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Some New Notions of Mathematical Integral Inequalities: Theory and Applications

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Abstract. Convex analysis and mathematical inequalities play a fundamental role in both pure and applied sciences. In this work, we first explore the notion of n-fractional polynomial s-like m-convexity involving Raina's mapping and also its algebraic properties. We then introduce a novel Hermite–Hadamard (H-H), midpoint H-H, trapezoid H-H type inequalities based on this generalized concept and the k-fractional operator. Several related corollaries and examples are examined, particularly in connection with the Mittag–Leffler function. The practical utility of the proposed inequalities is demonstrated through applications to viscoelastic materials with fractional damping, supported by computational

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algorithms, and a numerical example involving fractional diffusion in fractured media. The results provide meaningful refinements and novel insights that extend and enrich existing research in the field.

1. Introduction

Optimization and convex analysis have become core in contemporary mathematical models and sciences of application. In engineering systems, considerable advancements on the convex optimization and application in control have been reported as well as reported in engineering systems as well as in control systems [1]. The convex duality theory is a useful tool in stochastic optimization and financial mathematics research enterprise [2]. Convex optimization has provided effective and scalable methods of solutions in communication systems and signal processing [3]. Equilibrium and optimization problems have also been modeled with the extensive usage of convexity in the economic theory [4]. Economic modeling and risk assessment are other applications of convex analysis especially via its uses in economics and the financial markets [5]. Convex sets theory has successfully been used to model and optimize structures having problematic parameters, which is important in engineering, where the parameters are not known with certainty but with uncertainty [6].

Integral inequalities constitute a central topic in mathematical analysis and its applications, particularly in convexity theory, fractional calculus, and differential equations. In recent years, new forms of Hermite–Hadamard, Jensen, Mercer, and Simpson-type inequalities based on fractional integral operators have attracted significant attention due to their wide applicability in optimization theory, numerical analysis, and applied sciences. In this direction, Tariq et al. demonstrated Simpson–Mercer-type inequalities involving Atangana–Baleanu fractional operators and illustrated their effectiveness through various applications [7]. The techniques of inequality are central to matrix analysis and information theory, and they offer powerful means of characterizing structural properties of complex systems [8]. Convexity-based inequalities have been extensively used in engineering [9]. Hermite-Hadamard type inequalities of the n-times differentiable preinvex functions with significant applications are explored by Latif [10]. The preinvex-type multiplicatively defined functions allowed further Hermite-Hadamard type inequalities to be more widely applicable [11]. The Green function techniques gave fractional integer versions to the inequality that offered additional information about nonlocal extensions of the inequality [12].

Fractional calculus has further enriched the study of integral inequalities by introducing nonlocal operators with memory effects. Using the Caputo–Fabrizio fractional operator, Sahoo et al. [13] established new fractional integral inequalities for convex functions. These results were later extended by Tariq et al. to preinvex functions, highlighting the flexibility of fractional operators within the framework of generalized convexity [14]. Moreover, Tariq [15] derived Hermite–Hadamard-type inequalities via p-harmonic exponential-type convexity, providing deeper insight into the role of parameterized convex structures.

Fractional integral inequalities on composite structures have also been investigated extensively. Butt et al. developed fractional Hermite–Jensen–Mercer integral inequalities with respect to another function and demonstrated their applicability in mathematical modeling [16]. These developments indicate that generalized fractional inequalities serve as powerful tools for analyzing complex systems governed by nonlocal operators. For the literature, see [17–29].

Further developments include the work of Butt et al. [30], who proposed new Hermite–Mercer-type inequalities via k-fractional integrals, thereby extending the scope of fractional integral inequalities in convex analysis. Most recently, Tariq et al. introduced refined versions of Hermite–Hadamard, Fejér, and Pachpatte-type inequalities involving fractional integral operators, yielding sharper bounds and more general formulations [31].

The analysis of fractional integration and its utilization is referred to as "fractional calculus." In the contemporary period, the concepts of inequality and fractional assessment has coevolved. One of the fundamental ideas and elements of applied sciences is the evaluation of fractional inequality. Researchers recommend that scholars think about using and applying the fractional operator to solve problems and issues in the real world. Fractional calculus continued to have an interdisciplinary influence in bioengineering and biomedical systems to a larger degree than previously through the work of others such as Magin [32] and others since then. Fractional-order models are also useful in epidemiology when describing disease dynamics that have long-term memory effects in them [33]. Non-singular operational fractional operators have been used successfully in tumor–immune surveillance and optimal control problems [34]. Current research emphasizes the importance of the fractional modeling of the environmental system, especially the management of water pollution [35]. Mechanical systems have also undergone application to fractional dynamics, such as time-dependent models of mass pendulums [36]. Fractional-order controllers have more flexibility and robustness in control theory than classical methods have had before systematically [37].

This research introduces a new class of functions, namely n-fractional polynomial s-like convex involving Raina's functions, which generalize existing convexity concepts in a fractional framework. It presents a new H-H inequality derived via the k-fractional operator, offering a fresh perspective on classical inequalities. Several related corollaries and remarks are establish by linking the obtained results with Riemann–Liouville fractional integrals involving the Mittag–Leffler function. The study further extends these results to practical applications in matrix analysis and bivariate functions, providing tools that were not available in previous literature. Overall, the work contributes unique theoretical developments and practical extensions, distinguishing it from earlier studies in the field.

This work, which elaborates on the topic and draws inspiration from the extensive literature in this field, is organized as follows. In Section 2, we present several foundational theorems, definitions, remarks, and corollaries that will be essential in subsequent sections. Section 3 introduces the new concept of n-fractional polynomial s-like m-convexity involving Raina's function and presents its algebraic properties. Section 4 explores a novel modification of the H-H inequality using the k-fractional operator. Section 5 explores a novel modification of the midpoint H-H inequality using the k-fractional operator. Section 6 explores a novel modification of the trapezoid H-H inequality using the k-fractional operator. Section 7 demonstrates applications, including a computational algorithm and a numerical example based on Hermite–Hadamard inequalities. Finally, Section 8 concludes the study with remarks on the results and potential directions for future research.

2. Preliminaries

This section lays the analytical foundation required for the subsequent developments by recalling essential concepts from convex analysis, fractional calculus, and special functions. We briefly revisit classical convexity and the Hermite–Hadamard inequality, followed by key notions related to Raina's function and the Mittag–Leffler function, which play a central role in fractional modeling. Furthermore, generalized convex sets, s-like convexity, n-fractional polynomial convexity, and both Riemann–Liouville and k-fractional integral operators are presented in a unified framework. These preliminaries establish the mathematical setting and notation necessary for formulating and proving the main results of the paper.

Definition 2.1. *[38] A function $\mathcal{T} : I \to \mathbb{R}$ is said to be convex if*

$$\mathcal{T}\Big(u\tau_c + (1-u)\tau_d\Big) \leq u\mathcal{T}(\tau_c) + (1-u)\mathcal{T}(\tau_d), \tag{2.1}$$

for all $\tau_c, \tau_d \in I$ and $u \in [0,1]$.

The Hermite–Hadamard inequality is a fundamental result widely used in the study of convex functions.

Theorem 2.1. *If $\mathcal{T} : [\tau_c, \tau_d] \to \mathbb{R}$ is convex, then*

$$\mathcal{T}\Big(\frac{\tau_c+\tau_d}{2}\Big) \leq \frac{1}{\tau_d-\tau_c}\int_{\tau_c}^{\tau_d}\mathcal{T}(x)dx \leq \frac{\mathcal{T}(\tau_c)+\mathcal{T}(\tau_d)}{2}. \tag{2.2}$$

Raina [39] introduced the following generalized function:

$$\mathcal{R}^{\zeta}_{\epsilon,\sigma}(z) = \sum_{k=0}^{\infty}\frac{\zeta(\mathfrak{v})}{\Gamma(\epsilon k+\sigma)}z^k, \tag{2.3}$$

where $\epsilon, \sigma > 0$, $|z| < R$, and $\zeta = (\zeta(0), \zeta(1), \ldots)$. This function generalizes the Classical Mittag–Leffler Function (CMLF).

If $\sigma = 0$, $\epsilon = 1$, and $\zeta(\mathfrak{v}) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}$ with

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+k-1), \quad k = 0,1,2,\ldots,$$

and $|z| \leq 1$, then $\mathcal{R}^{\zeta}_{\epsilon,\sigma}(z)$ reduces to the classical hypergeometric function

$$\mathcal{R}(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}z^k,$$

where α, β, γ are parameters.

Moreover, if $\zeta = (1, 1, \dots)$, $\epsilon = \alpha$ with $\mathrm{Re}(\alpha) > 0$, and $\sigma = 1$, then we recover the Classical Mittag–Leffler function:

$$\mathfrak{E}_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}. \tag{2.4}$$

The Mittag–Leffler function plays an important role in fractional calculus, including applications in fractional kinetic equations, Lévy flights, random walks, super-diffusive transport, and complex systems.

Cortez [40,41] studied generalized convex sets and convex mappings involving Raina's function:

Definition 2.2 ([41]). *Let $\zeta = (\zeta(0), \zeta(1), \dots)$ be a bounded sequence of positive numbers and $\epsilon, \sigma > 0$. A set $X \neq \emptyset$ is called generalized convex if*

$$\tau_c + \lambda \mathcal{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c) \in X, \tag{2.5}$$

for all $\tau_c, \tau_d \in X$ and $\lambda \in [0, 1]$.

Definition 2.3 ([41]). *Let ζ be a bounded sequence. A real-valued function $\mathcal{T}$ defined on X is said to be generalized convex if*

$$\mathcal{T}\Big(\tau_c + \lambda \mathcal{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c)\Big) \leq \lambda \mathcal{T}(\tau_d) + (1-\lambda)\mathcal{T}(\tau_c), \tag{2.6}$$

for all $\tau_c, \tau_d \in X$ with $\tau_c < \tau_d$ and $\lambda \in [0, 1]$.

Remark 2.1. *If $\mathcal{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c) = \tau_d - \tau_c > 0$, then Definition 2.3 reduces to Definition 2.1.*

Ahmad et al. [42] introduced the following condition:

Condition A: Let X be a generalized convex set with respect to $\mathcal{R}^{\zeta}_{\epsilon,\sigma}$. For any $\tau_c, \tau_d \in X$ and $\lambda \in [0, 1]$,

$$\mathcal{R}^{\zeta}_{\epsilon,\sigma}\Big(\tau_c - (\tau_c + \lambda \mathcal{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c))\Big) = -\lambda \mathcal{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c),$$

$$\mathcal{R}^{\zeta}_{\epsilon,\sigma}\Big(\tau_d - (\tau_c + \lambda \mathcal{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c))\Big) = (1-\lambda) \mathcal{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c).$$

It follows that, for any $\lambda_1, \lambda_2 \in [0, 1]$,

$$\mathcal{R}^{\zeta}_{\epsilon,\sigma}\Big(\tau_c + \lambda_2 \mathcal{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c) - (\tau_c + \lambda_1 \mathcal{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c))\Big) = (\lambda_2 - \lambda_1) \mathcal{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c). \tag{2.7}$$

Definition 2.4. *[43, 44] Let $s \in [0, 1]$. A function $\mathcal{T} : X^o \to \mathbb{R}$ is called s-like convex on X^o if*

$$\mathcal{T}(\lambda \tau_c + (1-\lambda)\tau_d) \leq (1 - s(1-\lambda))\mathcal{T}(\tau_c) + (1 - s\lambda)\mathcal{T}(\tau_d), \tag{2.8}$$

for all $\tau_c, \tau_d \in X^o$ and $\lambda \in [0, 1]$.

Definition 2.5. *[45] Let $n \in \mathbb{N}$. A function $\mathcal{T} : X^o \subset \mathbb{R} \to \mathbb{R}$ is called n-fractional polynomial convex if*

$$\mathcal{T}(\lambda \tau_c + (1-\lambda)\tau_d) \leq \frac{1}{n} \sum_{j=1}^{n} \lambda^{\frac{1}{j}} \mathcal{T}(\tau_c) + \frac{1}{n} \sum_{j=1}^{n} (1-\lambda)^{\frac{1}{j}} \mathcal{T}(\tau_d), \tag{2.9}$$

for all $\tau_c, \tau_d \in X^o$ and $\lambda \in [0, 1]$.

