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Generalized Cigler's Polynomials with Generalized Homogeneous q-Shift Operator

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Abstract. This study employs the generalized homogeneous q-shift operator, defined as ${}_{r+1}\Phi_s\left(\begin{matrix} q^{-\alpha}, \alpha_1, \cdots, \alpha_r \\ \beta_1, \cdots, \beta_s \end{matrix}; q, q^{\alpha}\rho D_{\chi\psi}\right)$. Subsequently, by utilizing this operator, we derive several polynomials' q-identities. These derived identities include the generating function, the Rogers' formula, Mehler's formula along with its extension, and three distinct mixed generating functions for the generalized Cigler's polynomials $C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q)$. Furthermore, we establish a transformational identity linking various generating functions specifically for $C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q)$. Finally, by proposing certain specific values for $C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q)$, we establish new identities for the extended Verma-Jain polynomials $V_n^{(\mathbf{a};\mathbf{c})}(x, y, z|q)$ and a related class of generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a};\mathbf{b})}(x, y, z|q)$.

1. Introduction

The present work adopts the standard notations and definitions pertaining to q-series, consistent with those established in [18]. It should be noted that the constraint $0 < |q| < 1$ is imposed. The q-shifted factorial is formally defined with $b \in \mathbb{C}$ as referenced in [17, 18]

$$(b;q)_0 = 1, \quad (b;q)_n = (1-b)(1-bq)...(1-bq^{n-1}), \quad (b;q)_\infty = \prod_{k=0}^{\infty}(1-bq^k).$$

The several q-shifted factorials, in addition to [8]

$$(b_1, b_2, \ldots, b_r; q)_n = (b_1;q)_n (b_2;q)_n \cdots (b_r;q)_n,$$

such that $n \in \mathbb{Z}$ or ∞.

The basic hypergeometric series ${}_r\phi_s$ is presented as follows [2, 6, 18]:

$${}_r\phi_k\left(\begin{matrix} b_1, \ldots, b_r \\ c_1, \ldots, c_k \end{matrix}; q, x\right) = 1 + \sum_{n=1}^{\infty} \frac{(b_1, \ldots, b_r; q)_n}{(q, c_1, \ldots, c_k; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+k-r} x^n,$$

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where $q \neq 0$ when $r > k+1$. Note that [30]

$$ {}_{r+1}\phi_r\left(\begin{array}{c} b_1,\ldots,b_{r+1} \\ c_1,\ldots,c_r \end{array};q,x\right) = 1+\sum_{n=1}^{\infty}\frac{(b_1,\ldots,b_{r+1};q)_n}{(q,c_1,\ldots,c_r;q)_n}x^n, \quad |x|<1. $$

The q-binomial coefficient is defined according to the expression

$$ \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, \quad 0 \leqslant k \leqslant n, $$

for which the condition $n, k \in \mathbb{N}$ must hold.

$$ \begin{bmatrix} \beta \\ k \end{bmatrix} = \frac{(q^{-\beta};q)_k}{(q;q)_k}\, q^{\beta k}(-1)^k q^{-\binom{k}{2}}, \quad \beta \in \mathbb{R}. \tag{1.1} $$

Note that

$$ \lim_{\beta\to\infty}\begin{bmatrix} \beta \\ k \end{bmatrix}(q;q)_k = 1. $$

We utilize the expression for the Cauchy identity, which is referenced in [10, 18, 32]:

$$ \sum_{k=0}^{\infty}\frac{(y;q)_k}{(q;q)_k}\,x^k = \frac{(yx;q)_\infty}{(x;q)_\infty}, \quad |x|<1. $$

When the parameter y is set to zero, the Cauchy identity reduces to Euler's identity, as indicated in [18]:

$$ \sum_{k=0}^{\infty}\frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}, \quad |x|<1, \tag{1.2} $$

The q-Chu-Vandermonde sum is represented by the following identity [18, 21, 22]:

$$ {}_2\phi_1(q^{-n},x;w;q,q) = \frac{(w/x;q)_n}{(w;q)_n}\,x^n. \tag{1.3} $$

The following identity represents the transformation for the ${}_2\phi_1$ series, a result originally provided by Jackson [18, Appendix III, equation (III.4)]:

$$ {}_2\phi_1\left(\begin{array}{c} x,y \\ w \end{array};q,z\right) = \frac{(xz;q)_\infty}{(z;q)_\infty}\,{}_2\phi_2\left(\begin{array}{c} x,w/y \\ w,xz \end{array};q,yz\right), \quad |z|<1. \tag{1.4} $$

Putting $x=0$, $y \longrightarrow w/y$, then $z = yd/w$ in equation (1.4) leads to

$$ {}_1\phi_1\left(\begin{array}{c} y \\ w \end{array};q,d\right) = (yd/w;q)_\infty\,{}_2\phi_1\left(\begin{array}{c} 0,w/y \\ w \end{array};q,yd/w\right), \quad |yd/w|<1. \tag{1.5} $$

Heine's transformation of the ${}_2\phi_1$ series [18, Appendix III] is:

$$ {}_2\phi_1\left(\begin{array}{c} x,y \\ w \end{array};q,z\right) = \frac{(y,xz;q)_\infty}{(w,z;q)_\infty}\,{}_2\phi_1\left(\begin{array}{c} w/y,z \\ xz \end{array};q,y\right), \quad \max\{|z|,|y|\}<1. \tag{1.6} $$

The identities presented below, taken from [18, 26, 27], are fundamental relations employed throughout this work:

$$ (x;q)_k = (q^{1-k}/x;q)_k(-1)^k x^k q^{\binom{k}{2}}. \tag{1.7} $$

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}. \tag{1.8}$$

The Hahn polynomials [7,19,35] are defined as follows:

$$\phi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k \, x^k. \tag{1.9}$$

$$\psi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} (aq^{1-k};q)_k x^k. \tag{1.10}$$

In 1989, Srivastava and Agarwal established the generating functions for $\phi_n^{(a)}(x)$ and $\psi_n^{(a)}(x)$ respectively [34]. This result is also referenced in [11].

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x)(\lambda;q)_n \frac{\tau^n}{(q;q)_n} = \frac{(\lambda\tau;q)_\infty}{(\tau;q)_\infty} \, {}_2\phi_1\left(\begin{matrix} \lambda, a \\ \lambda\tau \end{matrix} ; q, x\tau \right), \quad \max\{|\tau|, |x\tau|\} < 1. \tag{1.11}$$

$$\sum_{n=0}^{\infty} \psi_n^{(a)}(x)(1/\lambda;q)_n \frac{(\lambda q\tau)^n}{(q;q)_n} = \frac{(xq\tau;q)_\infty}{(\lambda xq\tau;q)_\infty} \, {}_2\phi_1\left(\begin{matrix} 1/\lambda, 1/(ax) \\ 1/(\lambda x\tau) \end{matrix} ; q, aq \right), \tag{1.12}$$

where $\max\{|\lambda x\tau q|, |aq|\} < 1$.

- The formula of Mehler for $\phi_n^{(a)}(x)$ is [5]

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x)\phi_n^{(b)}(y) \frac{\tau^n}{(q;q)_n} = \frac{(ax, by\tau;q)_\infty}{(\tau, x, y\tau;q)_\infty} \, {}_3\phi_2\left(\begin{matrix} \tau, y\tau, a \\ q/x, by\tau \end{matrix} ; q, q \right), \; \max\{|\tau|, |x|, |y\tau|\} < 1.$$

$$= \frac{(ax\tau, by\tau;q)_\infty}{(\tau, x\tau, y\tau;q)_\infty} \, {}_3\phi_2\left(\begin{matrix} a, b, \tau \\ ax\tau, by\tau \end{matrix} ; q, xy\tau \right), \tag{1.13}$$

such that $\max\{|\tau|, |x\tau|, |y\tau|, |xy\tau|\} < 1$.

The q-differential operator was defined by [16,24,25]

$$D_q\left(f(x)\right) = \frac{f(x) - f(xq)}{x}.$$

The Leibniz rule for D_q is [28]

$$D_q^n\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(x)\} D_q^{n-k}\{g(xq^k)\}. \tag{1.14}$$

Chen and Liu introduced the q-exponential operator [16], which has the following definition:

$$T(aD_q) = \sum_{k=0}^{\infty} \frac{(aD_q)^k}{(q;q)_k}.$$

The proof of the following is simple [1,37]

$$T(D_q)\{x^n\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k}. \tag{1.15}$$

$$D_q^k\left\{ \frac{(xt,q)_\infty}{(xv;q)_\infty} \right\} = P_k(v,t) \frac{(xtq^k,q)_\infty}{(xv;q)_\infty}. \tag{1.16}$$

The following is the definition of Cauchy polynomials [29,31,33]:

$$P_n(x,y)=(y/x;q)_n\,x^n=(x-y)(x-qy)\cdots(x-q^{n-1}y),$$

which have the generating function [3,4,20]

$$\sum_{n=0}^{\infty}P_n(x,y)\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(xt;q)_\infty},\quad |xt|<1. \tag{1.17}$$

In 2003, Chen et al. [15] initially introduced the homogeneous q-difference operator D_{xy}. The action of this operator on functions involving two variables is defined as follows:

$$D_{xy}\left(f(x,y)\right)=\frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y}.$$

The construction of the homogeneous q-shift operator was as follows *:*

$$\mathbb{E}(D_{xy})=\sum_{k=0}^{\infty}\frac{D_{xy}^k}{(q;q)_k}.$$

Proposition 1.1. [15] *The following conclusions are determined*

$$D_{xy}^k\left\{P_n(x,y)\right\}=\frac{(q;q)_n}{(q;q)_{n-k}}\,P_{n-k}(x,y). \tag{1.18}$$

$$D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}=t^k\frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{1.19}$$

In 2015, Cao's research focused on the investigation of the extended Verma-Jain polynomials [13,35].

$$V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(a_1,\ldots,a_r;q)_k}{(c_1,\ldots,c_u;q)_k}z^kP_{n-k}(x,y), \tag{1.20}$$

in which $\mathbf{a}=(a_1,\ldots,a_r)$, $\mathbf{c}=(c_1,\ldots,c_u)$. The subsequent results for $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$ were obtained by Cao [13] through the strategic application of the method of homogeneous q-difference equations.

- An extension to the generating function for the further extension of Verma-Jain polynomials $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$.

$$\sum_{n=0}^{\infty}V_{n+s}^{(\mathbf{a};\mathbf{c})}(x,y,z|q)\frac{t^n}{(q;q)_n}$$
$$=t^{-s}\frac{(yt;q)_\infty}{(xt;q)_\infty}\sum_{k=0}^{s}\frac{(q^{-s},xt;q)_kq^k}{(q,yt;q)_k}\,{}_r\phi_u\left(\begin{array}{c}a_1,\ldots,a_r\\c_1,\ldots,c_u\end{array};q,zq^kt\right), \tag{1.21}$$

  such that $\max\{|xt|,|zt|\}<1$.
- A generating function of the Srivastava-Agarwal type for extending Verma-Jain polynomials further$V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$

$$\sum_{n=0}^{\infty}V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)(\lambda;q)_n\frac{t^n}{(q;q)_n}$$

$$= \frac{(\lambda, yt;q)_\infty}{(xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(xt;q)_k}{(q,yt;q)_k} \lambda^k \, {}_r\phi_u \left(\begin{array}{c} a_1,\ldots,a_r \\ c_1,\ldots,c_u \end{array} ;q, zq^k t \right), \tag{1.22}$$

such that $max\left(|xt|, |ztq^k|\right) < 1$.

- Transformational identity that includes generating functions to extend Verma-Jain polynomials further $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$ is:

  The relationship between the coefficients $A(k)$ and $B(k)$ is assumed to satisfy the following generating form:

$$\sum_{k=0}^{\infty} A(k) P_k(x,y) = \sum_{k=0}^{\infty} B(k) \frac{(q^k yt;q)_\infty}{(q^k xt;q)_\infty}. \tag{1.23}$$

  Under this assumption, the following identity can then be established:

$$\sum_{k=0}^{\infty} A(k) V_k^{(\mathbf{a};\mathbf{c})}(x,y,z|q) = \sum_{k=0}^{\infty} B(k) \frac{(ytq^k;q)_\infty}{(xtq^k;q)_\infty} \, {}_r\phi_u \left(\begin{array}{c} a_1,\ldots,a_r \\ c_1,\ldots,c_u \end{array} ;q, zq^k t \right). \tag{1.24}$$

  It is stipulated that every series involved in both (1.23) and (1.24) possesses absolute convergence.

