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Invariant Solutions, Conservation Laws, and Jacobi Elliptic Waves for a Fifth-Order Nonlinear Evolution Equation

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Abstract. A detailed structural analysis of a fifth-order nonlinear evolution equation is presented in this work. The study begins with the determination of classical Lie point symmetries, leading to the construction of an optimal system of one-dimensional subalgebras and the derivation of corresponding similarity reductions. These reductions transform the governing partial differential equation into a nonlinear ordinary differential equation that characterizes invariant solution profiles.The nonclassical symmetry approach is subsequently employed to uncover additional invariant structures that are not accessible through the classical framework, yielding complex exponential and traveling-wave type solutions. To further explore the intrinsic properties of the equation, conservation laws are systematically derived via Ibragimov's theorem. The existence of multiple conserved vectors highlights the internal structural consistency of the model and supports its integrable nature.In addition, explicit analytical solutions are obtained using the modified Jacobi elliptic expansion method. The resulting families of solutions include periodic and solitary-type wave forms expressed in terms of hyperbolic and elliptic functions. The combined application of symmetry analysis, conservation law construction, and elliptic function expansion provides a comprehensive understanding of the invariant structure and solution space of the considered fifth-order nonlinear model.

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1. Introduction

Nonlinear evolution equations of higher order arise naturally in the mathematical modeling of dispersive wave propagation, nonlinear lattices, plasma physics, and fluid dynamics. In particular, fifth-order nonlinear partial differential equations have attracted considerable attention due to their ability to describe complex interactions between nonlinearity and higher-order dispersion. Such equations frequently generalize classical models like the Korteweg–de Vries (KdV) equation by incorporating additional dispersive mechanisms or nonlinear coupling effects. The analytical investigation of these higher-order systems is essential for understanding their qualitative structure. Symmetry analysis provides a systematic framework for identifying invariant transformations, constructing similarity reductions, and revealing hidden geometric properties of differential equations [1–3]. Unlike purely computational solution methods, symmetry-based approaches often expose intrinsic structural features of the governing equation, including invariance under scaling, translation, or mixed transformations. Beyond classical Lie symmetries, nonclassical symmetry methods extend the reduction process by incorporating invariant surface conditions, allowing for the discovery of solutions that cannot be obtained through classical symmetry analysis alone [4–6]. These techniques frequently yield wave structures, complex exponential forms, and other nontrivial invariant solutions. Another important structural property of nonlinear evolution equations is the existence of conservation laws. Conservation laws play a central role in identifying physically meaningful invariants and may provide insight into the integrability or near-integrability of the system. Ibragimov's theorem offers a direct and systematic way to construct conservation laws even for equations that do not possess an obvious variational formulation [7–9]. In addition to symmetry-based reductions, explicit solution techniques such as the modified Jacobi elliptic expansion (MJEE) method provide families of periodic and solitary wave solutions. Jacobi elliptic functions serve as a bridge between periodic wave patterns and solitary wave limits, thereby enabling a unified treatment of different wave regimes [10–13].

In recent years, symmetry-based methods have played a crucial role in the analytical investigation of nonlinear evolution equations. In particular, classical and nonclassical Lie symmetry techniques provide systematic procedures for determining invariant transformations and reducing nonlinear partial differential equations to simpler forms. Masindi and Rundora analyzed the dynamics of the (1+1)-dimensional Fisher's equation using Lie symmetry analysis and obtained invariant solutions through similarity reductions [14]. Similarly, Yadav and Chauhan applied Lie symmetry analysis to the extended (2+1)-dimensional Camassa–Holm–Kadomtsev–Petviashvili equation and demonstrated the effectiveness of symmetry methods in analyzing complex nonlinear systems [15]. Mandal and Ghosh further investigated the Darcy–Brinkman model using Lie symmetry analysis and optimal classification techniques, leading to several invariant solutions [16]. Güngör and Özemir also examined the Lie symmetry structure of nonlinear wave equations in higher-dimensional space–time and emphasized the role of Lie groups in understanding their geometric properties [17]. In addition, nonclassical symmetry methods have been successfully applied to uncover additional invariant structures. For instance, Hosseini et al. derived nonclassical Lie symmetries for a generalized Hirota bilinear equation and obtained various resonant multi-wave and multi-complexiton solutions [18]. Exact analytical solutions of nonlinear evolution equations are essential for understanding the qualitative behavior of nonlinear wave phenomena. Various analytical techniques have been developed to obtain exact solutions such as traveling wave solutions, lump waves, and rogue waves. For example, Hussain et al. applied Lie symmetry analysis to the nonlinear integral Jimbo–Miwa equation and obtained rogue wave and lump wave solutions describing localized nonlinear structures [19]. Elbrolosy and Elmandouh also investigated a dynamical model of radial dislocations in microtubules using Painlevé analysis and Lie symmetry methods, revealing several analytical solutions and bifurcation structures [15]. Furthermore, Hosseini et al. demonstrated that combining symmetry analysis with bifurcation theory can provide deeper insights into the solution structure of nonlinear equations such as the nonlinear Kodama equation [20]. Another important aspect of nonlinear evolution equations is the construction of conservation laws, which reveal physically meaningful invariants and provide information about the intrinsic structure of the governing system. Chauhan et al. derived conservation laws for the Rosenau regularized long wave equation and analyzed its modulation instability properties [21]. Similarly, Alizadeh investigated the logarithmic modified nonlinear Schrödinger equation and constructed several conservation laws along with invariant solutions using Lie symmetry methods [22]. In a related study, Hosseini et al. obtained Lie vector fields and conservation laws for the Zakharov–Kuznetsov modified equal-width equation and analyzed its bifurcation behavior [23]. These studies demonstrate that conservation law analysis plays a fundamental role in understanding the integrability and physical interpretation of nonlinear models. In addition to symmetry-based techniques, Jacobi elliptic function methods have been widely used to construct periodic and solitary wave solutions of nonlinear evolution equations. Ma et al. obtained new periodic solutions for the modified KdV and BBM equations using Jacobi elliptic functions [10]. Later, Zayed et al. derived chirped and chirp-free soliton solutions for optical fiber Bragg gratings with dispersive reflectivity [11]. Zayed and Alngar further extended the generalized JEE method to obtain optical soliton solutions in birefringent fibers described by the Biswas–Arshed model [12]. Moreover, El-Sheikh et al. investigated higher-order nonlinear Boussinesq-like equations describing shallow water wave propagation and demonstrated the capability of analytical approaches in capturing dispersive wave dynamics [13]. These studies demonstrate that symmetry analysis, conservation law construction, and elliptic function expansion methods play a crucial role in revealing invariant structures and exact solutions of nonlinear evolution equations. Motivated by these developments, the present study focuses on the structural analysis of a new integrable fifth-order equation [24, 25], given by

$$\Delta : u_t + u_{xxt} + \mu(u_x - u_{xxxxx}) - \nu(uu_{xxx} + uu_x + 3u_x u_{xx}) = 0, \tag{1.1}$$

where μ and ν are real parameter. In order to further investigate the analytical properties of higher-order nonlinear evolution equations, we consider in this work the above fifth-order nonlinear partial differential equation. Equations of fifth order similar to the model considered here have been extensively investigated in the literature due to their rich mathematical structure and important physical applications. In recent years, several studies have investigated different types of fifth-order nonlinear equations and their solution structures. For example, Younas et al. examined a generalized fifth-order (2+1)-dimensional KdV equation and reported lump structures and collision phenomena associated with shallow water waves and nonlinear lattice systems [26]. Li et al. analyzed soliton dynamics in a perturbed fifth-order KdV equation and demonstrated the existence of stable solitary wave solutions [27]. Liu et al. studied a damped variable-coefficient fifth-order modified KdV equation arising in fluid mechanics and obtained various nonlinear wave structures such as breathers and multi-pole waves [28]. Furthermore, Liu et al. investigated localized wave interactions in a fifth-order nonlinear Schrödinger equation derived from Heisenberg ferromagnetism [29]. More recently, Ling et al. explored a time-fractional generalized fifth-order KdV equation and derived its Lie point symmetries, exact solutions, and conservation laws [30]. These investigations reveal the rich mathematical structures and diverse physical applications associated with fifth-order nonlinear evolution equations. Motivated by these developments, the present work aims to further explore the analytical properties of the proposed fifth-order nonlinear model using extended Lie symmetry techniques. The determination of the Lie symmetry structure of the governing equation is presented in the following section.

