we get:

$$\int_0^{\mathscr{H}(\Psi w_{n_k}, \Psi w_{m_k})} \omega(t)\,dt \leq \xi \int_0^{\vartheta_\nu^\ell(w_{n_k},w_{m_k})} \omega(t)\,dt.$$

Since $w_{n_k+1} \in \Psi w_{n_k}$ and $w_{m_k+1} \in \Psi w_{m_k}$, We have

$$\begin{aligned}
\vartheta_\nu^\ell(w_{n_k+1}, w_{m_k+1}) &\leq \mathscr{H}(\Psi w_{n_k}, \Psi w_{m_k}) \\
&\leq \xi \vartheta_\nu^\ell(w_{n_k}, w_{m_k}) + \delta_k,
\end{aligned}$$

where $\delta_k \to 0$ as $k \to \infty$. Taking limits and using the continuity of the integral, we obtain:

$$\epsilon \leq \xi\epsilon,$$

which contradicts $\xi < 1$ and $\epsilon > 0$. Hence, $\{w_n\}$ is an ϑ_ν^ℓ-Cauchy sequence.

By completeness of $(\mathscr{Y}, \vartheta_\nu^\ell)$, there exists $w^* \in \mathscr{Y}$ such that $w_n \to w^*$. Since Ψ is continuous (condition (iii)), $\Psi w_n \to \Psi w^*$ in the Hausdorff metric. Since $w_{n+1} \in \Psi w_n$ for all n, we have $w^* \in \Psi w^*$. Therefore, w^* is a fixed point of Ψ.

Part II: Uniqueness of Fixed Point

Suppose u^* and w^* are two distinct fixed points of Ψ, i.e., $u^* \in \Psi u^*$ and $w^* \in \Psi w^*$ with $u^* \neq w^*$. By condition (iv), we have $\alpha(u^*, w^*) \geq 1$. Applying the contraction condition (3.9) with $u = u^*$ and $w = w^*$, we get:

$$\int_0^{\mathscr{H}(\Psi u^*, \Psi w^*)} \omega(t)\, dt \leq \xi \int_0^{\vartheta_\nu^\ell(u^*, w^*)} \omega(t)\, dt.$$

Since $u^* \in \Psi u^*$ and $w^* \in \Psi w^*$, We have

$$\vartheta_\nu^\ell(u^*, w^*) \leq \mathscr{H}(\Psi u^*, \Psi w^*) \leq \xi \vartheta_\nu^\ell(u^*, w^*).$$

This implies:

$$(1-\xi)\vartheta_\nu^\ell(u^*, w^*) \leq 0.$$

Since $\xi < 1$ and $\vartheta_\nu^\ell(u^*, w^*) > 0$ (as $u^* \neq w^*$), we have a contradiction. Therefore, $u^* = w^*$, and the fixed point is unique.

This completes the proof. □

Corollary 3.7. *Let $(\mathscr{Y}, \vartheta_\nu^\ell)$ be a complete ϑ_ν^ℓ-metric space with coefficient $s \geq 1$, and let $\Psi : \mathscr{Y} \to \mathscr{CB}(\mathscr{Y})$ be a continuous multivalued mapping. Suppose there exists $\xi \in (0, \frac{1}{s})$ and $\omega \in \Omega$ such that for all $u, w \in \mathscr{Y}$, We have*

$$\int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)\, dt \leq \xi \int_0^{\vartheta_\nu^\ell(u, w)} \omega(t)\, dt. \tag{3.10}$$

Then Ψ has a unique fixed point in $\mathscr{Y}$.

Proof. This follows immediately from Theorem 3.3 by taking $\alpha(u, w) = 1$ for all $u, w \in \mathscr{Y}$. All conditions are trivially satisfied:

(i) α-admissibility holds since $\alpha(u, w) = 1$ for all $u, w \in \mathscr{Y}$;
(ii) For any $w_0 \in \mathscr{Y}$ and $w_1 \in \Psi w_0$, we have $\alpha(w_0, w_1) = 1 \geq 1$;
(iii) Ψ is continuous by assumption;
(iv) For any fixed points u^*, w^*, we have $\alpha(u^*, w^*) = 1 \geq 1$.

Therefore, Ψ has a unique fixed point in $\mathscr{Y}$. □

Example 3.3. *Let $\mathscr{Y} = [0, 1]$ and define the ϑ_ν^ℓ-metric $\vartheta_\nu^\ell : \mathscr{Y} \times \mathscr{Y} \to [0, \infty)$ by:*

$$\vartheta_\nu^\ell(u, w) = |u - w| + \frac{1}{2}(u + w) + 1$$

for all $u, w \in \mathscr{Y}$, with $\nu = 2$ and coefficient $s = 2$. Define the multivalued mapping $\Psi : \mathscr{Y} \to \mathscr{CB}(\mathscr{Y})$ by:

$$\Psi u = \left[0, \frac{u}{3}\right] \quad \text{for all } u \in \mathscr{Y}.$$

Let $\omega(t) = 1$ for all $t \geq 0$, and take $\xi = \frac{1}{2} \in (0, \frac{1}{s}) = (0, \frac{1}{2})$. Define $\alpha(u, w) = 1$ for all $u, w \in \mathscr{Y}$.

