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Constrained Problem of a Non Local Integral Problem of an Arbitrary Orders Differential Equation Subject to a Weighted Problem of a Delayed Fractional Differential Equation Constraint

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Abstract. This work presents a comprehensive theoretical framework proposing the existence of solution and its main properties of a constrained problem of an arbitrary orders differential equation with nonlocal-integral condition subject to a weighted problem of a delayed Riemann-Liouville fractional differential equation constraint. Firstly, we prove the existence of at least integrable solution of the constraint, then we prove that for every solution of the constraint there exists a unique continuous solution of the constrained problem it self. Moreover, examining the continuous dependence of the obtained solutions on some parameters and functions ensures the stability of the problem and establishes the existence and uniqueness of solutions. The Hyers–Ulam stability of the system is thoroughly analyzed, showing the strength of solutions against small perturbations in initial conditions or system parameters. The study develops comprehensive mathematical results based on the fixed point theorems in suitable spaces. We also deliver an example to demonstrate the validity of the theoretical findings and their potential application to dynamical systems and control problems.

1. Introduction

One of the most active research areas in applied mathematics is fractional calculus due to its outstanding capability to model phenomena exhibiting memory and hereditary properties. A powerful generalization of classical calculus is the non-integer order differentiation and integration which enable a more accurate representation of complex systems in physics, control theory, viscoelasticity, fluid dynamics and biological processes.

The foundations of fractional calculus trace back to the works of Liouville, Riemann, Caputo,

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and others, which has led to expanding research such as the work of Caputo–Fabrizio, Atangana–Baleanu, and others. These developments have opened new ways for modeling systems with distributed, nonlocal, or weighted memory effects. Fractional-order differential equations (FDEs) have become established as powerful mathematical tools for illustrating memory-dependent and hereditary behaviors in complex systems. As opposed to classical integer-order models, fractional calculus allows for more flexible and accurate modeling of real-world phenomena. The study of fractional integro-differential equations under nonlocal and weighted constraints has received increasing interest because of its wide applicability in problems involving distributed parameters and hereditary effects. For more details see [12]- [16].

In particular, [7] El-Sayed et al. studied a constrained problem of the Riemann-Liouville fractional differential equation with a nonlocal and weighted delay integral constraint. The authors demonstrated both the existence and uniqueness of the solution based on appropriate assumptions. Additionally, they explored the stability of the problem through the Hyers-Ulam stability, as well as the continuous dependence of the unique solution on certain parameters. Related works include studies in [1], [2], [4] - [9] and [16].

which highlight the significance of fractional modeling in constrained problem.

The present work aims to studying the existence of a solution of the constrained problem of a non local integral problem of an arbitrary orders differential equation subject to a weighted problem of a delayed fractional differential equation constraint.

Consider now the nonlocal integral problem of the functional integro-differential equation of arbitrary orders

$$\frac{dx}{dt} = f\Big(t, u(t), \lambda \int_0^{\phi(t)} g(s, D^{\alpha}x(s))ds\Big),\ t \in (0,T] \tag{1.1}$$

with the non local integral condition

$$x(0) + \int_{\tau}^{T} h(s, D^{\delta}x(s))ds = x_0,\ \tau \in [0,T] \tag{1.2}$$

and the delayed Riemann-Liouville (R-L) fractional order differential equation

$${}^{R}D^{\beta}u(t) = f_1(t, u(t), \lambda u(\gamma t)),\ t \in (0,T] \tag{1.3}$$

with each one of the weighted condition

$$t^{1-\beta}u(t)|_{t=0} = \frac{u_0}{\Gamma(\beta)} + \int_{\eta}^{T} h_1(s, u(s), \lambda u(\gamma t))ds,\ \eta \in (0,T] \tag{1.4}$$

or the non local condition

$$I^{1-\beta}u(t)|_{t=0} = u_0 + \Gamma(\beta)\int_{\eta}^{T} h_1(s, u(s), \lambda u(\gamma t))ds,\ \eta \in (0,T]. \tag{1.5}$$

Here, we study the constrained problem of (1.1)-(1.2) subject to the constraint (1.3)-(1.4) (or (1.3) and (1.5)).

Where D^{α} and D^{δ} are the the Caputo fractional derivative of orders $\alpha,\ \delta \in (0,1)$, and ${}^{R}D^{\beta}$ is the fractional derivative of Riemann-Liouville of order $\beta \in (0,1)$.

Here, by using the Schauder fixed point Theorem we proved the existence of the solution of the constraint (1.3)-(1.4) (or (1.3) and (1.5)) in $L_1[0,T]$. Furthermore, we prove that for any solution $u \in L_1[0,T]$ there exists a unique absolutely continuous solution $x \in AC[0,T]$ of the problem (1.1)-(1.2). Moreover, we present the continuous dependence of the solution $x \in AC[0,T]$ on the parameter λ and on the functions $u(t)$ and $\phi(t)$. Finally, we study the Hyers-Ulam stability of the problem (1.1)-(1.2). The theoretical developments are complemented by an example which shows the obtained results and highlights their mathematical consistency.

Our work in this article is organized as follows: Section 2, introduces the formulation of the constrained problem and presents the existence of solution. In Section 3, the main results concerning the existence and uniqueness of the solution to problem (1)–(2) are established. Section 4 is focuses on proving the continuous dependence of the obtained solutions with respect to the associatedon parameters. We investigate the Hyers–Ulam stability of the proposed system and provide sufficient conditions that guarantee this type of stability, as presented in Section 5. To illustrate the theoretical findings, a concrete numerical example is detailed in Section 6, which verifies the applicability and validity of the main results. Finally, the conclusions of our work in this paper are outlined in Section 7.

2. Solution of the constraint

In this section, we establish the existence of solution of at least one integrable solution $u \in L_1[0,T]$ of the problem (1.3)-(1.4) (or (1.3) and(1.5)).