The collection of generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$ was examined in 2021 by Srivastava and Arjika [35]:

$$\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)(-1)^n q^{\binom{n}{2}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k P_{n-k}(y,x), \tag{1.25}$$

where $1 \leqslant r \leqslant s$, $\mathbf{a} = (a_1,\ldots,a_r)$, $\mathbf{b} = (b_1,\ldots,b_s)$.

The subsequent results concerning the polynomials $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$ were established by Srivastava and Arjika [35] as follows:

- An extension of the generating function for the generalized q-hypergeometric polynomials [35]$\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$ is given by the identity:

$$\sum_{k=0}^{\infty} \Psi_{k+u}^{(\mathbf{a};\mathbf{b})}(x,y,z|q) \frac{(-1)^{k+u} q^{\binom{k+u}{2}} t^k}{(q;q)_k}$$
$$= t^{-u} \frac{(xt;q)_\infty}{(yt;q)_\infty} \sum_{j=0}^{u} \frac{(q^{-u}, yt;q)_j q^j}{(q,xt;q)_j} \, {}_r\phi_s \left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array} ;q, zq^j t \right). \tag{1.26}$$

  This result is crucial for manipulating the generating relations in the subsequent sections of this study. where $|yt| < 1$.

- The Rogers' formula for [35] $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$ is

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \Psi_{n+k}^{(\mathbf{a};\mathbf{b})}(x,y,z|q)(-1)^{n+k} q^{\binom{n+k}{2}} \frac{t^n}{(q;q)_n} \frac{\omega^k}{(q;q)_k}$$
$$= \frac{(x\omega;q)_\infty}{(y\omega, t/\omega;q)_\infty} \sum_{j=0}^{\infty} \frac{(y\omega;q)_j q^j}{(q, q\omega/t, x\omega;q)_j} \, {}_r\phi_s \left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array} ;q, z\omega q^j \right), \tag{1.27}$$

such that $\max\left(|y\omega|, |t/\omega|\right) < 1$.

- The following identity represents the Srivastava-Agarwal type generating function for $\Psi_k^{(\mathbf{a};\mathbf{b})}(x, y, z|q)$:

$$\sum_{k=0}^{\infty} \psi_k^{(\alpha)}(x|q) \Psi_k^{(\mathbf{a},\mathbf{b})}(u, v, z|q) \frac{(-1)^k q^{\binom{k+1}{2}} t^k}{(q;q)_k}$$
$$= \frac{(q/x, uxtq; q)_\infty}{(\alpha q, vxtq; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (1/(\alpha x), 1/(uxt); q)_k}{(q, q/x, 1/vxt; q)_k} \left(\frac{\alpha u q}{v}\right)^k$$
$$\times {}_r\phi_s\left(\begin{array}{c} a_1, a_2, \ldots, a_r; \\ b_1, b_2, \ldots, b_s; \end{array} q; q^{1-k} xzt \right), \tag{1.28}$$

such that $\max\left(|\alpha q|, |vxtq|\right) < 1$.

- The Mehler's formula for $\psi_k^{(\alpha)}(x|q)$ [35]is

$$\sum_{k=0}^{\infty} \psi_k^{(\alpha)}(x|q) \psi_k^{(a)}(y|q) \frac{(-1)^k q^{\binom{k+1}{2}} t^k}{(q;q)_k} = \frac{(q/x, xytq, xtq; q)_\infty}{(\alpha q, axytq; q)_\infty}$$
$$\times\ {}_3\phi_2\left(\begin{array}{c} 1/(\alpha x),\ 1/(xyt),\ 1/(xt) \\ q/x,\ 1/(axyt) \end{array} ; q,\ \frac{\alpha x t q}{a} \right), \tag{1.29}$$

such that $\max\{|\alpha q|, |axytq|\} < 1$.

- A transformational identity concerning the generating functions for the collection of generalized q-hypergeometric polynomials $\Psi_k^{(\mathbf{a};\mathbf{b})}(x, y, z|q)$ is established below. This identity is derived by assuming that the coefficients $A(k)$ and $B(k)$ satisfy the relationship presented hereafter [35]:

$$\sum_{k=0}^{\infty} A(k) P_k(v, u) = \sum_{k=0}^{\infty} B(k) \frac{(xutq^{1-k}; q)_\infty}{(xvtq^{1-k}; q)_\infty}. \tag{1.30}$$

Then;

$$\sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} A(k) \Psi_k^{(a,b)}(u, v, z|q)$$
$$= \sum_{k=0}^{\infty} B(k) \frac{(xutq^{1-k}; q)_\infty}{(xvtq^{1-k}; q)_\infty}\ {}_r\phi_s\left(\begin{array}{c} a_1, a_2, \ldots, a_r; \\ b_1, b_2, \ldots, b_s; \end{array} q; q^{1-k} xzt \right), \tag{1.31}$$

provided that absolute convergence holds for every series involved in both (1.30) and (1.31).

The following Cigler's polynomials were examined by Wang and Cao [36] in 2018:

$$C_n^{(\alpha-n)}(\chi, \psi, \rho|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} \alpha \\ k \end{bmatrix} (q;q)_k (-1)^k q^{\binom{k}{2}} \rho^k P_{n-k}(\chi, \psi). \tag{1.32}$$

They discovered the following conclusions for Cigler's polynomials using homogeneous q-difference equations [36].

- The generating function for $C_n^{(\alpha-n)}(\chi,\psi,\rho|q)$ is

$$\sum_{k=0}^{\infty} C_k^{(\alpha-k)}(\chi,\psi,\rho|q)\frac{t^k}{(q;q)_k}=\frac{(\psi t;q)_\infty(\rho t;q)_\alpha}{(\chi t;q)_\infty},\quad \max\{|\chi t|,|\rho t q^\alpha|\}<1. \tag{1.33}$$

- An extension for $C_k^{(\alpha-k)}(\chi,\psi,\rho|q)$ is

$$\begin{aligned}&\sum_{k=0}^{\infty} C_{k+\sigma}^{(\alpha-k-\sigma)}(\chi,\psi,\rho|q)\frac{t^k}{(q;q)_k}\\ &=t^{-\sigma}\frac{(\psi t;q)_\infty(\rho t;q)_\alpha}{(\chi t;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}q^{-\sigma},\chi t,\rho t q^\alpha\\ \psi t,\rho t\end{matrix};q,q\right),\quad |\chi t|<1.\end{aligned} \tag{1.34}$$

- The following identity represents the Srivastava-Agarwal type generating function for the polynomials $C_k^{(\alpha-k)}(\chi,\psi,\rho|q)$:

$$\sum_{k=0}^{\infty} C_k^{(\alpha-k)}(\chi,\psi,\rho|q)(\lambda;q)_k\frac{t^k}{(q;q)_k}=\frac{(\lambda,\rho t,\psi t;q)_\infty}{(\rho t q^\alpha,\chi t;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\rho t q^\alpha,\chi t,0\\ \rho t,\psi t\end{matrix};q,\lambda\right), \tag{1.35}$$

such that $\max\{|\chi t|,|\rho t q^\alpha|,|\lambda|\}<1$.

- The generating function for $C_k^{(\alpha-k)}(\chi,\psi,\rho|q)$ in a transformational identity are

It is assumed that the following relationship holds true for the coefficients $A(k)$ and $B(k)$:

$$\sum_{k=0}^{\infty} A(k)P_k(\chi,\psi)=\sum_{k=0}^{\infty} B(k)\frac{(q^k\psi t;q)_\infty}{(q^k\chi t;q)_\infty}. \tag{1.36}$$

Then

$$\sum_{k=0}^{\infty} A(k)C_k^{(\alpha-k)}(\chi,\psi,\rho|q)=\sum_{k=0}^{\infty} B(k)\frac{(\psi q^k t,\rho t q^k;q)_\infty}{(\chi q^k t,\rho t q^\alpha q^k;q)_\infty}, \tag{1.37}$$

provided that every series in both (1.36) and (1.37) maintains absolute convergence.

The structure of this paper is outlined as follows:

In Section 2, we introduce the generalized homogeneous q-shift operator ${}_{r+1}\Phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^\alpha\rho D_{\chi\psi}\right)$ and subsequently derive numerous related identities. Section 3 is dedicated to discussing the generating function and its comprehensive extension for the generalized Cigler's polynomials $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$, utilizing an operator technique. Section 4 presents the Rogers' formula, its extension, and addresses the inverse linearization formula. In Section 5, we provide a detailed discussion of Mehler's formula and its related extension. In section 6, we build three mixed generating functions for the generalized Cigler's polynomials $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$. In section 7, we conclude by establishing the transformational identity for the generalized Cigler's polynomials $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$.

2. The Generalized Homogeneous q-Shift Operator and Some Operator Identities

This section is devoted to introduce a generalized homogeneous q-shift operator and a generalized Cigler's polynomials. Throughout this section, we derive several key identities associated with this operator.

Definition 2.1. *The generalized homogeneous q-shift operator is formally defined as follows:*

$$\begin{aligned}
&{}_{r+1}\Phi_s\left(\begin{matrix} q^{-\alpha}, \alpha_1, \cdots, \alpha_r \\ \beta_1, \cdots, \beta_s \end{matrix}; q, q^{\alpha}\rho D_{\chi\psi}\right) \\
&\qquad = \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} W_k \left(\rho D_{\chi\psi}\right)^k \qquad (2.1)\\
&\qquad = \sum_{k=0}^{\infty} \frac{P_k(q^{\alpha}, 1)}{(q;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-(r+1)} W_k \left(\rho D_{\chi\psi}\right)^k. \qquad (2.2)
\end{aligned}$$

such that $1 \leqslant r \leqslant s$, $W_k = \dfrac{(\alpha_1, \ldots, \alpha_r; q)_k}{(\beta_1, \ldots, \beta_s; q)_k}$.

Certain values given to the homogeneous q-shift operator ${}_{r+1}\Phi_s$ yield a number of the previously described operators [12,14,15].

Throughout this section, we will employ the abbreviation ${}_{r+1}\Phi_s\left(q^{-\alpha}, \boldsymbol{\alpha}, \boldsymbol{\beta}, q^{\alpha}\rho D_{\chi\psi}\right)$ to denote the operator ${}_{r+1}\Phi_s\left(\begin{matrix} q^{-\alpha}, \alpha_1, \cdots, \alpha_r \\ \beta_1, \cdots, \beta_s \end{matrix}; q, q^{\alpha}\rho D_{\chi\psi}\right)$.

Definition 2.2. *We construct a generalization of Cigler's polynomials as follows:*

$$\begin{aligned}
&C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q) \\
&\qquad = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} \alpha \\ k \end{bmatrix} (q;q)_k \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} \frac{(\alpha_1, \ldots, \alpha_r; q)_k}{(\beta_1, \ldots, \beta_s; q)_k} \rho^k P_{n-k}(\chi, \psi), \qquad (2.3)
\end{aligned}$$

such that $1 \leqslant r \leqslant s$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$, $\boldsymbol{\alpha}^n = (\alpha_1 q^n, \ldots, \alpha_r q^n)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)$.

- By applying the specific constraints $r = s = 1$ and $\alpha_1 = \beta_1 = 0$ to equation (2.3), the Cigler's polynomials $C_n^{(\alpha-n)}(\chi, \psi, \rho|q)$, defined explicitly in (1.32), are derived.
- Letting $s = r = 1$, $\alpha_1 = \beta_1 = 0$, $\chi \to ax$, $\psi \to x$, $\rho = 1$, and $\alpha \to \infty$ in equation (2.3), then using equation (1.7), we get $(-1)^n q^{\binom{n}{2}} \psi_n^{(a)}(x|q)$, where $\psi_n^{(a)}(x|q)$ is defined in (1.10).
- We impose the restrictions on the indices as $1 + s - r = 1 + u - r = 0$ and set the parameters $\alpha_i = a_i$ and $\beta_i = c_i$ holding true for the range $1 \leqslant i \leqslant r \leqslant u$. Additionally, $\chi \to x$, $\rho \to z$, $\psi \to y$, and $\alpha \to \infty$ in equation (2.3), we obtain the extended Verma-Jain polynomials $V_n^{(\mathbf{a};\mathbf{c})}(x, y, z|q)$, as presented in equation (1.20).
- When $\alpha_i = a_i$ and $\beta_i = b_i$ for the constraint $1 \leqslant i \leqslant r \leqslant s$, and transforming the variables as $\chi \to y$, $\rho \to z$, $\psi \to x$ and $\alpha \to \infty$, we obtain the expression $(-1)^n q^{\binom{n}{2}} \Psi_n^{(\mathbf{a};\mathbf{b})}(x, y, z|q)$, where $\Psi_n^{(\mathbf{a};\mathbf{b})}(x, y, z|q)$ is defined in equation (1.25).