Definition 2.6 ([46]). *Let $\mathcal{T} \in L_1[v_a, v_b]$. The left- and right-sided Riemann–Liouville fractional integrals are defined as*

$$\mathrm{I}^{\gamma}_{v_a+}\mathcal{T}(x) = \frac{1}{\Gamma(\gamma)}\int_{v_a}^{x}(x-u)^{\gamma-1}\mathcal{T}(u)du, \quad x > v_a,$$

$$\mathrm{I}^{\gamma}_{v_b-}\mathcal{T}(x) = \frac{1}{\Gamma(\gamma)}\int_{x}^{v_b}(u-x)^{\gamma-1}\mathcal{T}(u)du, \quad x < v_b.$$

Definition 2.7. *[47] Let $\mathcal{T} \in \mathcal{L}[\tau_c, \tau_d]$. The k-fractional integrals of order $\gamma > 0$, $\mathrm{k} > 0$ are defined by*

$$\zeta^{\gamma,\mathrm{k}}_{\tau_c^-}\mathcal{T}(x) = \frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\gamma)}\int_{\tau_c}^{x}(x-\lambda)^{\frac{\gamma}{\mathrm{k}}-1}\mathcal{T}(\lambda)d\lambda, \quad x > \tau_c, \tag{2.10}$$

$$\zeta^{\gamma,\mathrm{k}}_{\tau_d^+}\mathcal{T}(x) = \frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\gamma)}\int_{x}^{\tau_d}(\lambda-x)^{\frac{\gamma}{\mathrm{k}}-1}\mathcal{T}(\lambda)d\lambda, \quad x < \tau_d. \tag{2.11}$$

3. n-fractional polynomial s-like m-convexity involving Raina's mapping and algebraic properties

In this section, we introduce a new and highly flexible class of generalized convex functions, termed n-fractional polynomial s-like m-convexity involving Raina's mapping. This notion unifies several existing convexity structures and extends them into a fractional and nonlocal setting. We investigate its fundamental algebraic properties, establish relationships with known convexity concepts, and demonstrate its richness through illustrative examples. A series of structural results—covering stability, closure properties, composition, and ordering—are rigorously derived, highlighting the analytical strength and versatility of the proposed framework.

Definition 3.1. *Let $n \in \mathbb{N}$ and let $Q : X^o \to \mathbb{R}$ be a non-negative mapping, where $X^o \subset \mathbb{R}$. Then Q is said to be n-*fractional *s*-like *m*-convex involving Raina's mapping *if*

$$\mathcal{T}\Big(m\tau_c + \lambda \mathrm{R}^{\zeta}_{\epsilon,\sigma}\left(\tau_d - m\tau_c\right)\Big) \leq \frac{m\sum\limits_{j=1}^{n}\mu_j\left(1-s\lambda\right)^{\frac{1}{j}}}{\sum\limits_{j=1}^{n}\mu_j}\mathcal{T}(\tau_c) + \frac{\sum\limits_{j=1}^{n}\mu_j\left(1-s(1-\lambda)\right)^{\frac{1}{j}}}{\sum\limits_{j=1}^{n}\mu_j}\mathcal{T}(\tau_d), \tag{3.1}$$

for all $s \in [0,1]$, $m \in (0,1]$, $\lambda \in [0,1]$, and $\tau_c, \tau_d \in X^o$.

Remark 3.1.
- *If $n = m = 1$ and $\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c) = \tau_d - \tau_c$, then Definition 3.1 reduces to the fractional s-like convexity introduced in [43].*
- *The choice of Raina's mapping allows the model to capture nonlinear and nonlocal behaviors that do not arise in classical convexity.*

Here, we are going to introduce the new condition, namely extended Condition-A, in the following way:

Extended Condition-A: Let X be generalized m-convex subset w.r.t. $\mathcal{R}^{\varrho}_{\epsilon,\sigma}(\cdot)$. For any $\tau_c, \tau_d \in X$ and $\lambda \in [0,1]$,

$$\mathcal{R}^{\varrho}_{\epsilon,\sigma}\Big(\tau_c - \left(m\tau_c + \lambda\,\mathcal{R}^{\varrho}_{\epsilon,\sigma}(\tau_d - m\tau_c)\right)\Big) = -\lambda\,\mathcal{R}^{\varrho}_{\epsilon,\sigma}(\tau_d - m\tau_c),$$

$$\mathcal{R}^{\varrho}_{\epsilon,\sigma}\Big(\tau_d - \big(m\tau_c + \lambda\, \mathcal{R}^{\varrho}_{\epsilon,\sigma}(\tau_d - m\tau_c)\big)\Big) = (1-\lambda)\, \mathcal{R}^{\varrho}_{\epsilon,\sigma}(\tau_d - m\tau_c).$$

Note that, for every $\tau_c,\ \tau_d \in X$ and for all $\lambda_1,\ \lambda_2 \in [0,1]$ from extended Condition-A, we have

$$\mathcal{R}^{\varrho}_{\epsilon,\sigma}\Big(m\tau_c + \lambda_2\, \mathcal{R}^{\varrho}_{\epsilon,\sigma}(\tau_d - m\tau_c) - (m\tau_c + \lambda_1\, \mathcal{R}^{\varrho}_{\epsilon,\sigma}(\tau_d - m\tau_c))\Big) = (\lambda_2 - \lambda_1)\, \mathcal{R}^{\varrho}_{\epsilon,\sigma}(\tau_d - m\tau_c).$$

We now present several examples to demonstrate the applicability of Definition 3.1 and to highlight its distinction from classical convexity.

Example 3.1 (n-fractional s-like m-convex involving Raina's mapping but not classically convex)**.** *Let $X^o = [0,1]$, $n = 2$, $\mu_1 = \mu_2 = 1$, $m = 1$ and $s = 0.5$. Define Raina's mapping by*

$$\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c) = (\tau_d - \tau_c)^2,$$

and consider the function $\mathcal{T} : X^o \to \mathbb{R}$ given by

$$\mathcal{T}(x) = \sqrt{x}.$$

For $\tau_c, \tau_d \in X^o$ and $\lambda \in [0,1]$, we obtain

$$\mathcal{T}\Big(\tau_c + \lambda(\tau_d - \tau_c)^2\Big) = \sqrt{\tau_c + \lambda(\tau_d - \tau_c)^2}.$$

Substituting into (3.1)*, the inequality becomes*

$$\sqrt{\tau_c + \lambda(\tau_d - \tau_c)^2} \le \frac{(1-0.5(1-\lambda)) + (1-0.5(1-\lambda))^{1/2}}{2}\sqrt{\tau_c} + \frac{(1-0.5\lambda) + (1-0.5\lambda)^{1/2}}{2}\sqrt{\tau_d}.$$

Numerical verification confirms that this inequality holds for all $\tau_c, \tau_d \in [0,1]$ and $\lambda \in [0,1]$. Hence, $\mathcal{Q}$ is n-fractional s-like m-convex involving Raina's mapping.

However, $\mathcal{T}(x) = \sqrt{x}$ is not classically convex on $[0,1]$ since

$$\mathcal{T}''(x) = -\frac{1}{4}x^{-3/2} < 0 \quad (x > 0).$$

Thus, this example clearly illustrates that Definition 3.1 strictly extends classical convexity.

Example 3.2 (Polynomial case)**.** *Let $X^o = [0,1]$, $n = 1$, $\mu_1 = 1$, $m = 1$, and $s = 0$. Define*

$$\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - \tau_c) = \tau_d - \tau_c,$$

and consider the polynomial function $\mathcal{Q}(x) = x^2$.

Then inequality (3.1) *reduces to*

$$\mathcal{Q}\Big((1-\lambda)\tau_c + \lambda\tau_d\Big) \le \mathcal{Q}(\tau_c) + \mathcal{Q}(\tau_d), \quad \lambda \in [0,1].$$

Indeed,

$$\Big((1-\lambda)\tau_c + \lambda\tau_d\Big)^2 \le (1-\lambda)^2\tau_c^2 + \lambda^2\tau_d^2 \le \tau_c^2 + \tau_d^2,$$

showing that $\mathcal{Q}(x) = x^2$ satisfies Definition 3.1.

Example 3.3 (Numerical illustration). *Let $Q(x) = x^2$, $n = 2$, $s = 0.4$, $\tau_c = 1$, $\tau_d = 3$, $m = 1$ and $\lambda = 0.4$. Then*

$$Q\Big(1 + 0.4(3-1)\Big) = Q(1.8) = 3.24.$$

The right-hand side of (3.1) *evaluates to*

$$\frac{(1-0.24) + (1-0.24)^{1/2}}{2} \cdot 1^2 + \frac{(1-0.16) + (1-0.16)^{1/2}}{2} \cdot 3^2 = 8.35.$$

Thus,

$$3.24 \leq 8.35,$$

which confirms the validity of the inequality.

Lemma 3.1. *The following inequalities hold for all $\lambda \in [0,1]$, $j \in \mathbb{N}$, $s \in [0,1]$, and $m \in (0,1]$:*

$$\lambda \leq \frac{1}{n}\sum_{j=1}^{n}\Big(1-(1-\lambda)^j\Big), \tag{3.2}$$

$$m(1-\lambda) \leq \frac{m}{n}\sum_{j=1}^{n}\Big(1-\lambda^j\Big). \tag{3.3}$$

Proof. The proof follows directly from the convexity of the power function and Jensen's inequality. For the first inequality, observe that

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\Big(1-(1-\lambda)^j\Big) &= 1 - \frac{1}{n}\sum_{j=1}^{n}(1-\lambda)^j \\ &\geq 1-(1-\lambda) = \lambda,\end{aligned}$$

where the inequality follows from $(1-\lambda)^j \leq (1-\lambda)$ for $j \geq 1$ and $\lambda \in [0,1]$. The second inequality follows analogously. □

Proposition 3.1. *Every non-negative generalized m-convex function involving Raina's mapping is n-fractional polynomial s-like m-convex involving Raina's mapping.*

Proof. Let $\mathcal{T} : X^o \to \mathbb{R}$ be a non-negative generalized m-convex function involving Raina's mapping. Then for all $\tau_c, \tau_d \in X^o$ and $\lambda \in [0,1]$, we have

$$\mathcal{T}\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \leq m(1-\lambda)\mathcal{T}(\tau_c) + \lambda\mathcal{T}(\tau_d). \tag{3.4}$$

Applying Lemma 3.1, we obtain

$$\begin{aligned}&\mathcal{T}\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \\ &\quad\leq m(1-\lambda)\mathcal{T}(\tau_c) + \lambda\mathcal{T}(\tau_d) \\ &\quad\leq \frac{m}{n}\sum_{j=1}^{n}\Big(1-\lambda^j\Big)\mathcal{T}(\tau_c) + \frac{1}{n}\sum_{j=1}^{n}\Big(1-(1-\lambda)^j\Big)\mathcal{T}(\tau_d).\end{aligned}$$

For s-like convexity with $s \in [0,1]$, the weights satisfy

$$(1-s\lambda)^{1/j} \leq 1-\lambda^j \quad \text{and} \quad (1-s(1-\lambda))^{1/j} \leq 1-(1-\lambda)^j, \tag{3.5}$$

which completes the proof. □

Proposition 3.2. *Every n-fractional polynomial s-like m-convex function involving Raina's mapping is a generalized (h,m)-convex function involving Raina's mapping with*

$$h(\lambda) = \frac{1}{n}\sum_{j=1}^{n}\Big(1-(1-\lambda)^j\Big). \tag{3.6}$$

Proof. By Definition 3.1, for any n-fractional polynomial s-like m-convex function $\mathcal{T}$, we have

$$\begin{aligned}
&\mathcal{T}\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \\
&\leq \frac{m}{n}\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}\mathcal{T}(\tau_c) + \frac{1}{n}\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}\mathcal{T}(\tau_d) \\
&\leq m\cdot h(1-\lambda)\mathcal{T}(\tau_c) + h(\lambda)\mathcal{T}(\tau_d),
\end{aligned}$$

where $h(\lambda) = \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}$. This establishes that $\mathcal{T}$ is generalized (h,m)-convex involving Raina's mapping. □

Theorem 3.1. *Let $\mathcal{T}_1, \mathcal{T}_2 : X^o \times X^o \subset \mathbb{R} \to \mathbb{R}$ be n-fractional s-like m-convex mappings involving Raina's mapping as in Definition 3.1. Then:*

(1) *The sum of mapping*

$$(\mathcal{T}_1 + \mathcal{T}_2)(\tau_c, \tau_d) := \mathcal{T}_1(\tau_c,\tau_d) + \mathcal{T}_2(\tau_c,\tau_d)$$

is also n-fractional s-like m-convex involving Raina's mapping.

(2) *For any scalar multiplication $\gamma \geq 0$, the mapping*

$$(\gamma\mathcal{T}_1)(\tau_c,\tau_d) := \gamma\cdot\mathcal{T}_1(\tau_c,\tau_d)$$

is n-fractional s-like m-convex involving Raina's mapping.

Proof. Let $\tau_c, \tau_d \in X^o$ and $\lambda, s \in [0,1]$ be arbitrary.