The following assumptions and Lemma will ply a crucial role in establishing this result.

First consider the following assumptions

(i)$f_1, h_1 : [0,T] = I \times R \times R \rightarrow R$ are measurable in $t \in I$ for any $x \in R$ and continuous in $x \in R$ for all $t \in I$ and there exist two integrable functions a_1, b_1 and two constants $K_1, L_1 > 0$, such that

$$|f_1(t,x_1,x_2)| \leq |a_1(t)| + K_1(|x_1| + |x_2|)$$
$$|h_1(t,x_1,x_2)| \leq |b_1(t)| + L_1(|x_1| + |x_2|),\ \forall t \in I, x_1, x_2 \in R.$$

(ii) $k^* = max\{L_1, K_1\}$ and $a = max\{||a||_1, ||b||_1\}$, $T^* = max\{\frac{T^{\beta}}{\beta}, \frac{T^{\beta}}{\Gamma(\beta+1)}\}$

(iii) $2T^*k^*(1 + \lambda/\gamma) < 1$.

Then, we have the following Lemma.

Lemma 2.1. *The problem (1.3)-(1.4) (or (1.3) and (1.5)) is equivalent to the integral equation*

$$u(t) = t^{\beta-1}\left(\frac{u_0}{\Gamma(\beta)} + \int_{\eta}^{T} h_1(s, u(s), \lambda u(\gamma s))ds\right) + I^{\beta} f_1(t, u(t), \lambda u(\gamma t)). \qquad (2.1)$$

Proof. Let $u \in L_1[0,T]$ be a solution of the problem (1.3)-(1.4) (or (1.3) and (1.5)), from (1.3) and by using the properties of the fractional calculus [14], we have

$$\frac{d}{dt}I^{1-\beta}u(t) = f_1(t,u(t),\lambda u(\gamma t)).$$

Integrating both sides of the above equation with respect to t, we get

$$I^{1-\beta}u(t) - I^{1-\beta}u(t)|_{t=0} = If_1(t,u(t),\lambda u(\gamma t)).$$

Let $I^{1-\beta}u(t)|_{t=0} = C$, we obtain

$$I^{1-\beta}u(t) = C + If_1(t,u(t),\lambda u(\gamma t)).$$

Operating both sides by I^{β}, then

$$I\,u(t) = \frac{Ct^{\beta}}{\Gamma(1+\beta)} + I^{1+\beta}f_1(t,u(t),\lambda u(\gamma t)).$$

By differentiating both sides with respect to t, we deduce that

$$u(t) = \frac{Ct^{\beta-1}}{\Gamma(\beta)} + I^{\beta}f_1(t,u(t),\lambda u(\gamma t)), \tag{2.2}$$

$$t^{1-\beta}u(t) = \frac{C}{\Gamma(\beta)} + t^{1-\beta}\,I^{\beta}f_1(t,u(t),\lambda u(\gamma t)). \tag{2.3}$$

To determine the value of C, we set $t=0$ in equation (2.3), yielding

$$t^{1-\beta}u(t)|_{t=0} = \frac{C}{\Gamma(\beta)}$$

From equation (1.5), we have

$$\frac{u_0}{\Gamma(\beta)} + \int_{\eta}^{T} h_1(s,u(s),\lambda u(\gamma t))ds = \frac{C}{\Gamma(\beta)}.$$

Substituting this expression for C into equatin (2.2), we obtain

$$u(t) = t^{\beta-1}\left(\frac{u_0}{\Gamma(\beta)} + \int_{\eta}^{T} h_1(s,u(s),\lambda u(\gamma t))ds\right) + I^{\beta}f_1(t,u(t),\lambda u(\gamma t)).$$

Conversely, let $u \in L_1[0,T]$ be a solution of (2.1), then we have

$$t^{1-\beta}u(t) = \frac{u_0}{\Gamma(\beta)} + \int_{\eta}^{T} h_1(s,u(s),\lambda u(\gamma t)))ds + t^{1-\beta}\,I^{\beta}f_1(t,u(t),\lambda u(\gamma t)),$$

which implies that

$$t^{1-\beta}u(t)|_{t=0} = \frac{u_0}{\Gamma(\beta)} + \int_{\eta}^{T} h_1(s,u(s),\lambda u(\gamma t))ds$$

and consequently,

$$\frac{d}{dt}I^{1-\beta}u(t) = f_1(t,u(t),\lambda u(\gamma t)).$$

Now, for the problem (1.3) and (1.5), we have

$$\frac{d}{dt}I^{1-\beta}u(t) = f_1(t,u(t),\lambda u(\gamma t)),\lambda u(\gamma t)).$$

By integrating both sides of the equation, we get

$$I^{1-\beta}u(t)-I^{1-\beta}u(t)|_{t=0}=If_1(t,u(t),\lambda u(\gamma t))$$

and from equation (1.5), we get

$$I^{1-\beta}u(t)=u_0+\Gamma(\beta)\int_{\eta}^{T}h_1(t,u(t),\lambda u(\gamma t)),\lambda u(\gamma t))ds+If_1(t,u(t),\lambda u(\gamma t)).$$

Applying the operator I^{β}, to both sides gives

$$Iu(t)=\frac{t^{\beta}}{\Gamma(1+\beta)}\Big(u_0+\Gamma(\beta)\int_{\eta}^{T}h_1(t,u(t),\lambda u(\gamma t)),\lambda u(\gamma t))ds\Big)+I^{1+\beta}f_1(t,u(t),\lambda u(\gamma t)).$$

and by differentiating both sides with respect to t, we get that

$$u(t)=\frac{t^{\beta-1}}{\Gamma(\beta)}\Big(u_0+\Gamma(\beta)\int_{\eta}^{T}h_1(t,u(t),\lambda u(\gamma t)),\lambda u(\gamma t))ds\Big)+I^{\beta}f_1(t,u(t),\lambda u(\gamma t)).$$

