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Fuzzy Linear Difference Equations and Ambiguity Propagation in Financial Volatility

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Abstract. This paper studies a class of fuzzy first-order non-homogeneous linear difference equations under the horizontal membership function (HMF) framework and derives explicit solution representations together with stability and oscillation conditions. Building on these theoretical results, we reinterpret financial volatility persistence through a fuzzy ambiguity perspective. Rather than replacing stochastic volatility models, we provide a complementary interpretation in which volatility clustering corresponds to slow convergence of uncertainty width within a stable fuzzy dynamic system. Using daily data from the Stock Exchange of Thailand (SET), the S&P 500 index, and the VIX over the period 1 January 2010 to 13 March 2026, we examine volatility persistence, shock transmission, and cross-market amplification. The empirical results indicate systematically higher persistence in the emerging market, consistent with slower ambiguity dissipation.

1. Introduction

Financial markets are characterized by persistent volatility clustering, shock amplification, and cross-market spillovers. Since the seminal contribution of Engle [1] and its generalization by Bollerslev [2], autoregressive conditional heteroskedasticity (ARCH/GARCH) models have become the dominant framework for describing time-varying volatility. These models interpret volatility persistence as second-moment dynamics driven by past squared innovations and conditional variance. Empirically, the persistence parameter is often found to be close to unity, particularly in emerging markets, indicating slow dissipation of shocks and near-integrated variance behavior [3].

Beyond purely stochastic interpretations, financial uncertainty may also contain an ambiguity component that is not fully reducible to probabilistic variance. Knight [5] distinguished measurable risk from unmeasurable uncertainty, a distinction that has received renewed attention in macro-finance and asset pricing [6]. In such settings, persistence in volatility may reflect not only stochastic feedback mechanisms but also prolonged ambiguity about underlying economic fundamentals.

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In parallel, fuzzy set theory provides a mathematical framework for modeling imprecision and ambiguity in a non-probabilistic manner [7,8]. Recent developments in horizontal membership function (HMF) representations allow fuzzy-valued dynamic systems to be analyzed using granular arithmetic and stability conditions [9]. In particular, fuzzy linear difference equations admit explicit solution representations and stability criteria formulated directly within the granular metric space.

Existing studies on fuzzy difference equations typically employ α-cut representations or embedding techniques, transforming fuzzy systems into families of deterministic equations. While these approaches are mathematically valid, they often rely on endpoint reductions or generalized Hukuhara differentiability assumptions. By contrast, the horizontal membership function framework operates intrinsically on fuzzy-valued mappings and permits stability analysis without decomposing the system into lower and upper bound dynamics. This structural distinction becomes relevant when linking fuzzy dynamics to economic persistence phenomena.

The present study retains the fuzzy linear difference equation framework and extends its interpretation to financial volatility dynamics. Instead of modeling asset returns themselves as fuzzy variables, we interpret the width component of a fuzzy dynamic system as a representation of uncertainty or ambiguity. Under the fuzzy linear evolution

$$P_{t+1} = A \otimes_{\mathrm{gr}} P_t \oplus_{\mathrm{gr}} B,$$

the center and width components evolve separately. While empirical returns typically exhibit weak serial dependence consistent with market efficiency [4], volatility measures display strong persistence. The propagation parameter governing the width dynamic is therefore mapped to empirical volatility persistence coefficients observed in autoregressive and GARCH-type models.

Importantly, we do not propose the fuzzy framework as a replacement for stochastic volatility models. Rather, we provide a complementary structural interpretation: volatility clustering can be viewed as slow convergence of uncertainty width within a stable fuzzy dynamic system. When the propagation parameter approaches unity, the system remains stable but exhibits prolonged ambiguity episodes, consistent with near-unit-root variance behavior [3].

To illustrate this perspective, we examine daily data from the Stock Exchange of Thailand (SET), the S&P 500 index, and the VIX over the period 1 January 2010 to 13 March 2026. The emerging market (SET) provides a useful contrast to the developed U.S. market. We estimate return dynamics, volatility persistence, and global uncertainty spillovers. The empirical findings show stronger volatility persistence in the emerging market and significant transmission from global uncertainty, consistent with slower ambiguity dissipation.

The contribution of this study is twofold. Mathematically, we formulate solution representations and stability properties of fuzzy linear difference equations within the granular metric framework, highlighting structural differences from α-cut and Hukuhara-based approaches. Economically, we demonstrate how these stability conditions provide an alternative lens for interpreting volatility persistence and cross-market amplification. By linking deterministic stability analysis with empirical volatility dynamics, the paper contributes to an interdisciplinary dialogue between fuzzy systems theory and financial econometrics.

2. Theoretical framework

In this section, we concisely review several fundamental notions from fuzzy analysis for the reader's convenience. Detailed expositions of these topics can be found in [8–11, 15] and the references cited therein.

Definition 2.1. Let $u \in \mathbb{E}$. The horizontal membership function (HMF) associated with u is defined by

$$\begin{aligned} u^{\mathrm{gr}} : [0,1] \times [0,1] &\to [a,b] \subseteq \mathbb{R} \\ (\xi, \alpha_u) &\mapsto u^{\mathrm{gr}}(\xi, \alpha_u) = u^-_\xi + \mathrm{len}([u]^\xi)\alpha_u. \end{aligned} \tag{2.1}$$

Remark 2.1. The function u^{gr} is a real-valued mapping depending on the variables ξ and α_u. Denoting

$$\mathscr{P}(u) := u^{\mathrm{gr}}(\xi, \alpha_u),$$

it follows from [9] that the ξ-level sets of $u \in \mathbb{E}$ can be reconstructed from its HMF via

$$\mathscr{P}^{-1}\Big(u^{\mathrm{gr}}(\xi, \alpha_u)\Big) = [u]^\xi = \left[\inf_{\beta \geq \xi} \min_{\alpha_u} u^{\mathrm{gr}}(\beta, \alpha_u),\ \sup_{\beta \geq \xi} \max_{\alpha_u} u^{\mathrm{gr}}(\beta, \alpha_u)\right]. \tag{2.2}$$

Definition 2.2. Let $u, v \in \mathbb{E}$ with corresponding HMFs $\mathscr{P}(u)$ and $\mathscr{P}(v)$. Arithmetic operations in $\mathbb{E}$ are symbolically defined by

$$\mathscr{P}(u \circledast_{gr} v) \triangleq \mathscr{P}(u) \circledast \mathscr{P}(v), \tag{2.3}$$

where "$\circledast_{gr}$" and "$\circledast$" denote arithmetic operations (addition, subtraction, multiplication, or division with $v^{\mathrm{gr}} \neq 0$) in $\mathbb{E}$ and $\mathbb{R}$, respectively.