2. Lie Symmetry Method

Consider a one-parameter Lie group of transformations acting on the variables of the governing equation. Let the independent and dependent variables transform according to

$$\hat{\hat{x}} \mapsto x + \epsilon\xi(x,t,u) + O(\epsilon^2),$$
$$\hat{\hat{t}} \mapsto t + \epsilon\tau(x,t,u) + O(\epsilon^2),$$
$$\hat{u} \mapsto u + \epsilon\eta(x,t,u) + O(\epsilon^2),$$

where ϵ is a very small quantity and $\xi, \tau,$ and η are denote the infinitesimal generators of the transformation. These transformations define the associated infinitesimal operator

$$\mathbf{V} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.1}$$

A differential equation is said to admit a Lie symmetry if it remains invariant under the action of the infinitesimal generator $\mathbf{V}$. Therefore, the invariance condition is given by

$$\mathcal{P}r^{(n)}\mathbf{V}(\Delta)\Big|_{\Delta=0} = 0, \tag{2.2}$$

where $\mathcal{P}r^{(n)}\mathbf{V}$ denotes the $n-$th prolongation of the vector field $\mathbf{V}$, and $\Delta = 0$ represents the governing differential equation. Applying Equation (2.2) to Equation (2.1) yields the following invariant conditions

$$\mathcal{P}r^{(n)}\mathbf{V} =\mathbf{V} + \eta^{x}\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + ... + \eta^{i_1 i_2 ... i_k}\frac{\partial}{\partial u_{i_1 i_2 ... i_k}},$$

Here, the symbol η^j denotes the prolonged components associated with the extended infinitesimals of integer order, given by

$$\eta^{i} = D_i\eta - (D_i\xi_j)u_j, \qquad i,j = 1,2,..,n,$$
$$\eta^{i_1 i_2 ... i_k} = D_{i_k}\eta^{i_1 i_2 ... i_{k-1}} - (D_{i_k}\xi_j)u_{i_1 ... i_{k-1} j},$$

where D represent the total derivative.

2.1. **Determination of Classical Symmetries of the Governing Equation.** Based on the preceding discussion, we now seek the symmetries of Equation (1.1) by following the previously outlined procedure. The infinitesimal invariance criterion for this equation is given by

$$\mathcal{P}r^{(5)}\mathbf{V}\bigg(u_t + u_{xxt} + \mu(u_x - u_{xxxxx}) - \nu(uu_{xxx} + uu_x + 3u_x u_{xx})\bigg)\bigg|_{Eq.(1.1)} = 0. \tag{2.3}$$

Hence, the following equation must be satisfied

$$-(\nu u_{xxx} + \nu u_x)\eta + \eta^{t} + (\mu - \nu u - 3\nu u_{xx})\eta^{x} - 3\nu u_x\eta^{xx} + \eta^{xxt} - \nu u\eta^{xxx} - \mu\eta^{xxxxx} = 0. \tag{2.4}$$

By collecting the coefficients of the various derivatives of u, we obtain the following system

$\eta_{uu} = 0,$

$\tau_{xx} = 0,$

$\tau_{xtu} = 0,$

$\tau_{tuu} = 0,$

$\xi_{tuu} = 0,$

$\eta_{xu} - 2\xi_{xx} = 0,$

$3\nu\tau_x - \tau_{tu} = 0,$

$2\tau_{tu} - 6\nu\tau_x = 0,$

$2\tau_{xt} + \xi_{xx} - 2\eta_{xu} = 0,$

$\tau_t - 3\nu u\tau_x - 3\xi_x = 0,$

$3\nu\eta_u + 6\nu\xi_x + 3\xi_{tu} = 0,$

$10\mu\eta_{xxu} - 10\mu\xi_{xxx} + 2\nu u\xi_x + \nu\eta + \xi_t = 0,$

$\mu\eta_{xxxxx} - \mu\eta_x + \nu u\eta_x + \nu u\eta_{xxx} - \eta_t - \eta_{xxt} = 0.$

According to the above system, the resulting vector fields are

$$V_1 = \frac{\partial}{\partial t}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = -\nu t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

Among these vector fields becomes

Table 1. Commutator table.

$[V_i, V_j]$	V_1	V_2	V_3
V_1	0	0	$-\nu V_2$
V_2	0	0	0
V_3	νV_2	0	0

The commutation relations presented in Table 1 describe the algebraic structure of the symmetry generators associated with the governing equation. These relations play an essential role in determining the structure of the Lie algebra and provide the necessary information for constructing the adjoint representation of the symmetry group. By utilizing these properties, the symmetry generators can be systematically classified into equivalence classes. This classification enables the construction of an optimal system of subalgebras, which forms the basis for obtaining distinct invariant reductions of the original equation.

2.2. **Optimal System of One-Dimensional Subalgebras.** In order to classify the invariant solutions associated with the obtained symmetry generators, it is necessary to construct an optimal system of one-dimensional subalgebras. The purpose of this procedure is to identify a minimal collection of inequivalent generators such that every other symmetry of the Lie algebra can be transformed into one of them through an adjoint transformation. Consider a general element of the Lie algebra written as a linear combination of the basis generators

$$V = a_1 V_1 + a_2 V_2 + a_3 V_3,$$

where a_1, a_2, and a_3 are arbitrary constants.

To determine the optimal system, we employ the adjoint representation of the Lie group acting on its Lie algebra. The adjoint action corresponding to the generator V_i is defined through the Lie series

$$\mathrm{Ad}\Big(\exp(\varepsilon V_i)\Big) V_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (\mathrm{adj}\, V_i)^n (V_j) = V_j - \varepsilon [V_i, V_j] + \frac{\varepsilon^2}{2!}[V_i, [V_i, V_j]] - \cdots ,$$

where ε is a parameter and $[V_i, V_j]$ represents the commutator of the Lie algebra. A real-valued function μ defined on the Lie algebra "L" is said to be invariant if the following condition holds:

$$\mu\Big(\mathrm{Ad}_g(M)\Big) \;=\; \mu(M), \qquad \forall\, M \in \mathcal{L},\; g \in G.$$

Table 2. Adjoint table.

$\mathrm{Ad}\Big(\exp(\varepsilon V_i)\Big)V_j$	V_1	V_2	V_3
V_1	V_1	V_2	$\nu V_2\varepsilon + V_3$
V_2	V_1	V_2	V_3
V_3	$-\nu V_2\varepsilon + V_1$	V_2	V_3

Let

$$V = a_1V_1 + a_2V_2 + a_3V_3, \qquad (a_1, a_2, a_3 \in \mathbb{R}).$$

Using the adjoint action we simplify V up to conjugacy:

- Acting by $Ad\left(e^{\alpha V_1}\right)$ sends V to $a_1V_1 + (a_2 + \nu\alpha a_3)\, V_2 + a_3V_3$.
- Acting by $Ad\left(e^{\beta V_3}\right)$ sends V to $a_1V_1 + (a_2 - \nu\beta a_1)\, V_2 + a_3V_3$.
- $Ad\left(e^{\gamma V_2}\right)$ is the identity.