Conversely, from (2.1), we have

$$I^{1-\beta}u(t)=u_0+\Gamma(\beta)\int_{\eta}^{T}h_1f_1(t,u(t),\lambda u(\gamma t)),\lambda u(\gamma t))ds+If_1(t,u(t),\lambda u(\gamma t)).$$

Hence, by differentiation both sides, we obtain

$$\frac{d}{dt}I^{1-\beta}u(t)=\frac{d}{dt}(u_0+\Gamma(\beta)\int_{\eta}^{T}h_1(t,u(t),\lambda u(\gamma t)),\lambda u(\gamma t))ds+If_1(t,u(t),\lambda u(\gamma t)),$$

then

$${}^{R}D^{\beta}u(t)=f_1(t,u(t),\lambda u(\gamma t))$$

and

$$I^{1-\beta}u(t)|_{t=0}=u_0+\Gamma(\beta)\int_{\eta}^{T}h_1(s,u(s),\lambda u(\gamma t))ds.$$

This completes the proof.

Now, we have the following existence theorem

Theorem 2.1. *Let the assumptions $(i)-(iii)$ be satisfied, then the equation (2.1) has at least one solution $u\in L_1(I)$.*

Proof. Let Q_{r_1} be the closed ball
$Q_{r_1}=\{u\in L_1(I):\|u\|_1\le r_1\}\subset L_1$, $r_1=\frac{T^*(2a+\frac{|u_0|}{\Gamma(\beta)})}{1-2T^*k^*(1+\lambda/\gamma)}$
and the operator F define by

$$Fu(t)=t^{\beta-1}\bigg(\frac{u_0}{\Gamma(\beta)}+\int_{\eta}^{T}h_1(s,u(s),\lambda u(\gamma t))ds\bigg)+I^{\beta}f_1(s,u(s),\lambda u(\gamma t)).$$

Now, let $u\in Q_{r_1}$, then for each $t\in I$ we have

$$|Fu(t)|=|t^{\beta-1}\bigg(\frac{u_0}{\Gamma(\beta)}+\int_{\eta}^{T}h_1(s,u(s),\lambda u(\gamma s))ds\bigg)+I^{\beta}(s,u(s),\lambda u(\gamma t))|$$

$$
\begin{aligned}
&\leq t^{\beta-1}\Big(\frac{|u_0|}{\Gamma(\beta)} + \int_\eta^T |h_1(s,u(s),\lambda u(\gamma s))|ds\Big) + I^\beta|f_1(s,u(s),\lambda u(\gamma t))| \\
&\leq t^{\beta-1}\Big(\frac{|u_0|}{\Gamma(\beta)} + \int_0^T (|b_1(s)| + L_1(|u(s)| + \lambda|u(\gamma s)|)ds\Big) + I^\beta|f_1(s,u(s),\lambda u(\gamma t))| \\
&\leq t^{\beta-1}\Big(\frac{|u_0|}{\Gamma(\beta)} + \|b_1\|_1 + L_1(1+\lambda/\gamma)\|u\|_1 + I^\beta|f_1(t,u(t),\lambda u(\gamma t))|.
\end{aligned}
$$

Hence, integrating both sides over $[0,T]$, we get

$$
\begin{aligned}
\int_0^T |Fu(t)|dt &\leq \frac{T^\beta}{\beta}\Big(\frac{|u_0|}{\Gamma(\beta)} + \|b_1\|_1 + L_1(1+\lambda/\gamma)\|u\|_1 + \int_0^T\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|f_1(s,u(s),\lambda u(\gamma s))|ds\, dt \\
&\leq \frac{T^\beta}{\beta}\Big(\frac{|u_0|}{\Gamma(\beta)} + \|b_1\|_1 + L_1(1+\lambda/\gamma)\|u\|_1 + \int_0^T |f_1(s,u(s),\lambda u(\gamma s))|\int_s^T \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}dt\, ds \\
&\leq \frac{T^\beta}{\beta}\Big(\frac{|u_0|}{\Gamma(\beta)} + \|b_1\|_1 + L_1(1+\lambda/\gamma)\|u\|_1 + \frac{T^\beta}{\Gamma(\beta+1)}\int_0^T (|a_1(s)| + K_1|u(s)| + \lambda u(\gamma s))|)ds \\
&\leq \frac{T^\beta}{\beta}\Big(\frac{|u_0|}{\Gamma(\beta)} + \|b_1\|_1 + L_1(1+\lambda/\gamma)\|u\|_1\Big) + \frac{T^\beta}{\Gamma(\beta+1)}(\|a_1\|_1 + K_1(1+\lambda/\gamma)\|u\|_1) \\
&\leq \frac{T^\beta}{\beta}\Big(\frac{|u_0|}{\Gamma(\beta)} + \|b_1\|_1 + L_1(1+\lambda/\gamma)r_1\Big) + \frac{T^\beta}{\Gamma(\beta+1)}(\|a_1\|_1 + K_1(1+\lambda/\gamma)r_1) = r_1,
\end{aligned}
$$

It followes that

$$\|Fu\|_1 \leq r_1.$$

which implies that $F : Q_{r_1} \to Q_{r_1}$, moreover $\{Fu\}$ is uniformly bounded.
Now, let $u \in Q_{r_1}$, then

$$
\begin{aligned}
|(Fu)_h(t) - Fu(t)| &= \Big|\frac{1}{h}\int_t^{t+h} Fu(\theta)d\theta - Fu(t)\Big| \\
&\leq \frac{1}{h}\int_t^{t+h} |Fu(\theta) - Fu(t)|d\theta
\end{aligned}
$$

and

$$\|(Fu)_h - Fu\|_1 \leq \int_0^T \frac{1}{h}\int_t^{t+h} |(Fu)_h(\theta) - Fu(t)|d\theta\, dt.$$

But the class of functions $\{Fu\} \in L_1(I)$, then from the properties of the Lebesgue point Theorem [**?**] we have

$$|(Fu)_h(\theta) - Fu(t)| \to 0 \qquad as \qquad h \to 0.$$

Hence,

$$\|(Fu)_h - Fu\|_1 \to 0$$

this means that $(Fu)_h(t) \to Fu(t)$ uniformly in $L_1(I)$. Thus, from the Kolmogorov compactness Theorem [11] , the class of functions $\{Fu\}$ is relatively compact and by Arzela's Theorem the operator F is compact [15].