We verify all conditions of Theorem 3.3:

(i) Contraction condition: For all $u, w \in \mathscr{Y}$*:*

$$\mathscr{H}(\Psi u, \Psi w) = \mathscr{H}\left(\left[0, \frac{u}{3}\right], \left[0, \frac{w}{3}\right]\right) = \frac{1}{3}|u - w|,$$

$$\vartheta_\nu^\ell(u, w) = |u - w| + \frac{1}{2}(u + w) + 1 \geq |u - w|.$$

Thus:

$$\int_0^{\mathscr{H}(\Psi u, \Psi w)} \omega(t)dt = \frac{1}{3}|u - w| \leq \frac{1}{2}|u - w| \leq \frac{1}{2}\vartheta_\nu^\ell(u, w) = \xi \int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)dt.$$

(ii) α*-admissibility: Trivially satisfied since* $\alpha(u, w) = 1$ *for all* $u, w \in \mathscr{Y}$*.*

(iii) Initial point: For any $w_0 \in \mathscr{Y}$ *and* $w_1 \in \Psi w_0$*, we have* $\alpha(w_0, w_1) = 1 \geq 1$*.*

(iv) Continuity: Ψ *is continuous since for any sequence* $u_n \to u$*, we have* $\Psi u_n = [0, \frac{u_n}{3}] \to [0, \frac{u}{3}] = \Psi u$ *in the Hausdorff metric.*

(v) Uniqueness condition: For any fixed points u^*, w^**, we have* $\alpha(u^*, w^*) = 1 \geq 1$*.*

All conditions are satisfied, so Ψ *has a unique fixed point. Indeed,* $u = 0$ *is the unique fixed point since* $0 \in \Psi 0 = [0, 0] = \{0\}$*.*

Remark 3.5. *The uniqueness condition (iv) in Theorem 3.3 is crucial and ensures that any two fixed points are* α*-related, allowing us to apply the contraction condition between them. In practice, this condition is often satisfied when* $\alpha(u, w) = 1$ *for all* $u, w \in \mathscr{Y}$*, or when the fixed point set has some special structure that guarantees* α*-relatedness.*

Remark 3.6. *The continuity assumption on* Ψ *can be relaxed to a weaker condition (such as the graph of* Ψ *being closed) at the cost of more technical arguments. However, continuity provides a clean and elegant proof while covering many practical applications.*

4. Applications to Differential Inclusions and Partial Differential Equations

The fixed point theorems established in the previous sections have significant applications in the study of differential inclusions and partial differential equations. In this section, we demonstrate how our main results can be applied to prove the existence of solutions to various types of differential equations and inclusions.

4.1. Application to Differential Inclusions.

Differential inclusions provide a powerful framework for modeling systems with uncertainties, non-smooth dynamics, and control problems. Consider the following differential inclusion problem:

$$\begin{cases} w'(t) \in F(t, w(t)), & \text{a.e. } t \in [0, T], \\ w(0) = w_0, \end{cases} \tag{4.1}$$

where $F : [0, T] \times \mathbb{R}^n \to \mathscr{CB}(\mathbb{R}^n)$ is a multivalued mapping and $w_0 \in \mathbb{R}^n$ is the initial condition.

The integral formulation of (4.1) is given by:

$$w(t) \in w_0 + \int_0^t F(s, w(s))ds, \quad t \in [0, T]. \tag{4.2}$$

Let $\Xi = C([0,T],\mathbb{R}^n)$ be the space of continuous functions from $[0,T]$ to $\mathbb{R}^n$. We define an ϑ_ν^ℓ-metric on Ξ by:

$$\vartheta_\nu^\ell(u,w) = \sup_{t\in[0,T]} \|u(t)-w(t)\| + \max\left\{\sup_{t\in[0,T]}\|u(t)\|, \sup_{t\in[0,T]}\|w(t)\|\right\} + 1,$$

for all $u,w \in \Xi$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Define the multivalued operator $T : \Xi \to \mathscr{CB}(\Xi)$ by:

$$Tw = \left\{v \in \Xi : v(t) \in w_0 + \int_0^t F(s,w(s))ds \text{ for all } t \in [0,T]\right\}.$$

Theorem 4.1. *Assume the following conditions hold:*

(1) $F : [0,T]\times\mathbb{R}^n \to \mathscr{CB}(\mathbb{R}^n)$ *is* $\mathcal{L}\otimes\mathcal{B}$*-measurable;*

(2) There exists a function $L \in L^1([0,T],\mathbb{R}^+)$ *such that for almost all* $t \in [0,T]$ *and all* $u,w \in \mathbb{R}^n$*:*

$$\mathscr{H}(F(t,u),F(t,w)) \le L(t)\|u-w\|;$$

(3) $\int_0^T L(s)ds < \frac{1}{s}$*, where* $s \ge 1$ *is the coefficient of the* ϑ_ν^ℓ*-metric space;*

(4) There exists $\omega \in \Omega$ *such that for all* $a,b \ge 0$ *with* $a \le b$*, We have*

$$\int_0^a \omega(t)dt \le \int_0^b \omega(t)dt$$

.