Hence, we can eliminate the V_2-component whenever a_1 or a_3 is nonzero.

- If $a_3 \neq 0$, choose $\alpha = -\dfrac{a_2}{\nu a_3}$ so that

$$\mathrm{Ad}(e^{\alpha V_1})V = a_1V_1 + \Big(a_2 + \nu\alpha a_3\Big)V_2 + a_3V_3 = a_1V_1 + a_3V_3.$$

  After scaling by a_3, we obtain the representative

$$\langle c\, V_1 + V_3\rangle, \qquad c = \frac{a_1}{a_3}.$$

- If $a_3 = 0$ and $a_1 \neq 0$, choose $\beta = \dfrac{a_2}{\nu a_1}$ so that

$$\mathrm{Ad}(e^{\beta V_3})V = a_1V_1 + \Big(a_2 - \nu\beta a_1\Big)V_2 = a_1V_1,$$

  hence the representative $\langle V_1\rangle$.
- If $a_1 = a_3 = 0$ and $a_2 \neq 0$, we directly have the representative $\langle V_2\rangle$.

Hence, the optimal 1-D classes: $\{\langle V_1\rangle, \langle V_2\rangle, \langle cV_1 + V_3\rangle\}$.

Case $\langle V_1\rangle$:

In the case of the invariant solution corresponding to the generator $\langle V_1\rangle$, where the solution is assumed in the form $u(x,t) = F(x)$, the substitution eliminates all time derivatives from the governing equation

$$\mu\left(F' - F^{(5)}\right) - \nu\left(FF''' + FF' + 3F'F''\right) = 0.$$

As a consequence, the partial differential equation reduces to an algebraic relation that can only be satisfied by constant functions. Therefore, the invariant solution obtained from this case is a trivial constant solution.

Case $\langle V_2 \rangle$:
In the case of the invariant solution associated with the generator $\langle V_2 \rangle$, the solution is assumed in the form $u(x,t) = F(t)$. Substituting this ansatz into the governing equation eliminates all spatial derivatives, which reduces the partial differential equation to the simple form

$$F' = 0. \tag{2.5}$$

This relation is satisfied only by constant functions, and thus, the invariant solution obtained from this case is also a trivial constant solution.

Case $W = \langle cV_1 + V_3 \rangle$:
The characteristic system

$$\frac{dx}{-\nu t} = \frac{dt}{c} = \frac{du}{1},$$

yields two independent invariants

$$\zeta \ = \ x + \frac{\nu}{2c}\,t^2, \qquad s \ = \ t - c\,u,$$

since $W(\zeta) = W(s) = 0$. Hence, we adopt the similarity ansatz

$$t - c\,u \ = \ F(\zeta) \qquad \Longleftrightarrow \qquad u(x,t) = \frac{t - F(\zeta)}{c}, \qquad \zeta = x + \frac{\nu}{2c}\,t^2.$$

Derivative identities. With $F = F(\zeta)$ and $\zeta_x = 1$, $\zeta_t = \frac{\nu}{c}t$, we have

$$u_x = -\frac{1}{c}F', \quad u_{xx} = -\frac{1}{c}F'', \quad u_{xxx} = -\frac{1}{c}F''', \quad u_{xxxxx} = -\frac{1}{c}F^{(5)},$$
$$u_t = \frac{1}{c} - \frac{\nu t}{c^2}F', \qquad u_{xt} = -\frac{\nu t}{c^2}F'', \qquad u_{xxt} = -\frac{\nu t}{c^2}F'''.$$

Reduced ODE. Substituting the above into the governing equation

$$u_t + u_{xxt} + \mu\,(u_x - u_{xxxxx}) - \nu(u\,u_{xxx} + u\,u_x + 3\,u_x u_{xx}) = 0,$$

and using $u = \frac{t - F(\zeta)}{c}$, all explicit t-dependent terms cancel and we obtain the fifth–order nonlinear ODE

$$1 + \mu\Big(F^{(5)} - F'\Big) - \frac{\nu}{c}\Big(F\,F''' + F\,F' + 3\,F'F''\Big) = 0, \qquad F = F(\zeta),\ \ \zeta = x + \frac{\nu}{2c}t^2. \tag{2.6}$$

Remark 2.1.

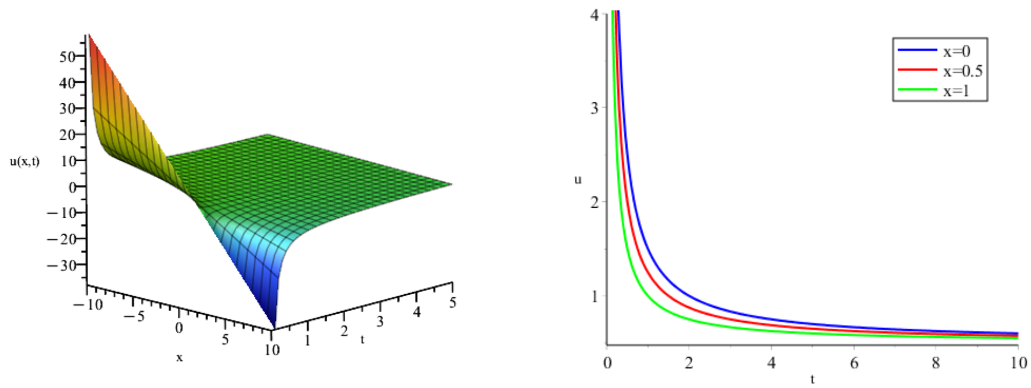
- *The parameter c cannot be removed by the adjoint action (it is a genuine modulus in the optimal system). The case $c = 0$ corresponds to the separate generator $\langle V_3 \rangle$ and must be treated on its own. For this generator the invariant solution is $u(x,t) = f(t) - \frac{x}{\nu t}$ which reduce the Equation* (1.1) *to the following equation*

$$f'\nu t + f\nu - \mu = 0,$$

and the solution of this ODE is $f(t) = \frac{\mu}{\nu} + \frac{c_1}{t}$. Therefore, the exact solution of the main equation is

$$u(x,t) = \frac{\mu t + c_1\nu - x}{\nu t}. \tag{2.7}$$

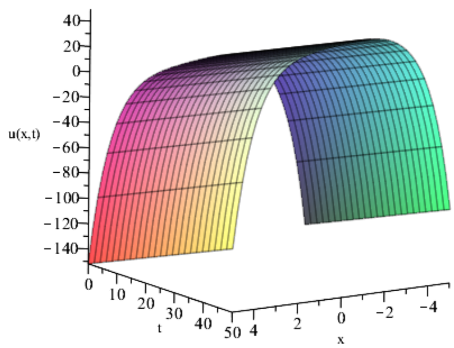
The plots of this solution is shown in Figure 1.



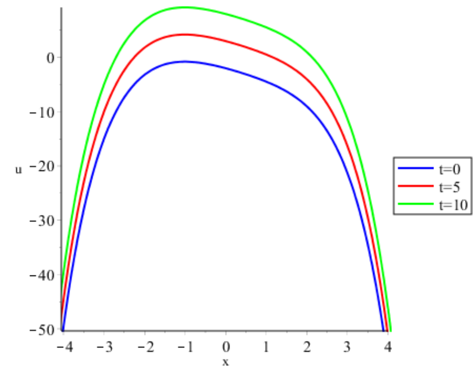
(A) (B)

FIGURE 1. Plots of the obtained invariant solution. (A) 3D surface of u, (B) Evolution of u with respect to time t for $x = 0$, $x = \frac{1}{2}$, and $x = 1$.