Now, let $\{u_n\} \subset Q_{r_1}$, and $u_n \to u$, then

$$Fu_n(t) = t^{\beta-1}\bigg(\frac{u_0}{\Gamma(\beta)} + \int_\eta^T h_1(s, u_n(s), \lambda u_n(\gamma s))ds\bigg) + I^\beta f_1(t, u_n(t), \lambda u_n(\gamma s))$$

and

$$\lim_{n\to\infty} Fu_n(t) = \lim_{n\to\infty}\Bigg(t^{\beta-1}\Big(\frac{u_0}{\Gamma(\beta)} + \int_\eta^T h_1(s, u_n(s), \lambda u_n(\gamma s))ds\Big) + I^\beta f_1(t, u_n(t), \lambda u_n(\gamma s))\Bigg).$$

Applying the Lebesgue Dominated Convergence Theorem [11], then from our assumptions we get

$$\begin{aligned}\lim_{n\to\infty} Fu_n(t) =& t^{\beta-1}\bigg(\frac{u_0}{\Gamma(\beta)} + \int_\eta^T h_1(s, \lim_{n\to\infty} u_n(s), \lambda \lim_{n\to\infty} u_n(\gamma s))ds\bigg) + I^\beta f_1(t, \lim_{n\to\infty} u_n(s), \lambda \lim_{n\to\infty} u_n(\gamma s)) \\ =& t^{\beta-1}\bigg(\frac{u_0}{\Gamma(\beta)} + \int_\eta^T h_1(s, u(s), \lambda u(\gamma s))ds\bigg) + I^\beta f_1(t, u(t), \lambda u(\gamma s)) = Fu(t).\end{aligned}$$

Then $Fu_n(t) \to Fu(t)$. This means that F is continuous.
Finally, by Schauder fixed point Theorem [3] and [11] we infer that there exist at least one solution $u \in L_1(I)$ of the functional integral equation (2.1).

3. Solution of the problem (1.1)-(1.2)

Now consider the following assumptions
(iv) $\phi : I \to I$ is continuous and $\phi(t) \le t$ on I.
(v)$f : I \times R \times R \to R$ is measurable in $t \in I$, $\forall\ x, y \in R$, and continuous in $x, y \in R$, $\forall\ t \in I$ and satisfies Lipschitz condition

$$|f(t,x,y) - |f(t,x_1,y_1)| \le K(|x - x_1| + |y - y_1|)$$

with $a(t) = f(t,0,0)$ is bounded and measurable on I. Then we can deduce that

$$|f(t,x,y)| \le |a(t)| + K(|x| + |y|).$$

(vi)$h : I \times R \to R$ is measurable in $t \in I$, $\forall\ x \in R$, and continuous in $x \in R$, $\forall\ t \in I$ and satisfies Lipschitz condition

$$|h(t,x) - h(t,y)| \le L|x - y| \, , \ L > 0.$$

(vii)$g : I \times R \to R$ is measurable in $t \in I$, $\forall\ x \in R$, and continuous in $x \in R$, $\forall\ t \in I$ and satisfies Lipschitz condition

$$|g(t,x) - g(t,y)| \le c|x - y| \, , \ c > 0$$

with $m(t) = g(t,0)$ is bounded and measurable on I. Then we can deduce that

$$|g(t,x)| \le |m(t)| + c|x|.$$

Under the above assumptions,we can now establish the following lemma

Lemma 3.1. *Let $x \in AC(I)$ be a solution of the problem (1.1)-(1.2), then for every solution $u \in L_1(I)$ of the constraint, it can be represented by the solution of*

$$x(t) = x_0 - \int_\tau^T h(s, I^{1-\gamma} y(s))ds + \int_0^t y(s)ds \tag{3.1}$$

where y is the solution of the functional integral equation.

$$y(t) = f(t, u(t), \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s))ds). \tag{3.2}$$

Proof. Assume that $x \in AC(I)$ satisfies the problem (1.1)-(1.2).
Let $\frac{dx}{dt} = y(t)$, then we obtain

$$x(t) = x(0) + \int_0^t y(s)ds$$

From equation (1.2), it follow that

$$x(t) = x_0 - \int_\tau^T h(s, D^\gamma x(s))ds + \int_0^t y(s)ds.$$

From the properties of the fractional order derivative, we can get

$$D^\alpha x(t) = I^{1-\alpha} y(t) \text{ and } D^\gamma x(t) = I^{1-\gamma} y(t).$$

Then

$$x(t) = x_0 - \int_\tau^T h(s, I^{1-\gamma} y(s))ds + \int_0^t y(s)ds$$

where y is the solution of the functional integral equation (3.2).
Conversely, let $x \in AC(I)$ be a solution of (3.1), then we have

$$\frac{dx}{dt} = \frac{d}{dt}(x_0 - \int_\tau^T h(s, I^{1-\gamma} y(s))ds + \int_0^t y(s)ds)$$

$$\frac{dx}{dt} = \frac{d}{dt} \int_0^t y(s)ds$$

$$\frac{dx}{dt} = f(t, u(t), \lambda \int_0^{\phi(t)} g(s, D^\alpha x(s))ds)$$

and

$$x(0) = x_0 - \int_\tau^T h(s, D^\gamma x(s))ds.$$

Finally, we are now in a position to state the following theorem

Theorem 3.1. *Let the assumptions $(iv) - (vii)$ be satisfied. If $K|\lambda| \frac{cT^{2-\alpha}}{\Gamma(3-\alpha)} < 1$, then for every solution $u \in L_1(I)$ of (1.3)-(1.4) (or (1.3) and (1.5)) there exists a unique solution $x \in AC(I)$ of the problem (1.1)-(1.2).*