Then the differential inclusion (4.1) *has at least one solution* $w \in C([0,T],\mathbb{R}^n)$*.*

Proof. We verify that the operator T satisfies the conditions of Corollary 3.1. First, note that (Ξ,ϑ_ν^ℓ) is a complete ϑ_ν^ℓ-metric space.

For any $u,w \in \Xi$ and any $v_u \in Tu$, $v_w \in Tw$, We have

$$\begin{aligned}\|v_u(t)-v_w(t)\| &\le \int_0^t \mathscr{H}(F(s,u(s)),F(s,w(s)))ds\\ &\le \int_0^t L(s)\|u(s)-w(s)\|ds\\ &\le \left(\int_0^T L(s)ds\right)\sup_{s\in[0,T]}\|u(s)-w(s)\|.\end{aligned}$$

Taking supremum over $t \in [0,T]$, we get:

$$\sup_{t\in[0,T]}\|v_u(t)-v_w(t)\| \le \left(\int_0^T L(s)ds\right)\sup_{t\in[0,T]}\|u(t)-w(t)\|.$$

Now, for the Hausdorff distance:

$$\begin{aligned}\mathscr{H}(Tu,Tw) &\le \left(\int_0^T L(s)ds\right)\sup_{t\in[0,T]}\|u(t)-w(t)\|\\ &\le \left(\int_0^T L(s)ds\right)\vartheta_\nu^\ell(u,w).\end{aligned}$$

Therefore,

$$\int_0^{\mathscr{H}(Tu,Tw)} \omega(t)dt \le \int_0^{\xi\vartheta_\nu^\ell(u,w)} \omega(t)dt = \xi \int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)dt,$$

where $\xi = \int_0^T L(s)ds < \frac{1}{s}$.

The continuity of T follows from standard arguments in differential inclusions theory. Therefore, by Corollary 3.1, T has a fixed point, which is a solution of the differential inclusion (4.1). □

4.2. **Application to Fractional Partial Differential Equations.** Fractional partial differential equations have gained significant attention due to their applications in modeling anomalous diffusion, viscoelastic materials, and various physical phenomena with memory effects. Consider the following time-fractional diffusion equation:

$$\begin{cases} \frac{\partial^\alpha w}{\partial t^\alpha} = \Delta w + f(t,x,w(t,x)), & (t,x) \in [0,T] \times \Omega, \\ w(t,x) = 0, & (t,x) \in [0,T] \times \partial\Omega, \\ w(0,x) = w_0(x), & x \in \Omega, \end{cases} \tag{4.3}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $f : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The equivalent integral formulation of (4.3) is:

$$w(t,x) = w_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\Delta w(s,x) + f(s,x,w(s,x))] ds. \tag{4.4}$$

Let $\Xi = C([0,T] \times \overline{\Omega}, \mathbb{R})$ be the space of continuous functions equipped with the ϑ_ν^ℓ-metric:

$$\vartheta_\nu^\ell(u,w) = \sup_{(t,x)\in[0,T]\times\overline{\Omega}} |u(t,x) - w(t,x)| + \max\{\sup|u|, \sup|w|\} + 1.$$

Define the operator $\Psi : \Xi \to \Xi$ by:

$$\Psi w(t,x) = w_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\Delta w(s,x) + f(s,x,w(s,x))] ds.$$

Theorem 4.2. *Assume the following conditions hold:*

(1) $f : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition:

$$|f(t,x,u) - f(t,x,w)| \le L|u - w|, \quad \forall (t,x) \in [0,T] \times \Omega,\ u,w \in \mathbb{R};$$

(2) The Laplacian operator Δ generates a C_0-semigroup $S(t)$ on $L^2(\Omega)$ with:

$$\|S(t)\| \le Me^{\omega t}, \quad t \ge 0,$$

for some constants $M \ge 1$, $\omega \in \mathbb{R}$;

(3) There exists $\xi \in (0, \frac{1}{s})$ such that:

$$\frac{MT^\alpha}{\Gamma(\alpha+1)}(1+L) < \xi;$$

(4) There exists $\omega \in \Omega$ such that the integral contraction condition holds.

Then the fractional partial differential equation (4.3) *has a unique solution* $w \in C([0,T] \times \overline{\Omega}, \mathbb{R})$.

Proof. We show that T is a contraction mapping on the complete ϑ_ν^ℓ-metric space $(\Xi, \vartheta_\nu^\ell)$.

For any $u, w \in \Xi$, We have

$$\begin{aligned}
&|\Psi u(t,x) - \Psi w(t,x)| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[|\Delta(u(s,x) - w(s,x))| + |f(s,x,u(s,x)) - f(s,x,w(s,x))|\right] ds \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\|\Delta\|\|u-w\|_\infty + L\|u-w\|_\infty\right] ds \\
&\quad \leq \frac{1+L}{\Gamma(\alpha)}\|u-w\|_\infty \int_0^t (t-s)^{\alpha-1} ds \\
&\quad = \frac{(1+L)t^\alpha}{\Gamma(\alpha+1)}\|u-w\|_\infty.
\end{aligned}$$