Remark 2.2. The operations $\circledast_{\mathrm{gr}}$, including addition ($\oplus_{\mathrm{gr}}$), subtraction ($\ominus_{\mathrm{gr}}$), multiplication ($\otimes_{\mathrm{gr}}$), and division ($\oslash_{\mathrm{gr}}$), are referred to as granular operations since they are performed through HMF representations. According to [9], the following properties hold for all $u, v, w \in \mathbb{E}$:

(i) $u \ominus_{\mathrm{gr}} v = (-1) \otimes_{\mathrm{gr}} (v \ominus_{\mathrm{gr}} u)$;
(ii) $v \ominus_{\mathrm{gr}} v = \hat{0}$, where $\hat{0}$ denotes the zero fuzzy number;
(iii) $u \otimes_{\mathrm{gr}} (v \oplus_{\mathrm{gr}} w) = u \otimes_{\mathrm{gr}} v \oplus_{\mathrm{gr}} u \otimes_{\mathrm{gr}} w$.

Definition 2.3. (i) For u, $v \in \mathbb{E}$, we say that u is less than or equal to v in the granular sense, denoted by $u \preceq_{\mathrm{gr}}$ (or $u \prec_{\mathrm{gr}}$), if and only if $\mathscr{P}(u) \leq$ (or $<$) $\mathscr{P}(v)$ for all $\alpha_u = \alpha_v \in [0,1]$ and $\xi \in [0,1]$.
(ii) The equality $u = v$ in the granular sense holds if and only if $u \preceq_{\mathrm{gr}} v$ and $v \preceq_{\mathrm{gr}} u$, equivalently $\mathscr{P}(u) = \mathscr{P}(v)$ for all $\alpha_u = \alpha_v \in [0,1]$ and $\xi \in [0,1]$.

Definition 2.4. A fuzzy-valued function v is a mapping

$$\begin{aligned} v : [a,b] \subseteq \mathbb{R} &\to \mathbb{E} \\ \tau &\mapsto v(\tau). \end{aligned} \tag{2.4}$$

Its horizontal membership function, denoted by $\mathscr{P}(v(\tau)) := v^{\text{gr}}(\xi, \tau, \alpha_v)$, is defined through

$$\begin{aligned} v^{\text{gr}} : [0,1] \times [a,b] \times [0,1] &\to \mathbb{R} \\ (\xi, \tau, \alpha_v) &\mapsto v^{\text{gr}}(\xi, \tau, \alpha_v). \end{aligned} \tag{2.5}$$

Definition 2.5. The distance between $u, v \in \mathbb{E}$ is measured by the mapping $\|\cdot\|_{\text{gr}} : \mathbb{E} \times \mathbb{E} \to \mathbb{R}^+ \cup \{0\}$ defined as

$$\|u \ominus_{\text{gr}} v\|_{\text{gr}} = \sup_{\xi \in [0,1]} \max_{\alpha_u, \alpha_v \in [0,1]} \left| u^{\text{gr}}(\xi, \alpha_u) - v^{\text{gr}}(\xi, \alpha_v) \right|. \tag{2.6}$$

This metric is referred to as the granular metric.

Remark 2.3. Equipped with the distance (2.6), the space $\mathbb{E}$ forms a complete metric space. Consequently, notions such as limits, convergence, and continuity for mappings $v : [a,b] \to \mathbb{E}$ can be rigorously formulated using the granular metric.

It is worth emphasizing that the granular framework differs structurally from classical approaches to fuzzy differential and difference equations. In the Seikkala solution concept, fuzzy-valued equations are typically reduced to coupled systems of endpoint equations via α-cuts, leading to deterministic dynamics for lower and upper bounds. In contrast, the horizontal membership function representation employed here operates directly on the fuzzy-valued mapping without decomposing the system into endpoint subsystems.

Similarly, formulations based on Hukuhara or generalized Hukuhara differentiability require the existence of the Hukuhara difference, which imposes structural restrictions and may fail in situations where subtraction of fuzzy numbers does not exist in the classical sense. The granular arithmetic framework avoids these limitations by defining operations directly through HMF representations. As a result, stability and oscillation properties can be derived intrinsically within the granular metric space without invoking generalized differentiability assumptions.

Therefore, the subsequent development of fuzzy linear difference equations is not merely a rewriting of classical linear difference equations in fuzzy notation, but a formulation carried out entirely within the granular metric structure, which provides a distinct analytical setting compared with α-cut and Hukuhara-based methodologies.

3. Main results

3.1. **Fuzzy linear difference equations.**

Theorem 3.1. Consider the nonhomogeneous fuzzy linear difference equation

$$x(n+1) = a(n) \otimes_{\text{gr}} x(n) \oplus_{\text{gr}} g(n), \quad x(n_0) = x_0, \tag{3.1}$$

where $a(n) \neq 0$ is a real-valued sequence, while $x(n)$ and $g(n)$ are fuzzy-valued sequences defined for all $n \geq n_0 \geq 0$, $n \in \mathbb{N}$, and x_0 is a fuzzy number.

Then problem (3.1) admits a unique solution expressed by

$$x(n) = \left(\prod_{i=n_0}^{n-1} a(i)\right) \otimes_{\text{gr}} x_0 \oplus_{\text{gr}} \sum_{r=n_0}^{n-1} \left(\prod_{i=r+1}^{n-1} a(i)\right) \otimes_{\text{gr}} g(r). \tag{3.2}$$

In addition, we adopt the standard conventions $\prod_{i=k+1}^{k} a(i) = 1$ and $\sum_{i=k+1}^{k} a(i) = 0$.