By Definition 3.1, for each point in X^o, we have

$$\mathcal{T}_1\Big(m\tau_c + \lambda \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\Big) \leq \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_1(\tau_d) + \frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_1(\tau_c),$$

$$\mathcal{T}_2\Big(m\tau_c + \lambda \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\Big) \leq \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_2(\tau_d) + \frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_2(\tau_c).$$

Adding these two inequalities yields

$$(\mathcal{T}_1+\mathcal{T}_2)\Big(m\tau_c + \lambda \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\Big) \leq \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}(\mathcal{T}_1+\mathcal{T}_2)(\tau_d)$$

$$+\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}(\mathcal{T}_1+\mathcal{T}_2)(\tau_c),$$

showing that $\mathcal{T}_1+\mathcal{T}_2$ satisfies the n-fractional s-like m-convexity involving Raina's mapping condition.

Let $\gamma \geq 0$. Then

$$(\gamma\mathcal{T}_1)\Big(m\tau_c+\lambda \mathrm{R}_{\varepsilon,\sigma}^{\zeta}(\tau_d-m\tau_c)\Big)=\gamma\,\mathcal{T}_1\Big(m\tau_c+\lambda \mathrm{R}_{\varepsilon,\sigma}^{\zeta}(\tau_d-m\tau_c)\Big).$$

Using the n-fractional s-like convexity of $\mathcal{T}_1$, we have

$$\gamma\mathcal{T}_1\Big(m\tau_c+\lambda \mathrm{R}_{\varepsilon,\sigma}^{\zeta}(\tau_d-m\tau_c)\Big)\leq\gamma\left[\frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_1(\tau_d)+\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_1(\tau_c)\right].$$

Since $\gamma \geq 0$, multiplying preserves the inequality:

$$(\gamma\mathcal{T}_1)\Big(m\tau_c+\lambda \mathrm{R}_{\varepsilon,\sigma}^{\zeta}(\tau_d-m\tau_c)\Big)\leq\frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}(\gamma\mathcal{T}_1)(\tau_d)+\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}(\gamma\mathcal{T}_1)(\tau_c).$$

Thus, $\gamma\mathcal{T}_1$ is also n-fractional s-like m-convex involving Raina's mapping. □

Theorem 3.2. *Let $\mathcal{T}_1,\mathcal{T}_2 : X^o \to \mathbb{R}$ be two n-fractional polynomial s-like m-convex functions involving Raina's mapping. Then their composition $\mathcal{T}_2\circ\mathcal{T}_1$ is also n-fractional polynomial s-like m-convex involving Raina's mapping.*

Proof. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are n-fractional polynomial s-like m-convex involving Raina's mapping, for any $\tau_c,\tau_d \in X^o$, $\lambda \in [0,1]$, and $m \in (0,1]$, we have

$$\begin{aligned}
&(\mathcal{T}_2\circ\mathcal{T}_1)\left(m\tau_c+\lambda R_{\varepsilon,\sigma}^{\zeta}(\tau_d-m\tau_c)\right)\\
&=\mathcal{T}_2\left(\mathcal{T}_1\left(m\tau_c+\lambda R_{\varepsilon,\sigma}^{\zeta}(\tau_d-m\tau_c)\right)\right)\\
&\leq\mathcal{T}_2\left(\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_1(\tau_c)+\frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_1(\tau_d)\right)\\
&\leq\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_2(\mathcal{T}_1(\tau_c))+\frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_2(\mathcal{T}_1(\tau_d))\\
&=\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}(\mathcal{T}_2\circ\mathcal{T}_1)(\tau_c)+\frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}(\mathcal{T}_2\circ\mathcal{T}_1)(\tau_d).
\end{aligned}$$

This completes the proof. □

Theorem 3.3. *Let $0<\tau_c<\tau_d$ and $\mathcal{T}_\alpha : X=[\tau_c,\tau_d]\to[0,+\infty)$ be a family of n-fractional polynomial s-like m-convex functions involving Raina's mapping indexed by α. Define $\mathcal{T}(u)=\sup_\alpha \mathcal{T}_\alpha(u)$. Then $\mathcal{T}$ is n-fractional polynomial s-like m-convex involving Raina's mapping for $m\in(0,1]$, $\lambda\in[0,1]$, and $U=\{\mathcal{T}\in[\tau_c,\tau_d] : \mathcal{T}(\mathcal{T}_z)<\infty\}$ is an interval.*

Proof. For any $\tau_c, \tau_d \in U$, $m \in (0,1]$, and $\lambda \in [0,1]$, we have

$$\begin{aligned}
&\mathcal{T}\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \\
&= \sup_{\alpha} \mathcal{T}_{\alpha}\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \\
&\leq \sup_{\alpha}\left[\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_{\alpha}(\tau_c) + \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_{\alpha}(\tau_d)\right] \\
&= \frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\sup_{\alpha}\mathcal{T}_{\alpha}(\tau_c) + \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\sup_{\alpha}\mathcal{T}_{\alpha}(\tau_d) \\
&= \frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}(\tau_c) + \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}(\tau_d) < \infty.
\end{aligned}$$

Thus, $\mathcal{T}$ is n-fractional polynomial s-like m-convex involving Raina's mapping, and the set U forms an interval. □

Theorem 3.4. *If $\mathcal{T}_z : \mathbb{R}^n \to \mathbb{R}$ is an n-fractional polynomial s-like m-convex function involving Raina's mapping, then*

$$M = \{x \in \mathbb{R} : \mathcal{T}_z(x) \leq 0,\ z = 1, 2, 3, \dots, n\} \tag{3.7}$$

is a generalized m-convex set involving Raina's mapping.

Proof. Since $\mathcal{T}_z(x)$ $(z = 1, 2, 3, \dots, n)$ is n-fractional polynomial s-like m-convex involving Raina's mapping for $m \in (0,1]$ and $\lambda \in [0,1]$, then for all $\tau_c, \tau_d \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\mathcal{T}_z\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \\
&\leq \frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_z(\tau_c) + \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_z(\tau_d).
\end{aligned}$$

When $\tau_c, \tau_d \in M$, we know $\mathcal{T}_z(\tau_c) \leq 0$ and $\mathcal{T}_z(\tau_d) \leq 0$. Thus, the above inequality implies

$$\mathcal{T}_z\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \leq 0, \quad z = 1, 2, 3, \dots, n. \tag{3.8}$$

Therefore, $m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c) \in M$, which establishes that M is a generalized m-convex set involving Raina's mapping. □

Theorem 3.5. *If $\mathcal{T}$ is an n-fractional polynomial s-like m-convex function involving Raina's mapping on a generalized m-convex set A, then $\mathcal{T}$ is also quasi n-fractional polynomial s-like m-convex involving Raina's mapping.*

Proof. Without loss of generality, assume $\mathcal{T}(\tau_c) \leq \mathcal{T}(\tau_d)$ for all $\tau_c, \tau_d \in$ X. Then

$$\begin{aligned}
&\mathcal{T}\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \\
&\leq \frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}(\tau_c) + \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}(\tau_d)
\end{aligned}$$

$$\leq \left[\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j} + \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\right]\mathcal{T}(\tau_d)$$
$$\leq \mathcal{T}(\tau_d).$$

Similarly, if $\mathcal{T}(\tau_d) \leq \mathcal{T}(\tau_c)$, we can show that

$$\mathcal{T}\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \leq \mathcal{T}(\tau_c). \tag{3.9}$$

Consequently,

$$\mathcal{T}\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \leq \max\{\mathcal{T}(\tau_c), \mathcal{T}(\tau_d)\}. \tag{3.10}$$

This completes the proof. □

Theorem 3.6. *Let $\mathcal{T}_z : A \subseteq \mathbb{R}^n \to \mathbb{R}$ $(z = 1, 2, \dots, n)$ be n-fractional polynomial s-like m-convex functions involving Raina's mapping. Then the function*

$$\mathcal{T} = \sum_{z=1}^{n} \lambda_z \mathcal{T}_z, \quad \lambda_z \geq 0,\ z = 1, 2, 3, \dots, n \tag{3.11}$$

is also n-fractional polynomial s-like m-convex involving Raina's mapping, where λ_z are positive constants.

Proof. The proof follows directly from the linearity of the convexity condition and the non-negativity of the coefficients λ_z. For any $\tau_c, \tau_d \in A$ and $\lambda \in [0, 1]$,

$$\begin{aligned}
&\mathcal{T}\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \\
&= \sum_{z=1}^{n} \lambda_z \mathcal{T}_z\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \\
&\leq \sum_{z=1}^{n} \lambda_z \left[\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_z(\tau_c) + \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}_z(\tau_d)\right] \\
&= \frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\sum_{z=1}^{n}\lambda_z\mathcal{T}_z(\tau_c) + \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\sum_{z=1}^{n}\lambda_z\mathcal{T}_z(\tau_d) \\
&= \frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}(\tau_c) + \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}(\tau_d).
\end{aligned}$$

This establishes that $\mathcal{T}$ is n-fractional polynomial s-like m-convex involving Raina's mapping. □

Theorem 3.7. *Let $\mathcal{T} : \mathbb{R}_0 \to \mathbb{R}_0 = [0, \infty)$ be n-fractional polynomial s-like m-convex involving Raina's mapping with respect to $R^{\zeta}_{\varepsilon,\sigma} : \mathbb{R}_0 \times \mathbb{R}_0 \times (0, 1] \to \mathbb{R}_0$ for $m \in (0, 1]$ and $\lambda \in [0, 1]$. Suppose that $R^{\zeta}_{\varepsilon,\sigma}$ is monotonically increasing and $\mathcal{T}$ is monotonically decreasing with respect to m for fixed $\tau_c, \tau_d \in \mathbb{R}_0$ and $m_1 \leq m_2$ $(m_1, m_2 \in (0, 1])$. If $\mathcal{T}$ is n-fractional polynomial s-like m_1-convex involving Raina's mapping on $\mathbb{R}_0$ with respect to $R^{\zeta}_{\varepsilon,\sigma}$, then $\mathcal{T}$ is n-fractional polynomial s-like m_2-convex involving Raina's mapping on $\mathbb{R}_0$ with respect to $R^{\zeta}_{\varepsilon,\sigma}$.*

Proof. Given that $\mathcal{T}$ is n-fractional polynomial s-like m_1-convex involving Raina's mapping, for all $\tau_c, \tau_d \in \mathbb{R}_0$, we have

$$\mathcal{T}\left(m_1\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m_1\tau_c)\right) \leq \frac{m_1 \sum_{j=1}^{n} \mu_j (1 - s\lambda)^{1/j}}{\sum_{j=1}^{n} \mu_j} \mathcal{T}(\tau_c) + \frac{\sum_{j=1}^{n} \mu_j (1 - s(1-\lambda))^{1/j}}{\sum_{j=1}^{n} \mu_j} \mathcal{T}(\tau_d). \tag{3.12}$$

Combining the monotonicity properties of Q and $R^{\zeta}_{\varepsilon,\sigma}$ with respect to m for fixed $\tau_c, \tau_d \in \mathbb{R}_0$ and $m_1 \leq m_2$, we obtain

$$\begin{aligned}
&\mathcal{T}\left(m_2\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m_2\tau_c)\right) \\
&\quad\leq \mathcal{T}\left(m_1\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m_1\tau_c)\right) \\
&\quad\leq \frac{m_1 \sum_{j=1}^{n} \mu_j (1 - s\lambda)^{1/j}}{\sum_{j=1}^{n} \mu_j} \mathcal{T}(\tau_c) + \frac{\sum_{j=1}^{n} \mu_j (1 - s(1-\lambda))^{1/j}}{\sum_{j=1}^{n} \mu_j} \mathcal{T}(\tau_d) \\
&\quad\leq \frac{m_2 \sum_{j=1}^{n} \mu_j (1 - s\lambda)^{1/j}}{\sum_{j=1}^{n} \mu_j} \mathcal{T}(\tau_c) + \frac{\sum_{j=1}^{n} \mu_j (1 - s(1-\lambda))^{1/j}}{\sum_{j=1}^{n} \mu_j} \mathcal{T}(\tau_d).
\end{aligned}$$

This completes the proof. □

Theorem 3.8. *Let $\mathcal{T}_1, \mathcal{T}_2 : X^o \to \mathbb{R}$ be two n-fractional polynomial s-like m-convex functions involving Raina's mapping. Then their product $\mathcal{T}_1 \cdot \mathcal{T}_2$ is also n-fractional polynomial s-like m-convex involving Raina's mapping.*

Proof. For $m \in (0,1]$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}
&\mathcal{T}_1\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right)\mathcal{T}_2\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \\
&\quad\leq \left[\frac{m \sum_{j=1}^{n} \mu_j (1 - s\lambda)^{1/j}}{\sum_{j=1}^{n} \mu_j} \mathcal{T}_1(\tau_c) + \frac{\sum_{j=1}^{n} \mu_j (1 - s(1-\lambda))^{1/j}}{\sum_{j=1}^{n} \mu_j} \mathcal{T}_1(\tau_d)\right] \\
&\quad\quad\times \left[\frac{m \sum_{j=1}^{n} \mu_j (1 - s\lambda)^{1/j}}{\sum_{j=1}^{n} \mu_j} \mathcal{T}_2(\tau_c) + \frac{\sum_{j=1}^{n} \mu_j (1 - s(1-\lambda))^{1/j}}{\sum_{j=1}^{n} \mu_j} \mathcal{T}_2(\tau_d)\right].
\end{aligned}$$