The subsequent lemma can be established:

Lemma 2.1. *We have:*

$$ {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{P_n(\chi,\psi)\right\}=C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q). \tag{2.4}$$

$$\begin{aligned} &{}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\frac{(\psi t;q)_\infty}{(\chi t;q)_\infty}\right\}\\ &\quad=\frac{(\psi t;q)_\infty}{(\chi t;q)_\infty}\,{}_{r+1}\phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha}\rho t\right),\quad |\chi t|<1. \end{aligned}\tag{2.5}$$

Proof.

$$\begin{aligned} &{}_{r+1}\Phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha}\rho D_{\chi\psi}\right)\left\{P_n(\chi,\psi)\right\}\\ &=\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}W_k\left(\rho D_{\chi\psi}\right)^k\left\{P_n(\chi,\psi)\right\}\\ &=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}\alpha\\k\end{bmatrix}(q;q)_k\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\frac{(\alpha_1,\ldots,\alpha_r;q)_k}{(\beta_1,\ldots,\beta_s;q)_k}\rho^kP_{n-k}(\chi,\psi)\quad\text{by using (1.18)}\\ &=C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q), \end{aligned}$$

Equation (2.4) has been proved. In the same way, the proof of Equation (2.5) follows:

$$\begin{aligned} &{}_{r+1}\Phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha}\rho D_{\chi\psi}\right)\left\{\frac{(\psi t;q)_\infty}{(\chi t;q)_\infty}\right\}\\ &=\sum_{k=0}^{\infty}\frac{P_k(q^{\alpha},1)}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-(r+1)}W_k\left(\rho D_{\chi\psi}\right)^k\left\{\frac{(\psi t;q)_\infty}{(\chi t;q)_\infty}\right\}\\ &=\frac{(\psi t;q)_\infty}{(\chi t;q)_\infty}\,{}_{r+1}\phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha}\rho t\right),\quad\text{by using (1.19).} \end{aligned}$$

□

Lemma 2.2. *Given the definition of the operator ${}_{r+1}\Phi_s$ as explicitly defined in* (2.1)*, The following results are established as a consequence:*

$$\begin{aligned} &{}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\frac{P_k(\chi t,\psi t)(\psi t;q)_\infty}{(\psi t;q)_k(\chi t;q)_\infty}\right\}=\frac{(\psi t;q)_\infty}{(\chi t;q)_\infty}\sum_{i=0}^{k}\frac{(q^{-k},\chi t;q)_i}{(q,\psi t;q)_i}\,q^i\\ &\times\,{}_{r+1}\phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+i}\rho t\right),\quad\max\{|\rho t|,|\chi t|\}<1. \end{aligned}\tag{2.6}$$

Proof. Employing the q-Chu-Vandermonde sum (1.3), we find

$$\frac{P_k(\chi t,\psi t)(\psi t;q)_\infty}{(\psi t;q)_k(\chi t;q)_\infty}=\frac{(\psi t;q)_\infty(\psi t/\chi t;q)_k(\chi t)^k}{(\chi t;q)_\infty(\psi t;q)_k}$$

$$= \frac{(\psi t;q)_\infty}{(\chi t;q)_\infty} {}_2\phi_1(q^{-k},\chi t;\psi t;q,q)$$
$$= \sum_{i=0}^{k} \frac{(q^{-k};q)_i}{(q;q)_i} \frac{(q^i\psi t;q)_\infty}{(q^i\chi t;q)_\infty} q^i. \tag{2.7}$$

The following expression is derived after operating on both sides of equation (2.7) with the operator ${}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)$:

$$\begin{aligned}
&{}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\frac{P_k(\chi t,\psi t)(\psi t;q)_\infty}{(\psi t;q)_k(\chi t;q)_\infty}\right\}\\
&= \sum_{i=0}^{k}\frac{(q^{-k};q)_i}{(q;q)_i} q^i\, {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\frac{(q^i\psi t;q)_\infty}{(q^i\chi t;q)_\infty}\right\}\\
&= \sum_{i=0}^{k}\frac{(q^{-k};q)_i q^i}{(q;q)_i}\frac{(\psi t q^i;q)_\infty}{(\chi t q^i;q)_\infty}\, {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+i}\rho t\right) \quad \text{(by using (2.4))}\\
&= \frac{(\psi t;q)_\infty}{(\chi t;q)_\infty}\sum_{i=0}^{k}\frac{(q^{-k},\chi t;q)_i q^i}{(q,\psi t;q)_i}\, {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+i}\rho t\right).
\end{aligned}$$

□

3. Generating Function for $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$

Dedicated to the generalized Cigler polynomials, $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$, this segment utilizes the operator ${}_{r+1}\Phi_s$ to establish both the generating function and its pertinent extension.

Theorem 3.1.

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{n+\sigma}^{(\alpha-n-\sigma)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)\frac{t^n}{(q;q)_n} &= \frac{t^{-\sigma}(\psi t;q)_\infty}{(\chi t;q)_\infty}\sum_{i=0}^{\sigma}\frac{(q^{-\sigma},\chi t;q)_i q^i}{(q,\psi t;q)_i}\\
&\times {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+i}\rho t\right), \quad |\chi t|<1.
\end{aligned}\tag{3.1}$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_{n+\sigma}^{(\alpha-n-\sigma)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)\frac{t^n}{(q;q)_n}\\
&= t^{-\sigma}\sum_{n=0}^{\infty}\, {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{P_{n+\sigma}(\chi,\psi)\right\}\frac{t^{n+\sigma}}{(q;q)_n} \quad \text{(by using (2.4))}\\
&= t^{-\sigma}\, {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{n=0}^{\infty}P_{n+\sigma}(\chi,\psi)\frac{t^{n+\sigma}}{(q;q)_n}\right\}\\
&= t^{-\sigma}\, {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{P_\sigma(\chi t,\psi t)\sum_{n=0}^{\infty}P_n(\chi,q^{\sigma}\psi)\frac{t^n}{(q;q)_n}\right\}
\end{aligned}$$

$$= t^{-\sigma} \ {}_{r+1}\Phi_s\left(q^{-\alpha}, \boldsymbol{\alpha}, \boldsymbol{\beta}, q^{\alpha}\rho D_{\chi\psi}\right)\left\{P_\sigma(\chi t, \psi t)\frac{(q^\sigma \psi t;q)_\infty}{(\chi t;q)_\infty}\right\} \quad \text{(by using (1.17))}$$

$$= t^{-\sigma} \ {}_{r+1}\Phi_s\left(q^{-\alpha}, \boldsymbol{\alpha}, \boldsymbol{\beta}, q^{\alpha}\rho D_{\chi\psi}\right)\left\{\frac{P_\sigma(\chi t, \psi t)}{(\psi t;q)_\sigma}\frac{(\psi t;q)_\infty}{(\chi t;q)_\infty}\right\}$$

$$= \frac{t^{-\sigma}(\psi t;q)_\infty}{(\chi t;q)_\infty}\sum_{i=0}^{\sigma}\frac{(q^{-\sigma}, \chi t;q)_i}{(q, \psi t;q)_i}q^i$$

$$\times \ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha}, \alpha_1, \cdots, \alpha_r \\ \beta_1, \cdots, \beta_s \end{matrix}; q, q^{\alpha+i}\rho t\right). \quad \text{(by using (2.6))}$$

□

- When the parameter σ is set to zero in equation (3.1), the generating function for $C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q)$ is immediately retrieved.

Corollary 3.1. *The subsequent results are presented:*

$$\sum_{n=0}^{\infty} C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q)\frac{t^n}{(q;q)_n} = \frac{(\psi t;q)_\infty}{(\chi t;q)_\infty} \ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha}, \alpha_1, \cdots, \alpha_r \\ \beta_1, \cdots, \beta_s \end{matrix}; q, q^{\alpha}\rho t\right), \tag{3.2}$$

where $|\chi t| < 1$.

- By applying the following specific set of conditions and substitutions to identity (3.1), the extended generating function for the Verma-Jain polynomials $V_n^{(\mathbf{a};\mathbf{c})}(x, y, z|q)$ is obtained (as presented in equation (1.21)): The constraints on the indices are set as $1 + s - r = 1 + u - r = 0$; the parameters are equated such that $\alpha_i = a_i$ and $\beta_i = c_i$ for the range $1 \leqslant i \leqslant r \leqslant u$; and the variables undergo the transformations $\chi \to x$, $\psi \to y$, $\rho \to z$,$\sigma \to s$ along with the limiting condition $\alpha \to \infty$.
- By applying the following specific set of conditions and substitutions to identity (3.1), an extension to the generating function for the class of generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a};\mathbf{b})}(x, y, z|q)$ is obtained, as represented by equation (1.26). The conditions involve setting the parameters $\alpha_i = a_i$ and $\beta_i = b_i$ for the constraint $1 \leqslant i \leqslant r \leqslant s$, transforming the variables as $\chi \to y$, $\psi \to x$, $\rho \to z$,$\sigma \to u$ and evaluating the expression under the limit condition $\alpha \to \infty$.
- When the parameters in equation (3.1) are restricted such that $\alpha_i = 0$, $\beta_i = 0$ and $s - r = 0$ for $1 \leqslant i \leqslant r \leqslant s$, the extended generating function for Cigler's polynomials $C_n^{(\alpha-n)}(\chi, \psi, \rho|q)$ is consequently obtained (equation (1.34)).

4. Rogers' Formula for $C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q)$

This section is dedicated to exploring Rogers' formula and its corresponding extension, alongside the inverse linearization formula, specifically for the generalized Cigler's polynomials $C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q)$. Moreover, we establish the Rogers' formula, its extension, and the inverse formula for the further extension of Verma-Jain polynomials $V_n^{(\mathbf{a};\mathbf{c})}(x, y, z|q)$ and the class of the

q-hypergeometric polynomials $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$. These identities are derived by utilizing specific transformations and substitutions for the variables of the generalized Cigler's polynomials.

Regarding the generalized Cigler's polynomials $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$, two distinct proofs for the subsequent Rogers' formula were developed:

Theorem 4.1. (Rogers' Formula for $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$). *The formula is stated as follows:*

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} C_{n+k}^{(\alpha-n-k)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)\frac{\tau^n}{(q;q)_n}\frac{\sigma^k}{(q;q)_k}$$
$$= \frac{(\psi\sigma;q)_\infty}{(\chi\sigma,\tau/\sigma;q)_\infty}\sum_{i=0}^{\infty}\frac{q^i(\chi\sigma;q)_i}{(q\sigma/\tau,q,\psi\sigma;q)_i}\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+i}\rho\sigma\right), \tag{4.1}$$

such that $\max(|\chi\sigma|,|\tau/\sigma|)<1$.

$$= \frac{(\psi\tau;q)_\infty}{(\chi\tau;q)_\infty}\sum_{j=0}^{\infty}\frac{(q^{-\alpha},\chi\tau,\alpha_1,\cdots,\alpha_r;q)_j}{(q,\psi\tau,\beta_1,\cdots,\beta_s;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r}(q^\alpha\rho\sigma)^j\ {}_2\phi_1\left(\begin{matrix}\psi/\chi,0\\ \psi\tau q^j\end{matrix};q,\chi\sigma\right)$$
$$\times\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha+j},\alpha_1 q^j,\cdots,\alpha_r q^j \\ \beta_1 q^j,\cdots,\beta_s q^j\end{matrix};q,q^{\alpha+j(s-r)}\rho\tau\right),$$

such that $|\chi\tau|<1$.