Proof. For all $n \geq n_0 \geq 0$, with $n \in \mathbb{N}$, Definition 2.4 allows the initial value problem (3.1) to be reformulated as follows:

$$\begin{cases} \mathscr{P}\left(x(n+1)\right) & = a(n)\mathscr{P}\left(x(n)\right) + \mathscr{P}\left(g(n)\right), \\ \mathscr{P}\left(x(n_0)\right) & = \mathscr{P}\left(x_0\right). \end{cases} \tag{3.3}$$

Consequently, for $n = n_0, n_0+1, \ldots,$ and $(\xi, \alpha_x) \in [0,1]^2$,

$$\begin{cases} x^{\text{gr}}\left(\xi, n+1, \alpha_x\right) & = a(n)x^{\text{gr}}\left(\xi, n, \alpha_x\right) + g^{\text{gr}}(\xi, n, \alpha_x), \\ x^{\text{gr}}\left(\xi, n_0, \alpha_x\right) & = x_0^{\text{gr}}\left(\xi, \alpha_x\right). \end{cases} \tag{3.4}$$

From the first relation in (3.4), it follows that for $n \geq n_0 \geq 0$, $n \in \mathbb{N}$, we obtain

$$x^{\text{gr}}\left(\xi, n_0+1, \alpha_x\right) = a(n_0)x_0^{\text{gr}}\left(\xi, \alpha_x\right) + g^{\text{gr}}(\xi, n_0, \alpha_x), \tag{3.5}$$

$$\begin{aligned} x^{\text{gr}}\left(\xi, n_0+2, \alpha_x\right) &= a(n_0+1)x^{\text{gr}}\left(\xi, n_0+1, \alpha_x\right) + g^{\text{gr}}(\xi, n_0+1, \alpha_x) \\ &= a(n_0+1)\left[a(n_0)x_0^{\text{gr}}\left(\xi, \alpha_x\right) + g^{\text{gr}}(\xi, n_0, \alpha_x)\right] + g^{\text{gr}}(\xi, n_0+1, \alpha_x) \\ &= a(n_0+1)a(n_0)x_0^{\text{gr}}\left(\xi, \alpha_x\right) + a(n_0+1)g^{\text{gr}}\left(\xi, n_0, \alpha_x\right) + g^{\text{gr}}\left(\xi, n_0+1, \alpha_x\right). \end{aligned} \tag{3.6}$$

Next, we apply mathematical induction to establish that, for every $n \in \mathbb{Z}^+$,

$$x^{\text{gr}}(\xi, n, \alpha_x) = \left[\prod_{i=n_0}^{n-1} a(i)\right] x_0^{\text{gr}}\left(\xi, \alpha_x\right) + \sum_{r=n_0}^{n-1} \left[\prod_{i=r+1}^{n-1} a(i)\right] g^{\text{gr}}(\xi, r, \alpha_x). \tag{3.7}$$

To verify this claim, suppose that expression (3.7) is valid for $n = k > n_0$. Then, by the first relation in (3.4), it follows that

$$x^{\text{gr}}\left(\xi, k+1, \alpha_x\right) = a(k)x^{\text{gr}}\left(\xi, k, \alpha_x\right) + g^{\text{gr}}(\xi, k, \alpha_x). \tag{3.8}$$

Invoking representation (3.7), it follows that

$$x^{\text{gr}}\left(\xi, k+1, \alpha_x\right) = a(k)\left(\left[\prod_{i=n_0}^{k-1} a(i)\right] x_0^{\text{gr}}\left(\xi, \alpha_x\right) + \sum_{r=n_0}^{k-1} \left[\prod_{i=r+1}^{k-1} a(i)\right] g^{\text{gr}}(\xi, r, \alpha_x)\right) + g^{\text{gr}}(\xi, k, \alpha_x) \tag{3.9}$$

$$= \left[\prod_{i=n_0}^{k} a(i)\right] x_0^{\text{gr}}\left(\xi, \alpha_x\right) + \sum_{r=n_0}^{k-1} \left[\prod_{i=r+1}^{k} a(i)\right] g^{\text{gr}}(\xi, r, \alpha_x) + \prod_{i=k+1}^{k} a(i) g^{\text{gr}}(\xi, k, \alpha_x) \tag{3.10}$$

$$= \left[\prod_{i=n_0}^{k} a(i)\right] x_0^{\text{gr}}(\xi, \alpha_x) + \sum_{r=n_0}^{k} \left[\prod_{i=r+1}^{k} a(i)\right] g^{\text{gr}}(\xi, r, \alpha_x). \tag{3.11}$$

Therefore, expression (3.7) is valid for every $n \geq n_0 \geq 0$, $n \in \mathbb{N}$. Consequently, by Definition 2.4, we obtain

$$\mathscr{P}(x(n)) = \mathscr{P}\left(\left[\prod_{i=n_0}^{n-1} a(i)\right] \otimes_{\text{gr}} x_0 \oplus_{\text{gr}} \sum_{r=n_0}^{n-1} \left[\prod_{i=r+1}^{n-1} a(i)\right] \otimes_{\text{gr}} g(r)\right). \tag{3.12}$$

Applying Definition (2.3)-(ii) to (3.12) yields the representation stated in (3.2). □

Definition 3.1. A point x^* in the domain of f is called an equilibrium point of the difference equation $x(n+1) = f(x(n)), \forall n = 0, 1, 2, \ldots,$ if it is a fixed point of f, that is, $f(x^*) = x^*$.

We now examine a particular case of problem (3.1) in which $a(n) = A$ and $g(n) = \widetilde{B}$ for all $n \geq 0$, $n \in \mathbb{N}$, where A is a real constant and $\widetilde{B}$ is a fuzzy number:

$$x(n+1) = A \otimes_{\text{gr}} x(n) \oplus_{\text{gr}} \widetilde{B}, \quad x(0) = x_0, \tag{3.13}$$

The corresponding equilibrium point is given by

$$x^* = \frac{1}{1-A} \otimes_{\text{gr}} \widetilde{B}. \tag{3.14}$$

Corollary 3.1. Problem (3.13) admits a unique solution given by

$$x(n) = A^n \otimes_{\text{gr}} x_0 \oplus_{\text{gr}} \frac{A^n - 1}{A - 1} \otimes_{\text{gr}} \widetilde{B}. \tag{3.15}$$

Proof. By applying formula (3.2) to equation (3.13), for $n = 1, 2, 3, \ldots,$ we obtain

$$\begin{aligned} x(n) &= \left(\prod_{i=0}^{n-1} A\right) \otimes_{\text{gr}} x_0 \oplus_{\text{gr}} \sum_{r=0}^{n-1} \left(\prod_{i=r+1}^{n-1} A\right) \otimes_{\text{gr}} \widetilde{B} \\ &= A^n \otimes_{\text{gr}} x_0 \oplus_{\text{gr}} \sum_{r=0}^{n-1} A^{n-r-1} \otimes_{\text{gr}} \widetilde{B} \\ &= A^n \otimes_{\text{gr}} x_0 \oplus_{\text{gr}} \frac{A^n - 1}{A - 1} \otimes_{\text{gr}} \widetilde{B}. \end{aligned} \tag{3.16}$$

□

Definition 3.2. A nontrivial solution $x(n)$ of problem (3.13) is called oscillatory about the equilibrium point x^* if, for every positive integer N, there exists an integer $n \geq N$ such that

$$\left(x(n+1) \ominus_{\text{gr}} x^*\right) \otimes_{\text{gr}} \left(x(n) \ominus_{\text{gr}} x^*\right) \leq_{\text{gr}} \hat{0}. \tag{3.17}$$

If this condition is not satisfied, then the solution is regarded as nonoscillatory.