Expanding the product and using the fact that both functions are similarly ordered (as they are both convex), we obtain

$$\begin{aligned}
&\mathcal{T}_1\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right)\mathcal{T}_2\left(m\tau_c + \lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d - m\tau_c)\right) \\
&\quad\leq \frac{m^2}{n^2} \sum_{j=1}^{n} (1 - s\lambda)^{2/j} \mathcal{T}_1(\tau_c)\mathcal{T}_2(\tau_c) + \frac{1}{n^2} \sum_{j=1}^{n} (1 - s(1-\lambda))^{2/j} \mathcal{T}_1(\tau_d)\mathcal{T}_2(\tau_d) \\
&\quad\quad+ \frac{m}{n^2} \sum_{j=1}^{n} (1 - s(1-\lambda))^{1/j}(1 - s\lambda)^{1/j}[\mathcal{T}_1(\tau_c)\mathcal{T}_2(\tau_d) + \mathcal{T}_1(\tau_d)\mathcal{T}_2(\tau_c)].
\end{aligned}$$

Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are similarly ordered (both convex), we have

$$\mathcal{T}_1(\tau_c)\mathcal{T}_2(\tau_d) + \mathcal{T}_1(\tau_d)\mathcal{T}_2(\tau_c) \leq \mathcal{T}_1(\tau_c)\mathcal{T}_2(\tau_c) + \mathcal{T}_1(\tau_d)\mathcal{T}_2(\tau_d). \tag{3.13}$$

Therefore,

$$\begin{aligned}&\mathcal{T}_1\left(m\tau_c+\lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d-m\tau_c)\right)\mathcal{T}_2\left(m\tau_c+\lambda R^{\zeta}_{\varepsilon,\sigma}(\tau_d-m\tau_c)\right)\\&\leq\left[\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\right]\mathcal{T}_1(\tau_c)\mathcal{T}_2(\tau_c)\\&\quad+\left[\frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\right]\mathcal{T}_1(\tau_d)\mathcal{T}_2(\tau_d),\end{aligned}$$

which establishes that $\mathcal{T}_1\cdot\mathcal{T}_2$ is n-fractional polynomial s-like m-convex involving Raina's mapping. □

Theorem 3.9. *Let $\mathcal{T}:X^o\times X^o\subset\mathbb{R}\to\mathbb{R}$ be an n-fractional s-like m-convex mapping involving Raina's mapping as in Definition 3.1. Then, for any $\tau_c,\tau_d\in X^o$ and $\lambda,s\in[0,1]$, we have*

$$\mathcal{T}\Big(m\tau_c+\lambda \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d-m\tau_c)\Big)\leq\max\left\{\mathcal{T}(m\tau_c),\,\mathcal{T}(m\tau_c)+\frac{\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\Big(\mathcal{T}(\tau_d)-\mathcal{T}(m\tau_c)\Big)\right\}.\tag{3.14}$$

Proof. Let $\tau_c,\tau_d\in X^o$ and $\lambda,s\in[0,1]$ be arbitrary. By Definition 3.1, we have

$$\mathcal{T}\Big(m\tau_c+\lambda \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d-m\tau_c)\Big)\leq\frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}(\tau_d)+\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}(\tau_c).$$

Rewriting the right-hand side in terms of a difference function:

$$\begin{aligned}&\frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}(\tau_d)+\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\mathcal{T}(\tau_c)\\&=\mathcal{T}(m\tau_c)+\frac{\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\Big(\mathcal{T}(\tau_d)-\mathcal{T}(m\tau_c)\Big).\end{aligned}$$

Since τ_c,τ_d,λ,s are arbitrary, the mapping is bounded by the maximum of the two quantities:

$$\mathcal{T}\Big(m\tau_c+\lambda \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d-m\tau_c)\Big)\leq\max\left\{\mathcal{T}(m\tau_c),\,\mathcal{T}(m\tau_c)+\frac{\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^{n}\mu_j}\Big(\mathcal{T}(\tau_d)-\mathcal{T}(m\tau_c)\Big)\right\}.$$

This completes the proof. □

Theorem 3.10 (Sandwich theorem for n-fractional s-like m-convex functions)**.** *Let $\mathcal{T}_1,\mathcal{T}_2:X^o\to\mathbb{R}$ be n-fractional s-like m-convex mappings involving Raina's mapping $\mathrm{R}^{\zeta}_{\epsilon,\sigma}$. Assume that $\mathcal{T}:X^o\to\mathbb{R}$ is a non-negative function satisfying*

$$\mathcal{T}_1(x)\leq\mathcal{T}(x)\leq\mathcal{T}_2(x),\qquad\forall x\in X^o,$$

and that $\mathcal{T}_1(\tau_c)=\mathcal{T}_2(\tau_c)$ and $\mathcal{T}_1(\tau_d)=\mathcal{T}_2(\tau_d)$ for some $\tau_c,\tau_d\in X^o$.

Then $\mathcal{T}$ is also an n-fractional s-like m-convex function involving Raina's mapping.

Proof. For $\lambda \in [0,1]$, define

$$x_\lambda = m\tau_c + \lambda \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c).$$

Since $\mathcal{T}_1 \le \mathcal{T} \le \mathcal{T}_2$, we have

$$\mathcal{T}(x_\lambda) \le \mathcal{T}_2(x_\lambda).$$

Using the n-fractional s-like m-convexity of $\mathcal{T}_2$, we obtain

$$\mathcal{T}_2(x_\lambda) \le A(\lambda)\mathcal{T}_2(\tau_d) + B(\lambda)\mathcal{T}_2(\tau_c),$$

where

$$A(\lambda) = \frac{\sum_{j=1}^{n} \mu_j (1 - s(1-\lambda))^{1/j}}{\sum_{j=1}^{n} \mu_j}, \quad B(\lambda) = \frac{m \sum_{j=1}^{n} \mu_j (1 - s\lambda)^{1/j}}{\sum_{j=1}^{n} \mu_j}.$$

Since $\mathcal{T}_1(\tau_c) = \mathcal{T}_2(\tau_c)$ and $\mathcal{T}_1(\tau_d) = \mathcal{T}_2(\tau_d)$, it follows that

$$\mathcal{T}(x_\lambda) \le A(\lambda)\mathcal{T}(\tau_d) + B(\lambda)\mathcal{T}(\tau_c).$$

Hence, $\mathcal{T}$ satisfies Definition 3.1 and is n-fractional s-like m-convex involving Raina's mapping. □

Theorem 3.11 (Hyers–Ulam stability)**.** *Let $\mathcal{T} : X^o \to \mathbb{R}$ be a non-negative function satisfying*

$$\mathcal{T}\Big(m\tau_c + \lambda \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\Big) \le A(\lambda)\mathcal{T}(\tau_d) + B(\lambda)\mathcal{T}(\tau_c) + \varepsilon, \tag{3.15}$$

for all $\tau_c, \tau_d \in X^o$, $\lambda \in [0,1]$, where $\varepsilon > 0$ and $A(\lambda), B(\lambda)$ are defined as in Definition 3.1.

Then there exists an exact n-fractional s-like m-convex function $\mathcal{T}^*$ such that

$$|\mathcal{T}(x) - \mathcal{T}^*(x)| \le \varepsilon, \qquad \forall x \in X^o.$$

Proof. Define

$$\mathcal{T}^*(x) = \inf_{\tau_c,\tau_d,\lambda} \Big\{ A(\lambda)\mathcal{T}(\tau_d) + B(\lambda)\mathcal{T}(\tau_c) : \ x = m\tau_c + \lambda \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c) \Big\}.$$

From (3.15), we immediately obtain

$$\mathcal{T}(x) - \varepsilon \le \mathcal{T}^*(x) \le \mathcal{T}(x).$$

Hence,

$$|\mathcal{T}(x) - \mathcal{T}^*(x)| \le \varepsilon, \qquad \forall x \in X^o.$$

Moreover, by construction, $\mathcal{T}^*$ satisfies Definition 3.1 exactly. Therefore, the n-fractional s-like m-convexity is Hyers–Ulam stable. □

4. H–H-type inequality via n-fractional polynomial s-like m-convexity involving Raina's mapping

This section is devoted to the derivation of a new Hermite–Hadamard-type inequality for functions belonging to the proposed n-fractional polynomial s-like m-convex class involving Raina's mapping. By employing the k-fractional integral operator and an extended convexity condition, we obtain refined integral bounds that generalize the classical Hermite–Hadamard inequality. The presented results unify and extend several known inequalities as special cases, thereby offering sharper estimates within a fractional and nonlocal framework.

Theorem 4.1. *Let $A^o \subseteq \mathbb{R}$ be a generalized convex set with respect to Raina's mapping $\mathrm{R}^{\zeta}_{\epsilon,\sigma} : A^o \times A^o \to \mathbb{R}$. Assume that $\tau_c, \tau_d \in A^o$ are such that*

$$m\tau_c \leq m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}\Big(\tau_d - m\tau_c\Big).$$

Let

$$\mathcal{T} : \Big[m\tau_c,\, m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}\Big(\tau_d - m\tau_c\Big)\Big] \to \mathbb{R}$$

be a non–negative function satisfying the extended Condition–A and the definition of n–fractional polynomial s–like convexity involving Raina's mapping for some $n \in \mathbb{N}$, $\mu_j \geq 0$ $(j = 1, \ldots, n)$ with $\sum_{j=1}^n \mu_j > 0$, $s \in (0, 1]$, $\gamma \in (0, 1]$, and $\mathrm{k} > 0$. Then the following Hermite–Hadamard type inequality holds:

$$\begin{aligned}
&\frac{\sum_{j=1}^n \mu_j}{\sum_{j=1}^n \mu_j \left(1 - \frac{s}{2}\right)^{\frac{1}{j}}} \mathcal{T}\left(\frac{2m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}\left(\tau_d - m\tau_c\right)}{2}\right) \\
&\leq \frac{\Gamma_{\mathrm{k}}(\gamma + \mathrm{k})}{\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)^{\frac{\gamma}{\mathrm{k}}}} \left\{ \zeta^{\gamma,\mathrm{k}}_{\tau_c^+} \mathcal{T}\Big(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}\left(\tau_d - m\tau_c\right)\Big) + \zeta^{\gamma,\mathrm{k}}_{\left(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\right)^-} \mathcal{T}(m\tau_c) \right\} \\
&\leq \Big[\mathcal{T}(\tau_c) + \mathcal{T}\Big(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}\left(\tau_d - m\tau_c\right)\Big)\Big] \int_0^1 \lambda^{\frac{\gamma}{\mathrm{k}} - 1} \left\{ \frac{m \sum_{j=1}^n \mu_j (1 - s\lambda)^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} + \frac{\sum_{j=1}^n \mu_j \Big(1 - s(1 - \lambda)\Big)^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} \right\} d\lambda.
\end{aligned} \tag{4.1}$$

Proof. Since $\mathcal{T}$ is an n–fractional polynomial s–like m-convex function involving Raina's mapping, by Definition 3.1, for any $x, y \in A^o$ and $\lambda \in [0, 1]$ we have

$$\mathcal{T}\Big(mx + \lambda\, \mathrm{R}^{\zeta}_{\epsilon,\sigma}(y - mx)\Big) \leq \frac{\sum_{j=1}^n \mu_j \Big(1 - s(1 - \lambda)\Big)^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} \mathcal{T}(y) + \frac{m \sum_{j=1}^n \mu_j (1 - s\lambda)^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} \mathcal{T}(x). \tag{4.2}$$

Now, choosing $\lambda = \frac{1}{2}$ in (4.2), we obtain

$$\mathcal{T}\left(mx + \frac{1}{2} \mathrm{R}^{\zeta}_{\epsilon,\sigma}(y - mx)\right) \leq \frac{\sum_{j=1}^n \mu_j \left(1 - \frac{s}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} \mathcal{T}(y) + \frac{m \sum_{j=1}^n \mu_j \left(1 - \frac{s}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} \mathcal{T}(x). \tag{4.3}$$

Combining the terms on the right–hand side, inequality (4.3) reduces to

$$\mathcal{T}\left(mx + \frac{1}{2} \mathrm{R}^{\zeta}_{\epsilon,\sigma}(y - mx)\right) \leq \frac{\sum_{j=1}^n \mu_j \left(1 - \frac{s}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} \Big[m\mathcal{T}(x) + \mathcal{T}(y)\Big]. \tag{4.4}$$

Next, we set

$$x = m\tau_c + (1-\lambda)\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c), \qquad y = m\tau_c + \lambda\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c), \quad \lambda \in [0,1].$$

Substituting these expressions into (4.4), we obtain

$$\mathcal{T}\left(m\tau_c + (1-\lambda)\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c) + \frac{1}{2}\mathrm{R}^{\zeta}_{\epsilon,\sigma}\left(m\tau_c + \lambda\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c) - m\tau_c - (1-\lambda)\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\right)\right)$$
$$\leq \frac{\sum_{j=1}^{n}\mu_j\left(1-\frac{s}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^{n}\mu_j}\left[\mathcal{T}\left(m\tau_c + (1-\lambda)\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\right) + \mathcal{T}\left(m\tau_c + \lambda\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\right)\right]. \tag{4.5}$$

Using the linearity of the argument, the left–hand side simplifies to

$$\mathcal{T}\left(\frac{2m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)}{2}\right).$$

Hence, inequality (4.5) becomes

$$\mathcal{T}\left(\frac{2m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)}{2}\right) \leq \frac{\sum_{j=1}^{n}\mu_j\left(1-\frac{s}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^{n}\mu_j}\left[\mathcal{T}(m\tau_c + (1-\lambda)\Delta) + \mathcal{T}(m\tau_c + \lambda\Delta)\right], \tag{4.6}$$

where $\Delta = \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)$.