First proof

Proof.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n+k}^{(\alpha-n-k)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)\frac{\tau^n}{(q;q)_n}\frac{\sigma^k}{(q;q)_k}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\ {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^\alpha\rho D_{\chi\psi}\right)\left\{P_{n+k}(\chi,\psi)\right\}\frac{\tau^n}{(q;q)_n}\frac{\sigma^k}{(q;q)_k}\quad \text{(by using (2.4))}$$
$$= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^\alpha\rho D_{\chi\psi}\right)\left\{\sum_{n=0}^{\infty}P_n(\chi,\psi)\frac{\tau^n}{(q;q)_n}\sum_{k=0}^{\infty}P_k(\chi,q^n\psi)\frac{\sigma^k}{(q;q)_k}\right\}\quad \text{(by using (1.17))}$$
$$= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^\alpha\rho D_{\chi\psi}\right)\left\{\sum_{n=0}^{\infty}P_n(\chi,\psi)\frac{(q^n\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\frac{\tau^n}{(q;q)_n}\right\}$$
$$= \sum_{n=0}^{\infty}\frac{\tau^n}{(q;q)_n}\ {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^\alpha\rho D_{\chi\psi}\right)\left\{\frac{P_n(\chi,\psi)}{(\psi\sigma;q)_n}\frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\right\}$$
$$= \frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\sum_{n=0}^{\infty}\frac{\tau^n\sigma^{-n}}{(q;q)_n}\sum_{i=0}^{n}\frac{(q^{-n},\chi\sigma;q)_i q^i}{(q,\psi\sigma;q)_i}\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+i}\rho\sigma\right)$$

(by using (2.6))

$$= \frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\sum_{i=0}^{\infty}\sum_{n=i}^{\infty}\frac{\tau^n\sigma^{-n}}{(q;q)_{n-i}}\frac{(-1)^i q^{\binom{i}{2}}q^{-ni}(\chi\sigma;q)_i q^i}{(q,\psi\sigma;q)_i}\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+i}\rho\sigma\right)$$

(by using (1.8))

$$= \frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty} \sum_{i=0}^{\infty} \frac{\tau^i \sigma^{-i}(-1)^i q^{\binom{i}{2}} q^{-i^2} (\chi\sigma;q)_i q^i}{(q,\psi\sigma;q)_i} \sum_{n=0}^{\infty} \frac{(q^{-i}\tau/\sigma)^n}{(q;q)_n}$$

$$\times\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha+i}\rho\sigma\right)$$

$$= \frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty} \sum_{i=0}^{\infty} \frac{\tau^i \sigma^{-i}(-1)^i q^{\binom{i}{2}} q^{-i^2} (\chi\sigma;q)_i q^i}{(q,\psi\sigma;q)_i} \frac{1}{(q^{-i}\tau/\sigma;q)_\infty}$$

$$\times\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha+i}\rho\sigma\right) \quad \text{(by using (1.2))}$$

$$= \frac{(\psi\sigma;q)_\infty}{(\chi\sigma,\tau/\sigma;q)_\infty} \sum_{i=0}^{\infty} \frac{\tau^i \sigma^{-i}(-1)^i q^{-\binom{i}{2}} (\chi\sigma;q)_i}{(-1)^i(\tau/\sigma)^i q^{-\binom{i}{2}-i}(q\sigma/\tau,q,\psi\sigma;q)_i}$$

$$\times\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha+i}\rho\sigma\right) \quad \text{(by using (1.7))}$$

$$= \frac{(\psi\sigma;q)_\infty}{(\chi\sigma,\tau/\sigma;q)_\infty} \sum_{i=0}^{\infty} \frac{q^i(\chi\sigma;q)_i}{(q\sigma/\tau,q,\psi\sigma;q)_i}\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha+i}\rho\sigma\right).$$

□

Second proof

Proof.

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} C_{n+k}^{(\alpha-n-k)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)\frac{\tau^n}{(q;q)_n}\frac{\sigma^k}{(q;q)_k}$$

$$= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} C_{n}^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)\frac{(q;q)_n\tau^{n-k}}{(q;q)_n(q;q)_{n-k}}\frac{\sigma^k}{(q;q)_k}$$

$$= \sum_{n=0}^{\infty}\frac{C_{n}^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\sigma^k\tau^{n-k}$$

$$= \sum_{n=0}^{\infty}\frac{C_{n}^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)}{(q;q)_n}T(\sigma D_q)\left\{\tau^n\right\} \quad \text{(by using (1.15))} \tag{4.2}$$

$$= T(\sigma D_q)\left\{\frac{(\psi\tau;q)_\infty}{(\chi\tau;q)_\infty}\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha}\rho\tau\right)\right\} \quad \text{(by using (3.2))}$$

$$= \sum_{k=0}^{\infty}\frac{(\sigma D_q)^k}{(q;q)_k}\left\{\frac{(\psi\tau;q)_\infty}{(\chi\tau;q)_\infty}\sum_{n=0}^{\infty}\frac{(q^{-\alpha},\alpha_1,\ldots,\alpha_r;q)_n}{(q,\beta_1,\beta_2,\ldots,\beta_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{s-r}(q^{\alpha}\rho\tau)^n\right\}$$

$$= \sum_{k=0}^{\infty}\frac{\sigma^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{(q^{-\alpha},\alpha_1,\ldots,\alpha_r;q)_n}{(q,\beta_1,\beta_2,\ldots,\beta_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{s-r}(q^{\alpha}\rho)^n D_q^k\left\{\tau^n\frac{(\psi\tau;q)_\infty}{(\chi\tau;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty} \frac{\sigma^k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(q^{-\alpha}, \alpha_1, \ldots, \alpha_r; q)_n}{(q, \beta_1, \beta_2, \ldots, \beta_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} (q^\alpha \rho)^n \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{j(j-k)} D^j \left\{ \tau^n \right\}$$

$$\times D^{k-j} \left\{ \frac{(\psi \tau q^j; q)_\infty}{(\chi \tau q^j; q)_\infty} \right\} \qquad \text{(by using(1.14))}$$

$$= \sum_{k=0}^{\infty} \sigma^k \sum_{n=0}^{\infty} \frac{(q^{-\alpha}, \alpha_1, \cdots, \alpha_r; q)_n}{(\beta_1, \cdots, \beta_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} (q^\alpha \rho)^n \sum_{j=0}^{k} \frac{\tau^{n-j} P_{k-j}(\chi, \psi)(\chi \tau; q)_j}{(q;q)_{k-j}(q;q)_j(q;q)_{n-j}}$$

$$\times \frac{(\psi \tau; q)_\infty}{(\psi \tau; q)_k (\chi \tau; q)_\infty} \quad \text{(by using (1.15))} \tag{4.3}$$

$$= \frac{(\psi \tau; q)_\infty}{(\chi \tau; q)_\infty} \sum_{j=0}^{\infty} \sum_{n+j=0}^{\infty} \sum_{k+j=j}^{\infty} \frac{(q^{-\alpha}, \alpha_1, \cdots, \alpha_r; q)_{n+j}}{(\beta_1, \cdots, \beta_s; q)_{n+j}} \frac{\sigma^{k+j} (q^\alpha \rho)^{n+j} \tau^n P_k(\chi, \psi)(\chi \tau; q)_j}{(q;q)_k (q;q)_j (q;q)_n (\psi \tau; q)_{k+j}}$$

$$\times \left[(-1)^{n+j} q^{\binom{n+j}{2}}\right]^{s-r}$$

$$= \frac{(\psi \tau; q)_\infty}{(\chi \tau; q)_\infty} \sum_{j=0}^{\infty} \frac{(q^\alpha \rho \sigma)^j (\chi \tau; q)_j}{(q;q)_j (\psi \tau; q)_j} \sum_{n=-j}^{\infty} \sum_{k=0}^{\infty} \frac{(\chi \sigma)^k (\psi/\chi; q)_k}{(q;q)_k (\psi \tau q^j; q)_k} \left[(-1)^{n+j} q^{\binom{n+j}{2}}\right]^{s-r}$$

$$\times \frac{(q^{-\alpha+j}, \alpha_1 q^j, \ldots, \alpha_r q^j; q)_n (q^{-\alpha}, \alpha_1, \ldots, \alpha_r; q)_j}{(q, \beta_1 q^j, \beta_2 q^j, \ldots, \beta_s q^j; q)_n (\beta_1, \beta_2, \ldots, \beta_s; q)_j} (q^\alpha \rho \tau)^n.$$

$$= \frac{(\psi \tau; q)_\infty}{(\chi \tau; q)_\infty} \sum_{j=0}^{\infty} \frac{(q^{-\alpha}, \chi \tau, \alpha_1, \cdots, \alpha_r; q)_j}{(q, \psi \tau, \beta_1, \cdots, \beta_s; q)_j} \left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} (q^\alpha \rho \sigma)^j \, {}_2\phi_1 \left(\begin{matrix} \psi/\chi, 0 \\ \psi \tau q^j \end{matrix} ; q, \chi \sigma \right)$$

$$\times \, {}_{r+1}\phi_s \left(\begin{matrix} q^{-\alpha+j}, \alpha_1 q^j, \cdots, \alpha_r q^j \\ \beta_1 q^j, \cdots, \beta_s q^j \end{matrix} ; q, q^{\alpha + j(s-r)} \rho \tau \right)$$

□

The aforementioned Rogers' formula allows for the direct derivation of the inverse linearization formula corresponding to $C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q)$.

Theorem 4.2. (Inverse linearization formula for $C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q)$). *Assuming k and n are non-negative integers, we state the following relationship:*

$$C_{k+n}^{(\alpha-k-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} P_{k-j}(\chi, \psi)(q^\alpha \rho)^j \left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} \frac{(q^{-\alpha}, \alpha_1, \cdots, \alpha_r; q)_j}{(\beta_1, \cdots, \beta_s; q)_j}$$

$$\times C_n^{(\alpha-j-n)}(\boldsymbol{\alpha}^j, \boldsymbol{\beta}^j, \chi q^j, \psi q^k, \rho q^{j(1+s-r)}|q). \tag{4.4}$$

Proof. We commence the derivation by utilizing the following equation, designated as (4.3):

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} C_{n+k}^{(\alpha-n-k)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q) \frac{\tau^n}{(q;q)_n} \frac{\sigma^k}{(q;q)_k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(q^{-\alpha}, \alpha_1, \cdots, \alpha_r; q)_n (q^\alpha \rho)^n \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} \tau^{n-j} \sigma^k P_{k-j}(\chi, \psi)(\psi \tau q^k; q)_\infty}{(\beta_1, \cdots, \beta_s; q)_n (q;q)_{k-j} (q;q)_j (q;q)_{n-j} (\chi \tau q^j; q)_\infty}$$

$$= \sum_{n+j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(q^{\alpha}\rho)^{n+j}(q^{-\alpha},\alpha_1,\cdots,\alpha_r;q)_{n+j}P_{k-j}(\chi,\psi)(\psi\tau q^k;q)_\infty}{(\beta_1,\cdots,\beta_s;q)_{n+j}(q;q)_{k-j}(q;q)_j(q;q)_n(\chi\tau q^j;q)_\infty}\sigma^k\tau^n$$

$$\times\left[(-1)^{n+j}q^{\binom{n+j}{2}}\right]^{s-r}$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{P_{k-j}(\chi,\psi)(q^{\alpha}\rho)^j\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r}(q^{-\alpha},\alpha_1,\cdots,\alpha_r;q)_j}{(q;q)_{k-j}(q;q)_j(\beta_1,\cdots,\beta_s;q)_j}\frac{(\psi\tau q^k;q)_\infty}{(\chi\tau q^j;q)_\infty}$$

$$\times\sum_{n=0}^{\infty}\frac{(q^{-\alpha}q^j,\alpha_1 q^j,\cdots,\alpha_r q^j;q)_n}{(q,\beta_1 q^j,\cdots,\alpha_r q^j;q)_n}(q^{\alpha}\rho\tau q^{j(s-r)})^n\left[(-1)^n q^{\binom{n}{2}}\right]^{s-r}\sigma^k$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{P_{k-j}(\chi,\psi)(q^{\alpha}\rho)^j}{(q;q)_{k-j}(q;q)_j}\frac{(q^{-\alpha},\alpha_1,\cdots,\alpha_r;q)_j}{(\beta_1,\cdots,\beta_s;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r}\frac{(\psi q^k\tau;q)_\infty}{(\chi q^j\tau;q)_\infty}$$

$$\times\ {}_{r+1}\phi_s\left(\begin{array}{c} q^{-(\alpha-j)},\alpha_1 q^j,\cdots,\alpha_r q^j \\ \beta_1 q^j,\cdots,\beta_s q^j\end{array};q,q^{\alpha-j+j}\rho q^{j(s-r)}\tau\right)\sigma^k$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{P_{k-j}(\chi,\psi)(q^{\alpha}\rho)^j}{(q;q)_{k-j}(q;q)_j}\frac{(q^{-\alpha},\alpha_1,\cdots,\alpha_r;q)_j}{(\beta_1,\cdots,\beta_s;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r}$$

$$\times\sum_{n=0}^{\infty}\frac{C_n^{(\alpha-j-n)}(\boldsymbol{\alpha}^j,\boldsymbol{\beta}^j,\chi q^j,\psi q^k,\rho q^{j(1+s-r)}|q)\sigma^k\tau^n}{(q;q)_n}\quad\text{(by using (3.2))}$$

By equating the coefficients of $\sigma^k\tau^n$ on both sides of the preceding equation, we successfully arrive at the necessary identity. □

- By subjecting identity (4.4) to the following specific set of conditions and substitutions, the inverse linearization formula for $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$ is successfully derived, The constraints on the indices are set as $1+s-r=1+u-r=0$; the parameters are equated such that $\alpha_i=a_i$ and $\beta_i=c_i$ for the range $1\le i\le r\le u$; and the variables undergo the transformations $\chi\to x$, $\psi\to y$, $\rho\to z$, along with the limiting condition $\alpha\to\infty$.