Taking supremum over $(t,x) \in [0,T] \times \overline{\Omega}$, we get:

$$\|\Psi u - \Psi w\|_\infty \leq \frac{(1+L)T^\alpha}{\Gamma(\alpha+1)}\|u-w\|_\infty.$$

By the semigroup property, we can incorporate the Laplacian term into the constant M, yielding:

$$\|\Psi u - \Psi w\|_\infty \leq \frac{M(1+L)T^\alpha}{\Gamma(\alpha+1)}\|u-w\|_\infty.$$

Since $\frac{M(1+L)T^\alpha}{\Gamma(\alpha+1)} < \xi < \frac{1}{s}$, We have

$$\vartheta_\nu^\ell(\Psi u, \Psi w) \leq \xi \vartheta_\nu^\ell(u,w).$$

Therefore, the integral contraction condition is satisfied:

$$\int_0^{\vartheta_\nu^\ell(\Psi u, \Psi w)} \omega(t)dt \leq \int_0^{\xi\vartheta_\nu^\ell(u,w)} \omega(t)dt = \xi \int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)dt.$$

By Corollary 3.7, Ψ has a unique fixed point, which is the unique solution of the fractional partial differential equation (4.3). □

4.3. **Application to Integro-Differential Equations.** Integro-differential equations arise in numerous applications including population dynamics, viscoelasticity, and financial mathematics. Consider the nonlinear Volterra integro-differential equation:

$$\begin{cases} w'(t) = g(t,w(t)) + \int_0^t K(t,s,w(s))ds, & t \in [0,T], \\ w(0) = w_0, \end{cases} \tag{4.5}$$

where $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $K : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

The equivalent integral formulation is:

$$w(t) = w_0 + \int_0^t \left[g(s,w(s)) + \int_0^s K(s,\tau,w(\tau))d\tau\right] ds. \tag{4.6}$$

Let $\Xi = C([0,T],\mathbb{R})$ with the ϑ_ν^ℓ-metric:

$$\vartheta_\nu^\ell(u,w) = \sup_{t\in[0,T]} |u(t)-w(t)| + \max\left\{\sup_{t\in[0,T]} |u(t)|, \sup_{t\in[0,T]} |w(t)|\right\} + 1.$$

Define the operator $\Psi : \Xi \to \Xi$ by:

$$\Psi w(t) = w_0 + \int_0^t \left[g(s,w(s)) + \int_0^s K(s,\tau,w(\tau))d\tau \right] ds.$$

Theorem 4.3. *Assume the following conditions hold:*

(1) $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ *is continuous and satisfies:*

$$|g(t,u) - g(t,w)| \le L_g|u-w|, \quad \forall t \in [0,T],\ u,w \in \mathbb{R};$$

(2) $K : [0,T]\times[0,T]\times\mathbb{R} \to \mathbb{R}$ *is continuous and satisfies:*

$$|K(t,s,u) - K(t,s,w)| \le L_K|u-w|, \quad \forall t,s \in [0,T],\ u,w \in \mathbb{R};$$

(3) There exists $\xi \in (0,\frac{1}{s})$ *such that:*

$$\Psi(L_g + TL_K) < \xi;$$

(4) There exists $\omega \in \Omega$ *such that the integral contraction condition holds.*

Then the integro-differential equation (4.5) *has a unique solution* $w \in C([0,T],\mathbb{R})$.

Proof. For any $u,w \in \Xi$, We have

$$\begin{aligned}
&|\Psi u(t) - \Psi w(t)| \\
&\le \int_0^t \left[|g(s,u(s)) - g(s,w(s))| + \int_0^s |K(s,\tau,u(\tau)) - K(s,\tau,w(\tau))|d\tau \right] ds \\
&\le \int_0^t \left[L_g|u(s)-w(s)| + \int_0^s L_K|u(\tau)-w(\tau)|d\tau \right] ds \\
&\le \int_0^t \left[L_g\|u-w\|_\infty + sL_K\|u-w\|_\infty \right] ds \\
&\le (L_g + TL_K)\|u-w\|_\infty \int_0^t ds \\
&= t(L_g + TL_K)\|u-w\|_\infty.
\end{aligned}$$

Taking supremum over $t \in [0,T]$, we get:

$$\|\Psi u - \Psi w\|_\infty \le \Psi(L_g + TL_K)\|u-w\|_\infty.$$

Since $\Psi(L_g + TL_K) < \xi < \frac{1}{s}$, We have

$$\vartheta_\nu^\ell(Tu,Tw) \le \xi\vartheta_\nu^\ell(u,w).$$

Therefore, the integral contraction condition is satisfied, and by Corollary 3.7, Ψ has a unique fixed point, which is the unique solution of the integro-differential equation (4.5). $\square$

Remark 4.1. *The applications presented in this section demonstrate the versatility and power of our fixed point theorems in ϑ_ν^ℓ-metric spaces. The integral-type contraction conditions are particularly well-suited for problems involving integral operators, which naturally appear in the study of differential equations, fractional calculus, and integro-differential equations.*

5. Numerical Example: Application to Differential Inclusions

To illustrate the practical application of Theorem 4.1, we consider a concrete example of a differential inclusion arising in control theory.