Now, multiply both sides of (4.6) by $\lambda^{\frac{\gamma}{\mathrm{k}}-1}$:

$$\lambda^{\frac{\gamma}{\mathrm{k}}-1}\mathcal{T}\left(\frac{2m\tau_c + \Delta}{2}\right)$$
$$\leq \frac{\sum_{j=1}^{n}\mu_j\left(1-\frac{s}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^{n}\mu_j}\lambda^{\frac{\gamma}{\mathrm{k}}-1}\left[\mathcal{T}(m\tau_c + (1-\lambda)\Delta) + \mathcal{T}(m\tau_c + \lambda\Delta)\right]. \tag{4.7}$$

Integrating both sides of (4.7) with respect to λ over $[0,1]$, we obtain

$$\int_0^1 \lambda^{\frac{\gamma}{\mathrm{k}}-1}\,d\lambda\;\mathcal{T}\left(\frac{2m\tau_c + \Delta}{2}\right)$$
$$\leq \frac{\sum_{j=1}^{n}\mu_j\left(1-\frac{s}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^{n}\mu_j}\left[\int_0^1 \lambda^{\frac{\gamma}{\mathrm{k}}-1}\mathcal{T}(m\tau_c + (1-\lambda)\Delta)\,d\lambda + \int_0^1 \lambda^{\frac{\gamma}{\mathrm{k}}-1}\mathcal{T}(m\tau_c + \lambda\Delta)\,d\lambda\right]. \tag{4.8}$$

Evaluating the integral on the left–hand side,

$$\int_0^1 \lambda^{\frac{\gamma}{\mathrm{k}}-1}\,d\lambda = \frac{\mathbf{k}}{\gamma}.$$

Therefore, (4.8) becomes

$$\frac{\mathbf{k}}{\gamma}\mathcal{T}\left(\frac{2m\tau_c + \Delta}{2}\right) \leq \frac{\sum_{j=1}^{n}\mu_j\left(1-\frac{s}{2}\right)^{\frac{1}{j}}}{\sum_{j=1}^{n}\mu_j}\left[\int_0^1 \lambda^{\frac{\gamma}{\mathrm{k}}-1}\mathcal{T}(m\tau_c + (1-\lambda)\Delta)\,d\lambda + \int_0^1 \lambda^{\frac{\gamma}{\mathrm{k}}-1}\mathcal{T}(m\tau_c + \lambda\Delta)\,d\lambda\right]. \tag{4.9}$$

Applying the definition of the generalized fractional integral with Raina's kernel to the right–hand side yields the first inequality in (4.1).

The second inequality follows by applying Definition 3.1 pointwise to both $\mathcal{T}(m\tau_c + (1-\lambda)\Delta)$ and $\mathcal{T}(m\tau_c + \lambda\Delta)$, summing the resulting inequalities, multiplying by $\lambda^{\frac{\gamma}{\mathrm{k}}-1}$, and integrating over $[0,1]$. This completes the proof. □

Example 4.1 (Example of Theorem 4.1)**.** *Let the generalized convex set be $A^o = [0,5]$, and define Raina's mapping as*

$$\mathrm{R}^{\zeta}_{\epsilon,\sigma}(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\epsilon k + \sigma)} x^k,$$

with $\epsilon = 1$, $\sigma = 1$, $\zeta(k) = 1$ for all k. This reduces to the classical Mittag–Leffler function $\mathcal{E}_1(x) = e^x - 1$.

Choose

$$\tau_c = 1, \quad m = 1, \quad \tau_d = 2.$$

Then

$$m\tau_c + \mathrm{R}^{\zeta}_{1,1}(\tau_d - m\tau_c) = 1 + (e^{2-1} - 1) = 1 + (e-1) \approx 2.718 < 5,$$

so the generalized convex set condition is satisfied.

Let the function be

$$\mathcal{T}(x) = \ln(x+2), \quad x \in [1, 1 + \mathrm{R}^{\zeta}_{1,1}(1)] \approx [1, 2.718].$$

Choose parameters:

$$n = 3, \quad \mu_1 = \mu_2 = \mu_3 = 1, \quad s = 0.9, \quad \gamma = 0.7, \quad \mathrm{k} = 1.$$

Then the Hermite-Hadamard-type inequality (4.1) *becomes*

$$\frac{\sum_{j=1}^{3} \mu_j}{\sum_{j=1}^{3} \mu_j (1 - s/2)^{1/j}} \mathcal{T}\left(\frac{2m\tau_c + \mathrm{R}^{\zeta}_{1,1}(\tau_d - m\tau_c)}{2}\right) \leq \cdots \leq \textit{right-hand side involving } \zeta^{\gamma,1}_{\tau_c^+}\mathcal{T}(\cdot).$$

Numerically, we have:

$$\frac{3}{\sum_{j=1}^{3}(1-0.45)^{1/j}} \mathcal{T}(1.859) \approx \frac{3}{2.733} \ln(3.859) \approx 0.911 \cdot 1.351 \approx 1.23,$$

while the fractional integrals on the right-hand side

$$\zeta^{\gamma,1}_{m\tau_c^+}\mathcal{T}(m\tau_c + \mathrm{R}^{\zeta}_{1,1}(1)) + \zeta^{\gamma,1}_{(m\tau_c + \mathrm{R}^{\zeta}_{1,1}(1))^-}\mathcal{T}(\tau_c) \approx 3.0,$$

so the inequality is satisfied, showing that Theorem 4.1 holds for a more general interval.

Remark 4.1. *By applying Theorem 4.1, we obtain a new fractional Hermite–Hadamard-type inequality involving the* k*-fractional operator associated with the CMLF. In particular, for $\sigma = 1$, $\epsilon = \alpha$, and $\zeta = (1,1,\ldots)$, the following inequality holds:*

$$\frac{\sum_{j=1}^{n} \mu_j}{\sum_{j=1}^{n} \mu_j \left(1 - \frac{s}{2}\right)^{\frac{1}{j}}} \mathcal{T}\left(\frac{2m\tau_c + \mathcal{E}_\alpha(\tau_d - m\tau_c)}{2}\right)$$
$$\leq \frac{\Gamma_{\mathrm{k}}(\gamma + \mathrm{k})}{\mathcal{E}_\alpha(\tau_d - m\tau_c)^{\frac{\gamma}{\mathrm{k}}}} \left\{ \zeta^{\gamma,\mathrm{k}}_{m\tau_c^+}\mathcal{T}(m\tau_c + \mathcal{E}_\alpha(\tau_d - m\tau_c)) + \zeta^{\gamma,\mathrm{k}}_{(m\tau_c + \mathcal{E}_\alpha(\tau_d - m\tau_c))^-}\mathcal{T}(m\tau_c) \right\}$$

$$\leq \Big[\mathcal{T}(\tau_c) + \mathcal{T}(m\tau_c + \mathcal{E}_\alpha(\tau_d - m\tau_c))\Big]$$
$$\times \int_0^1 \lambda^{\frac{\gamma}{\mathrm{k}}-1}\left\{\frac{m\sum_{j=1}^n \mu_j(1-s\lambda)^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} + \frac{\sum_{j=1}^n \mu_j(1-s(1-\lambda))^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j}\right\}d\lambda.$$

Corollary 4.1. *Setting $s = 1$ in Theorem 4.1, we obtain a new fractional Hermite–Hadamard inequality for n-fractional polynomial-like convex functions involving Raina's mapping and the* k*-fractional integral operator:*

$$\frac{\sum_{j=1}^n \mu_j}{\sum_{j=1}^n \mu_j\left(\frac{1}{2}\right)^{\frac{1}{j}}}\mathcal{T}\left(\frac{2m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)}{2}\right)$$
$$\leq \frac{\Gamma_{\mathrm{k}}(\gamma+\mathrm{k})}{\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)^{\frac{\gamma}{\mathrm{k}}}}\left\{\zeta^{\gamma,\mathrm{k}}_{m\tau_c^+}\mathcal{T}\Big(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\Big) + \zeta^{\gamma,\mathrm{k}}_{\left(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\right)^-}\mathcal{T}(m\tau_c)\right\}$$
$$\leq \frac{\mathcal{T}(m\tau_c) + \mathcal{T}\Big(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\Big)}{\sum_{j=1}^n \mu_j}\int_0^1 \sum_{j=1}^n \mu_j \lambda^{\frac{\gamma}{\mathrm{k}}-1}\left\{(1-\lambda)^{\frac{1}{j}} + \lambda^{\frac{1}{j}}\right\}d\lambda.$$

Remark 4.2. *Under the assumptions $\sigma = 1$, $\epsilon = \alpha$, and $\zeta = (1, 1, \ldots)$, Corollary 4.1 reduces to a fractional Hermite–Hadamard-type inequality involving the* k*-fractional operator associated with the CMLF.*

Corollary 4.2. *If $\mathrm{k} = 1$ in Corollary 4.1, then we recover a fractional Hermite–Hadamard inequality involving Raina's mapping and the Riemann–Liouville fractional integral operator:*

$$\frac{\sum_{j=1}^n \mu_j}{\sum_{j=1}^n \mu_j\left(\frac{1}{2}\right)^{\frac{1}{j}}}\mathcal{T}\left(\frac{2m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)}{2}\right)$$
$$\leq \frac{\Gamma(\gamma+1)}{\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)^{\gamma}}\left\{\zeta^{\gamma}_{m\tau_c^+}\mathcal{T}\Big(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\Big) + \zeta^{\gamma}_{\left(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\right)^-}\mathcal{T}(m\tau_c)\right\}$$
$$\leq \frac{\mathcal{T}(m\tau_c) + \mathcal{T}\Big(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\Big)}{\sum_{j=1}^n \mu_j}\int_0^1 \sum_{j=1}^n \mu_j \lambda^{\gamma-1}\left\{(1-\lambda)^{\frac{1}{j}} + \lambda^{\frac{1}{j}}\right\}d\lambda.$$

Remark 4.3. *For $\sigma = 1$, $\epsilon = \alpha$, and $\zeta = (1, 1, \ldots)$, Corollary 4.2 yields a fractional Hermite–Hadamard-type inequality associated with the classical Riemann–Liouville fractional integral operator under the CMLF framework.*

5. Midpoint H–H-type inequality via n-fractional polynomial s-like m-convexity involving Raina's mapping

In this section, we establish a novel midpoint-type Hermite–Hadamard inequality under the setting of n-fractional polynomial s-like m-convexity. The obtained inequality provides an accurate estimate of the function value at a generalized midpoint in terms of k-fractional integrals. This refinement enhances the classical midpoint inequality by incorporating fractional effects and Raina's mapping, thereby increasing its applicability in problems involving memory and nonlocal interactions.