- *In the linear limit* $\nu = 0$*, the reduction becomes* $\mu\left(F^{(5)} - F'\right) + 1 = 0$. *Integrating once gives*
$$\mu\,(F^{(4)} - F) = -\zeta + C_1,$$
whose general solution is the sum of the homogeneous solution
$$F_h = Ae^{\zeta} + Be^{-\zeta} + C\cos\zeta + D\sin\zeta,$$
and a particular linear polynomial $F_p(\zeta) = \frac{1}{\mu}\,\zeta - C_1$. *Hence, the general solution in this limit is a fourth-degree polynomial*
$$F(\xi) = Ae^{\zeta} + Be^{-\zeta} + C\cos\zeta + D\sin\zeta + \frac{1}{\mu}\,\zeta - C_1, \tag{2.8}$$
and the corresponding invariant solution of the original PDE becomes
$$u(x,t) = \frac{t - Ae^{x} + Be^{-x} + C\cos(x) + D\sin(x) + \dfrac{1}{\mu}\,(x) - C_1}{C}, \tag{2.9}$$
which can be verified by direct substitution. The plots of the above solution is shown in Figure 2.



(A)



(B)

FIGURE 2. Plots of the obtained invariant solution. (A) 3D surface of u, (B) Evolution of u with respect to time x for $t = 0$, $t = 5$, and $t = 10$.

- *For $\nu \neq 0$, Equation* (2.6) *above provides a precise starting point for analytic or numerical study (e.g., shooting/collocation in ζ), and admits travelling-type profiles through $\zeta = x + \frac{\nu}{2c}t^2$.*

2.3. **Determination of Non-Classical Symmetries for the Main Equation.** To explore the non-classical symmetries of the Lie group, besides satisfying $\mathcal{P}r^5(\mathbf{V}) = 0$, the following invariant surface condition must also hold

$$\Omega_1 \equiv \xi(x,t,u)u_x + \tau(x,,t,u)u_t - \eta(x,t,u) = 0.$$

In the first scenario, by setting $\xi = 1$ and $\tau = 0$, the invariant surface condition reduces to

$$u_x = \eta,$$

then

$$\begin{aligned} u_{xt} &= \eta_u u_t + \eta_t, \\ u_{xx} &= \eta_u \eta + \eta_x, \\ u_{xxx} &= \eta_{xx} + \eta_x \eta_u + 2\eta\eta_{xu} + \eta^2\eta_{uu} + \eta\eta_u^2, \\ &\vdots \end{aligned}$$

Inserting these expressions into Equation (2.4) and organizing terms by their respective powers leads to the corresponding infinitesimal generator

$$W_1 = \frac{\partial}{\partial x} + \frac{i}{2}u\frac{\partial}{\partial u},$$

which generates the invariant solution of Equation (1.1) as $u(x,t) = \phi(t)e^{\frac{i}{2}x}$ we have

$$15i\mu\phi(t) + 24\phi'(t) = 0. \tag{2.10}$$

Solving Equation (2.10) provides the complex solution

$$\phi(t) = Ae^{-\frac{5}{8}i\mu t}.$$

Therefore, the resulting solution of the governing equation is

$$u(x,t) = Ae^{\frac{1}{2}i(-\frac{5}{4}\mu t + x)}.$$

In the second scenario, with $\xi = 1$ and $\zeta = \tau \neq 0$, three infinitesimal symmetries are investigated. The related vector field is expressed as

$$W_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}.$$

The corresponding invariant reduction leads to a different functional form. Vector W_2 yields traveling-wave reductions $u = f(t-x)$, which can admit nontrivial and even complex solutions, and they are important explicit solutions.

By substituting $u = f(\xi)$ where $\xi = t - x$ into the main equation, following ODE yields

$$f' + f''' - \mu f' + \mu f^{(5)} + \nu f f''' + \nu f f' + 3\nu f' f'' = 0. \tag{2.11}$$

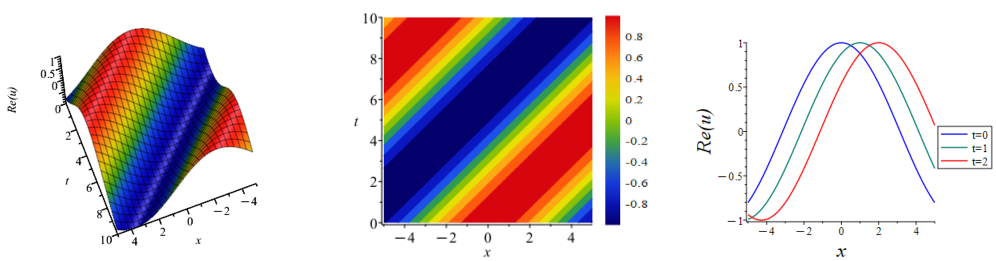
Let $\mu = \frac{4}{5}$, then, the exact solution of the Equation (2.11) is

$$f = A\cos(\frac{\xi}{2}) + iB\sin(\frac{\xi}{2}).$$

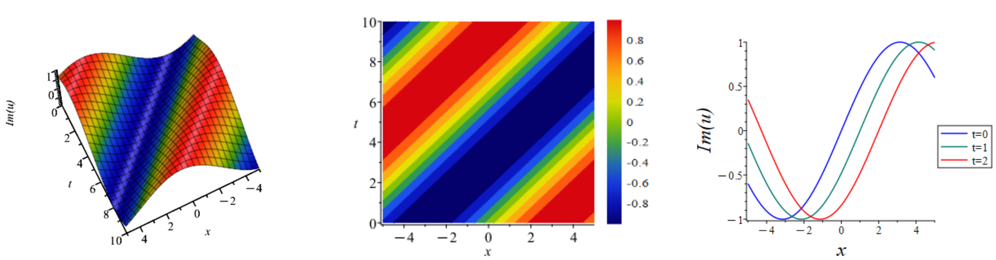
Thus, the exact solution of Equation (1.1) is

$$u(x,t) = A\cos(\frac{t-x}{2}) + iB\sin(\frac{t-x}{2}). \tag{2.12}$$

The plots of the solution in Eq. (2.12) is shown in Figure 3.



(A)



(B)

FIGURE 3. The plots, derived from the nonclassical vector field W_2, show (A) the real part and (B) the imaginary part of u.

3. Conservation Laws Associated with the Governing Equation

Conservation laws provide important insights into the intrinsic properties of nonlinear differential equations. In order to determine the conserved quantities associated with the governing equation, we employ the theorem proposed by Ibragimov [7–9]. This approach establishes a systematic connection between symmetries of differential equations and their corresponding conservation laws through the concept of adjoint equations. By constructing a formal Lagrangian and utilizing the admitted Lie symmetries, the conserved vectors corresponding to the equation can be derived. For clarity, we begin by revisiting Ibragimov's theorem.

Theorem 3.1. *If a vector field,*

$$X = \xi^{i}\Big(x, u, u_{(1)}, \dots\Big)\frac{\partial}{\partial x^{i}} + \eta\Big(x, u, u_{(1)}, \dots\Big)\frac{\partial}{\partial u},$$

represents a Lie point symmetry of the differential equation

$$\boldsymbol{\mathcal{U}}(x, u, u_{(1)}, \cdots, u_{(s)}) = 0, \tag{3.1}$$

with a set of independent variables $x = (x^1, \cdots, x^n)$ and dependent variable u, one can derive a conserved vector

$$\begin{aligned} C^i =& \xi^i L + W[\frac{\partial L}{\partial f_i} - D_j(\frac{\partial L}{\partial f_{ij}}) + D_j D_k(\frac{\partial L}{\partial f_{ijk}}) - \cdots] + D_j(W)[\frac{\partial}{\partial f_{ij}} - D_k(\frac{\partial L}{\partial f_{ijk}}) + \cdots] \\ &+ D_j D_k(W)[\frac{\partial}{\partial f_{ijk}} - \cdots] + \cdots, \end{aligned}$$

with characteristic

$$W = \eta - \xi^j u_j,$$

and formal Lagrangian

$$L = \psi\, \boldsymbol{\mathcal{U}}(x, u, u_{(1)}, \cdots, u_{(s)}),$$

such that $D_i(C^i) = 0$. The adjoint equations defined by

$$\boldsymbol{\mathcal{U}}^*(x, u, \psi, u_{(1)}, \psi_{(1)} \cdots, u_{(s)}, \psi_{(s)}) = \frac{\delta(\psi \boldsymbol{\mathcal{U}})}{\delta u}, \tag{3.2}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}},$$

represents the Euler–Lagrange operator.