Theorem 3.2. If $A < 0$, then the solution of problem (3.13) is oscillatory about its equilibrium point x^*.

Proof. Using Corollary 3.1 together with the expression of the equilibrium point x^* in (3.14), we obtain

$$\begin{aligned}
&\left(x(n+1) \ominus_{\mathrm{gr}} x^*\right) \otimes_{\mathrm{gr}} \left(x(n) \ominus_{\mathrm{gr}} x^*\right) \\
&= \left(A^{n+1} \otimes_{\mathrm{gr}} x_0 \oplus_{\mathrm{gr}} \frac{A^{n+1}-1}{A-1} \otimes_{\mathrm{gr}} \widetilde{B} \ominus_{\mathrm{gr}} \frac{1}{1-A} \otimes_{\mathrm{gr}} \widetilde{B}\right) \\
&\quad \otimes_{\mathrm{gr}} \left(A^{n} \otimes_{\mathrm{gr}} x_0 \oplus_{\mathrm{gr}} \frac{A^{n}-1}{A-1} \otimes_{\mathrm{gr}} \widetilde{B} \ominus_{\mathrm{gr}} \frac{1}{1-A} \otimes_{\mathrm{gr}} \widetilde{B}\right) \\
&= \left(A^{n+1} \otimes_{\mathrm{gr}} \left(x_0 \oplus_{\mathrm{gr}} \frac{1}{A-1} \otimes_{\mathrm{gr}} \widetilde{B}\right)\right) \otimes_{\mathrm{gr}} \left(A^{n} \otimes_{\mathrm{gr}} \left(x_0 \oplus_{\mathrm{gr}} \frac{1}{A-1} \otimes_{\mathrm{gr}} \widetilde{B}\right)\right) \\
&= A \otimes_{\mathrm{gr}} \left[A^{n} \otimes_{\mathrm{gr}} \left(x_0 \oplus_{\mathrm{gr}} \frac{1}{A-1} \otimes_{\mathrm{gr}} \widetilde{B}\right)\right]^2 \preceq_{\mathrm{gr}} \hat{0}, \quad \forall n \geq 0,\ n \in \mathbb{N}.
\end{aligned} \tag{3.18}$$

Since $A < 0$, the above inequality holds for all $n \geq 0$. Therefore, by Definition 3.2, the solution of problem (3.13) is oscillatory around the equilibrium point x^*. □

Corollary 3.2. In the case $A = -1$, the solution of problem (3.13) oscillates about its equilibrium point x^* between $\widetilde{B} \ominus_{\mathrm{gr}} x_0$ and x_0.

Proof. Starting from formula (3.15), for $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned}
x(n) &= A^{n} \otimes_{\mathrm{gr}} x_0 \oplus_{\mathrm{gr}} \frac{A^{n}-1}{A-1} \otimes_{\mathrm{gr}} \widetilde{B} \\
&= \left(x_0 \oplus_{\mathrm{gr}} \frac{1}{A-1} \otimes_{\mathrm{gr}} \widetilde{B}\right) \otimes_{\mathrm{gr}} A^{n} \ominus_{\mathrm{gr}} \frac{1}{A-1} \otimes_{\mathrm{gr}} \widetilde{B}.
\end{aligned} \tag{3.19}$$

When $A = -1$, it follows that

$$\begin{aligned}
x(n) &= \left(x_0 \oplus_{\mathrm{gr}} \left(-\frac{1}{2}\right) \otimes_{\mathrm{gr}} \widetilde{B}\right) \otimes_{\mathrm{gr}} (-1)^n \ominus_{\mathrm{gr}} \left(-\frac{1}{2}\right) \otimes_{\mathrm{gr}} \widetilde{B} \\
&= \begin{cases} x_0, & n = 2k, \\ \widetilde{B} \ominus_{\mathrm{gr}} x_0, & n = 2k+1, \end{cases} \qquad k = 0, 1, 2, \ldots
\end{aligned} \tag{3.20}$$

which confirms that the solution alternates between the two fuzzy values x_0 and $\widetilde{B} \ominus_{\mathrm{gr}} x_0$. Hence, the proof is complete. □

Theorem 3.3. If $|A| < 1$, then the solution of problem (3.13) converges to the equilibrium point x^* as $n \to \infty$.

Proof. Using formula (3.15), we derive

$$\begin{aligned}
\|x(n) \ominus_{\mathrm{gr}} x^*\|_{\mathrm{gr}} &= \left\| A^{n} \otimes_{\mathrm{gr}} x_0 \oplus_{\mathrm{gr}} \frac{A^{n}-1}{A-1} \otimes_{\mathrm{gr}} \widetilde{B} \ominus_{\mathrm{gr}} \frac{1}{1-A} \otimes_{\mathrm{gr}} \widetilde{B} \right\|_{\mathrm{gr}} \\
&= \left\| A^{n} \otimes_{\mathrm{gr}} \left(x_0 \oplus_{\mathrm{gr}} \frac{1}{A-1} \otimes_{\mathrm{gr}} \widetilde{B}\right) \right\|_{\mathrm{gr}} \\
&\leq |A|^n \left(\|x_0\|_{\mathrm{gr}} + \frac{\|\widetilde{B}\|_{\mathrm{gr}}}{|A-1|} \right).
\end{aligned} \tag{3.21}$$

Since $|A| < 1$, the right-hand side approaches zero as $n \to \infty$. Therefore, the solution of problem (3.13) is asymptotically stable and converges to the equilibrium point x^*. □

Example 3.1. Consider problem (3.13) with the initial value $x_0 = \widetilde{e_1}$ and constant term $\widetilde{B} = \widetilde{e_2}$ for all $n \geq 0$, $n \in \mathbb{N}$, where $\widetilde{e_1}$ and $\widetilde{e_2}$ are triangular fuzzy numbers. For example, let $\widetilde{e_1} = \widetilde{-2.5} = (-3, -2.5, -2)$ and $\widetilde{e_2} = \widetilde{1} = (0.9, 1, 1.1)$. According to formula (3.15), the solution is expressed as

$$x(n) = A^n \otimes_{\text{gr}} \widetilde{e_1} \oplus_{\text{gr}} \frac{A^n - 1}{A - 1} \otimes_{\text{gr}} \widetilde{e_2}. \tag{3.22}$$

When $A = -0.65$, the assumptions of Theorem 3.2 and Theorem 3.3 are satisfied. Hence, the solution of problem (3.13) oscillates while remaining stable around the equilibrium point (see Fig. 1).