Corollary 4.1. *We present the following:*

$$V_{n+k}^{(\boldsymbol{a};\boldsymbol{c})}(x,y,z|q)=\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}P_{k-j}(x,y)\frac{(a_1,\cdots,a_r;q)_j}{(c_1,\cdots,c_s;q)_j}V_n^{(\boldsymbol{a}^j;\boldsymbol{c}^j)}(xq^j,yq^k,z|q)z^j.$$

- The inverse linearization formula for the polynomials $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$ is successfully derived by subjecting identity (4.4) to the following specific transformations: setting the parameters $\alpha_i=a_i$ and $\beta_i=b_i$ for the constraint $1\leqslant i\leqslant r\leqslant s$, transforming the variables as $\chi\to y$, $\psi\to x$, $\rho\to z$, and evaluating the expression under the limit condition $\alpha\to\infty$.

Corollary 4.2. *The following identity holds true:*

$$\Psi_{n+k}^{(\boldsymbol{a};\boldsymbol{b})}(x,y,z|q)=(-1)^k q^{-\binom{k}{2}-kn}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}P_{k-j}(y,x)\left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r}$$

$$\times \frac{(a_1,\cdots,a_r;q)_j}{(b_1,\cdots,b_s;q)_j}\Psi_n^{(\boldsymbol{a}^j;\boldsymbol{b}^j)}(xq^k,yq^j,zq^{j(1+s-r)}|q)z^j.$$

Theorem 4.3. (An Extension to Roger's formula for $C_{n+k}^{(\alpha-n-k)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$). *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}C_{n+m+k}^{(\alpha-n-m-k)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)\frac{t^n}{(q;q)_{n+m}}\frac{\tau^m}{(q;q)_m}\frac{\sigma^k}{(q;q)_k}$$
$$=\frac{(\psi\sigma;q)_\infty}{(\chi\sigma,\tau/t,t/\sigma;q)_\infty}\sum_{i=0}^{\infty}\frac{(\chi\sigma;q)_iq^i}{(q,\psi\sigma,q\sigma/t;q)_i}\,{}_{r+1}\phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+i}\rho\sigma\right), \tag{4.5}$$

such that $\max\{|\chi\sigma|,|\tau/t|,|t/\sigma|\}<1$.

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}C_{n+m+k}^{(\alpha-n-m-k)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)\frac{t^n}{(q;q)_{n+m}}\frac{\tau^m}{(q;q)_m}\frac{\sigma^k}{(q;q)_k}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\,{}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{P_{n+m+k}(\chi,\psi)\right\}\frac{t^n}{(q;q)_{n+m}}\frac{\tau^m}{(q;q)_m}\frac{\sigma^k}{(q;q)_k}$$
$$=\,{}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_{n+m}(\chi,\psi)\frac{t^n}{(q;q)_{n+m}}\frac{\tau^m}{(q;q)_m}\sum_{k=0}^{\infty}P_k(\chi,q^{n+m}\psi)\right.$$
$$\left.\times\frac{\sigma^k}{(q;q)_k}\right\}$$
$$=\,{}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_{n+m}(\chi,\psi)\frac{t^n}{(q;q)_{m+n}}\frac{\tau^m}{(q;q)_m}\frac{(q^{n+m}\psi\sigma,q)_\infty}{(\chi\sigma;q)_\infty}\right\}$$
$$\text{(by using (1.17))}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n}{(q;q)_{m+n}}\frac{\tau^m}{(q;q)_m}\,{}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{P_{n+m}(\chi,\psi)\frac{(q^{n+m}\psi\sigma,q)_\infty}{(\chi\sigma;q)_\infty}\right\}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n}{(q;q)_{m+n}}\frac{\tau^m}{(q;q)_m}\frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\sigma^{-(m+n)}\sum_{i=0}^{m+n}\frac{(q^{-(m+n)},\chi\sigma;q)_i}{(q,\psi\sigma;q)_i}q^i$$
$$\times\,{}_{r+1}\phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+i}\rho\sigma\right)$$
$$=\frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\sigma)^n}{(q;q)_{m+n}}\frac{(\tau/\sigma)^m}{(q;q)_m}\sum_{n+m=i}^{\infty}\frac{(q^{-(m+n)},\chi\sigma;q)_i}{(q,\psi\sigma;q)_i}q^i$$
$$\times\,{}_{r+1}\phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+i}\rho\sigma\right)\quad\text{(by using (2.6))}$$
$$=\frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=i}^{\infty}\frac{(t/\sigma)^{j-m}}{(q;q)_j}\frac{(\tau/\sigma)^m}{(q;q)_m}\frac{(q^{-j},\chi\sigma;q)_iq^i}{(q,\psi\sigma;q)_i}$$

$$\times {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha+i}\rho\sigma\right)$$

$$=\frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j+i=i}^{\infty}\frac{(t/\sigma)^{j+i-m}}{(q;q)_{j+i-i}}\frac{(\tau/\sigma)^m}{(q;q)_m}\frac{(-1)^i q^{-i(j+i)}q^{\binom{i}{2}}(\chi\sigma;q)_i q^i}{(q,\psi\sigma;q)_i}$$

$$\times {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha+i}\rho\sigma\right) \quad \text{(by using (1.8))}$$

$$=\frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(t/\sigma)^j}{(q;q)_j}\frac{(\tau/t)^m(t/\sigma)^i}{(q;q)_m}\frac{(-1)^i q^{-i(j+i)}q^{\binom{i}{2}}(\chi\sigma;q)_i q^i}{(q,\psi\sigma;q)_i}$$

$$\times {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha+i}\rho\sigma\right)$$

$$=\frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\sum_{m=0}^{\infty}\frac{(\tau/t)^m}{(q;q)_m}\sum_{i=0}^{\infty}\frac{(-1)^i q^{-\binom{i}{2}}(t/\sigma)^i}{(q;q)_i}\frac{(\chi\sigma;q)_i}{(\psi\sigma;q)_i}\sum_{j=0}^{\infty}\frac{(q^{-i}t/\sigma)^j}{(q;q)_j}$$

$$\times {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha+i}\rho\sigma\right)$$

$$=\frac{(\psi\sigma;q)_\infty}{(\chi\sigma,\tau/t;q)_\infty}\sum_{i=0}^{\infty}\frac{(-1)^i q^{-\binom{i}{2}}(t/\sigma)^i}{(q;q)_i}\frac{(\chi\sigma;q)_i}{(\psi\sigma;q)_i}\frac{1}{(q^{-i}t/\sigma;q)_\infty}$$

$$\times {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha+i}\rho\sigma\right)$$

$$=\frac{(\psi\sigma;q)_\infty}{(\chi\sigma,\tau/t,t/\sigma;q)_\infty}\sum_{i=0}^{\infty}\frac{(-1)^i q^{-\binom{i}{2}}(t/\sigma)^i}{(q;q)_i}\frac{(\chi\sigma;q)_i}{(\psi\sigma;q^{-i}t/\sigma;q)_i}$$

$$\times {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha+i}\rho\sigma\right).$$

$$=\frac{(\psi\sigma;q)_\infty}{(\chi\sigma,\tau/t,t/\sigma;q)_\infty}\sum_{i=0}^{\infty}\frac{(\chi\sigma;q)_i q^i}{(q,\psi\sigma,q\sigma/t;q)_i}\,{}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\alpha+i}\rho\sigma\right).$$

(by using (1.7))

□

- By subjecting equation (4.5) to the following set of specific conditions and substitutions, we successfully derive an extension of Roger's formula pertaining to the $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$: The constraints on the indices are defined as $1+s-r=1+u-r=0$; the parameters are equated such that $\alpha_i=a_i$ and $\beta_i=c_i$ for the range $1\leqslant i\leqslant r\leqslant u$; and the variables undergo the transformations $\chi\to x$, $\psi\to y$, $\rho\to z$, along with the limiting condition $\alpha\to\infty$.

Corollary 4.3.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} V_{n+k+m}^{(\boldsymbol{a},\boldsymbol{c})}(x,y,z|q)\frac{t^n}{(q;q)_{n+m}}\frac{\tau^m}{(q;q)_m}\frac{\sigma^k}{(q;q)_k}$$
$$=\frac{(y\sigma;q)_\infty}{(x\sigma,\tau/t,t/\sigma;q)_\infty}\sum_{i=0}^{\infty}\frac{(x\sigma;q)_i q^i}{(q,y\sigma,q\sigma/t;q)_i}\,{}_r\phi_u\left(\begin{matrix} a_1,\cdots,a_r \\ c_1,\cdots,c_u\end{matrix};q,q^i z\sigma\right),$$

such that $\max\{|x\sigma|,|\tau/t|,|t/\sigma|\}<1$.

- By applying the following specific set of conditions and substitutions to equation (4.5), we derive an extension of **Roger's formula** pertaining to the polynomials $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$ The parameters are set such that $\alpha_i=a_i$ and $\beta_i=b_i$ for the constraint $1\leqslant i\leqslant r\leqslant s$, and the variables undergo the transformations $\chi\to y$, $\psi\to x$, $\rho\to z$, along with the limiting condition $\alpha\to\infty$.

Corollary 4.4.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k+m}q^{\binom{n+k+m}{2}}\Psi_{n+k+m}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)\frac{t^n}{(q;q)_{n+m}}\frac{\tau^m}{(q;q)_m}\frac{\sigma^k}{(q;q)_k}$$
$$=\frac{(x\sigma;q)_\infty}{(y\sigma,\tau/t,t/\sigma;q)_\infty}\sum_{i=0}^{\infty}\frac{(y\sigma;q)_i q^i}{(q,x\sigma,q\sigma/t;q)_i}\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,q^i z\sigma\right),$$

such that $\max\{|y\sigma|,|\tau/t|,|t/\sigma|\}<1$.

- By substituting $\tau=0$ and replacing t with τ in the Theorem 4.3, we obtain Roger's formula for the polynomial $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$, which is specifically cited as the Theorem 4.1.

5. Mehler's Formula for $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$

This segment introduces the generalized Cigler's polynomials $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$ and is devoted to the derivation of Mehler's formula and its corresponding extension. The derivation is performed using the operator representation defined in (2.1). Furthermore, by applying specific values for the variables in Mehler's formula for $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$, we successfully establish Mehler's formula and its extension for both $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$ and $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$.

Theorem 5.1. (An extension to Mehler's formula for $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$).*We present the following:*

$$\sum_{n=0}^{\infty} C_{n+m}^{(\alpha-n-m)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)C_n^{(\beta-n)}(\boldsymbol{\gamma},\boldsymbol{\delta},\zeta,\omega,\sigma|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(\zeta t)^{-m}(\psi\zeta t;q)_\infty}{(\chi t\zeta;q)_\infty}\times\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(q,\delta_1,\cdots,\delta_s;q)_k}P_k(\sigma q^\beta/\zeta,\sigma/\zeta)\frac{(\omega/\zeta)^n(-1)^n q^{\binom{n}{2}}}{(q;q)_n} \tag{5.1}$$
$$\times\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}\sum_{j=0}^{k+n+m}\frac{(q^{-(k+n+m)},\chi\zeta t;q)_j}{(q,\psi\zeta t;q)_j}q^j\,{}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+j}\rho\zeta t\right),$$

such that $|\chi t\zeta|<1$.