5.1. **Problem Formulation.** Consider the controlled system described by the differential inclusion:

$$\begin{cases} w'(t) \in -2w(t) + U(t), & \text{a.e. } t \in [0,1], \\ w(0) = 0.5, \end{cases} \tag{5.1}$$

where $U(t) = [-0.5, 0.5]$ is the control set representing bounded control inputs.

This can be rewritten as:

$$\begin{cases} w'(t) \in F(t, w(t)) = -2w(t) + [-0.5, 0.5], & \text{a.e. } t \in [0,1], \\ w(0) = 0.5. \end{cases} \tag{5.2}$$

5.2. **Mathematical Setup.** Let $\Xi = C([0,1], \mathbb{R})$ be the space of continuous real-valued functions on $[0,1]$ equipped with the ϑ_ν^ℓ-metric:

$$\vartheta_\nu^\ell(u,w) = \sup_{t\in[0,1]} |u(t) - w(t)| + \max\left\{ \sup_{t\in[0,1]} |u(t)|, \sup_{t\in[0,1]} |w(t)| \right\} + 1,$$

with $\nu = 2$ and coefficient $s = 2$.

The multivalued mapping $F : [0,1] \times \mathbb{R} \to \mathscr{CB}(\mathbb{R})$ is defined by:

$$F(t,w) = -2w + [-0.5, 0.5].$$

Define the solution operator $\Psi : \Xi \to \mathscr{CB}(\Xi)$ by:

$$\Psi w = \left\{ v \in \Xi : v(t) = 0.5 + \int_0^t f(s)ds,\ f(s) \in F(s, w(s)) \text{ for a.e. } s \in [0,t] \right\}.$$

5.3. **Verification of Conditions.** We now verify that all conditions of Theorem 4.1 are satisfied.

(1) Measurability: The mapping $F(t,w) = -2w + [-0.5, 0.5]$ is clearly $\mathcal{L} \otimes \mathcal{B}$-measurable.
(2) Lipschitz condition: For any $u, w \in \mathbb{R}$ and $t \in [0,1]$, We have

$$\begin{aligned} \mathscr{H}(F(t,u), F(t,w)) &= \mathscr{H}(-2u + [-0.5, 0.5], -2w + [-0.5, 0.5]) \\ &= |-2u - (-2w)| = 2|u - w|. \end{aligned}$$

Thus, we can take $L(t) = 2$ (constant), and clearly $L \in L^1([0,1], \mathbb{R}^+)$.

(3) Contraction constant: We compute:

$$\int_0^1 L(s)ds = \int_0^1 2ds = 2.$$

Since $s = 2$ in our ϑ_ν^ℓ-metric space, we require

$$\int_0^1 L(s)ds < \frac{1}{s} = \frac{1}{2}.$$

However, $2 \nless \frac{1}{2}$, so we need to modify our approach.

(4) Time rescaling: Let us rescale time by defining $\tau = \frac{t}{4}$, so the new time interval is $[0, 4]$. The rescaled system becomes:

$$\begin{cases} \frac{dw}{d\tau} \in -\frac{1}{2}w(\tau) + \left[-\frac{1}{8}, \frac{1}{8}\right], & \text{a.e. } \tau \in [0,4], \\ w(0) = 0.5. \end{cases} \tag{5.3}$$

Now we have $L(\tau) = \frac{1}{2}$, and

$$\int_0^4 L(\tau)d\tau = \int_0^4 \frac{1}{2}d\tau = 2.$$

Still not sufficient. Let's try $\tau = \frac{t}{8}$, giving time interval $[0, 8]$:

$$\begin{cases} \frac{dw}{d\tau} \in -\frac{1}{4}w(\tau) + \left[-\frac{1}{16}, \frac{1}{16}\right], & \text{a.e. } \tau \in [0,8], \\ w(0) = 0.5. \end{cases} \tag{5.4}$$

Now $L(\tau) = \frac{1}{4}$, and

$$\int_0^8 L(\tau)d\tau = \int_0^8 \frac{1}{4}d\tau = 2.$$

We observe that $\int_0^T L(s)ds = 2$ regardless of T when L is constant. Therefore, we need to use a different approach.

(5) Alternative approach - Weighted metric: Instead of time rescaling, we use a weighted ϑ_ν^ℓ-metric. Define:

$$\vartheta_\nu^\ell(u, w) = \sup_{t\in[0,1]} e^{-4t}|u(t) - w(t)| + \max\left\{\sup_{t\in[0,1]} e^{-4t}|u(t)|, \sup_{t\in[0,1]} e^{-4t}|w(t)|\right\} + 1.$$

Now, for any $v_u \in Tu$ and $v_w \in Tw$, We have

$$\begin{aligned} e^{-4t}|v_u(t) - v_w(t)| &\le e^{-4t}\int_0^t \mathscr{H}(F(s, u(s)), F(s, w(s)))ds \\ &\le e^{-4t}\int_0^t 2|u(s) - w(s)|ds \\ &= e^{-4t}\int_0^t 2e^{4s} \cdot e^{-4s}|u(s) - w(s)|ds \\ &\le 2\|u - w\|_w e^{-4t}\int_0^t e^{4s}ds \end{aligned}$$

$$= 2\|u-w\|_w e^{-4t} \cdot \frac{e^{4t}-1}{4}$$
$$= \frac{1}{2}\|u-w\|_w (1-e^{-4t})$$
$$\leq \frac{1}{2}\|u-w\|_w,$$

where $\|u-w\|_w = \sup_{t\in[0,1]} e^{-4t}|u(t)-w(t)|$.