If $A = -1$, the condition in Corollary 3.2 holds, implying that the solution of (3.13) alternates between the two fuzzy values $\widetilde{e_1}$ and $\widetilde{e_2} \ominus_{\text{gr}} \widetilde{e_1}$ (see Fig. 2).

For $A = 0.6$, the requirement of Theorem 3.3 is fulfilled, and therefore the solution of (3.13) is asymptotically stable (see Fig. 3).

Finally, when $A = 1.5$, neither Theorem 3.2 nor Theorem 3.3 applies. In this situation, the solution of (3.13) is neither oscillatory nor stable (see Fig. 4).

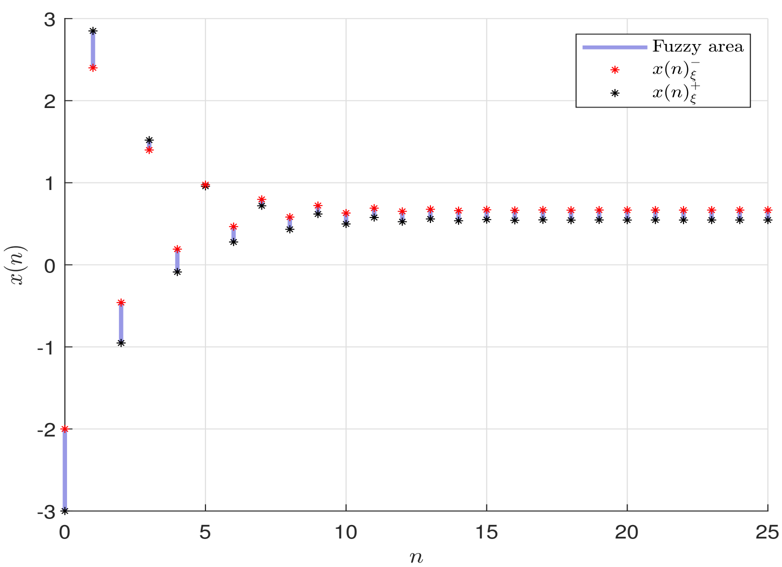

FIGURE 1. Trajectory of the solution of problem (3.13) in the case where $A = -0.65$.

3.2. **Fuzzy cobweb model and its applications.** In this subsection, we investigate the price formation of a specific commodity. Let $\mathcal{S}(\mathfrak{l})$ denote the quantity supplied in period $\mathfrak{l}$, $\mathcal{D}(\mathfrak{l})$ the quantity demanded in the same period, and $\mathfrak{p}(\mathfrak{l})$ a fuzzy-valued function representing the unit price at time $\mathfrak{l}$, where $\mathfrak{l} = 0, 1, 2, \ldots$.

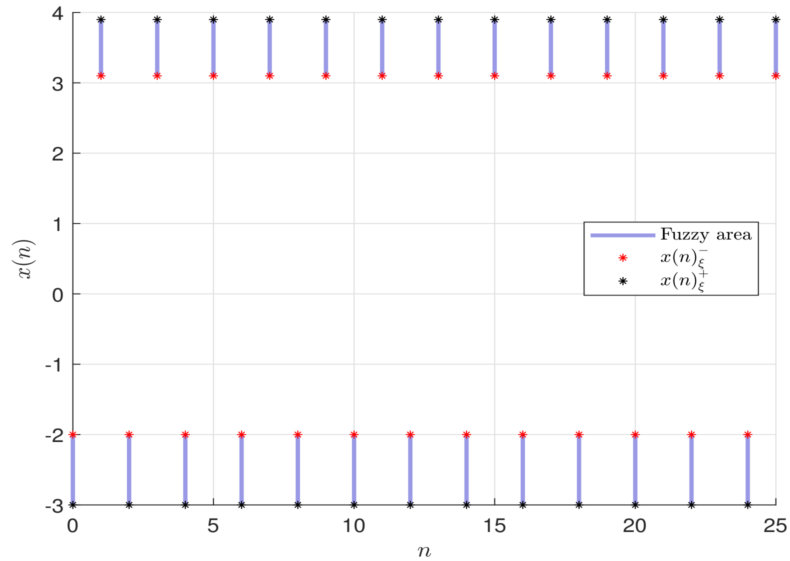

FIGURE 2. Trajectory of the solution of problem (3.13) in the case where $A = -1$.

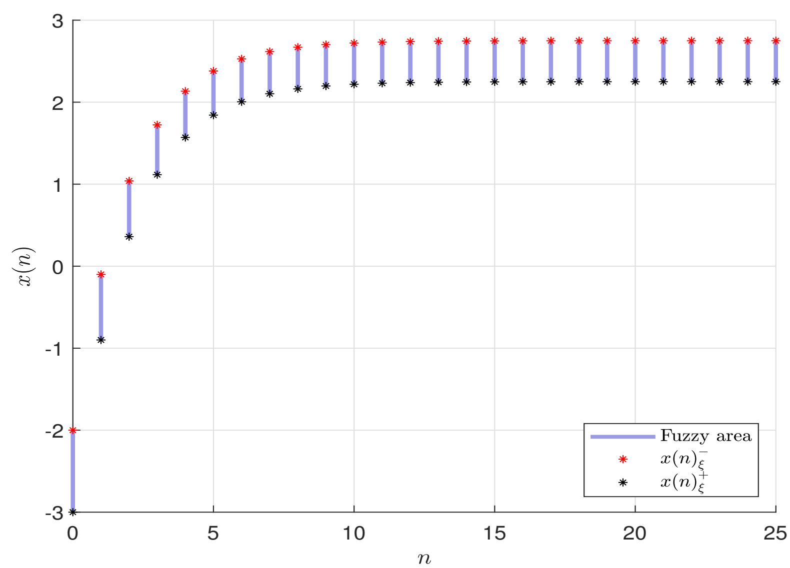

FIGURE 3. Trajectory of the solution of problem (3.13) in the case where $A = 0.6$.