Proof. Define the operator F by

$$Fy(t) = f(t, u(t), \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s))ds).$$

Let u be a solution of (1.3)-(1.4) (or (1.3) and (1.5)) and $y \in L_1(I)$, then we have

$$\begin{aligned}
|Fy(t)| =& |f(t, u(t), \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s))ds)| \\
\leq& |a(t)| + K|u(t)| + K|\lambda| |\int_0^{\phi(t)} g(s, I^{1-\alpha} y(s))ds| \\
\leq& |a(t)| + K|u(t)| + K|\lambda| \int_0^{t} (|m(s)| + cI^{1-\alpha}|y(s)|)ds \\
\leq& |a(t)| + K|u(t)| + K|\lambda| (\|m\|_1 + \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)}\|y\|_1),
\end{aligned}$$

By integrating both sides, we get

$$\begin{aligned}
\int_0^T |Fy(t)|dt \leq& \int_0^T (|a(t)| + K|u(t)| + K|\lambda| (\|m\|_1 + \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)}\|y\|_1))dt \\
\leq& \|a\|_1 + K\|u\|_1 + K|\lambda| (\|m\|_1 T + \frac{cT^{2-\alpha}}{\Gamma(3-\alpha)}\|y\|_1) \\
\|Fy\|_1 \leq& \|a\|_1 + Kr_1 + K|\lambda| (\|m\|_1 T + \frac{cT^{2-\alpha}}{\Gamma(3-\alpha)}\|y\|_1).
\end{aligned}$$

This proves that $F : L_1(I) \to L_1(I)$.
Now, let $y_1, y_2 \in L_1(I)$, then

$$\begin{aligned}
|Fy_2(t) - Fy_1(t)| =& |f(t, u(t), \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y_2(s))ds) - f(t, u(t), \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y_1(s))ds)| \\
\leq& K|\lambda| \int_0^{\phi(t)} |g(s, I^{1-\alpha} y_2(s)) - g(s, I^{1-\alpha} y_1(s))|ds \\
\leq& K|\lambda|\, c \int_0^{t} I^{1-\alpha}|y_2(s) - y_1(s)|ds \\
\leq& K|\lambda|\, \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)}\|y_2 - y_1\|_1.
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^T |Fy_2(t) - Fy_1(t)|dt \leq& \int_0^T (K|\lambda|\, \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)}\|y_2 - y_1\|_1)dt \\
\leq& K|\lambda|\, \frac{cT^{2-\alpha}}{\Gamma(3-\alpha)}\|y_2 - y_1\|_1 \\
\|Fy_2 - Fy_1\|_1 \leq& K|\lambda|\, \frac{cT^{2-\alpha}}{\Gamma(3-\alpha)}\|y_2 - y_1\|_1 \\
(1 - K|\lambda|\, \frac{cT^{2-\alpha}}{\Gamma(3-\alpha)})\|y_2 - y_1\|_1 \leq& 0.
\end{aligned}$$

This implies that F is a contraction operator. By Banach fixed point Theorem [3] for every solution $u \in L_1(I)$ of the problem (1.3)-(1.4) or (or (1.3) and (1.5)), then there exists a unique solution $y \in L_1(I)$ of the integral equation (3.2).

Lemma 3.2. *From Lemma 3.1 for every solution $u \in L_1(I)$ of the problem (1.3)-(1.4) or (or (1.3) and (1.5)), then there exists a unique solution $x \in AC(I)$ of the problem (1.1)-(1.5) given by equation (3.1).*

4. CONTINUOUS DEPENDENCE

In this section, we establish the continuous dependence of the solutions on the given data. To this end, two theorems are presented together with their detailed proofs.

Theorem 4.1. *Let the assumptions of Theorem(3.1) be satisfied. Then the unique solution of (3.2) depends continuously on λ, ϕ, u in the sense that $\forall\ \epsilon > 0$, $\exists\ \delta(\epsilon)$ such that*

$$max\{|\lambda - \lambda^*|, |\phi(t) - \phi^*(t)|, ||u - u^*||_1\} < \delta,$$

implies $||y - y^*||_1 < \epsilon$,
where y^* is the solution of

$$y^*(t) = f(t, u^*(t), \lambda^* \int_0^{\phi^*(t)} g(s, I^{1-\alpha} y^*(s))ds).$$

Proof.

$$\begin{aligned}
|y(t) - y^*(t)| =& |f(t, u(t), \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s))ds) - f(t, u^*(t), \lambda^* \int_0^{\phi^*(t)} g(s, I^{1-\alpha} y^*(s))ds)| \\
\leq& K|u(t) - u^*(t)| + |K\lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s))ds - K\lambda^* \int_0^{\phi^*(t)} g(s, I^{1-\alpha} y^*(s))ds| \\
\leq& K|u(t) - u^*(t)| + K|\lambda| \int_0^{\phi(t)} |g(s, I^{1-\alpha} y(s))|ds - K|\lambda| \int_0^{\phi(t)} |g(s, I^{1-\alpha} y^*(s))|ds \\
+& K|\lambda| \int_0^{\phi(t)} |g(s, I^{1-\alpha} y^*(s))|ds - K|\lambda^*| \int_0^{\phi(t)} |g(s, I^{1-\alpha} y^*(s))|ds \\
+& K|\lambda^*| \int_0^{\phi(t)} |g(s, I^{1-\alpha} y^*(s))|ds - K|\lambda^*| \int_0^{\phi^*(t)} |g(s, I^{1-\alpha} y^*(s))|ds \\
\leq& K|u(t) - u^*(t)| + K|\lambda|c \int_0^t I^{1-\alpha}|y(s) - y^*(s)|ds + K|\lambda - \lambda^*| \int_0^{\phi(t)} |g(s, I^{1-\alpha} y^*(s))|ds \\
+& K|\lambda^*| \int_{\phi^*(t)}^{\phi(t)} |g(s, I^{1-\alpha} y^*(s))|ds \\
\leq& K|u(t) - u^*(t)| + K|\lambda| \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)} ||y - y^*||_1 + K\delta \int_0^t (|m(s) + cI^{1-\alpha}|y^*(s))|)ds \\
+& K|\lambda^*| \int_{\phi^*(t)}^{\phi(t)} |g(s, I^{1-\alpha} y^*(s))|ds
\end{aligned}$$