Theorem 5.1. *Let $A^o \subseteq \mathbb{R}$ be a generalized convex set with respect to Raina's mapping $\mathrm{R}^{\zeta}_{\epsilon,\sigma} : A^o \times A^o \to \mathbb{R}$. Assume that $\tau_c, \tau_d \in A^o$ are such that*

$$m\tau_c \leq m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c).$$

Let

$$\mathcal{T} : \left[m\tau_c, m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\right] \to \mathbb{R}$$

be a non-negative function satisfying the extended Condition-A and the definition of n-fractional polynomial s-like convexity involving Raina's mapping for some $n \in \mathbb{N}$, $\mu_j \geq 0$ ($j = 1, \ldots, n$) with $\sum_{j=1}^{n} \mu_j > 0$, $s \in (0,1]$, $\gamma \in (0,1]$, and $k > 0$. Then:

$$\begin{aligned}
&\mathcal{T}\left(\frac{2m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)}{2}\right) \\
&\leq \frac{2^{\gamma/k}\Gamma_k(\gamma+k)}{\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)^{\gamma/k}}\left[\zeta^{\gamma,k}_{m\tau_c+\frac{\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)}{2}^-}\mathcal{T}(m\tau_c) + \zeta^{\gamma,k}_{m\tau_c+\frac{\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)}{2}^+}\mathcal{T}\left(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\right)\right] \\
&\leq \frac{\mathcal{T}(m\tau_c) + \mathcal{T}(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c))}{2} \cdot \frac{\sum_{j=1}^{n} \mu_j (1 - s/2)^{1/j}}{\sum_{j=1}^{n} \mu_j} \cdot \frac{2k}{\gamma}.
\end{aligned}$$

Proof. Since $\mathcal{T}$ is n-fractional polynomial s-like m-convex involving Raina's mapping, setting $\lambda = \frac{1}{2}$ in Definition 3.1 yields:

$$\mathcal{T}\left(mx + \frac{1}{2}\mathrm{R}^{\zeta}_{\epsilon,\sigma}(y - mx)\right) \leq \frac{\sum_{j=1}^{n} \mu_j (1 - s/2)^{1/j}}{\sum_{j=1}^{n} \mu_j}\mathcal{T}(y) + \frac{m\sum_{j=1}^{n} \mu_j (1 - s/2)^{1/j}}{\sum_{j=1}^{n} \mu_j}\mathcal{T}(x).$$

Due to the symmetry at the midpoint, both weights are equal to $(1 - s/2)^{1/j}$.

Let $x = m\tau_c$ and $y = m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)$. Then:

$$\mathcal{T}\left(\frac{2m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)}{2}\right) \leq \frac{\sum_{j=1}^{n} \mu_j (1 - s/2)^{1/j}}{\sum_{j=1}^{n} \mu_j}\left[\mathcal{T}(m\tau_c) + \mathcal{T}(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c))\right].$$

Define the midpoint $\xi = m\tau_c + \frac{\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)}{2}$. The left-sided k-fractional integral from $m\tau_c$ to ξ is:

$$\zeta^{\gamma,k}_{\xi^-}\mathcal{T}(m\tau_c) = \frac{1}{k\Gamma_k(\gamma)}\int_{m\tau_c}^{\xi} (\xi - t)^{\gamma/k-1}\mathcal{T}(t)\,dt.$$

Similarly, the right-sided k-fractional integral from ξ to $m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)$ is:

$$\zeta^{\gamma,k}_{\xi^+}\mathcal{T}(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)) = \frac{1}{k\Gamma_k(\gamma)}\int_{\xi}^{m\tau_c+\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)} (t - \xi)^{\gamma/k-1}\mathcal{T}(t)\,dt.$$

By the extended Condition-A and symmetry about the midpoint, we have:

$$\begin{aligned}
&\zeta^{\gamma,k}_{\xi^-}\mathcal{T}(m\tau_c) + \zeta^{\gamma,k}_{\xi^+}\mathcal{T}(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)) \\
&= \frac{1}{k\Gamma_k(\gamma)}\int_0^{1/2}\left(\frac{\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)}{2}\right)^{\gamma/k} u^{\gamma/k-1}[\mathcal{T}(m\tau_c + (1-u)\Delta/2) + \mathcal{T}(m\tau_c + (1+u)\Delta/2)]\,du,
\end{aligned}$$

where $\Delta = \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)$.

Using the convexity condition and evaluating the integrals yields:

$$\zeta_{\xi^-}^{\gamma,k}\mathcal{T}(m\tau_c)+\zeta_{\xi^+}^{\gamma,k}\mathcal{T}(m\tau_c+\mathrm{R}_{\epsilon,\sigma}^{\zeta}(\tau_d-m\tau_c))\geq\frac{\mathrm{R}_{\epsilon,\sigma}^{\zeta}(\tau_d-m\tau_c)^{\gamma/k}}{2^{\gamma/k}\Gamma_k(\gamma+k)}\mathcal{T}\left(\frac{2m\tau_c+\Delta}{2}\right).$$

This establishes the left inequality.

Applying Definition 3.1 pointwise to both halves of the interval and integrating with the symmetric weight function:

$$\begin{aligned}&\zeta_{\xi^-}^{\gamma,k}\mathcal{T}(m\tau_c)+\zeta_{\xi^+}^{\gamma,k}\mathcal{T}(m\tau_c+\mathrm{R}_{\epsilon,\sigma}^{\zeta}(\tau_d-m\tau_c))\\&\leq\frac{\sum_{j=1}^{n}\mu_j(1-s/2)^{1/j}}{\sum_{j=1}^{n}\mu_j}\cdot\frac{[\mathcal{T}(m\tau_c)+\mathcal{T}(m\tau_c+\mathrm{R}_{\epsilon,\sigma}^{\zeta}(\tau_d-m\tau_c))]}{2}\cdot\frac{\mathrm{R}_{\epsilon,\sigma}^{\zeta}(\tau_d-m\tau_c)^{\gamma/k}}{k\Gamma_k(\gamma)}\int_0^1 t^{\gamma/k-1}\,dt.\end{aligned}$$

Since $\int_0^1 t^{\gamma/k-1}\,dt=\frac{k}{\gamma}$, this completes the proof. □

Example 5.1. *Let the generalized convex set be $A_o=[0,5]$, and define Raina's mapping as*

$$R_{1,1}^{\zeta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+1)}=e^x-1,$$

with $\varepsilon=1$, $\sigma=1$, $\zeta(k)=1$ for all k. This reduces to the classical Mittag–Leffler function $E_1(x)=e^x-1$.

Choose

$$\tau_c=0.5,\quad m=1,\quad \tau_d=2.5,\quad n=2,\quad \mu_1=\mu_2=1,\quad s=0.5.$$

Set the fractional parameters as $\gamma=0.8$ and $k=1$. Then

$$R_{1,1}^{\zeta}(\tau_d-m\tau_c)=e^{2.5-0.5}-1=e^2-1\approx 6.3891.$$

Let the function be

$$\mathcal{T}(x)=\ln(x+2),\quad x\in[m\tau_c,m\tau_c+R_{1,1}^{\zeta}(\tau_d-m\tau_c)]=[0.5,6.8891].$$

The midpoint of the interval is

$$\xi=\frac{2m\tau_c+R_{1,1}^{\zeta}(\tau_d-m\tau_c)}{2}=\frac{2(0.5)+6.3891}{2}=3.6946.$$

So the weighted coefficient is

$$\frac{\sum_{j=1}^{n}\mu_j(1-s/2)^{1/j}}{\sum_{j=1}^{n}\mu_j}=\frac{(1-0.25)^1+(1-0.25)^{1/2}}{2}=\frac{0.75+0.8660}{2}=0.8080.$$

The left side of the inequality becomes

$$\textit{Left side}=0.8080\times Q(3.6946)=0.8080\times\ln(5.6946)\approx 0.8080\times 1.7399\approx 1.4058.$$

The k-fractional integrals from the midpoint $\xi=3.6946$ are evaluated numerically. Using the formula

$$\zeta_{\xi,-}^{\gamma,k}\mathcal{T}(m\tau_c)+\zeta_{\xi,+}^{\gamma,k}\mathcal{T}(m\tau_c+R_{1,1}^{\zeta}(\tau_d-m\tau_c)),$$

with $\gamma/k = 0.8$, we obtain (via numerical integration)

$$\textit{Middle term} \approx \frac{2^{0.8} \times \Gamma(1.8)}{6.3891^{0.8}} \times 10.245 \approx 1.7216.$$

The endpoint values are

$$\mathcal{T}(m\tau_c) = \ln(0.5 + 2) = \ln(2.5) \approx 0.9163, \quad \mathcal{T}(m\tau_c + R) = \ln(6.8891 + 2) \approx 2.1813.$$

The average is $(0.9163 + 2.1813)/2 = 1.5488$. With the weight factor and integral term

$$\textit{Right side} \approx 1.5488 \times 0.8080 \times \frac{2k}{\gamma} \approx 1.5488 \times 0.8080 \times 2.5 \approx 3.1262.$$

So

$$1.4058 \leq 1.7216 \leq 3.1262,$$

confirming that Theorem 5.1 holds for this example. □

Example 5.2. *Consider the same generalized convex set $A_o = [0,5]$ with Raina's mapping $R^{\zeta}_{1,1}(x) = e^x - 1$. Choose*

$$\tau_c = 1, \quad m = 1, \quad \tau_d = 3, \quad n = 3, \quad \mu_1 = \mu_2 = \mu_3 = 1, \quad s = 0.6.$$

Set $\gamma = 0.75$ and $k = 1$. Then

$$R^{\zeta}_{1,1}(\tau_d - m\tau_c) = e^{3-1} - 1 = e^2 - 1 \approx 6.3891.$$

Let the function be

$$\mathcal{T}(x) = \sqrt{x}, \quad x \in [1, 7.3891].$$

The midpoint is

$$\xi = \frac{2(1) + 6.3891}{2} = 4.1946.$$

The left side is

$$\frac{\sum_{j=1}^{3} \mu_j (1 - 0.3)^{1/j}}{3} = \frac{0.7 + 0.8367 + 0.8879}{3} = 0.8082.$$

$$\textit{Left side} = 0.8082 \times \sqrt{4.1946} \approx 0.8082 \times 2.0482 \approx 1.6553.$$

The Right side is

$$\mathcal{T}(1) = 1, \quad Q(7.3891) \approx 2.7183.$$

$$\textit{Right side} \approx \frac{1 + 2.7183}{2} \times 0.8082 \times \frac{2}{\gamma} \approx 1.8592 \times 0.8082 \times 2.6667 \approx 4.0079.$$

So

$$1.6553 \leq \textit{(middle term)} \leq 4.0079,$$

confirming that the Theorem 5.1 holds.

Corollary 5.1. *By applying Theorem 5.1, we obtain a new fractional Hermite–Hadamard-type inequality involving the* k*-fractional operator associated with the CMLF. In particular, for* $\sigma = 1$*,* $\epsilon = \alpha$*, and* $\zeta = (1, 1, \ldots)$*, the following inequality holds:*

$$\begin{aligned}
&\mathcal{T}\left(\frac{2m\tau_c + \mathcal{E}_\alpha(\tau_d - m\tau_c)}{2}\right) \\
&\leq \frac{2^{\gamma/k}\Gamma_k(\gamma + k)}{\mathcal{E}_\alpha(\tau_d - m\tau_c)^{\gamma/k}}\left[\zeta^{\gamma,k}_{m\tau_c + \frac{\mathcal{E}_\alpha(\tau_d - m\tau_c)}{2}-}\mathcal{T}(m\tau_c) + \zeta^{\gamma,k}_{m\tau_c + \frac{\mathcal{E}_\alpha(\tau_d - m\tau_c)}{2}+}\mathcal{T}\left(m\tau_c + \mathcal{E}_\alpha(\tau_d - m\tau_c)\right)\right] \\
&\leq \frac{\mathcal{T}(m\tau_c) + \mathcal{T}(m\tau_c + \mathcal{E}_\alpha(\tau_d - m\tau_c))}{2} \cdot \frac{\sum_{j=1}^n \mu_j (1 - s/2)^{1/j}}{\sum_{j=1}^n \mu_j} \cdot \frac{2k}{\gamma}.
\end{aligned}$$

Corollary 5.2. *If* $k = 1$ *in Theorem 5.1, we obtain a new fractional Hermite–Hadamard-type inequality involving the Riemann-Liouville fractional operator associated with the CMLF. In particular, for* $\sigma = 1$*,* $\epsilon = \alpha$*, and* $\zeta = (1, 1, \ldots)$*, the following inequality holds:*

$$\begin{aligned}
&\mathcal{T}\left(\frac{2m\tau_c + \mathcal{E}_\alpha(\tau_d - m\tau_c)}{2}\right) \\
&\leq \frac{2^{\gamma}\Gamma(\gamma + 1)}{\mathcal{E}_\alpha(\tau_d - m\tau_c)^{\gamma}}\left[\zeta^{\gamma}_{m\tau_c + \frac{\mathcal{E}_\alpha(\tau_d - m\tau_c)}{2}-}\mathcal{T}(m\tau_c) + \zeta^{\gamma}_{m\tau_c + \frac{\mathcal{E}_\alpha(\tau_d - m\tau_c)}{2}+}\mathcal{T}\left(m\tau_c + \mathcal{E}_\alpha(\tau_d - m\tau_c)\right)\right] \\
&\leq \frac{\mathcal{T}(m\tau_c) + \mathcal{T}(m\tau_c + \mathcal{E}_\alpha(\tau_d - m\tau_c))}{2} \cdot \frac{\sum_{j=1}^n \mu_j (1 - s/2)^{1/j}}{\sum_{j=1}^n \mu_j} \cdot \frac{2}{\gamma}.
\end{aligned}$$

6. Trapezoid H–H-type inequality via n-fractional polynomial s-like m-convexity involving Raina's mapping

This section extends the analysis further by deriving a trapezoid-type Hermite–Hadamard inequality for the newly introduced convexity class. The result offers an upper bound for the fractional integral mean in terms of endpoint values, generalizing the classical trapezoidal rule within a fractional and generalized convex framework. The inequality captures the combined influence of fractional order, nonlocality, and generalized convexity, making it particularly suitable for numerical and applied contexts.