For simplicity, the demand function is assumed to depend linearly on the price and is expressed as

$$\mathcal{D}(\mathfrak{l}) = -m_d \otimes_{\text{gr}} \mathfrak{p}(\mathfrak{l}) \oplus_{\text{gr}} b_d, \qquad m_d > 0, \quad b_d \succ_{\text{gr}} \hat{0}. \tag{3.23}$$

This relation is commonly known as the *price–demand curve*, where the constant m_d characterizes the responsiveness of consumers to price variations.

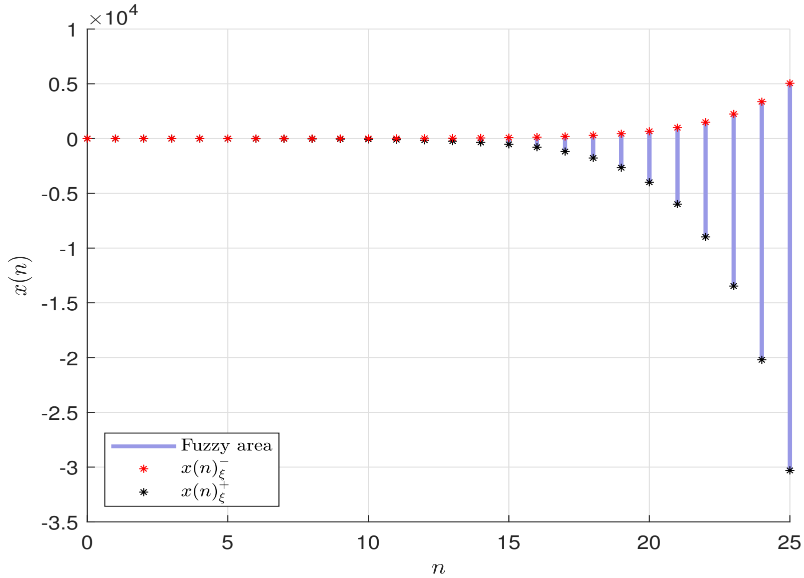

Figure 4. Trajectory of the solution of problem (3.13) in the case where $A = 1.5$.

Similarly, the supply in each period is assumed to depend on the price from the previous period, namely

$$\mathcal{S}(\mathfrak{l}+1) = m_s \otimes_{\mathrm{gr}} \mathfrak{p}(\mathfrak{l}) \oplus_{\mathrm{gr}} b_s, \qquad m_s > 0, \quad b_s >_{\mathrm{gr}} \hat{0}. \tag{3.24}$$

Here, m_s measures the sensitivity of producers to price changes. The negative slope of the demand curve reflects the decrease in demand when price increases, whereas the positive slope of the supply curve indicates that higher prices stimulate greater production.

A further assumption is that the market price is determined by the equality between supply and demand, that is, $\mathcal{D}(\mathfrak{l}+1) = \mathcal{S}(\mathfrak{l}+1)$. Consequently,

$$-m_d \otimes_{\mathrm{gr}} \mathfrak{p}(\mathfrak{l}+1) \oplus_{\mathrm{gr}} b_d = m_s \otimes_{\mathrm{gr}} \mathfrak{p}(\mathfrak{l}) \oplus_{\mathrm{gr}} b_s, \tag{3.25}$$

which can be rewritten as the first-order fuzzy linear difference equation

$$\mathfrak{p}(\mathfrak{l}+1) = \Upsilon_1 \otimes_{\mathrm{gr}} \mathfrak{p}(\mathfrak{l}) \oplus_{\mathrm{gr}} \Upsilon_2, \tag{3.26}$$

where

$$\begin{cases} \Upsilon_1 := -\dfrac{m_s}{m_d}, \\ \Upsilon_2 := \dfrac{b_d - b_s}{m_d}. \end{cases} \tag{3.27}$$

Equation (3.26) represents a linear difference equation whose equilibrium price $\mathfrak{p}^*$ corresponds to the intersection of the supply and demand curves. Since $\mathfrak{p}^*$ is the unique fixed point of (3.26), it follows that

$$\mathfrak{p}^* = \Upsilon_2 \oslash_{\mathrm{gr}} (1 - \Upsilon_1). \tag{3.28}$$

Corollary 3.3. For $\mathfrak{l} = 1, 2, 3, \dots$ with fuzzy initial value $\mathfrak{p}(0) = \mathfrak{p}_0$, equation (3.26) admits the unique solution

$$\mathfrak{p}(\mathfrak{l}) = \Big(\mathfrak{p}_0 \ominus_{gr} Y_2 \oslash_{gr} (1 - Y_1)\Big) \otimes_{gr} Y_1^{\mathfrak{l}} \oplus_{gr} Y_2 \oslash_{gr} (1 - Y_1)\,. \tag{3.29}$$

Proof. The result follows directly from Corollary 3.1. □

Because Y_1 represents the ratio between the slopes of the supply and demand curves, it governs the qualitative behavior of the price sequence. According to Theorem 3.2, Theorem 3.3, and Corollary 3.2, three distinct situations arise:

(1) $-1 < Y_1 < 0$,
(2) $Y_1 = -1$,
(3) $Y_1 < -1$.

- In case (1), the price sequence alternates around the equilibrium level $\mathfrak{p}^*$ while gradually converging to it. In economic terminology, this equilibrium is stable; mathematically, it is asymptotically stable.
- In case (2), the price oscillates between two fixed values only. If $\mathfrak{p}(0) = \mathfrak{p}_0$, then $\mathfrak{p}(1) = Y_2 \ominus_{gr} \mathfrak{p}_0$ and $\mathfrak{p}(2) = \mathfrak{p}_0$, implying stability of the equilibrium point $\mathfrak{p}^*$.
- In case (3), the oscillations occur around $\mathfrak{p}^*$ with increasing amplitude, so the trajectory diverges from equilibrium. Therefore, the equilibrium price is unstable.

4. Financial Market Applications

This section connects the stability structure of fuzzy linear difference equations to empirical volatility dynamics in financial markets. The theoretical results in Section 3 show that when the propagation parameter of a fuzzy difference equation approaches unity, the system remains stable but converges slowly toward equilibrium. In a financial interpretation, such slow convergence corresponds to prolonged persistence of uncertainty.

Mapping Empirical Persistence to Fuzzy Propagation. Consider the homogeneous fuzzy linear difference equation

$$x(n+1) = A \otimes_{gr} x(n).$$

Let $w(n)$ denote the width of the fuzzy number $x(n)$ under the granular representation, defined by

$$w(n) = \sup_{\xi \in [0,1]} \operatorname{len}([x(n)]^{\xi}).$$

By properties of granular scalar multiplication, the width evolves according to

$$w(n+1) = |A|\, w(n).$$

Hence, the stability condition $|A| < 1$ guarantees asymptotic contraction of the uncertainty width, while $|A|$ close to unity implies slow decay of ambiguity.