In the following, Theorem 3.1 will be used to identify and calculate the conservative vectors associated with Equation (1.1). To achieve this goal, we examine the Lagrangian associated with Equation (1.1).

$$L = \psi(x, t)\Big(u_t + u_{xxt} + \mu(u_x - u_{xxxxx}) - \nu(u u_{xxx} + u u_x + 3 u_x u_{xx})\Big).$$

The adjoint of Equation (1.1) is

$$\mathcal{U}^* \equiv \psi_t + \psi_{xxt} + \mu\left(\psi_x - \psi_{xxxxx}\right) - \nu\left(\psi_x u + \psi_{xxx} u\right) = 0,$$

$\psi = 1$ satisfy the obtained adjoint equation. Then

$$L = u_t + u_{xxt} + \mu(u_x - u_{xxxxx}) - \nu(uu_{xxx} + uu_x + 3u_x u_{xx}).$$

Applying Theorem 3.1 to each of the Lie point symmetries, we obtain the following conserved vectors associated with the symmetries

Conserved Quantities Generated by V_1

For the symmetry generator V_1, the characteristic equation is given by

$$W = -u_t.$$

Substituting this characteristic into the general conservation formula obtained from the formal Lagrangian yields the components of the conserved vector

$$\begin{aligned} C^t &= L - u_t - u_{xxt}, \\ C^x &= -u_t(\mu - \nu u - \nu u_{xx}) + 2\nu u_x u_{xt} + \nu u u_{xxt} + \mu u_{xxxxt}. \end{aligned}$$

Also, a direct calculation yields

$$D_t(C^t) + D_x(C^x) = 0,$$

which confirms that (C^t, C^x) constitutes a valid conserved vector associated with the generator V_1.

Conservation Laws Associated with V_2

Similarly, by considering the symmetry generator V_2, the characteristic equation is

$$W = -u_x,$$

so, the conservation vectors can be written as follows

$$\begin{aligned} C^t &= -u_x - u_{xxx}, \\ C^x &= u_t + u_{xxt}. \end{aligned}$$

Conservation Laws Derived from the Symmetry V_3

Applying the same procedure for the generator V_3 we obtain

$$W = 1 + \nu t u_x,$$

then, the conservation vectors can be written as follows

$$C^t = 1 + \nu t u_x + \nu t u_{xxx},$$
$$C^x = -\nu t L + (1 + \nu t u_x)(\mu - \nu u - \nu u_{xx}) - 2\nu^2 t u_x u_{xx} - \nu^2 t u u_{xxx} - \mu \nu t u_{xxxxx}.$$

Conservation Laws Associated with W_1

Applying the same procedure

$$W = -u_t - u_x,$$

Consequently, the conservation law associated with W_1 can be written as follows

$$C^t = \mu(u_x - u_{xxxxx}) - \nu(u u_{xxx} + u u_x + 3u_x u_{xx}) - u_x - u_{xxx},$$
$$C^x = L - (u_t + u_x)(\mu - \nu u - \nu u_{xx}) + 2\nu u_x(u_{xt} + u_{xx}) + \nu u(u_{xxt} + u_{xxx}) + \mu(u_{xxxxt} + u_{xxxxx}).$$

Conserved Quantities Generated by W_2

Similarly

$$W = \frac{i}{2}u - u_x,$$

so, the conservation vectors can be written as follows

$$C^t = \frac{i}{2}(u + u_{xx}) - (u_x + u_{xxx}),$$
$$C^x = L + (\frac{i}{2}u - u_x)(\mu - \nu u - \nu u_{xx}) - 2\nu u_x(\frac{i}{2}u_x - u_{xx}) - \nu u(\frac{i}{2}u_{xx} - u_{xxx}) - \mu(\frac{i}{2}u_{xxxx} - u_{xxxxx}).$$

For all the considered cases, a direct calculation shows that $D_t(C^t) + D_x(C^x) = 0$, holds identically. Hence, every case satisfies the conservation law condition, and no gauge modification is necessary.

4. Analytical Solutions Using the Modified Jacobi Elliptic Expansion Method

One of the most prevalent methods is the MJEE method. In this method, a group of remarkable functions known as the Jacobi elliptic functions is used to represent the solution. The unique properties of each function, along with the relations between them, make solving the equation significantly easier (the Jacobi elliptic functions and their properties can be found in [31]). These functions play a key role in the MJEE method, which can be applied to solve differential equations through the following steps:

Starting with a nonlinear partial differential equation

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0, \tag{4.1}$$

where P is a function of u and its derivatives, and u is an unknown function. Assume that this equation can be converted to an ordinary differential equation using an appropriate transformation:

$$O(U, U', U'', \cdots) = 0, \quad \text{where} \quad ' = \frac{d}{d\xi}. \tag{4.2}$$

Once the balance number N is determined, the nontrivial solution of Equation (4.2) can be represented as:

$$U(\xi) = c_0 + \sum_{k=1}^{N} \left(\frac{J(\xi)}{1 + J^2(\xi)} \right)^{k-1} \left(c_k \frac{J(\xi)}{1 + J^2(\xi)} + b_k \frac{1 - J^2(\xi)}{1 + J^2(\xi)} \right), \tag{4.3}$$

where c_0, c_k, and b_k are constants to be determined later, and $J(\xi)$ is a known function satisfying the following Jacobi elliptic equation [32]

$$(J'(\xi))^2 = FJ^4(\xi) + EJ^2(\xi) + D. \tag{4.4}$$

This equation admits the exact solutions shown in Table 3.

Table 3. Jacobi elliptic function solutions of Equation (4.4) [33]

No.	$J(\xi)$	F	E	D
1	$\mathrm{sn}(\xi)$	m^2	$-(1+m^2)$	1
2	$\mathrm{cn}(\xi)$	$-m^2$	$-(1-2m^2)$	$(1-m^2)$
3	$\mathrm{ns}(\xi)$	1	$-(1+m^2)$	m^2

After substituting the solution (4.3) and the Jacobi elliptic equation (4.4) into Equation (4.2) and performing multiple calculations, we will obtain a set of nonlinear algebraic equations, and solving these equations will provide exact solutions of the nonlinear partial differential equation (4.1). As mentioned earlier, the Jacobi elliptic functions exhibit several distinctive properties that are fundamental in problem solving. The fundamental identities of Jacobi elliptic functions are considered here. In particular, the relation

$$\mathrm{sn}^2(\xi, m) + \mathrm{cn}^2(\xi, m) = 1,$$

holds for any modulus m. In the limiting case as m approaches unity, the elliptic functions reduce to hyperbolic functions: $\mathrm{sn}(\xi, m)$ converges to $\tanh(\xi)$, $\mathrm{cn}(\xi, m)$ approaches $\mathrm{sech}(\xi)$, and $\mathrm{ns}(\xi, m) = 1/\mathrm{sn}(\xi, m)$ tends to $\coth(\xi)$. With these properties established, the MJEE method can be employed to the Wazwaz New Integrable Fifth-Order Equation, enabling the derivation of its exact solutions.