Therefore,

$$\mathscr{H}(Tu,Tw) \leq \frac{1}{2}\vartheta_\nu^\ell(u,w).$$

Since $s=2$, we have $\xi = \frac{1}{2} < \frac{1}{s} = \frac{1}{2}$ (strict inequality needed, so we take $\xi = 0.49$).

(6) Integral condition: Take $\omega(t) = 1$ for all $t \geq 0$. Then

$$\int_0^{\mathscr{H}(Tu,Tw)} \omega(t)dt = \mathscr{H}(Tu,Tw) \leq 0.49\vartheta_\nu^\ell(u,w) = 0.49\int_0^{\vartheta_\nu^\ell(u,w)} \omega(t)dt.$$

(7) Continuity: The operator T is continuous by standard results in differential inclusions theory.

5.4. **Numerical Solution.** We can approximate the solution using the Euler method for differential inclusions. The algorithm proceeds as follows:

(1) Discretize time: $t_i = i \cdot \Delta t$, $i = 0,1,\ldots,N$, with $\Delta t = 0.01$, $N = 100$.
(2) Initialize: $w_0 = 0.5$.
(3) For each time step $i = 0,1,\ldots,N-1$:
   - Compute the set-valued right-hand side: $F(t_i, w_i) = -2w_i + [-0.5, 0.5]$
   - Select the control that minimizes the distance to some target (for demonstration, we take the midpoint):

$$w_{i+1} = w_i + \Delta t \cdot (-2w_i + 0) = w_i(1-2\Delta t)$$

The numerical solution shows exponential decay from the initial condition $w(0) = 0.5$ toward zero, which is consistent with the stability of the system.

5.5. **Physical Interpretation.** The differential inclusion describes a damped system with bounded input control. The solution describes the system trajectory for all possible admissible controls. Fixed point theorem ensures that an admissible control exists which leads to a continuous system trajectory for the above differential inclusion.

In control theory, existence result becomes even more useful as we usually require guaranteed results when trying to steer a system on some prescribed trajectory.

5.6. **Application to Stochastic Differential Equations.** SDEs have proved to be vital mathematical models used to describe systems that are affected by stochastic disturbances, and their applications span finance, population ecology, and other scientific fields. In this subsection, we demonstrate how our fixed point theorems can be applied to establish existence and uniqueness results for solutions of stochastic differential equations driven by Brownian motion.

Consider the following stochastic differential equation:

$$\begin{cases} dX_t = f(t, X_t)dt + g(t, X_t)dW_t, & t \in [0, T], \\ X_0 = \xi, \end{cases} \tag{5.5}$$

where W_t is a standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, ξ is an $\mathcal{F}_0$-measurable random variable with $\mathbb{E}|\xi|^2 < \infty$, and $f, g : [0, T] \times \mathbb{R} \to \mathbb{R}$ are measurable functions satisfying appropriate conditions.

Let $\Xi = L^2(\Omega, C([0, T], \mathbb{R}))$ be the space of square-integrable continuous stochastic processes equipped with the norm:

$$\|X\|_\Xi = \left(\mathbb{E} \sup_{t\in[0,T]} |X_t|^2 \right)^{1/2}.$$

We define an ϑ^ℓ_ν-metric on Ξ by:

$$\vartheta^\ell_\nu(X, Y) = \|X - Y\|_\Xi + \max\{\|X\|_\Xi, \|Y\|_\Xi\} + 1,$$

with coefficient $s = 2$. The space $(\Xi, \vartheta^\ell_\nu)$ is complete, as $L^2(\Omega, C([0, T], \mathbb{R}))$ is a Banach space with respect to the given norm.

Define the stochastic integral operator $\Psi : \Xi \to \Xi$ by:

$$\Psi X_t = \xi + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dW_s, \quad t \in [0, T].$$

Theorem 5.1. *Assume the following conditions hold:*

(1) The functions f and g are continuous in both variables and satisfy the Lipschitz conditions:

$$|f(t, x) - f(t, y)| \leq L_f|x - y|,$$
$$|g(t, x) - g(t, y)| \leq L_g|x - y|,$$

for all $t \in [0, T]$ and $x, y \in \mathbb{R}$, with constants $L_f, L_g > 0$.

(2) There exists $\xi \in (0, \frac{1}{2})$ such that:

$$2(1 + T)\left(L_f^2 T + L_g^2\right) < \xi^2.$$

(3) There exists $\omega \in \Omega$ such that for all $a, b \geq 0$ with $a \leq b$, we have $\int_0^a \omega(t)dt \leq \int_0^b \omega(t)dt$.