In the empirical specification, volatility dynamics are modeled as

$$vol_{t+1} = \phi\, vol_t + \varepsilon_t,$$

where ϕ measures persistence. Taking conditional expectations and focusing on the propagation component yields

$$\mathbb{E}[vol_{t+1} \mid vol_t] \approx \phi\, vol_t.$$

Thus, the autoregressive coefficient ϕ governs the deterministic propagation of volatility in the same way that $|A|$ governs the contraction of uncertainty width in the fuzzy dynamic system.

We therefore establish the correspondence

$$\phi \longleftrightarrow |A|.$$

Under this identification, the empirical condition $\phi < 1$ corresponds to the fuzzy stability condition $|A| < 1$, and near-unit-root volatility behavior corresponds to propagation parameters approaching the stability boundary in the granular metric space.

The empirical analysis employs daily data from 1 January 2010 to 13 March 2026 for the Stock Exchange of Thailand (SET), the S&P 500 index, and the CBOE Volatility Index (VIX). The sample consists of 4,199 trading days for each market and includes major global episodes such as the European sovereign debt crisis, the COVID-19 pandemic, and the subsequent tightening cycle. Log returns are computed as $r_t = \log(P_t/P_{t-1})$, and volatility is proxied by a 20-day rolling standard deviation of returns, which serves as an empirical counterpart to the uncertainty width in the fuzzy framework.

Descriptive statistics confirm standard stylized facts. Mean daily returns are economically small, at 0.000159 for the SET and 0.000435 for the S&P 500. Both markets exhibit pronounced excess kurtosis (18.5 and 17.8, respectively), indicating heavy tails and extreme observations. Volatility measures are positively skewed and display substantial dispersion, reflecting episodic spikes during crisis periods.

Stationarity tests strongly reject the presence of unit roots in both returns and volatility series, while Ljung–Box and ARCH LM tests confirm weak serial dependence in raw returns but strong conditional heteroskedasticity. These findings justify focusing on volatility dynamics rather than mean dynamics, consistent with the separation between center and width components in the fuzzy interpretation.

Estimating the volatility propagation equation yields persistence parameters of $\phi_{SET} = 0.925$ and $\phi_{SP500} = 0.901$. Both coefficients are highly significant and economically large. A coefficient of 0.925 implies that approximately 92.5% of yesterday's volatility level carries over into the next trading day for the SET, while 90.1% carries over for the S&P 500. Although the numerical difference appears modest, its economic implications are substantial in near-unit-root environments.

The half-life of volatility shocks is computed as $HL = \ln(0.5)/\ln(\phi)$. For the SET, the half-life is approximately nine trading days, compared to roughly seven trading days for the S&P 500. This implies that uncertainty shocks dissipate more slowly in the emerging market. In practical terms, episodes of heightened volatility persist longer in the SET, increasing risk exposure horizons and potentially affecting portfolio rebalancing strategies, margin requirements, and liquidity provision.

The coefficient on the absolute return term is positive and highly significant in both markets, indicating strong shock amplification. Large price movements immediately widen the volatility band. The VIX coefficient is also positive and statistically significant, demonstrating that global uncertainty acts as an external driver of domestic volatility. Economically, this suggests that emerging and developed markets are both sensitive to global risk sentiment, but the persistence mechanism differs in magnitude.

For robustness, GARCH(1,1) models are estimated. The persistence measure $(\alpha + \beta)$ equals approximately 0.991 for the SET and 0.967 for the S&P 500. These values confirm near-integrated variance behavior, particularly in the emerging market. The higher persistence in the SET implies that variance shocks are nearly permanent over short horizons. Within the fuzzy framework, this corresponds to a propagation parameter close to the stability boundary, where uncertainty converges slowly but remains bounded.

The economic interpretation of these findings is consistent with structural differences between emerging and developed markets. Lower liquidity depth, greater informational asymmetry, and higher exposure to external shocks in emerging markets may amplify uncertainty propagation. In the fuzzy dynamic interpretation, such structural characteristics manifest as a larger propagation coefficient governing the width component.

Figure 5 visually illustrates the rolling volatility series. Periods of global stress, particularly during the COVID-19 crisis, show pronounced spikes in both markets, with visibly more persistent decay in the SET. This visual evidence supports the quantitative persistence estimates.

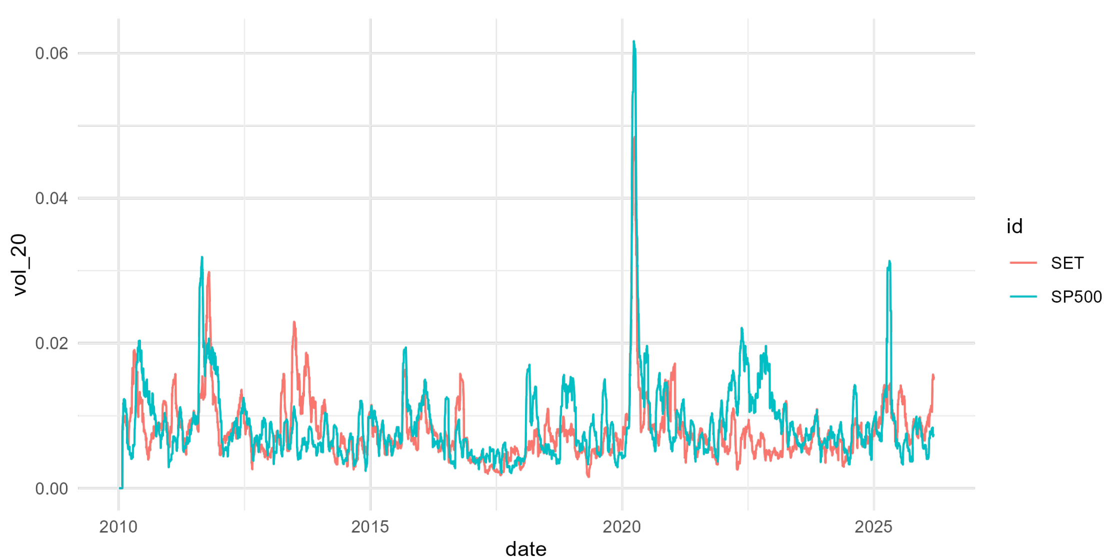

Figure 5. Rolling 20-day volatility of the SET and S&P 500 indices (2010–2026).