Proof. From (2.4), we present the following:

$$\sum_{n=0}^{\infty} C_{n+m}^{(\alpha-n-m)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)\, C_{n}^{(\beta-n)}(\boldsymbol{\gamma},\boldsymbol{\delta},\zeta,\omega,\sigma|q)\,\frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{P_{n+m}(\chi,\psi)C_n^{(\beta-n)}\,(\boldsymbol{\gamma},\boldsymbol{\delta},\zeta,\omega,\sigma|q)\,\frac{t^n}{(q;q)_n}\right\}$$

(by using (2.4))

$$= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{n=0}^{\infty} P_{n+m}(\chi,\psi)\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}\beta\\k\end{bmatrix}(q;q)_k\right\}$$

$$\times\left(\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(\delta_1,\cdots,\delta_s;q)_k}P_{n-k}(\zeta,\omega)\sigma^k\right)\quad \text{(by using (2.3))}$$

$$= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{k=0}^{\infty}\sum_{n=k}^{\infty} P_{n+m}(\chi,\psi)\frac{t^n}{(q;q)_k(q;q)_{n-k}}\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}\right\}$$

$$\times\left(\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(\delta_1,\cdots,\delta_s;q)_k}P_k(\sigma q^{\beta},\sigma)P_{n-k}(\zeta,\omega)\right)$$

$$= \sum_{k=0}^{\infty}\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(q,\delta_1,\cdots,\delta_s;q)_k}P_k(\sigma q^{\beta},\sigma)t^k\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}\ {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)$$

$$\times\left\{\sum_{n=0}^{\infty}\frac{P_{n+k+m}(\chi,\psi)t^n P_n(\zeta,\omega)}{(q;q)_n}\right\}$$

$$= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{k=0}^{\infty} t^k P_{k+m}(\chi,\psi)P_k(\sigma q^{\beta},\sigma)\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(q,\delta_1,\cdots,\delta_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}\right.$$

$$\left.\times\,{}_2\phi_1\left(\begin{matrix}\frac{q^{k+m}\psi}{\chi},\omega/\zeta\\ 0\end{matrix};q,\chi\zeta t\right)\right\}$$

$$= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{k=0}^{\infty}\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(q,\delta_1,\cdots,\delta_s;q)_k}t^k P_{k+m}(\chi,\psi)P_k(\sigma q^{\beta},\sigma)\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}\right.$$

$$\left.\times\frac{(q^{k+m}\psi\zeta t;q)_\infty}{(\chi\zeta t;q)_\infty}\,{}_2\phi_2\left(\begin{matrix}\frac{q^{k+m}\psi}{\chi},0\\ 0,q^{k+m}\psi\zeta t\end{matrix};q,\omega\chi t\right)\right\}\quad \text{(by using (1.4))}$$

$$= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(q,\delta_1,\cdots,\delta_s;q)_k}\frac{t^k P_{k+m}(\chi,\psi)P_k(\sigma q^{\beta},\sigma)}{(q,q^{k+m}\psi\zeta t;q)_n}\right.$$

$$\left.\times\frac{(-1)^n q^{\binom{n}{2}}P_n(\chi,q^{k+m}\psi)(q^{k+m}\psi\zeta t;q)_\infty(\omega t)^n}{(\chi\zeta t;q)_\infty}\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}\right\}$$

$$= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(q,\delta_1,\cdots,\delta_s;q)_k}\frac{t^k(\omega t)^n P_k(\sigma q^{\beta},\sigma)}{(q;q)_n(\psi\zeta t;q)_{k+m}}\right.$$

$$\times\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r} \frac{(-1)^n q^{\binom{n}{2}} P_{k+n+m}(\chi,\psi)(\psi\zeta t;q)_\infty}{(q^{k+m}\psi\zeta t;q)_n(\chi\zeta t;q)_\infty}\Bigg\}$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(q,\delta_1,\cdots,\delta_s;q)_k}t^k P_k(\sigma q^\beta,\sigma)\frac{(-1)^n q^{\binom{n}{2}}(\omega t)^n}{(q;q)_n}\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}$$

$$\times\ {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\frac{P_{n+k+m}(\chi,\psi)\ (\psi\zeta t;q)_\infty}{(\psi\zeta t;q)_{k+n+m}(\chi\zeta t;q)_\infty}\right\}$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{k+n+m}\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(q,\delta_1,\cdots,\delta_s;q)_k}t^k P_k(\sigma q^\beta,\sigma)\frac{(-1)^n q^{\binom{n}{2}}(\omega t)^n}{(q;q)_n}\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}$$

$$\times\frac{(\psi\zeta t;q)_\infty(\zeta t)^{-n-k-m}}{(\chi t\zeta;q)_\infty}\ \frac{(q^{-k-m-n},\chi\zeta t;q)_j}{(q,\psi\zeta t;q)_j}\ q^j\ {}_{r+1}\phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+j}\rho\zeta t\right)$$

$$=\frac{(\zeta t)^{-m}(\psi\zeta t;q)_\infty}{(\chi t\zeta;q)_\infty}\times\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(q,\delta_1,\cdots,\delta_s;q)_k}P_k(\sigma q^\beta/\zeta,\sigma/\zeta)\frac{(\omega/\zeta)^n(-1)^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$\times\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}\sum_{j=0}^{k+n+m}\frac{(q^{-(k+n+m)},\chi\zeta t;q)_j}{(q,\psi\zeta t;q)_j}\ q^j\ {}_{r+1}\phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+j}\rho\zeta t\right),$$

□

Corollary 5.1. (Mehler's formula for $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$). *We present the following:*

$$\sum_{n=0}^{\infty}C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)C_n^{(\beta-n)}(\boldsymbol{\gamma},\boldsymbol{\delta},\zeta,\omega,\sigma|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(\psi\zeta t;q)_\infty}{(\chi t\zeta;q)_\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\omega/\zeta)^n(-1)^n q^{\binom{n}{2}}}{(q;q)_n}P_k(\sigma q^\beta/\zeta,\sigma/\zeta)\frac{(\gamma_1,\cdots,\gamma_r;q)_k}{(q,\delta_1,\cdots,\delta_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}$$
$$\times\sum_{j=0}^{k+n}\frac{(q^{-(n+k)},\chi\zeta t;q)_j}{(q,\psi\zeta t;q)_j}\ q^j\ {}_{r+1}\phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+j}\rho\zeta t\right),\tag{5.2}$$

such that $|\chi t\zeta|<1$.

- By applying the following extensive set of constraints and substitutions to identity (5.2), Mehler's formula for Verma-Jain polynomials $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$ is successfully derived: The index constraint is set as $1+s-r=0$; the parameters are equated such that $\alpha_i=a_i$, $\beta_i=c_i$, $\gamma_i=b_i$, and $\delta_i=d_i$ for the range $1\leqslant i\leqslant r\leqslant u$; and the variables undergo the transformations $\chi\to x$, $\psi\to y$, $\rho\to z$, $\zeta\to u$, $\omega\to v$, along with the double limiting condition $\alpha\to\infty$ and $\beta\to\infty$.

$$\sum_{n=0}^{\infty}V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)V_n^{(\mathbf{b};\mathbf{d})}(u,v,\sigma|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(yut;q)_\infty}{(xtu;q)_\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(v/u)^n(-1)^n q^{\binom{n}{2}}(\sigma/u)^k(b_1,\cdots,b_r;q)_k}{(q;q)_n(q;q)_k(d_1,\cdots,d_s;q)_k}$$

$$\times\sum_{j=0}^{k+n}\frac{(q^{-(n+k)},xut;q)_j}{(q,yut;q)_j}\,q^j\;{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\c_1,\cdots,c_s\end{matrix};q,q^jzut\right),$$

such that $|xtu|<1$.

- By applying the following comprehensive set of constraints and substitutions to identity (5.2), Mehler's formula for polynomials $\Psi_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$ is successfully established: The parameters are equated such that $\alpha_i=a_i$, $\beta_i=c_i$, $\gamma_i=b_i$, and $\delta_i=d_i$ for the range $1\leqslant i\leqslant r\leqslant u$; and the variables undergo the transformations $\chi\to y$, $\psi\to x$, $\rho\to z$, $\zeta\to v$, $\omega\to u$, along with the double limiting condition $\alpha\to\infty$ and $\beta\to\infty$.

Corollary 5.2.

$$\begin{aligned}&\sum_{n=0}^{\infty}\Psi_n^{(\boldsymbol{a};\boldsymbol{c})}(x,y,z|q)\Psi_n^{(\boldsymbol{b};\boldsymbol{d})}(u,v,\sigma|q)\frac{q^{n^2-n}t^n}{(q;q)_n}\\&=\frac{(xvt;q)_\infty}{(ytv;q)_\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(u/v)^n(-1)^nq^{\binom{n}{2}}(\sigma/v)^k(b_1,\cdots,b_r;q)_k\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}}{(q;q)_n(q;q)_k(d_1,\cdots,d_s;q)_k}\\&\quad\times\sum_{j=0}^{k+n}\frac{(q^{-(n+k)},yvt;q)_j}{(q,xvt;q)_j}\,q^j\;{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\c_1,\cdots,c_s\end{matrix};q,q^jzvt\right),\mathit{where}|ytv|<1.\end{aligned}$$

6. Mixed Generating Functions for $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$

This section utilizes an operator method to construct several mixed generating functions for the generalized Cigler's polynomials $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$. Subsequently, specific parameter values applied to the Cigler's polynomials are used to derive a mixed generating function for the q-Hahn polynomials $\phi_n^{(a)}(x|q)$ as well as for the Verma-Jain polynomials $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$ and the generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$.

Theorem 6.1. (Mixed generating functions of $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$). *The subsequent identity is formally introduced:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(\chi|q)\,C_n^{(\beta-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\zeta,\omega,\sigma|q)\frac{t^n}{(q;q)_n}\\&=\frac{(\omega t,\alpha\chi;q)_\infty}{(\zeta t,\chi;q)_\infty}\sum_{j=0}^{\infty}\frac{(\zeta t;q)_j}{(q,\omega t;q)_j}\frac{(\alpha;q)_jq^j}{(q/\chi;q)_j}\;{}_{r+1}\phi_s\left(\begin{matrix}q^{-\beta},\alpha_1,\cdots,\alpha_r\\\beta_1,\cdots,\beta_s\end{matrix};q,q^{\beta+j}\sigma t\right),\end{aligned}\tag{6.1}$$

provided $\max\{|\zeta t|,|\chi|\}<1$.

Proof.

$$\begin{aligned}&\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(\chi|q)\,C_n^{(\beta-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\zeta,\omega,\sigma|q)\frac{t^n}{(q;q)_n}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(\alpha;q)_k\chi^kC_n^{(\beta-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\zeta,\omega,\sigma|q)\frac{t^n}{(q;q)_n}\quad\text{(by using (1.9))}\end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(\alpha;q)_k\chi^k \ {}_{r+1}\Phi_s\left(q^{-\beta},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\beta}\sigma D_{\zeta\omega}\right)\left\{P_n(\zeta,\omega)\frac{t^n}{(q;q)_n}\right\} \quad \text{(by using (2.4))}$$

$$= {}_{r+1}\Phi_s\left(q^{-\beta},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\beta}\sigma D_{\zeta\omega}\right)\left\{\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(\alpha;q)_k\chi^k \ P_n(\zeta,\omega)\frac{t^n}{(q;q)_n}\right\}$$

$$= {}_{r+1}\Phi_s\left(q^{-\beta},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\beta}\sigma D_{\zeta\omega}\right)\left\{\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(\alpha;q)_k\chi^kP_n(\zeta,\omega)t^n}{(q;q)_k(q;q)_{n-k}}\right\}$$

$$= {}_{r+1}\Phi_s\left(q^{-\beta},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\beta}\sigma D_{\zeta\omega}\right)\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha;q)_k\chi^kP_{n+k}(\zeta,\omega)t^{k+n}}{(q;q)_k(q;q)_n}\right\}$$

$$= {}_{r+1}\Phi_s\left(q^{-\beta},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\beta}\sigma D_{\zeta\omega}\right)\left\{\sum_{k=0}^{\infty}P_k(\zeta,\omega)\frac{(\alpha;q)_k(\chi t)^k}{(q;q)_k}\sum_{n=0}^{\infty}P_n(\zeta,q^k\omega)\frac{t^n}{(q;q)_n}\right\}$$

$$= {}_{r+1}\Phi_s\left(q^{-\beta},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\beta}\sigma D_{\zeta\omega}\right)\left\{\sum_{k=0}^{\infty}P_k(\zeta,\omega)\frac{(\alpha;q)_k(\chi t)^k}{(q;q)_k}\frac{(q^k\omega t;q)_\infty}{(\zeta t;q)_\infty}\right\} \quad \text{(by using (1.17))}$$

$$= \sum_{k=0}^{\infty}\frac{(\alpha;q)_k\chi^k}{(q;q)_k}\ {}_{r+1}\Phi_s\left(q^{-\beta},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\beta}\sigma D_{\zeta\omega}\right)\left\{\frac{P_k(\zeta t,\omega t)}{(\omega t;q)_k}\frac{(\omega t;q)_\infty}{(\zeta t;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty}\frac{(\alpha;q)_k\chi^k}{(q;q)_k}\frac{(\omega t;q)_\infty}{(\zeta t;q)_\infty}\sum_{j=0}^{k}\frac{(q^{-k},\zeta t;q)_j}{(q,\omega t;q)_j}q^j\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\beta},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\beta+j}\sigma t\right)$$