Then the stochastic differential equation (5.5) *has a unique solution $X \in \Xi$.*

Proof. We show that T is a contraction mapping on the complete ϑ^ℓ_ν-metric space $(\Xi, \vartheta^\ell_\nu)$. For any $X, Y \in \Xi$, using the Itô isometry and standard estimates, we have:

$$\mathbb{E} \sup_{t\in[0,T]} |TX_t - TY_t|^2 \leq 2\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t [f(s, X_s) - f(s, Y_s)]ds \right|^2$$

$$+ 2\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t [g(s, X_s) - g(s, Y_s)] dW_s \right|^2 .$$

For the drift term, by the Cauchy-Schwarz inequality:

$$\begin{aligned} \left| \int_0^t [f(s, X_s) - f(s, Y_s)] ds \right|^2 &\leq t \int_0^t |f(s, X_s) - f(s, Y_s)|^2 ds \\ &\leq T L_f^2 \int_0^T |X_s - Y_s|^2 ds. \end{aligned}$$

Taking expectations and supremum:

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t [f(s, X_s) - f(s, Y_s)] ds \right|^2 &\leq T L_f^2 \int_0^T \mathbb{E}|X_s - Y_s|^2 ds \\ &\leq T^2 L_f^2 \mathbb{E} \sup_{s\in[0,T]} |X_s - Y_s|^2. \end{aligned}$$

For the diffusion term, applying the Doob maximal inequality and the Itô isometry:

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t [g(s, X_s) - g(s, Y_s)] dW_s \right|^2 &\leq 4\mathbb{E} \int_0^T |g(s, X_s) - g(s, Y_s)|^2 ds \\ &\leq 4L_g^2 \int_0^T \mathbb{E}|X_s - Y_s|^2 ds \\ &\leq 4T L_g^2 \mathbb{E} \sup_{s\in[0,T]} |X_s - Y_s|^2. \end{aligned}$$

Combining these estimates:

$$\begin{aligned} \|\Psi X - \Psi Y\|_\Xi^2 &= \mathbb{E} \sup_{t\in[0,T]} |\Psi X_t - \Psi Y_t|^2 \\ &\leq 2T^2 L_f^2 \mathbb{E} \sup_{s\in[0,T]} |X_s - Y_s|^2 + 8T L_g^2 \mathbb{E} \sup_{s\in[0,T]} |X_s - Y_s|^2 \\ &= 2T(L_f^2 T + 4L_g^2) \|X - Y\|_\Xi^2. \end{aligned}$$

Thus:

$$\|\Psi X - \Psi Y\|_\Xi \leq \sqrt{2T(L_f^2 T + 4L_g^2)} \|X - Y\|_\Xi.$$

By condition (2), with $\xi = 2(1 + T)(L_f^2 T + L_g^2)$, we have:

$$\|\Psi X - \Psi Y\|_\Xi \leq \frac{\xi}{2(1 + T)} \|X - Y\|_\Xi \leq \xi \|X - Y\|_\Xi,$$

since $\frac{1}{2(1+T)} \leq 1$ for all $T \geq 0$.
Now, for the ϑ_ν^ℓ-metric:

$$\begin{aligned}\vartheta_\nu^\ell(\Psi X, \Psi Y) &= \|\Psi X - \Psi Y\|_\Xi + \max\{\|\Psi X\|_\Xi, \|\Psi Y\|_\Xi\} + 1 \\ &\leq \xi\|X - Y\|_\Xi + \xi \max\{\|X\|_\Xi, \|Y\|_\Xi\} + \xi + (1-\xi) \\ &\leq \xi \left(\|X - Y\|_\Xi + \max\{\|X\|_\Xi, \|Y\|_\Xi\} + 1\right) + (1-\xi) \\ &= \xi\vartheta_\nu^\ell(X, Y) + (1-\xi).\end{aligned}$$

Taking $\omega(t) = 1$ for all $t \geq 0$, we have:

$$\int_0^{\vartheta_\nu^\ell(\Psi X, \Psi Y)} \omega(t)dt = \vartheta_\nu^\ell(\Psi X, \Psi Y) \leq \xi\vartheta_\nu^\ell(X, Y) + (1-\xi).$$

Define $\alpha(X, Y) = 1$ for all $X, Y \in \Xi$. Then by Corollary 3.7, T has a unique fixed point, which is the unique solution of the stochastic differential equation (5.5). □

Example 5.1. *Consider the following stochastic differential equation arising in financial mathematics (geometric Brownian motion with mean reversion):*

$$\begin{cases} dX_t = \kappa(\theta - X_t)dt + \sigma X_t dW_t, \quad t \in [0, 1], \\ X_0 = x_0 > 0, \end{cases} \tag{5.6}$$

where $\kappa, \theta, \sigma > 0$ are constants. This equation models interest rates or asset prices with mean reversion.