The empirical results demonstrate that volatility persistence can be interpreted as slow convergence of uncertainty width within a stable fuzzy dynamic system. The fuzzy framework does not replace stochastic volatility models; rather, it provides a structural interpretation that links mathematical stability conditions with economically meaningful persistence behavior.

5. Discussion

The objective of this study is not to propose a competing volatility estimator, but to demonstrate how the stability structure of fuzzy linear difference equations can provide a complementary interpretation of volatility persistence in financial markets. The empirical findings show weak serial dependence in returns, strong persistence in volatility, and significant spillover effects from global uncertainty. These stylized facts are well documented in the ARCH/GARCH literature initiated by Engle [1] and Bollerslev [2], and further analyzed in the context of near-unit-root variance behavior by Nelson [3]. The contribution of the present paper lies in interpreting these empirical regularities through the lens of fuzzy dynamic stability.

Section 3 established that fuzzy linear difference equations admit explicit solution representations with convergence governed by a propagation parameter. When this parameter is strictly less than unity in magnitude, the system remains stable; however, when it approaches unity, convergence becomes slow and uncertainty dissipates gradually. In the financial application, this stability condition is mapped to volatility persistence. The estimated propagation coefficient in the volatility equation plays a role analogous to the fuzzy stability parameter. High persistence in volatility thus corresponds to slow contraction of uncertainty width within a stable dynamic system.

Importantly, the fuzzy framework operates at the structural and interpretative level rather than at the level of statistical optimization. The empirical model is estimated using standard econometric techniques, yet its interpretation is enriched by distinguishing between center dynamics and width dynamics. The center component, represented by returns, exhibits near-random behavior consistent with market efficiency [4]. In contrast, the width component, proxied by rolling volatility, displays strong persistence. This separation provides a conceptual distinction between efficient price adjustment and persistent ambiguity.

The cross-market comparison yields economically meaningful implications. The emerging market (SET) exhibits a higher volatility persistence parameter than the developed market (S&P 500), implying slower dissipation of uncertainty shocks. Within the fuzzy interpretation, this indicates stronger ambiguity propagation. Such a pattern is consistent with structural characteristics commonly attributed to emerging markets, including lower liquidity, greater informational asymmetry, and higher exposure to external shocks. The persistence differential, though numerically modest, translates into meaningful differences in half-life calculations and long-run variance behavior in near-integrated environments.

The significance of the VIX coefficient further reinforces the ambiguity interpretation. Global uncertainty shocks widen domestic volatility bands, acting as external forces in the dynamic system. Rather than viewing spillovers purely as stochastic variance shocks, the fuzzy perspective interprets them as external widening mechanisms affecting the uncertainty width. This view complements the probabilistic treatment of spillovers in conventional models and aligns with broader discussions of Knightian uncertainty [5] and ambiguity in asset pricing [6].

Moreover, recent empirical studies in emerging markets suggest that market inefficiencies, exchange rate exposure, and nonlinear predictive structures may amplify uncertainty propagation [12–14]. These findings are consistent with the interpretation that structural frictions and informational imperfections can strengthen ambiguity persistence within a dynamic system.

The novelty of the paper, therefore, resides in establishing a bridge between deterministic stability conditions in fuzzy difference equations and empirical volatility persistence across markets. While stochastic models describe how variance evolves conditionally on past innovations, the fuzzy framework offers a structural interpretation of why persistence near unity corresponds to prolonged ambiguity. The two approaches are mathematically compatible but conceptually distinct. The fuzzy model does not aim to outperform GARCH forecasting; rather, it broadens the interpretative framework within which volatility clustering can be understood.

Several limitations warrant acknowledgment. Volatility is proxied by rolling standard deviations rather than estimated as a genuine fuzzy-valued process, so the fuzzy component remains conceptual rather than directly observed. The empirical specification is linear and does not account for potential regime shifts or nonlinear propagation effects during crises. Furthermore, the analysis is conducted at the aggregate index level and does not explore firm-level heterogeneity. These limitations suggest directions for future research, including hybrid stochastic–fuzzy volatility models, regime-dependent propagation parameters, and nonlinear fuzzy dynamics capable of capturing structural breaks.

The results indicate that volatility persistence in financial markets can be interpreted as slow convergence of uncertainty width within a stable fuzzy dynamic system. This interpretation complements established stochastic volatility models and contributes to an interdisciplinary dialogue between fuzzy systems theory and financial econometrics. By reframing persistence as ambiguity propagation, the study expands the conceptual toolkit available for understanding financial uncertainty without challenging the empirical success of existing econometric approaches.

6. Conclusion

This paper develops and preserves the fuzzy linear difference equation framework and demonstrates its relevance in interpreting financial volatility dynamics. While stochastic volatility models such as GARCH effectively capture conditional variance persistence, the present study shows that the stability structure of fuzzy difference equations provides a complementary conceptual interpretation of volatility clustering as slow convergence of uncertainty width.

Using daily data for the Stock Exchange of Thailand, the S&P 500, and the VIX over the period 2010–2026, we document strong volatility persistence in both markets and systematically higher persistence in the emerging market. The implied half-life calculations indicate slower dissipation of uncertainty shocks in the SET relative to the S&P 500. Global uncertainty, proxied by the VIX, significantly amplifies domestic volatility, suggesting that ambiguity propagation is influenced by both internal dynamics and external spillovers.

The contribution of this study is interpretative rather than predictive. The fuzzy framework does not seek to replace established stochastic models; instead, it links deterministic stability conditions with empirical persistence behavior in financial markets. By distinguishing between center dynamics and width dynamics, the approach offers a structural perspective on how uncertainty evolves over time.

From an economic standpoint, the findings imply that markets with structural frictions or lower informational efficiency may exhibit stronger ambiguity propagation, leading to longer-lasting volatility episodes. Such persistence has implications for risk management, portfolio allocation horizons, and regulatory monitoring during periods of global stress.

Several avenues for future research emerge from this framework. Hybrid stochastic–fuzzy volatility models could formalize the interaction between probabilistic variance and ambiguity width. Nonlinear or regime-dependent fuzzy difference equations may better capture structural breaks during crises. Extending the analysis to firm-level or sector-level data could further illuminate how uncertainty propagation varies across economic structures.

The study contributes to an interdisciplinary dialogue between fuzzy systems theory and financial econometrics. By reframing volatility persistence as ambiguity propagation within a stable dynamic system, it broadens the conceptual toolkit for understanding financial uncertainty while remaining consistent with established empirical evidence.

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