$$\begin{aligned}
&\leq K|u(t)-u^*(t)| + K|\lambda|\, \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)}\|y-y^*\|_1 + K\delta(\|m\|_1 + \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)}\|y^*\|_1) \\
&+K|\lambda^*|\int_{\phi^*(t)}^{\phi(t)} |g(s, I^{1-\alpha}y^*(s))|ds \\
&\leq K|u(t)-u^*(t)| + K|\lambda|\, \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)}\|y-y^*\|_1 + \delta_1.
\end{aligned}$$

Hence

$$\int_0^T |y(t)-y^*(t)|dt \leq \int_0^T (K|u(t)-u^*(t)| + K|\lambda|\frac{ct^{1-\alpha}}{\Gamma(2-\alpha)}\|y-y^*\|_1 + \delta_1)dt$$

$$\begin{aligned}
\|y-y^*\|_1 &\leq K\|u-u^*\|_1 + K|\lambda|\, \frac{cT^{2-\alpha}}{\Gamma(3-\alpha)}\|y-y^*\|_1 + \delta_1 T \\
\|y-y^*\|_1 &\leq \frac{K\delta + \delta_1 T}{1 - K|\lambda|\, \frac{cT^{2-\alpha}}{\Gamma(3-\alpha)}} = \epsilon.
\end{aligned}$$

Then

$$\|y-y^*\|_1 \leq \epsilon.$$

Where $\delta_1 = K\delta(\|m\|_1 + \frac{cT^{1-\alpha}}{\Gamma(2-\alpha)}\|y^*\|_1) + K|\lambda^*|\int_{\phi^*(t)}^{\phi(t)} |g(s, I^{1-\alpha}y^*(s))|ds.$

Theorem 4.2. *Let the assumptions of Theorem(3.1) be satisfied. Then the unique solution of (3.1) depends continuously on* x_0, y *in the sense that* $\forall\ \epsilon > 0,\ \exists\ \delta(\epsilon)$ *such that*

$$max\{|x_0 - x_0^*|, \|y-y^*\|_1\} < \delta$$

implies

$$\|x - x^*\|_c < \epsilon.$$

where x^* *is the solution of*

$$x^*(t) = x_0^* - \int_\tau^T h(s, I^{1-\gamma}y^*(s))ds + \int_0^t y^*(s)ds.$$

Proof.

$$\begin{aligned}
|x(t)-x^*(t)| &= |x_0 - \int_\tau^T h(s, I^{1-\gamma}y(s))ds + \int_0^t y(s)ds - x_0^* + \int_\tau^T h(s, I^{1-\gamma}y^*(s))ds - \int_0^t y^*(s)ds| \\
&\leq |x_0 - x_0^*| + \int_\tau^T |h(s, I^{1-\gamma}y(s)) - h(s, I^{1-\gamma}y^*(s))|ds + \int_0^t |y(s)-y^*(s)|ds \\
&\leq \delta + L\int_0^T I^{1-\gamma}|y(s)-y^*(s)|ds + \|y-y^*\|_1 \\
&\leq \delta + L\int_0^T \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}|y(s)-y^*(s)|ds\ dt + \|y-y^*\|_1 \\
&\leq \delta + L\int_0^T |y(s)-y^*(s)| \int_s^T \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}dt\ ds + \|y-y^*\|_1 \\
&\leq \delta + \frac{LT^{1-\gamma}}{\Gamma(2-\gamma)}\int_0^T |y(s)-y^*(s)|ds + \|y-y^*\|_1
\end{aligned}$$

$$\leq \delta + \frac{LT^{1-\gamma}}{\Gamma(2-\gamma)} \|y - y^*\|_1 + \|y - y^*\|_1.$$

Hence

$$\|x - x^*\|_c \leq \delta + \frac{LT^{1-\gamma}}{\Gamma(2-\gamma)}\delta + \delta = \epsilon,$$

then

$$\|x - x^*\|_c \leq \epsilon.$$

Corollary 4.1. *Let the assumptions of Theorems 4.1-4.2 be satisfied, then the solution $x \in AC(I)$ of the problem (1.1)-(1.2) depends continuously on x_0, λ, ϕ and u.*

5. Hyers-Ulam stability

Numerous researchers have extensively investigated and developed the concept of Hyers-Ulam stability in connection with various classes of functional and differential equations, see for instance, [6]- [10]. This notion provides an important framework for analyzing the robustness of exact solutions under small perturbations in the data or functional arguments.
In the following, we introduce the formal definition and establish the theorem proving that the problems (1.1)-(1.5) possess the property of Hyers-Ulam stability.