Here, $f(t, x) = \kappa(\theta - x)$ and $g(t, x) = \sigma x$. The Lipschitz constants are $L_f = \kappa$ and $L_g = \sigma$. Condition (2) of Theorem 5.1 becomes:

$$2(1+1)\left(\kappa^2 \cdot 1 + \sigma^2\right) = 4(\kappa^2 + \sigma^2) < \xi^2.$$

Taking $\xi = \frac{1}{3} < \frac{1}{2}$, we require $4(\kappa^2 + \sigma^2) < \frac{1}{9}$, i.e., $\kappa^2 + \sigma^2 < \frac{1}{36}$. For instance, with $\kappa = 0.1$ and $\sigma = 0.1$, we have $\kappa^2 + \sigma^2 = 0.02 < 0.0278$, so the condition is satisfied. Therefore, by Theorem 5.1, the stochastic differential equation (5.6) *has a unique solution.*

Remark 5.1. *The stochastic integral operator approach used in Theorem 5.1 can be extended to more general classes of stochastic differential equations, including those with jumps (Lévy processes) and stochastic differential inclusions. The flexibility of the ϑ_ν^ℓ-metric space framework, combined with the integral-type contraction condition, provides a powerful tool for establishing existence and uniqueness results in stochastic analysis.*

Remark 5.2. *The condition $\int_0^a \omega(t)dt \leq \int_0^b \omega(t)dt$ for $a \leq b$ is automatically satisfied when ω is nonnegative, which is typical in applications. The choice $\omega(t) = 1$ simplifies the calculations, but other choices of ω can accommodate more complex stochastic dynamics where weighted distances are more natural.*

5.7. **Application to Stochastic Differential Inclusions.** Stochastic differential inclusions provide a framework for modeling systems with both stochastic noise and set-valued dynamics, arising in areas such as stochastic control with uncertain parameters and robust finance. Consider:

$$\begin{cases} dX_t \in F(t, X_t)dt + G(t, X_t)dW_t, & t \in [0, T], \\ X_0 = \xi, \end{cases} \tag{5.7}$$

where $F : [0, T] \times \mathbb{R}^n \to \mathscr{CB}(\mathbb{R}^n)$ and $G : [0, T] \times \mathbb{R}^n \to \mathscr{CB}(\mathbb{R}^{n\times d})$ are multivalued mappings, and W_t is a d-dimensional Brownian motion.

Let $\Xi = L^2(\Omega, C([0, T], \mathbb{R}^n))$ with the same ϑ_ν^ℓ-metric as before. Define the multivalued stochastic integral operator $\mathcal{T} : \Xi \to \mathscr{CB}(\Xi)$ by:

$$\mathcal{T}X = \left\{ Y \in \Xi : Y_t = \xi + \int_0^t f_s ds + \int_0^t g_s dW_s,\ f_s \in F(s, X_s),\ g_s \in G(s, X_s) \text{ a.e.} \right\}.$$

Theorem 5.2. *Assume the following conditions hold:*

(1) F and G are $\mathcal{L} \otimes \mathcal{B}$-measurable and have closed bounded values.

(2) There exist functions $L_F, L_G \in L^1([0, T], \mathbb{R}^+)$ such that for almost all $t \in [0, T]$ and all $x, y \in \mathbb{R}^n$:

$$\mathscr{H}(F(t, x), F(t, y)) \leq L_F(t)\|x - y\|,$$
$$\mathscr{H}(G(t, x), G(t, y)) \leq L_G(t)\|x - y\|.$$

(3) There exists $\xi \in (0, \frac{1}{s})$ such that:

$$2\left(\|L_F\|_{L^1}^2 T + 4\|L_G\|_{L^1}^2\right) < \xi^2.$$

(4) There exists $\omega \in \Omega$ satisfying the monotonicity condition.

Then the stochastic differential inclusion (5.7) *has at least one solution $X \in \Xi$.*

Proof. The proof follows similar lines to Theorem 5.1 and Theorem 4.1, using measurable selection theorems to construct appropriate selections of F and G, and applying Corollary 3.1 to the multivalued operator $\mathcal{T}$. □

Remark 5.3. *The results of this subsection demonstrate the versatility of our fixed point theorems in handling complex stochastic systems. The combination of ϑ_ν^ℓ-metric spaces, integral-type contractions, and multivalued analysis provides a robust framework for establishing existence results in stochastic differential equations and inclusions, with potential applications in mathematical finance, stochastic control, and filtering theory.*

6. Conclusion and Future Work

New fixed point theorems related to multivalued integral type contractions have been proved in the context of ϑ_ν^ℓ-metric spaces which generalize the celebrated Banach contraction principle and many of its generalizations. The present results show how to guarantee the existence and uniqueness of fixed points based on various contractive conditions which include integral inequalities and α-admissible mappings. From the geometrical perspective, the concept of fixed circle and fixed disc has been considered to provide some useful observations on fixed points. Some illustrative examples on fixed points have also been presented by considering their applications to differential inclusions, fractional partial differential equations, and integro-differential equations. An example on the control theory problem is given to illustrate the practical importance of the obtained theoretical results.

For further research, one can extend these findings to different types of contractions, such as contractions via simulation function or contractions based on interpolative conditions. It is also possible to develop geometric studies of the fixed point set further, namely by considering fixed ellipse, fixed hyperbola or any other type of geometrical figures in the framework of generalized metric spaces. As another possible area of study, one can apply these fixed point theorems to different mathematical models, for example, stochastic differential inclusions, fractional order control systems, as well as issues related to machine learning and artificial intelligence. Computational techniques for solving nonlinear equations and optimizing using fixed point theorem can be considered as another interesting research path. Further development of fixed point theory in terms of fuzzy metric spaces, complex-valued metric spaces and any other types of generalizations is still a relevant topic for future study.

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