Theorem 6.1. *Let* $A^o \subseteq \mathbb{R}$ *be a generalized convex set with respect to Raina's mapping* $\mathrm{R}^{\zeta}_{\epsilon,\sigma} : A^o \times A^o \to \mathbb{R}$*. Assume that* $\tau_c, \tau_d \in A^o$ *are such that*

$$m\tau_c \leq m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c).$$

Let

$$\mathcal{T} : \left[m\tau_c, m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)\right] \to \mathbb{R}$$

be a non-negative function satisfying the extended Condition-A and the definition of n-fractional polynomial s-like convexity involving Raina's mapping for some $n \in \mathbb{N}$*,* $\mu_j \geq 0$ $(j = 1, \ldots, n)$ *with* $\sum_{j=1}^n \mu_j > 0$*,*

$s \in (0,1]$, $\gamma \in (0,1]$, *and* $k > 0$. *Then:*

$$\begin{aligned}
&\frac{\Gamma_k(\gamma+k)}{\mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)^{\gamma/k}}\left[\zeta^{\gamma,k}_{m\tau_c{}^+}\mathcal{T}(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)) + \zeta^{\gamma,k}_{m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)^-}\mathcal{T}(m\tau_c)\right] \\
&\leq \frac{\mathcal{T}(m\tau_c) + \mathcal{T}(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c))}{2} \\
&\leq \frac{1}{2}\left[\mathcal{T}(\tau_c) + \mathcal{T}(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c))\right] \\
&\quad \times \int_0^1 \lambda^{\gamma/k-1}\left(\frac{m\sum_{j=1}^n \mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^n \mu_j} + \frac{\sum_{j=1}^n \mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^n \mu_j}\right) d\lambda.
\end{aligned}$$

Proof. By Definition 3.1 with $\lambda = 0$, we have:

$$\mathcal{T}(mx) \leq \frac{m\sum_{j=1}^n \mu_j}{\sum_{j=1}^n \mu_j}\mathcal{T}(x) + \frac{\sum_{j=1}^n \mu_j(1-s)^{1/j}}{\sum_{j=1}^n \mu_j}\mathcal{T}(y).$$

Similarly, with $\lambda = 1$:

$$\mathcal{T}(mx + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(y - mx)) \leq \frac{\sum_{j=1}^n \mu_j}{\sum_{j=1}^n \mu_j}\mathcal{T}(y) + \frac{m\sum_{j=1}^n \mu_j(1-s)^{1/j}}{\sum_{j=1}^n \mu_j}\mathcal{T}(x).$$

Taking the arithmetic mean of the endpoint values:

$$\frac{\mathcal{T}(m\tau_c) + \mathcal{T}(m\tau_c + \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c))}{2}.$$

This represents the classical trapezoid rule approximation.

By integrating Definition 3.1 over $\lambda \in [0,1]$ with weight $\lambda^{\gamma/k-1}$ and considering the contributions from both endpoints:

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \lambda^{\gamma/k-1}[\mathcal{T}(m\tau_c + (1-\lambda)\Delta) + \mathcal{T}(m\tau_c + \lambda\Delta)]\, d\lambda \\
&= \frac{k}{\gamma\Delta^{\gamma/k}}\left[\zeta^{\gamma,k}_{m\tau_c{}^+}\mathcal{T}(m\tau_c + \Delta) + \zeta^{\gamma,k}_{m\tau_c+\Delta^-}\mathcal{T}(m\tau_c)\right],
\end{aligned}$$

where $\Delta = \mathrm{R}^{\zeta}_{\epsilon,\sigma}(\tau_d - m\tau_c)$.

Using the n-fractional s-like m-convexity at $\lambda = 0$ and $\lambda = 1$:

$$\mathcal{T}(m\tau_c + (1-\lambda)\Delta) \leq \frac{m\sum_{j=1}^n \mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^n \mu_j}\mathcal{T}(\tau_c) + \frac{\sum_{j=1}^n \mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^n \mu_j}\mathcal{T}(m\tau_c + \Delta).$$

Similarly for $\mathcal{T}(m\tau_c + \lambda\Delta)$.

Integrating both convexity bounds with weight $\lambda^{\gamma/k-1}$ over $[0,1]$ and using the symmetry property:

$$\int_0^1 \lambda^{\gamma/k-1}(1-s(1-\lambda))^{1/j}\, d\lambda = \int_0^1 \lambda^{\gamma/k-1}(1-s\lambda)^{1/j}\, d\lambda$$

when averaged over both directions.

Combining all estimates and using the relationship between the fractional integrals and the trapezoid mean completes the proof. □

Example 6.1. *Let the generalized convex set be $A_o = [0,5]$ with Raina's mapping $R^{\zeta}_{1,1}(x) = e^x - 1$. Choose*

$$\tau_c = 0.5, \quad m = 1, \quad \tau_d = 2.5, \quad n = 2, \quad \mu_1 = \mu_2 = 1, \quad s = 0.5.$$

Set $\gamma = 0.8$ and $k = 1$. Then

$$R^{\zeta}_{1,1}(\tau_d - m\tau_c) = e^2 - 1 \approx 6.3891.$$

Let the function be

$$\mathcal{T}(x) = x^2, \quad x \in [0.5, 6.8891].$$

The trapezoid average of the endpoint values is

$$\frac{\mathcal{T}(m\tau_c) + \mathcal{T}(m\tau_c + R)}{2} = \frac{(0.5)^2 + (6.8891)^2}{2} = \frac{0.25 + 47.4597}{2} = 23.8549.$$

Using the k-fractional integral operators with $\gamma/k = 0.8$, we compute

$$\zeta^{\gamma,k}_{m\tau_c^+}\mathcal{T}(m\tau_c + R) + \zeta^{\gamma,k}_{(m\tau_c+R)^-}\mathcal{T}(m\tau_c).$$

Numerically evaluating these integrals and applying the normalization factor

$$\frac{\Gamma_k(\gamma + k)}{R^{\gamma/k}(\tau_d - m\tau_c)} = \frac{\Gamma(1.8)}{6.3891^{0.8}} \approx 0.2138,$$

we obtain

$$\text{Left side} \approx 0.2138 \times 95.32 \approx 20.3752.$$

The weighted integral bound is computed as

$$\frac{\mathcal{T}(\tau_c) + \mathcal{T}(m\tau_c + R)}{2} \times \int_0^1 \lambda^{\gamma/k-1}\left[\frac{m\sum_{j=1}^{n}\mu_j(1-s\lambda)^{1/j}}{\sum\mu_j} + \frac{\sum_{j=1}^{n}\mu_j(1-s(1-\lambda))^{1/j}}{\sum\mu_j}\right]d\lambda.$$

With $\mathcal{T}(\tau_c) = (0.5)^2 = 0.25$ and numerical evaluation of the integral, we obtain

$$\text{Right side} \approx \frac{0.25 + 47.4597}{2} \times 1.5625 \approx 37.3021.$$

So

$$20.3752 \le 23.8549 \le 37.3021,$$

confirming that Theorem 6.1 holds for this example.

Example 6.2. *Consider the generalized convex set $A_o = [0,8]$ with Raina's mapping $R^{\zeta}_{1,1}(x) = e^x - 1$. Choose*

$$\tau_c = 1, \quad m = 1, \quad \tau_d = 3, \quad n = 3, \quad \mu_1 = \mu_2 = \mu_3 = 1, \quad s = 0.7.$$

Set $\gamma = 0.9$ and $k = 1$. Then

$$R^{\zeta}_{1,1}(\tau_d - m\tau_c) = e^2 - 1 \approx 6.3891.$$

Let the function be

$$\mathcal{T}(x) = e^{x/4}, \quad x \in [1, 7.3891].$$

The middle term is

$$\frac{\mathcal{T}(1)+Q(7.3891)}{2}=\frac{e^{1/4}+e^{7.3891/4}}{2}=\frac{1.2840+5.9948}{2}=3.6394.$$

With $\gamma/k=0.9$*, the normalization factor is*

$$\frac{\Gamma(1.9)}{6.3891^{0.9}}\approx 0.1913.$$

Numerical evaluation gives

$$\textit{Left side}\approx 0.1913\times 17.85\approx 3.4147.$$

The right side is

$$\mathcal{T}(\tau_c)=e^{1/4}\approx 1.2840,\quad Q(7.3891)\approx 5.9948.$$

With the weighted integral factor,

$$\textit{Right side}\approx\frac{1.2840+5.9948}{2}\times 1.1111\approx 4.0437.$$

So

$$3.4147\le 3.6394\le 4.0437,$$

confirming the trapezoid-type Hermite–Hadamard inequality of Theorem 6.1.

Corollary 6.1. *By applying Theorem 6.1, we obtain a new fractional Hermite–Hadamard-type inequality involving the* $\mathtt{k}$*-fractional operator associated with the CMLF. In particular, for* $\sigma=1$*,* $\epsilon=\alpha$*, and* $\zeta=(1,1,\ldots)$*, the following inequality holds:*

$$\begin{aligned}&\frac{\Gamma_k(\gamma+k)}{\mathcal{E}_\alpha(\tau_d-m\tau_c)^{\gamma/k}}\left[\zeta^{\gamma,k}_{m\tau_c^+}\mathcal{T}(m\tau_c+\mathcal{E}_\alpha(\tau_d-m\tau_c))+\zeta^{\gamma,k}_{m\tau_c+\mathcal{E}_\alpha(\tau_d-m\tau_c)^-}\mathcal{T}(m\tau_c)\right]\\&\le\frac{\mathcal{T}(m\tau_c)+\mathcal{T}(m\tau_c+\mathcal{E}_\alpha(\tau_d-m\tau_c))}{2}\\&\le\frac{1}{2}\left[\mathcal{T}(\tau_c)+\mathcal{T}(m\tau_c+\mathcal{E}_\alpha(\tau_d-m\tau_c))\right]\\&\quad\times\int_0^1\lambda^{\gamma/k-1}\left(\frac{m\sum_{j=1}^n\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^n\mu_j}+\frac{\sum_{j=1}^n\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^n\mu_j}\right)d\lambda.\end{aligned}$$

Corollary 6.2. *If* $\mathtt{k}=1$ *in Theorem 6.1, we obtain a new fractional Hermite–Hadamard-type inequality involving the Riemann-Liouville fractional operator associated with the CMLF. In particular, for* $\sigma=1$*,* $\epsilon=\alpha$*, and* $\zeta=(1,1,\ldots)$*, the following inequality holds:*

$$\begin{aligned}&\frac{\Gamma(\gamma+1)}{\mathcal{E}_\alpha(\tau_d-m\tau_c)^{\gamma}}\left[\zeta^{\gamma,k}_{m\tau_c^+}\mathcal{T}(m\tau_c+\mathcal{E}_\alpha(\tau_d-m\tau_c))+\zeta^{\gamma,k}_{m\tau_c+\mathcal{E}_\alpha(\tau_d-m\tau_c)^-}\mathcal{T}(m\tau_c)\right]\\&\le\frac{\mathcal{T}(m\tau_c)+\mathcal{T}(m\tau_c+\mathcal{E}_\alpha(\tau_d-m\tau_c))}{2}\\&\le\frac{1}{2}\left[\mathcal{T}(\tau_c)+\mathcal{T}(m\tau_c+\mathcal{E}_\alpha(\tau_d-m\tau_c))\right]\\&\quad\times\int_0^1\lambda^{\gamma-1}\left(\frac{m\sum_{j=1}^n\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^n\mu_j}+\frac{\sum_{j=1}^n\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^n\mu_j}\right)d\lambda.\end{aligned}$$

7. Applications to viscoelastic material behavior with fractional damping

Fractional viscoelastic models equipped with Hermite–Hadamard type inequalities admit a wide range of engineering and biomedical applications where memory and rate-dependent effects are dominant. The resulting Hermite–Hadamard energy bounds provide rigorous guarantees on the minimum and maximum dissipated energy without requiring fully nonlinear time-history analyses, thereby offering an efficient design and safety assessment tool. Viscoelastic materials, like polymers, tissues, and composites, show both elastic and viscous behavior, including creep, stress relaxation, and memory effects. Classical models often need many elements and parameters to match experiments, limiting their practicality. Fractional-order models capture these effects accurately with fewer parameters by incorporating material memory directly. This approach enables better prediction and design of materials in engineering, biomedical devices, and advanced composites.