Definition. [6]- [10] Let the solution $x \in AC(I)$ of the problem (1.1)-(1.5) be exists, then the problem (1.1)-(1.5) is Hyers-Ulam stable if $\forall\ \epsilon > 0$, $\exists\ \delta(\epsilon)$ such that for any $\delta-$ approximate solution $x_s \in AC(I)$ satisfies,

$$\left|\frac{dx_s}{dt} - f\left(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, D^\alpha x_s(\theta))d\theta\right)\right| < \delta, \tag{5.1}$$

then $\|x - x_s\|_c < \epsilon$.

Theorem 5.1. *Let the assumptions of Theorem (3.1) be satisfied, then for every solution $u \in L_1(I)$ of (1.3)-(1.4) (or (1.3) and (1.5)) the constrained problem (1.1)-(1.5) is Hyers-Ulam stable.*

Proof. *Let $\frac{dx_s}{dt} = y_s(t)$ then substituting this relation into inequality (5.1), we obtain*

$$|y_s(t) - f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)| < \delta,$$

$$-\delta < y_s(t) - f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta) < \delta,$$

$$|\int_0^t y_s(t)dt - \int_0^t f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)dt| < \delta t.$$

Furthermore, from inequality (5.1), it follows that

$$-\delta < \frac{dx_s}{dt} - f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta) < \delta$$

$$-\delta t < x_s(t) - x_s(0) - \int_0^t f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)dt < \delta t$$

$$\begin{aligned}-\delta t <& x_s(t) - (x_0 - \int_\tau^T h(\theta, I^{1-\gamma} y_s(\theta))d\theta) - \int_0^t f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)dt < \delta t\\ -\delta T <& x_s(t) - (x_0 - \int_\tau^T h(\theta, I^{1-\gamma} y_s(\theta))d\theta) - \int_0^t f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)dt < \delta T\end{aligned}$$

and

$$|x_s(t) - x_0 + \int_\tau^T h(\theta, I^{1-\gamma} y_s(\theta))d\theta - \int_0^t f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)dt| < \delta T.$$

We first observe that

$$\begin{aligned}|y(t) - y_s(t)| =& |f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y(\theta))d\theta) - y_s(t)|\\ =& |f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y(\theta))d\theta) - f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)\\ &+ f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta) - y_s(t)|\\ \leq& K|\lambda| \int_0^t |g(\theta, I^{1-\alpha} y(\theta)) - g(\theta, I^{1-\alpha} y_s(\theta))|d\theta\\ &+ |y_s(t) - f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)|\\ \leq& K|\lambda|\, c \int_0^t I^{1-\alpha} |y(\theta) - y_s(\theta)|d\theta + \delta\\ \leq& K|\lambda|\, \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)} \|y - y_s\|_1 + \delta.\end{aligned}$$

Integrating both side, we obtain

$$\begin{aligned}\int_0^T |y(t) - y_s(t)|dt \leq& \int_0^T (K|\lambda|\, \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)} \|y - y_s\|_1 + \delta)dt\\ \|y - y_s\|_1 \leq& K|\lambda|\, \frac{cT^{2-\alpha}}{\Gamma(3-\alpha)} \|y - y_s\|_1 + \delta\, T\\ \|y - y_s\|_1 \leq& \frac{\delta\, T}{1 - K|\lambda|\, \frac{cT^{2-\alpha}}{\Gamma(3-\alpha)}} = \epsilon,\end{aligned}$$

thereafter,

$$\|y - y_s\|_1 \leq \epsilon.$$

Secondly, we have

$$\begin{aligned}|x(t) - x_s(t)| =& |x_0 - \int_\tau^T h(\theta, I^{1-\alpha} y(\theta))d\theta + \int_0^t y(\theta)d\theta - x_s(t)|\\ =& |x_0 - \int_\tau^T h(\theta, I^{1-\alpha} y(\theta))d\theta + \int_0^t y(\theta)d\theta\\ &+ \int_\tau^T h(\theta, I^{1-\gamma} y_s(\theta))d\theta - \int_0^t y_s(\theta)d\theta\end{aligned}$$

$$\begin{aligned}
&- \int_\tau^T h(\theta, I^{1-\gamma} y_s(\theta))d\theta + \int_0^t y_s(\theta)d\theta - x_s(t)| \\
&\leq \int_0^T |h(\theta, I^{1-\gamma} y(\theta)) - h(\theta, I^{1-\gamma} y_s(\theta))|d\theta + \int_0^t |y(\theta) - y_s(\theta)|d\theta \\
&+ |x_0 - \int_\tau^T h(\theta, I^{1-\gamma} y_s(\theta))d\theta - \int_0^t f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)dt - x_s(t)| \\
&+ |\int_0^t f(t, u(t), \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)dt - \int_0^t y_s(\theta)d\theta| \\
&\leq L \int_0^T I^{1-\gamma} |y(\theta) - y_s(\theta)|d\theta + \int_0^t |y(\theta) - y_s(\theta)|d\theta + \delta T + \delta T \\
&\leq \int_0^T \int_0^t \frac{L(t-s)^{-\gamma}}{\Gamma(1-\gamma)} |y(\theta) - y_s(\theta)|d\theta \, dt + \int_0^t |y(\theta) - y_s(\theta)|d\theta + \delta T + \delta T \\
&\leq \int_0^T \frac{L(t-s)^{-\gamma}}{\Gamma(1-\gamma)} \int_s^T |y(\theta) - y_s(\theta)|dt d\theta + \|y - y_s\|_1 + \delta T + \delta T \\
&\leq \frac{L(t-s)^{1-\gamma}}{\Gamma(2-\gamma)} \int_0^T |y(\theta) - y_s(\theta)|d\theta + \|y - y_s\|_1 + \delta T + \delta T \\
&\leq \frac{L(t-s)^{1-\gamma}}{\Gamma(2-\gamma)} \|y - y_s\|_1 + \|y - y_s\|_1 + \delta T + \delta T.
\end{aligned}$$

therefore

$$\|x - x_s\|_c \leq \left(\frac{L(t-s)^{1-\gamma}}{\Gamma(2-\gamma)} + 1\right) \frac{\delta \; T}{1 - K|\lambda| \, \frac{cT^{2-\alpha}}{\Gamma(3-\alpha)}} + \delta T + \delta T = \epsilon$$

finally

$$\|x - x_s\|_c \leq \epsilon.$$

This complete the proof.