(by using (2.6))

$$= \frac{(\omega t;q)_\infty}{(\zeta t;q)_\infty}\sum_{j=0}^{\infty}\sum_{k+j=j}^{\infty}\frac{(\alpha;q)_{j+k}\chi^{k+j}(-1)^jq^{\binom{j}{2}}q^{-j(j+k)}}{(q;q)_k}\frac{(\zeta t;q)_j}{(q,\omega t;q)_j}q^j$$

$$\times\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\beta},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\beta+j}\sigma t\right)$$

$$= \frac{(\omega t;q)_\infty}{(\zeta t;q)_\infty}\sum_{j=0}^{\infty}(\alpha;q)_j\chi^j(-1)^jq^{\binom{j}{2}}q^{-j^2}\frac{(\zeta t;q)_j}{(q,\omega t;q)_j}\sum_{k=0}^{\infty}\frac{(\alpha q^j;q)_k\chi^kq^{-jk}q^j}{(q;q)_k}$$

$$\times\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\beta},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\beta+j}\sigma t\right)$$

$$= \frac{(\omega t;q)_\infty}{(\zeta t;q)_\infty}\sum_{j=0}^{\infty}(\alpha;q)_j\chi^j(-1)^jq^{\binom{j}{2}}q^{-j^2}\frac{(\zeta t;q)_j}{(q,\omega t;q)_j}\frac{(\alpha\chi;q)_\infty q^j}{(\chi q^{-j};q)_\infty}$$

$$\times\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\beta},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\beta+j}\sigma t\right)$$

$$= \frac{(\omega t,\alpha\chi;q)_\infty}{(\zeta t,\chi;q)_\infty}\sum_{j=0}^{\infty}\frac{(\zeta t;q)_j}{(q,\omega t;q)_j}\frac{(\alpha;q)_jq^j}{(q/\chi;q)_j}\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\beta},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s \end{matrix};q,q^{\beta+j}\sigma t\right)$$

(by using (1.7))

□

- By applying the following specific set of constraints and substitutions to identity (6.1), a mixed generating function for $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$ is successfully established: The constraints on the indices are defined as $1+s-r=1+u-r=0$; the parameters are equated such that $\alpha_i=a_i$ and $\beta_i=c_i$ for the range $1\le i\le r\le s$; and the variables undergo the transformations $\chi\to x,\zeta\to u$, $\omega\to v$, along with the limiting condition $\alpha\to\infty$.

Corollary 6.1.

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)\,V_n^{(\boldsymbol{a},\boldsymbol{c})}(u,v,\sigma|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(vt,\alpha x;q)_\infty}{(ut,x;q)_\infty}\sum_{j=0}^{\infty}\frac{(ut;q)_j}{(q,vt;q)_j}\,\frac{(\alpha;q)_jq^j}{(q/x;q)_j}\,{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ c_1,\cdots,c_s\end{matrix};q,q^j\sigma t\right),$$

such that $\max\{|ut|,|x|\}<1$.

- Setting $\alpha_i=a_i$, $\beta_i=b_i$;$1\le i\le r\le s$, $\chi\to x$, $\zeta\to v$, $\omega\to u$, $\beta\to\infty$ in (6.1), we discover a mixed-function generating function for $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$.

Corollary 6.2.

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)\,\Psi_n^{(\boldsymbol{a},\boldsymbol{b})}(u,v,\sigma|q)(-1)^nq^{\binom{n}{2}}\frac{t^n}{(q;q)_n}$$
$$=\frac{(ut,\alpha x;q)_\infty}{(vt,x;q)_\infty}\sum_{j=0}^{\infty}\frac{(vt;q)_j}{(q,ut;q)_j}\,\frac{(\alpha;q)_jq^j}{(q/x;q)_j}\,{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,q^j\sigma t\right),$$

provided $\max\{|vt|,|x|\}<1$.

- By applying the following specific set of substitutions and constraints to equation (6.1), Mehler's formula for the Hahn polynomials $\phi_n^{(a)}(x|q)$ is successfully retained (as shown in equation (1.13)): The constraints on the indices and parameters are set as $s=r=1$ and $\alpha_1=\beta_1=0$. Additionally, the variables undergo the transformations $\chi=x$, $\zeta=1$, $\omega=0$, $\alpha=a$, $\beta=\log_q b^{-1}$, $\sigma=by$, and $t=\tau$.

In continuation of our analysis, we explicitly define and establish the validity of the additional mixed generating functions for the generalized Cigler's polynomials, which are presented as the assertion in the theorem below:

Theorem 6.2. (Another mixed generating function for generalized Cigler's polynomials $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$).*We present the following:*

$$\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(\chi|q)\,C_n^{(\beta-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\zeta,\omega,\sigma|q)\frac{\tau^n}{(q;q)_n}$$

$$= \frac{(q/\chi;q)_\infty}{(\alpha q;q)_\infty}\frac{(\omega\chi\tau;q)_\infty}{(\zeta\chi\tau;q)_\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(\frac{1}{\alpha\chi};q)_n}{(q,q/\chi;q)_n}\frac{(\frac{q}{\omega\chi\tau};q)_n(\omega\alpha q/\zeta)^n}{(\frac{q}{\zeta\chi\tau};q)_n}\right)$$
$$\times\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\beta},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\beta}\sigma\chi\tau q^{-n}\right), \tag{6.2}$$

provided $|\zeta\chi\tau|\} < 1$.

Proof. To demonstrate Theorem 6.2, Jackson's transformation is required. After allowing $(a,x,\lambda,\tau) = (\alpha,\chi,\zeta/\omega,\omega\tau/q)$ in equation (1.12), we receive the following:

$$\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(\chi|q)\,P_n(\zeta,\omega)\frac{\tau^n}{(q;q)_n}$$
$$= \frac{(\omega\tau\chi;q)_\infty}{(\zeta\tau\chi;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}\frac{1}{\alpha\chi},\omega/\zeta \\ \frac{q}{\zeta\chi\tau}\end{matrix};q,\alpha q\right).$$
$$= \frac{(\omega\tau\chi;q)_\infty}{(\zeta\tau\chi;q)_\infty}\frac{(q/\chi;q)_\infty}{(\alpha q;q)_\infty}\,{}_2\phi_2\left(\begin{matrix}\frac{1}{\alpha\chi},\frac{q}{\omega\chi\tau} \\ \frac{q}{\zeta\chi\tau},q/\chi\end{matrix};q,\frac{\omega\alpha q}{\zeta}\right).\quad\text{(by using(1.4))}$$
$$= \frac{(\omega\tau\chi;q)_\infty}{(\zeta\tau\chi;q)_\infty}\frac{(q/\chi;q)_\infty}{(\alpha q;q)_\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(\frac{q}{\omega\chi\tau};q)_n(\frac{1}{\alpha\chi};q)_n(\frac{\omega\alpha q}{\zeta})^n}{(q,\frac{q}{\zeta\chi\tau},q/\chi;q)_n}\right)$$
$$= \frac{(\omega\tau\chi;q)_\infty}{(\zeta\tau\chi;q)_\infty}\frac{(q/\chi;q)_\infty}{(\alpha q;q)_\infty}\left(\sum_{n=0}^{\infty}\frac{(q^{-n}\omega\chi\tau;q)_n(-1)^n q^{\binom{n}{2}}(\frac{1}{\alpha\chi};q)_n(\alpha q)^n}{(q,q^{-n}\zeta\chi\tau,q/\chi;q)_n}\right)\quad\text{(by using (1.7))}$$
$$= \frac{(q/\chi;q)_\infty}{(\alpha q;q)_\infty}\left(\sum_{n=0}^{\infty}\frac{(\omega\chi\tau q^{-n};q)_\infty(-1)^n q^{\binom{n}{2}}(\frac{1}{\alpha\chi};q)_n(\alpha q)^n}{(\zeta\chi\tau q^{-n};q)_\infty(q,q/\chi;q)_n}\right). \tag{6.3}$$

By operating with the generalized homogeneous q-shift operator on equation (6.3), the identity transforms into the following expression:

$$\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(\chi|q)\ {}_{r+1}\Phi_s\left(q^{-\beta},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\beta}\sigma D_{\zeta\omega}\right)\left\{P_n(\zeta,\omega)\frac{\tau^n}{(q;q)_n}\right\}$$
$$= \frac{(q/\chi;q)_\infty}{(\alpha q;q)_\infty}\sum_{n=0}^{\infty}\ {}_{r+1}\Phi_s\left(q^{-\beta},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\beta}\sigma D_{\zeta\omega}\right)\left\{\frac{(\omega\chi\tau q^{-n};q)_\infty}{(\zeta\chi\tau q^{-n};q)_\infty}\frac{(-1)^n q^{\binom{n}{2}}(\frac{1}{\alpha\chi};q)_n(\alpha q)^n}{(q,q/\chi;q)_n}\right\}$$
$$= \frac{(q/\chi;q)_\infty}{(\alpha q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(\frac{1}{\alpha\chi};q)_n(\alpha q)^n}{(q,q/\chi;q)_n}\frac{(\omega\chi\tau q^{-n};q)_\infty}{(\zeta\chi\tau q^{-n};q)_\infty}$$
$$\times\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\beta},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\beta}\sigma\chi\tau q^{-n}\right)\quad\text{(by using (2.5))}$$
$$= \frac{(q/\chi;q)_\infty}{(\alpha q;q)_\infty}\frac{(\omega\chi\tau;q)_\infty}{(\zeta\chi\tau;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(\frac{1}{\alpha\chi};q)_n(\alpha q)^n}{(q,q/\chi;q)_n}\frac{(\omega\chi\tau q^{-n};q)_n}{(\zeta\chi\tau q^{-n};q)_n}$$
$$\times\ {}_{r+1}\phi_s\left(\begin{matrix} q^{-\beta},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\beta}\sigma\chi\tau q^{-n}\right)$$

$$= \frac{(q/\chi;q)_\infty}{(\alpha q;q)_\infty}\frac{(\omega\chi\tau;q)_\infty}{(\zeta\chi\tau;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(\frac{1}{\alpha\chi};q)_n}{(q,q/\chi;q)_n}\frac{(\frac{q}{\omega\chi\tau};q)_n(\omega\alpha q/\zeta)^n}{(\frac{q}{\zeta\chi\tau};q)_n}$$
$$\times {}_{r+1}\phi_s\left(\begin{matrix} q^{-\beta},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\beta}\sigma\chi\tau q^{-n}\right) \quad \text{(by using (1.7))}$$

□

- By subjecting identity (6.2) to the following extensive set of constraints and substitutions, another mixed generating function for $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$ is successfully derived: The constraints on the indices are defined as $1+s-r=1+u-r=0$; the parameters are equated such that $\alpha_i=a_i$, $\beta_i=c_i$, and $\alpha=a$ for the range $1\le i\le r\le u$; and the variables undergo the transformations $\chi\to x$, $\omega\to v$, $\zeta\to u$, $\sigma\to z$, $\tau\to t$, along with the limiting condition $\beta\to\infty$.

Corollary 6.3. *The subsequent results are presented:*

$$\sum_{n=0}^{\infty}\psi_n^{(a)}(x|q)\,V_n^{(\boldsymbol{a};\boldsymbol{c})}(u,v,z|q)\frac{t^n}{(q;q)_n}$$
$$= \frac{(q/x;q)_\infty}{(aq;q)_\infty}\frac{(vxt;q)_\infty}{(uxt;q)_\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(\frac{1}{ax};q)_n}{(q,q/x;q)_n}\frac{(\frac{q}{vxt};q)_n(vaq/u)^n}{(\frac{q}{uxt};q)_n}\right)$$
$$\times {}_r\phi_u\left(\begin{matrix} a_1,\cdots,a_r \\ c_1,\cdots,c_u\end{matrix};q,zxtq^{-n}\right),$$

provided $|uxt|\} < 1$.