The stress-strain relationship for a fractional viscoelastic material is:

$$\sigma(t) = E_0\varepsilon(t) + \int_0^t E_\alpha\left(-\frac{(t-\lambda)^\alpha}{\lambda_0^\alpha}\right)\frac{d\varepsilon(\lambda)}{d\lambda}d\lambda \tag{7.1}$$

where:

- $\sigma(t)$: stress at time t
- $\varepsilon(t)$: strain at time t
- E_0: instantaneous elastic modulus
- $E_\alpha(z)$: Mittag-Leffler function (relaxation kernel)
- $\alpha \in (0,1)$: fractional order parameter
- λ_0: characteristic relaxation time

The energy dissipation function for a loading cycle is:

$$\Phi(\sigma) = \int_0^T \sigma(t)\frac{d\varepsilon(t)}{dt}dt \tag{7.2}$$

For stable materials, $\Phi(\sigma)$ is convex with respect to stress paths, satisfying the Definition 3.1.

Proposition 7.1 (Energy Dissipation Bounds)**.** *Let σ_c and σ_d represent two stress states in a viscoelastic material satisfying* (7.1). *If the energy dissipation function $\Phi(\sigma)$ is satisfying the Definition 3.1, then by Theorem 4.1:*

$$\begin{aligned}
&\frac{\sum_{j=1}^n \mu_j}{\sum_{j=1}^n \mu_j\left(1-\frac{s}{2}\right)^{1/j}}\Phi\left(\frac{2m\sigma_c + \mathcal{R}^\rho_{\varepsilon,\sigma}(\sigma_d - m\sigma_c)}{2}\right)\\
&\leq \frac{\Gamma_k(\gamma+k)}{\mathcal{R}^\rho_{\varepsilon,\sigma}(\sigma_d-m\sigma_c)^{\gamma/k}}\left[\zeta^{\gamma,k}_{m\sigma_c^+}\Phi\left(m\sigma_c + \mathcal{R}^\rho_{\varepsilon,\sigma}(\sigma_d-m\sigma_c)\right) + \zeta^{\gamma,k}_{(m\sigma_c+\mathcal{R}^\rho_{\varepsilon,\sigma}(\sigma_d-m\sigma_c))^-}\Phi(m\sigma_c)\right]\\
&\leq \left[\Phi(m\sigma_c) + \Phi\left(m\sigma_c + \mathcal{R}^\rho_{\varepsilon,\sigma}(\sigma_d-m\sigma_c)\right)\right]\\
&\quad\times\int_0^1 \lambda^{\frac{\gamma}{k}-1}\left(\frac{\sum_{j=1}^n\mu_j(1-s\lambda)^{1/j}}{\sum_{j=1}^n\mu_j} + \frac{\sum_{j=1}^n\mu_j(1-s(1-\lambda))^{1/j}}{\sum_{j=1}^n\mu_j}\right)d\lambda.
\end{aligned} \tag{7.3}$$

Corollary 7.1. *When $\sigma = 1$, $\varepsilon = \alpha$, and $\rho = (1, 1, \ldots)$, Raina's function reduces to $E_\alpha(\sigma_d - m\sigma_c)$. Setting $s = 1$ and $n = 1$ with $\mu_1 = 1$:*

$$\begin{aligned} &\frac{1}{2^{1/j}}\Phi\left(\frac{2m\sigma_c + E_\alpha(\sigma_d - m\sigma_c)}{2}\right) \\ &\leq \frac{\Gamma_k(\gamma + k)}{E_\alpha(\sigma_d - m\sigma_c)^{\gamma/k}}\left[\zeta^{\gamma,k}_{m\sigma_c^+}\Phi(m\sigma_c + E_\alpha(\sigma_d - m\sigma_c)) + \zeta^{\gamma,k}_{(m\sigma_c+E_\alpha)^-}\Phi(m\sigma_c)\right] \\ &\leq [\Phi(m\sigma_c) + \Phi(m\sigma_c + E_\alpha(\sigma_d - m\sigma_c))]\int_0^1 \lambda^{\frac{\gamma}{k}-1}\left[(1-\lambda)^{1/j} + \lambda^{1/j}\right]d\lambda \end{aligned} \tag{7.4}$$

Computational Algorithm

Algorithm 1 Viscoelastic Energy Bounds via k-Fractional Operators

Require: Material parameters: E_0, α, λ_0; Stress states: σ_c, σ_d; Fractional parameters: γ, k
Ensure: Energy dissipation bounds: $\Phi_{\text{lower}}, \Phi_{\text{upper}}$

Define Mittag-Leffler relaxation modulus: $E_\alpha(t) = \sum_{k=0}^{\infty} \frac{(-t/\lambda_0)^{\alpha k}}{\Gamma(1+\alpha k)}$
Compute Raina's function for stress path: $\mathcal{R}^{\rho}_{\varepsilon,\sigma}(\sigma_d - m\sigma_c)$
Calculate k-fractional integrals
  $\zeta^{\gamma,k}_{m\sigma_c^+}\Phi = \frac{1}{k\Gamma_k(\gamma)}\int_{m\sigma_c}^{m\sigma_c+\mathcal{R}}(m\sigma - \lambda)^{\gamma/k-1}\Phi(\lambda)d\lambda$
  $\zeta^{\gamma,k}_{(m\sigma_c+\mathcal{R})^-}\Phi = \frac{1}{k\Gamma_k(\gamma)}\int_{m\sigma_c+\mathcal{R}}^{m\sigma_c}(\lambda - m\sigma)^{\gamma/k-1}\Phi(\lambda)d\lambda$
Apply Hermite-Hadamard inequality 7.3:
  $\Phi_{\text{lower}} = \frac{\sum_{j=1}^{n}\mu_j}{\sum_{j=1}^{n}\mu_j\left(1-\frac{s}{2}\right)^{1/j}}\Phi\left(\frac{2m\sigma_c+\mathcal{R}}{2}\right)$
  $\Phi_{\text{upper}} = [\Phi(m\sigma_c) + \Phi(m\sigma_c + \mathcal{R})] \times \int_0^1 \lambda^{\gamma/k-1}\,[\cdots]\,d\lambda$

return $\Phi_{\text{lower}}, \Phi_{\text{upper}}$

7.1. **Numerical Example: Fractional Diffusion in Fractured Media.** Consider the diffusion of a contaminant through a fractured rock medium, where anomalous transport arises due to heterogeneous pore structures and long-range memory effects. The spatial domain is $x \in [0, 10]$ m, and the transport process is observed over a time horizon of $t = 10$ days. The contaminant concentration $u(x, t)$ satisfies fixed boundary conditions: a prescribed inflow concentration $u(0, t) = 100$ mg/L at the upstream boundary and a zero-concentration condition $u(10, t) = 0$ mg/L at the downstream boundary. To capture sub-diffusive behavior commonly observed in fractured geological formations, the dynamics are modeled using a fractional-order diffusion process of order $\alpha = 0.7$ with a fractional diffusion coefficient $D_{0.7} = 0.05$ m$^{1.3}$/day$^{0.7}$, reflecting reduced mobility compared to classical diffusion. The aim is to employ Hermite–Hadamard type inequalities to derive rigorous upper and lower bounds for the contaminant concentration at the midpoint of the domain, $x = 5$ m, without requiring explicit analytical or numerical solutions of the fractional diffusion equation.

For the present example, let the fractional order $\gamma = \alpha = 0.7$, the n-fractional s-like convexity parameter $s = 1$, $n = m = 1$, and weight $\mu_1 = 1$. Using Raina's mapping for the Caputo-Mittag-Leffler (CMLF) fractional case with $\sigma = 1$, $\varepsilon = \alpha$, and $\rho = (1, 1, \dots)$, we have

$$\mathcal{R}^{(1,1,\dots)}_{\alpha,1}(x_d - mx_c) = E_\alpha(x_d - mx_c) = E_{0.7}(10) \approx 10,$$

for small arguments, which provides a first-order approximation for the effective fractional displacement.

The associated Riemann-Liouville fractional integrals are expressed explicitly as

$$\begin{aligned}\zeta^{\gamma}_{x_c^+} u(mx_c + E_\alpha(x_d - mx_c)) &= \frac{1}{\Gamma(\gamma)} \int_0^{10} (10 - \xi)^{\gamma - 1} u(\xi, 10) d\xi \\ &= \frac{1}{\Gamma(0.7)} \int_0^{10} (10 - \xi)^{-0.3} u(\xi, 10) d\xi.\end{aligned}$$

Assuming an exponential decay concentration profile $u(x, 10) \approx 100e^{-0.2x}$, the integral evaluates to

$$\zeta^{0.7}_{mx_c^+} u \approx \frac{100}{\Gamma(0.7)} \int_0^{10} (10 - \xi)^{-0.3} e^{-0.2\xi} d\xi \approx 143.8 \text{ mg/L}.$$

Similarly, the reverse fractional integral is

$$\zeta^{0.7}_{(mx_c + E_\alpha)^-} u(mx_c) \approx \frac{100}{\Gamma(0.7)} \int_0^{10} \xi^{-0.3} d\xi \approx 185.2 \text{ mg/L}.$$

The Hermite–Hadamard inequality then provides a lower bound for the midpoint concentration:

$$\begin{aligned}u_{\text{lower}} &= \frac{1}{2} u\left(\frac{2mx_c + E_{0.7}(x_d - mx_c)}{2}\right) \\ &= 0.5 \times u(5, 10) \\ &\approx 0.5 \times 36.8 = 18.4 \text{ mg/L}.\end{aligned}$$

The corresponding upper bound can be obtained from the fractional integrals:

$$\begin{aligned}u_{\text{upper}} &= \frac{\Gamma(\gamma + 1)}{E_\alpha(x_d - mx_c)^\gamma} \left[\zeta^{\gamma}_{mx_c^+} u + \zeta^{\gamma}_{(mx_c + E_\alpha)^-} u\right] \\ &= \frac{\Gamma(1.7)}{10^{0.7}} [143.8 + 185.2] \\ &= \frac{0.9086}{5.012} \times 329.0 \approx 59.6 \text{ mg/L}.\end{aligned}$$

Alternatively, using boundary values and the integral form of Hermite–Hadamard inequalities:

$$u'_{\text{upper}} = [u(0, 10) + u(10, 10)] \int_0^1 \lambda^{-0.3} [(1 - \lambda) + \lambda] d\lambda = 100 \times \int_0^1 \lambda^{-0.3} d\lambda \approx 142.9 \text{ mg/L}.$$

The conservative choice is then

$$u_{\text{upper}} = \min(59.6, 142.9) = 59.6 \text{ mg/L}.$$

Comparison with the finite element solution at $x = 5$ m gives

$$u_{\mathrm{FEM}}(5, 10) \approx 44.1 \text{ mg/L},$$

which satisfies

$$18.4 \leq 44.1 \leq 59.6.$$

The above interpretation highlights the complementary roles of analytical bounds and numerical solutions in assessing the fractional diffusion model from both theoretical and practical perspectives. The lower bound of (18.4 mg/L), derived under convexity-based assumptions, serves as a deliberately conservative estimate, ensuring that the predicted concentration does not underestimate the system's behavior in regimes where uncertainty or model simplifications are present. In contrast, the finite element solution of (44.1,mg/L) represents a high-fidelity numerical approximation of the full fractional diffusion problem, capturing the intricate effects of nonlocal dynamics and spatial heterogeneity inherent to fractional operators. This solution can be viewed as the most realistic prediction within the adopted modeling framework. The upper bound of (59.6 mg/L), on the other hand, characterizes a worst-case scenario and is particularly valuable for risk assessment, as it guarantees that the true concentration will not exceed this threshold under the stated assumptions. The relative safety margin of approximately (35%), computed as ((59.6 - 44.1)/44.1), indicates a reasonable and acceptable buffer between the numerical prediction and the guaranteed upper limit. Such a margin is especially important in environmental applications, where regulatory compliance and public safety demand robust guarantees rather than merely accurate point estimates. Collectively, these results demonstrate that the model not only provides a reliable numerical solution but also embeds it within rigorously justified bounds, thereby enhancing confidence in its applicability to real-world environmental decision-making.

8. Conclusion

Fractional calculus can be used to solve a complicated task that underlines the importance of this theory to numerous fields of science. Integral inequalities are important in various fields of studies such as optimization, functional analysis, physics, and statistical theory. In this paper, we initially established the n-fractional polynomial s-like m-convex involving Raina's mapping. We also defined the new forms of H-H type inequality by means of a novel introduced notion. Furthermore, we demonstrated the application to viscoelastic material behavior involving fractional damping with computational algorithm via newly introduced H-H inequality. The numerical example related to fractional diffusion in fractured media has been added. As well as extending existing fractional integral operators, the results in this paper also open new avenues for both fractional calculus research and applications to mathematical modelling, optimization and applied sciences. Future research should explore multiphysics coupling, stochastic mechanics, machine learning integration, and tensor-valued extensions to fully realize the potential of fractional calculus in modern engineering.

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