To obtain a Jacobi elliptic function solutions for the main integrable fifth-order equation we need first to apply a transformation in the following form

$$u(x,t) = U(\xi), \qquad \xi = x - ct, \tag{4.5}$$

where c is a constant. After applying this transformation, the result will be

$$\begin{aligned} &- c(\frac{d}{d\xi}U(\xi)) - c(\frac{d^3}{d\xi^3})U(\xi)) + \mu(\frac{d}{d\xi}U(\xi)) - U(\xi)(\frac{d^3}{d\xi^3}U(\xi)) - U(\xi)(\frac{d}{d\xi}U(\xi)) \\ &- 3(\frac{d}{d\xi}U(\xi))(\frac{d^2}{d\xi^2}U(\xi)) - \mu(\frac{d^5}{d\xi^5}U(\xi)) = 0. \end{aligned} \tag{4.6}$$

After that, upon substituting Equations (4.3) and (4.4) into Equation (4.6), and utilizing a symbolic computation system such as Maple, yields the following set of nonlinear algebraic equations

$$\begin{aligned}
&E^2\mu c_1 - E^2\mu c_4 - 60EF\mu c_1 + 180EF\mu c_4 + 12DF\mu c_1 - 12DF\mu c_4 + 120F^2\mu c_1 - 600F^2\mu c_4\\
&+ Ec\,c_1 + \mu c_4 - Ec\,c_4 + Ec_0c_1 - Ec_0c_4 - Ec_1c_2 + Ec_2c_4 - 6cF\,c_1 + 18cF\,c_4 + 6Fc_1c_3 + cc_1\\
&- 30Fc_2c_4 - 6Fc_3c_4 - cc_4 - \mu c_1 + c_0c_1 - c_0c_4 - c_1c_2 - 6Fc_0c_1 + 18Fc_0c_4 + c_2c_4 + 18Fc_1c_2 = 0,\\
&64E^2\mu c_2 + 32E^2\mu c_3 - 960EF\mu c_2 - 960EF\mu c_3 + 288DF\mu c_2 + 144DF\mu c_3 + 1440F^2\mu c_2 + c_1^2\\
&+ 4Ec_4^2 - 48cF\,c_2 + c_4^2 - 48cF\,c_3 - 48Fc_0c_2 - 48Fc_0c_3 - 24Fc_1^2 + 96Fc_1c_4 + 96Fc_2^2 - 72Fc_4^2\\
&- 4c_2^2 - 2c_2c_3 + 16Ec\,c_2 + 8Ec\,c_3 + 16Ec_0c_2 + 8Ec_0c_3 + 4Ec_1^2 - 8Ec_1c_4 - 16Ec_2^2 - 8Ec_2c_3\\
&+ 96Fc_2c_3 + 12Fc_3^2 - 2c_1c_4 + 4cc_2 + 2cc_3 - 4\mu c_2 - 2\mu c_3 + 4c_0c_2 + 2c_0c_3 + 2160F^2\mu c_3 = 0,\\
&- 236E^2\mu c_1 + 722E^2\mu c_4 + 60ED\mu c_1 - 60ED\mu c_4 + 1620EF\mu c_1 - 672DF\mu c_1 + 6Dc_2c_4 + 4c_0c_1\\
&- 27Ec_3c_4 + 6cD\,c_1 - 6cD\,c_4 + 18cF\,c_1 - 174cF\,c_4 + 6Dc_0c_1 - 6Dc_0c_4 - 6Dc_1c_2 + 2184DF\mu c_4\\
&+ 18Fc_0c_1 - 174Fc_0c_4 - 174Fc_1c_2 - 138Fc_1c_3 + 570Fc_2c_4 + 258Fc_3c_4 + 4cc_1 + 2cc_4 - 20Ec_0c_1\\
&- 1680F^2\mu c_1 + 13440F^2\mu c_4 - 20ac\,c_1 + 74Ec\,c_4 + 74Ec_0c_4 + 74Ec_1c_2 + 27Ec_1c_3 - 128Ec_2c_4\\
&+ 2c_1c_2 + 3c_1c_3 - 8c_2c_4 - 3c_3c_4 - 8940EF\mu c_4 - 2\mu c_4 - 4\mu c_1 + 2c_0c_4 = 0,\\
&+ 200Ec_2c_3 + 32Ec_3^2 - 164Ec_4^2 + 48cD\,c_2 + 24cD\,c_3 - 96cF\,c_2 + 24cF\,c_3 + 48Dc_0c_2 + 24Dc_0c_3\\
&- 3360F^2\mu c_2 - 11760F^2\mu c_3 - 16Ec\,c_2 - 72Ec\,c_3 - 16Ec_0c_2 - 72Ec_0c_3 - 36Ec_1^2 + 200Ec_1c_4\\
&- 24Dc_1c_4 - 48Dc_2^2 - 24Dc_2c_3 + 12Dc_4^2 - 96Fc_0c_2 + 24Fc_0c_3 + 12Fc_1^2 - 408Fc_1c_4 - 48Fc_2^2\\
&- 156Fc_3^2 + 636Fc_4^2 + 20cc_2 + 6cc_3 - 20\mu c_2 - 6\mu c_3 + 20c_0c_2 + 6c_0c_3 + 3c_1^2 + 2c_1c_4 + 2c_2c_3\\
&- 1600E^2\mu c_2 - 1824E^2\mu c_3 + 960ED\mu c_2 + 480ED\mu c_3 + 5760EF\mu c_2 + 12000EF\mu c_3 - 5c_4^2\\
&- 4368DF\mu c_3 + 144Ec_2^2 + 12Dc_1^2 - 408Fc_2c_3 + 2c_3^2 - 3360DF\mu c_2 - 12c_2^2 = 0,\\
&1445E^2\mu c_1 - 10543E^2\mu c_4 - 1620ED\mu c_1 + 5700ED\mu c_4 - 2520EF\mu c_1 + 41160EF\mu c_4 + 84cF\,c_1\\
&+ 2460DF\mu c_1 - 21876Df\mu c_4 - 25200F^2\mu c_4 - 43Ec\,c_1 - 79Ec\,c_4 - 43Ec_0c_1 - 79Ec_0c_4 + 17cc_4\\
&- 60Dc_3c_4 + 84Fc_0c_1 - 252Fc_0c_4 - 252Fc_1c_2 + 126Fc_1c_3 - 420Fc_2c_4 - 120D^2\mu c_4 - 186Ec_1c_3\\
&- 17\mu c_4 + 5c_0c_1 + 17c_0c_4 + 6c_1c_3 - 19c_2c_4 + 4c_3c_4 + 120D^2\mu c_1 + 17c_1c_2 - 79Ec_1c_2 + 138Dc_1c_2\\
&+ 701Ec_2c_4 + 436Ec_3c_4 - 18cD\,c_1 + 138cD\,c_4 - 252cF\,c_4 - 18Dc_0c_1 + 138Dc_0c_4 - 258Dc_2c_4\\
&- 5\mu c_1 - 966Fc_3c_4 + 5cc_1 + 60Dc_1c_3 = 0,\\
&2560E^2\mu c_2 + 9664E^2\mu c_3 - 5760ED\mu c_2 - 9120ED\mu c_3 - 23520EF\mu c_3 + 1440D^2\mu c_2 + 720D^2\mu c_3\\
&+ 18528DF\mu c_3 - 3360F^2\mu c_2 + 8400F^2\mu c_3 - 128Ec\,c_2 - 80Ec\,c_3 - 128Ec_0c_2 - 80Ec_0c_3 - 40Ec_1^2\\
&+ 160Ec_2^2 - 320Ec_2c_3 - 208Ec_3^2 + 792Ec_4^2 + 96cD\,c_2 - 72cD\,c_3 + 168cF\,c_3 + 96Dc_0c_2 - 72Dc_0c_3\\
&+ 312Dc_1c_4 + 144Dc_2^2 + 312Dc_2c_3 + 60Dc_3^2 - 276Dc_4^2 + 168Fc_0c_3 + 84Fc_1^2 - 168fc_1c_4 - 336Fc_2^2
\end{aligned}$$