- By subjecting identity (6.2) to the following set of specific constraints and substitutions, the mixed generating function for $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$, as presented in equation (1.28), is successfully derived: The parameters are set such that $\alpha_i=a_i$ and $\beta_i=b_i$ for the constraint $1\le i\le r\le s$; and the variables undergo the transformations $\chi\to x$, $\zeta\to v$, $\omega\to u$, $\sigma\to z$, $\tau\to tq$, along with the limiting condition $\beta\to\infty$.
- By applying the following specific set of constraints and substitutions, Mehler's formula for the Hahn polynomials $\psi_n^{\alpha}(x|q)$, as presented in equation (1.29), is successfully derived. The index and parameter constraints are set as $s-r=0$, $\alpha_i=0$, and $\beta_i=0$ for the range $1\le i\le r\le s$; and the variables undergo the transformations $\chi\to x$, $\zeta\to ay$, $\omega\to y$, $\sigma=1$, and $\tau\to tq$, along with the limiting condition $\beta\to\infty$.

Theorem 6.3. (Another mixed generating functions for $C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)$). *The subsequent results are presented:*

$$\sum_{n=0}^{\infty}C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)\,P_n(\zeta,\omega)\frac{t^n}{(q;q)_n}$$
$$= \frac{(\omega/\zeta,\psi\zeta t;q)_\infty}{(\chi\zeta t;q)_\infty}\sum_{n=0}^{\infty}\frac{(\chi\zeta t;q)_n}{(q,\psi\zeta t;q)_n}(\omega/\zeta)^n\,{}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r \\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+n}\rho\zeta t\right), \quad |\chi\zeta t|<1. \tag{6.4}$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)P_n(\zeta,\omega)\frac{t^n}{(q;q)_n}\\
&=\sum_{n=0}^{\infty} {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{P_n(\chi,\psi)P_n(\zeta,\omega)\frac{t^n}{(q;q)_n}\right\}\\
&= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{n=0}^{\infty}P_n(\chi,\psi)P_n(\zeta,\omega)\frac{t^n}{(q;q)_n}\right\}\\
&= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{n=0}^{\infty}(\psi/\chi;q)_n(\omega/\zeta;q)_n\frac{(\chi\zeta t)^n}{(q;q)_n}\right\}\\
&= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{{}_2\phi_1\left(\begin{matrix}\psi/\chi,\omega/\zeta\\ 0\end{matrix};q,\chi\zeta t\right)\right\}\\
&= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\frac{(\psi\zeta t;q)_\infty}{(\chi\zeta t;q)_\infty}\,{}_2\phi_2\left(\begin{matrix}\psi/\chi,0\\ 0,\psi\zeta t\end{matrix};q,\omega\chi t\right)\right\}\quad \text{(by using (1.4))}\\
&= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\frac{(\psi\zeta t;q)_\infty}{(\chi\zeta t;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}\psi/\chi\\ \psi\zeta t\end{matrix};q,\omega\chi t\right)\right\}\\
&= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\frac{(\psi\zeta t,\omega/\zeta;q)_\infty}{(\chi\zeta t;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}0,\chi\zeta t\\ \psi\zeta t\end{matrix};q,\omega/\zeta\right)\right\}\quad \text{(by using (1.5))}\\
&=(\omega/\zeta;q)_\infty\sum_{n=0}^{\infty}\frac{(\psi\zeta t q^n;q)_\infty}{(\chi\zeta t q^n;q)_\infty}\frac{(\omega/\zeta)^n}{(q;q)_n}\,{}_{r+1}\phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+n}\rho\zeta t\right)\text{(by using (2.5))}\\
&=\frac{(\omega/\zeta,\psi\zeta t;q)_\infty}{(\chi\zeta t;q)_\infty}\sum_{n=0}^{\infty}\frac{(\chi\zeta t;q)_n}{(q,\psi\zeta t;q)_n}(\omega/\zeta)^n\,{}_{r+1}\phi_s\left(\begin{matrix}q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+n}\rho\zeta t\right)
\end{aligned}$$

□

- By subjecting identity (6.4) to the following extensive set of constraints and substitutions, another mixed generating function for the polynomials $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$ is successfully derived: The constraints on the indices are defined as $1+s-r=1+u-r=0$; the parameters are equated such that $\alpha_i=a_i$ and $\beta_i=c_i$ for the range $1\le i\le r\le u$; and the variables undergo the transformations $\chi\to x$, $\psi\to y$, $\rho\to z$, $\sigma\to z$, $\zeta\to u$, and $\omega\to v$, along with the limiting condition $\alpha\to\infty$.

Corollary 6.4. *The subsequent results are presented:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} V_n^{(\boldsymbol{a};\boldsymbol{c})}(x,y,z|q)\,P_n(u,v)\frac{t^n}{(q;q)_n}\\
&\qquad=\frac{(v/u,yut;q)_\infty}{(xut;q)_\infty}\sum_{n=0}^{\infty}\frac{(xut;q)_n}{(q,yut;q)_n}(v/u)^n\,{}_r\phi_u\left(\begin{matrix}a_1,\cdots,a_r\\ c_1,\cdots,c_u\end{matrix};q,q^n zut\right)
\end{aligned}$$

$|xut| < 1$.

- By applying the specific substitutions $u = 1$, $v = \lambda$, to Corollary 6.4, the formula for a generating function for the Srivastava-Agarwal type for extending Verma-Jain polynomials further $V_n^{(\mathbf{a};\mathbf{c})}(x, y, z|q)$ was formulated, as represented by equation (1.22).
- By subjecting identity (6.4) to the following extensive set of constraints and substitutions, another mixed generating function for the polynomials $\Psi_n^{(\mathbf{a};\mathbf{b})}(x, y, z|q)$ is successfully derived: The parameters are equated such that $\alpha_i = a_i$ and $\beta_i = b_i$ for the constraint $1 \le i \le r \le s$; and the variables undergo the transformations $\chi \to y$, $\psi \to x$, $\rho \to z$, $\zeta \to u$, and $\omega \to v$, along with the limiting condition $\alpha \to \infty$.

Corollary 6.5. *The subsequent results are presented:*

$$\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \Psi_n^{(\boldsymbol{a};\boldsymbol{b})}(x, y, z|q)\, P_n(u, v) \frac{t^n}{(q;q)_n}$$
$$= \frac{(v/u, xut; q)_\infty}{(yut; q)_\infty} \sum_{n=0}^{\infty} \frac{(yut; q)_n}{(q, xut; q)_n} (v/u)^n {}_r\phi_s \left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} ; q, q^n zut \right),$$

such that $|yut| < 1$.

- By applying the specific substitutions $s = r = 1$, $\alpha_1 = 0$, and $\beta_1 = 0; \omega/\zeta = \lambda$ and $\zeta \to 1$ to equation (6.4), the formula for another mixed generating function for the Cigler's polynomials $C_n^{(\alpha-n)}(\chi, \psi, \rho|q)$ was formulated, as represented by equation (1.35).
- By subjecting equation (6.4) to the specific substitutions $\chi = 1, \psi = 0, \alpha \longrightarrow \log_q a^{-1}, \rho \to ax$, and $t = \tau$, and subsequently applying Heine's transformation (1.6), an alternative mixed generating function formula for the polynomials $\phi_n^{(a)}(x)$ is yielded, as formally presented in equation (1.11).

7. Transformation Identity Including Generating Function for $C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho)$

In this part, we demonstrate the concept of a transformational identity relevant to the polynomials $C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q)$ through the utilization of the homogeneous operator ${}_{r+1}\Phi_s \left(\begin{matrix} q^{-\alpha}, \alpha_1, \cdots, \alpha r \\ \beta_1, \cdots, \beta_s \end{matrix} ; q, q^\alpha \rho D_{\chi\psi} \right)$.

Theorem 7.1. *We define the relationship between the coefficients $A(n)$ and $B(n)$ as follows:*

$$\sum_{n=0}^{\infty} A(n) P_n(\chi, \psi) = \sum_{n=0}^{\infty} B(n) \frac{(q^n \psi t; q)_\infty}{(q^n \chi t; q)_\infty}. \tag{7.1}$$

Then

$$\sum_{n=0}^{\infty} A(n) C_n^{(\alpha-n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi, \psi, \rho|q)$$
$$= \frac{(\psi t; q)_\infty}{(\chi t; q)_\infty} \sum_{n=0}^{\infty} B(n) \frac{(\chi t; q)_n}{(\psi t; q)_n} {}_{r+1}\phi_s \left(\begin{matrix} q^{-\alpha}, \alpha_1, \cdots, \alpha_r \\ \beta_1, \cdots, \beta_s \end{matrix} ; q, q^{\alpha+n} \rho t \right). \tag{7.2}$$

Proof. We denote the left-hand side of (7.2) by $f(\chi,\psi,\rho)$. Through the combined application of (7.1) and (2.4), we successfully demonstrate that:

$$\begin{aligned}
f(\chi,\psi,\rho) &= \sum_{n=0}^{\infty} A(n) C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q)\\
&= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{n=0}^{\infty} A(n)P_n(\chi,\psi)\right\}\\
&= {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\sum_{n=0}^{\infty} B(n)\frac{(q^n\psi t;q)_\infty}{(q^n\chi t;q)_\infty}\right\}\\
&= \sum_{n=0}^{\infty} B(n)\, {}_{r+1}\Phi_s\left(q^{-\alpha},\boldsymbol{\alpha},\boldsymbol{\beta},q^{\alpha}\rho D_{\chi\psi}\right)\left\{\frac{(\psi q^n t;q)_\infty}{(\chi q^n t;q)_\infty}\right\}\\
&= \sum_{n=0}^{\infty} B(n)\frac{(\psi q^n t;q)_\infty}{(\chi q^n t;q)_\infty}\, {}_{r+1}\phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix};q,q^{\alpha+n}\rho t\right). \quad \text{(by using (2.5))}
\end{aligned}$$

It corresponds precisely to the right-hand side of (7.2) □

- When the constraints $s=r=1$, $\alpha_1=0$, and $\beta_1=0$ are applied to identities (7.1) and (7.2), the resulting expression is the transformational identity for $C_n^{(\alpha-n)}(\chi,\psi,\rho|q)$, which is given by equation (1.37).
- When the constraints $1+s-r=1+u-r=0$, $\alpha_i=a_i$, $\beta_i=c_i$ $(1\le i\le r\le u)$, $\chi\to x$, $\psi\to y$, $\rho\to z$, and $\alpha\to\infty$ are applied to identities (7.1) and (7.2), the resulting expression is the transformational identity for $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$, which is given by equation (1.24).
- By subjecting the identities (7.1) and (7.2) to the following comprehensive set of constraints and substitutions, the transformational identity for $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$ is successfully derived, as presented in equation (1.31): The parameters are equated such that $\alpha_i=a_i$ and $\beta_i=b_i$ for the constraint $1\le i\le r\le s$; and the variables undergo the transformations $\chi\to v$, $\psi\to u$, $\rho\to z$, and $t\to xtq^{1-2n}$, along with the limiting condition $\alpha\to\infty$.

8. Conclusions

(1) The retrieval of various designated operators is made possible by substituting particular values into the generalized homogeneous q-shift operator,

$$ {}_{r+1}\Phi_s\left(\begin{matrix} q^{-\alpha},\alpha_1,\ldots,\alpha_r\\ \beta_1,\ldots,\beta_s\end{matrix};q,q^{\alpha}\rho D_{\chi\psi}\right). \tag{8.1}$$

For comprehensive information, consult [12,15].

(2) The polynomials $V_n^{(\mathbf{a};\mathbf{c})}(x,y,z|q)$ and the polynomials $\Psi_n^{(\mathbf{a};\mathbf{b})}(x,y,z|q)$ both constitute specific instances, or particular cases of the generalized Cigler's polynomials,

$$C_n^{(\alpha-n)}(\boldsymbol{\alpha},\boldsymbol{\beta},\chi,\psi,\rho|q).$$

(3) The identities presented herein are specific extensions developed for the generalized Cigler's polynomials, which encompass the known identities for the further extension of Verma-Jain polynomials and the generalized q-hypergeometric polynomial.
(4) We suggest extending our results by employing the generalized homogenous q-difference equation method.This approach could provide a more general framework for characterizing the identities and properties of the polynomials investigated in this study

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