6. Example

Consider the following constrained problem of the initial value problem of the functional integral differential equation

$$\frac{dx}{dt} = \frac{t^4}{1+t} + \frac{1}{2}u(t) + \frac{1}{2} \int_0^t \left(\frac{s^3}{2} - \frac{1}{4} D^{\frac{1}{2}} x(s)\right)ds, \text{ a.e. } t \in (0, 1]$$

with the nonlocal integral condition

$$x(0) = 1 - \int_\tau^1 \left(\frac{s^2}{3} + \frac{1}{2} D^{\frac{1}{3}} x(s)\right)ds,$$

subject to the fractional order differential equation

$${}^R D^{\frac{1}{3}} u(t) = te^{-t} - \frac{1}{4}u(t),$$

with each of the following nonlocal condition

$$t^{\frac{2}{3}}u(t)|_{t=0} = \frac{1}{\Gamma(\frac{1}{3})} + \int_{\eta}^{1} (\frac{s^2}{3} - \frac{1}{3}u(s))ds,$$

or

$$I^{\frac{2}{3}}u(t)|_{t=0} = \frac{1}{\Gamma(\frac{1}{3})} + \int_{\eta}^{1} (\frac{s^2}{3} - \frac{1}{3}u(s))ds.$$

Set

$$f_1(t,z) = te^{-t} - \frac{1}{4}z$$

$$|f_1(t,z))| \leq te^{-t} + \frac{1}{4}|z|, \ \forall\ t \in I,\ z \in R.$$

Where $a_1(t) = te^{-t}$ *and* $\|a_1\|_1 = \int_0^1 te^{-t}dt = 1 - \frac{1}{e}$.
Similarly,

$$h_1(t,z) = \frac{t^2}{3} - \frac{1}{3}z$$

$$|h_1(t,z)| \leq \frac{t^2}{3} + \frac{1}{3}|z|, \ \forall\ t \in I,\ z \in R.$$

Where $b_1(t) = \frac{t^2}{3}$ *and* $\|b_1\|_1 = \int_0^1 \frac{t^2}{3}dt = 0.111111$.

Now, we have $T = 1,\ \beta = \frac{1}{3},\ L_{=}\frac{1}{3},\ K_1 = \frac{1}{4}$.
Then

$$L_1\frac{T^{\beta}}{\beta} - K_1\frac{T^{\beta}}{\Gamma(1+\beta)} = 0.71982 < 1.$$

Now all the conditions of Theorem 3.1 are satisfied, then the problems (1.3)-(1.4) or (1.3)-(1.5) has at least one solution $u \in L_1[0,1]$. And

$$f(t,z_1,z_2) = \frac{t^4}{1+t} + \frac{1}{2}z_1 + \frac{1}{2}z_2,$$

$$|f(t,z_1,z_2) - f(t,z_3,z_4)| \leq \frac{1}{2}(|z_1 - z_3| + |z_2 - z_4|), \ \forall\ t \in I,\ z_1,\ z_2 \in R.$$

Where $f(t,0,0) = \frac{t^4}{1+t} \in L_1[0,1]$.

In a similar manner, consider

$$h(t,z) = \frac{t^2}{3} + \frac{1}{2}z.$$

which satisfies

$$|h(t,z) - h(t,z_1)| \leq \frac{1}{2}|z - z_1|, \ \forall\ t \in I,\ z,\ z_1 \in R.$$

Moreover, $h(t,0) = \frac{t^2}{3} \in L_1[0,1]$.
we obtain $T = 1,\ \lambda = \frac{1}{2},\ \alpha = \frac{1}{2},\ c = \frac{1}{4},\ K = L = \frac{1}{2},\ \beta = \frac{1}{3}$.
Then

$$K|\lambda|\frac{cT^{2-\alpha}}{\Gamma(3-\alpha)} = 0.0625 < 1.$$

Consequently, all the conditions of Theorem 3.1 are satisfied, then the problem (1.1)-(1.2) has a unique solution $x \in AC[0,1]$.

7. Conclusion

The fractional order derivatives extend the concept of classical derivatives to non-integer orders. In this paper, we have considered the constrained problem of a nonlocal integral problem of a functional integro- differential (mixed integer and fractional (1.1)-(1.2) subject to the Riemann-Liouville fractional order nonlinear constraint (1.3)-(1.4) (or (1.3) and (1.5)). By using fixed point theorems within suitable spaces, we clearly established the existence and uniqueness of solutions. Furthermore, we have proved that for every solution $u \in L_1(I)$ of the constraint, there exists a unique solution x in the class $AC(I)$ of the initial value problem of the non-local functional integro-differential equation. Our analysis confirmed the continuous dependence of these solutions on initial data, parameters, and functional terms, thus preserving model consistency and reliability. Moreover, we analyzed the Hyers-Ulam stability and the continuous dependence of the solution on the initial condition x_0, the parameter λ and some functions $u(t)$ and $\phi(t)$. Additionally, the investigation of Hyers–Ulam stability further confirms the strength of the fractional model, proving that small disturbances in parameters or initial values do not considerably affect the solution trajectory, which is crucial for the practical implementation of fractional models in real-world applications.

The presented example verifies the usefulness of the theoretical framework, which generalizes existing results in the field of fractional differential equations with constraints. Overall, this study not only adds to the fundamental theory of fractional calculus but also opens the way for further contributions to the advancement of fractional differential theory in both theoretical and applied mathematical contexts by presenting new analytical tools for understanding complex dynamical behavior in nonlocal and memory-dependent systems.

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