$$+252Fc_3^2-924Fc_4^2+40cc_2+4cc_3-40\mu c_2+40c_0c_2+4c_0c_3+2c_1^2+16c_1c_4-6c_4^2+3840Df\mu c_2$$
$$+16c_2c_3-320Ec_1c_4-168Fc_2c_3-36Dc_1^2+2c_3^2-4\mu c_3-8c_2^2=0,$$

$$1680F^2\mu c_1+630Fc_3c_4-28\mu c_4+28c_0c_4+28c_1c_2+14c_3c_4+44016DF\mu c_4+84cF\,c_4+84cF\,c_1$$
$$-756Fc_2c_4+6720F^2\mu c_4-308Ec\,c_4-308Ec_1c_2+756Dc_2c_4-994Ec_3c_4-84cD\,c_1+84cD\,c_4$$
$$-84Dc_0c_1+84Dc_0c_4+84Dc_1c_2-36120ED\mu c_4+630Dc_3c_4+84Fc_0c_1+84Fc_0c_4+84Fc_1c_2$$
$$+23548E^2\mu c_4+2520ED\mu c_1-210Dc_1c_3-2520EF\mu c_1-36120EF\mu c_4+6720D^2\mu c_4+28cc_4$$
$$-1680D^2\mu c_1-308Ec_0c_4+210Fc_1c_3=0,$$
$$-60Fc_3^2+276Fc_4^2+40cc_2-4cc_3+40c_0c_2+8c_2^2-40\mu c_2+16c_2c_3-128Ec_0c_2-8400D^2\mu c_3$$
$$+2560E^2\mu c_2-9664E^2\mu c_3+23520ED\mu c_3-5760EF\mu c_2+9120EF\mu c_3-3360D^2\mu c_2+6c_4^2$$
$$-160Ec_2^2-320Ec_2c_3+208Ec_3^2-792Ec_4^2-168cD\,c_3+96cF\,c_2-168Dc_0c_3-84Dc_1^2-2c_1^2$$
$$+336Dc_2^2-168Dc_2c_3-252Dc_3^2+924Dc_4^2+96Fc_0c_2+72Fc_0c_3+36Fc_1^2+312Fc_1c_4-2c_3^2$$
$$-18528DF\mu c_3+1440F^2\mu c_2-720F^2\mu c_3-128Ec\,c_2+80Ec\,c_3+80Ec_0c_3-144Fc_2^2+4\mu c_3$$
$$-4c_0c_3+312Fc_2c_3+16c_1c_4+72cF\,c_3-320Ec_1c_4-168Dc_1c_4+40Ec_1^2+3840DF\mu c_2=0,$$

$$-252Dc_1c_2-126Dc_1c_3+420Dc_2c_4-966Dc_3c_4+18Fc_0c_1+138Fc_0c_4+138Fc_1c_2+17cc_4$$
$$+186Ec_1c_3-701Ec_2c_4+436Ec_3c_4-84cD\,c_1-252cD\,c_4+18cF\,c_1+138cF\,c_4-84Dc_0c_1$$
$$-21876DF\mu c_4-10543E^2\mu c_4+2520ED\mu c_1+41160ED\mu c_4+1620EF\mu c_1+5700EF\mu c_4$$
$$-2460DF\mu c_1-120F^2\mu c_1-120F^2\mu c_4+43Ec\,c_1-79Ec\,c_4+43Ec_0c_1-79Ec_0c_4-5cc_1$$
$$+5\mu c_1-17\mu c_4-5c_0c_1+17c_0c_4+17c_1c_2-6c_1c_3+19c_2c_4-60Fc_3c_4-79Ec_1c_2+4c_3c_4$$
$$+258Fc_2c_4-252Dc_0c_4-1445E^2\mu c_1-25200D^2\mu c_4-60Fc_1c_3=0,$$

$$\vdots$$

The above set of nonlinear algebraic equations can be solved to identify the following cases:

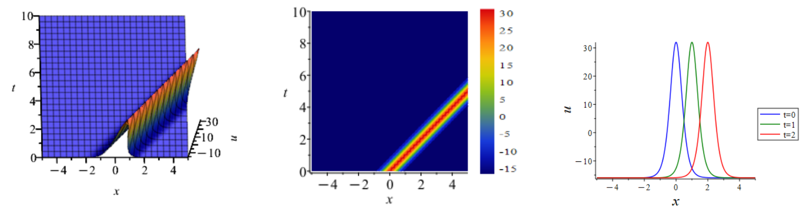
(I) By setting $E=-(m^2+1)$, $F=m^2$, and $D=1$, we obtain

- $c_0=-c+33\mu,\quad c_1=0,\quad c_2=0,\quad c_3=-192\mu,\quad c_4=0,\quad m=1,$
- $c_0=-c+21\mu,\quad c_1=0,\quad c_2=0,\quad c_3=-96\mu,\quad c_4=\pm 48i\mu,\quad m=1.$

Thus, we have

$$u_1(x,t)=-c+33\mu-\frac{192\,\mu\,\tanh^2(x-ct)}{(1+\tanh^2(x-ct))^2}. \tag{4.7}$$

The plots of the solution in Eq. (4.7) is shown in Figure 4.



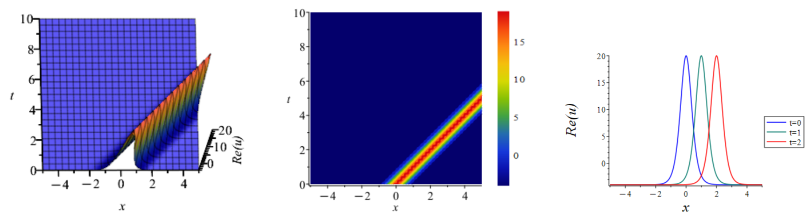
(A) (B) (C)

Figure 4. The plots show (A) 3D surface, (B) Contour plot, and (C) 2D plot of u_1 for values of $t = 0, 1, 2$.

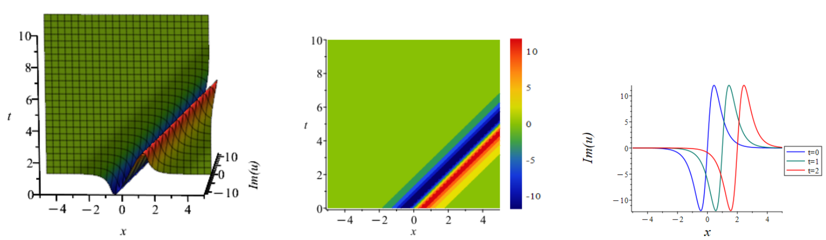
and

$$u_2(x,t) = -c + 21\mu + \frac{\tanh(x-ct)\left(-\frac{96\mu\tanh(x-ct)}{1+\tanh^2(x-ct)} + \frac{48i\mu(1-\tanh^2(x-ct))}{1+\tanh^2(x-ct)}\right)}{1+\tanh^2(x-ct)}. \tag{4.8}$$

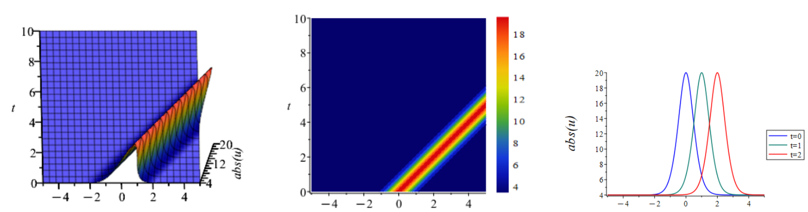
The plots of the solution in Eq. (4.8) is shown in Figure 5.



(A)



(B)



(C)

Figure 5. The plots show (A) the real part, (B) the imaginary part, and (C) the absolute value of u_2 for values of $t = 0, 1, 2$.

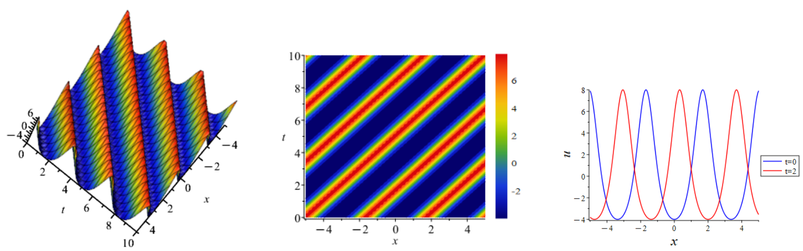
(II) Setting $E = 2m^2 - 1, F = -m^2$, and $D = 1 - m^2$, gives

$$c_0 = 6\mu - c, \quad c_1 = 0, \quad c_2 = 12\mu, \quad c_3 = -48\mu, \quad c_4 = 0, \quad m = 1/2. \tag{4.9}$$

This gives the solution:

$$u_3(x,t) = 6\mu - c + \frac{12\mu(1 - cn^2((x-ct),1/2))/(1 + cn^2((x-ct),1/2)) - 48\mu(cn^2((x-ct),1/2))}{(1 + cn^2((x-ct),1/2))^2}. \tag{4.10}$$

The plots of this solution is shown in Figure 6.



(A) (B) (C)

Figure 6. The plots show (A) 3D surface, (B) Contour plot, and (C) 2D plot of u_3 for values of $t = 0, 2$.

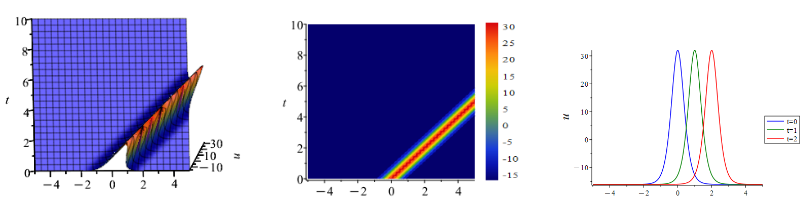
(III) When $D = m^2$, $E = -(m^2 + 1)$, and $F = 1$, the following is obtained:

- $c_0 = 33\mu - c, \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = -192\mu, \quad c_4 = 0, \quad m = 1,$
- $c_0 = 21\mu - c, \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = -96\mu, \quad c_4 = 48i\mu, \quad m = 1.$

Thus, the Jacobi elliptic function solutions can be defined as:

$$u_4(x,t) = 33\mu - c - \frac{192\coth^2(x-ct)\mu}{\left(1 + \coth^2(x-ct)\right)^2}. \tag{4.11}$$

The solution in Eq. (4.11) can be plotted in Figure 7.



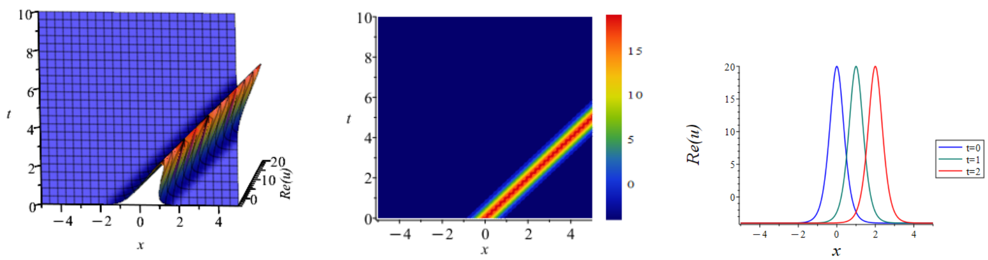
(A) (B) (C)

Figure 7. The plots show (A) 3D surface, (B) Contour plot, and (C) 2D plot of u_4 for values of $t = 0, 1, 2$ (blue, green, red), respectively.

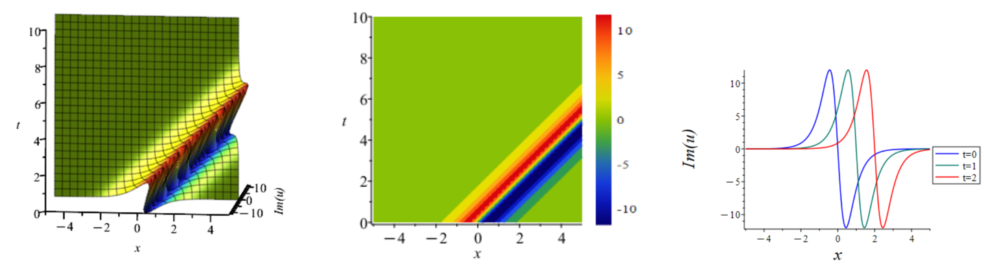
and

$$u_5(x,t) = 21\mu - c + \frac{\coth(x-ct)\left(-\dfrac{96\mu\coth(x-ct)}{1+\coth^2(x-ct)} + \dfrac{48i\mu(1-\coth^2(x-ct))}{1+\coth^2(x-ct)}\right)}{1+\coth^2(x-ct)}. \tag{4.12}$$

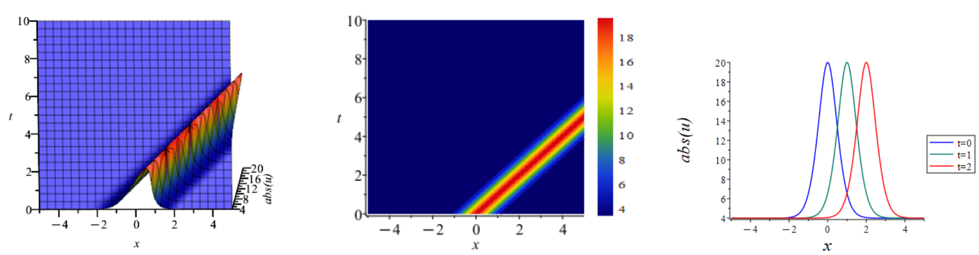
Figure 8 shows the 3-dimensional graphs and their related plots of the above solution.



(A)



(B)



(C)

Figure 8. The plots show (A) the real part, (B) the imaginary part, and (C) the absolute value of u_5 for values of $t = 0, 1, 2$.

5. Conclusion

In this study, a comprehensive structural investigation of a fifth-order nonlinear evolution equation has been carried out using extended symmetry-based techniques. The classical Lie symmetry analysis revealed a three-dimensional Lie algebra, and the construction of the optimal system enabled systematic similarity reductions. These reductions led to invariant solutions and a reduced fifth-order nonlinear ordinary differential equation governing similarity profiles. The nonclassical symmetry analysis further enriched the solution space by generating additional invariant structures, including complex exponential and traveling-wave forms. The presence of such solutions highlights the richer symmetry structure of the equation beyond the classical framework. Through Ibragimov's theorem, multiple conservation laws were derived without requiring a traditional variational formulation. The existence of these conserved vectors indicates an underlying structural coherence of the equation and supports its integrable character. Finally, the MJEE method produced families of exact solutions expressed in terms of hyperbolic and elliptic functions. These solutions describe both periodic and solitary-type wave structures, demonstrating the diversity of dynamical behaviors admitted by the equation. Overall, the combination of symmetry analysis, conservation law construction, and explicit solution techniques provides a unified framework for understanding the qualitative and analytical properties of the considered fifth-order model. The methodology presented here can be extended to other higher-order nonlinear evolution equations arising in mathematical physics.

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