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Hyperbolic Kenmotsu Manifolds Admitting Schouten-Van Kampen Connection

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Abstract. In this paper we study certain curvature identities on hyperbolic Kenmotsu manifolds associated with the Schouten-Van Kampen connection. Also, we study hyperbolic Kenmotsu manifolds associated with the Schouten-Van Kampen connection that fulfills concircularly flat, ξ-concircularly flat, pseudo-concircularly flat, ϕ-concircularly flat, ϕ-concircularly semi-symmetric and Ricci semi-symmetric conditions. Moreover, we examine hyperbolic Kenmotsu manifolds with the Schouten-Van Kampen connection admitting Ricci solitons and η-Ricci solitons. To conclude, we present an illustration of a hyperbolic Kenmotsu manifold to validate certain results.

1. Introduction

The Schouten-van Kampen connection was introduced to analyze non-holomorphic manifolds. It represents one of the most natural connections that align with a pair of complementary distributions on a differentiable manifold with an affine connection [1–3]. Olszak [4] explored the Schouten-Van Kampen connection in 2014, to modify it for compatibility with an almost contact metric structure. He classified various types of almost contact metric manifolds through this connection and studied certain curvature properties. Bejancu [5] examined the Schouten–Van Kampen connection on foliated manifolds in 2006. In 2018, Ghosh [6] investigated the Schouten-Van Kampen connection in the context of Sasakian manifolds. Singh, Mishra and Kumar [7] investigated the non-symmetric non-metric connection in Kenmotsu Manifolds.

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Recently, Ramanaik and Nagaraja [8] investigated this connection in Kenmotsu manifold, while Yildiz [9] examined it in the f-Kenmotsu manifold in 2017. In 2021, Mondal [10] explored the Schouten-Van Kampen connection on f-Kenmotsu manifolds, while Zeren and Yildiz [11], characterize Lorentzian para-Sasakian manifolds with the Schouten-Van Kampen connection. Most recently, Mandal, Shahid and Yadav [12] have studied conformal Ricci solitons on para-contact metric (k, μ)-manifolds with Schouten–Van Kampen connection. A new type of semi-symmetric non-metric connection in the Kenmotsu manifolds has explored by Singh et al. [13].

Hyperbolic Kenmotsu manifolds represent a significant area of study within the broader context of differential geometry, particularly, in the exploration of almost contact metric structures. These manifolds extend the classical notions of Kenmotsu manifolds by incorporating hyperbolic geometry, offering unique insights into curvature properties and geometric structures. In 1972, Kenmotsu [14] investigated a class of contact Riemannian manifolds that fulfill certain special conditions, which are referred to as Kenmotsu manifolds. Kenmotsu demonstrated that a local Kenmotsu manifold can be expressed as a warped product $\mathcal{I} \times_f \mathcal{M}$, where $\mathcal{I}$ is an interval, $\mathcal{M}$ is a Kahler manifold, and the warping function is given by $f(t) = \mathfrak{s}e^t$; with $\mathfrak{s}(\neq 0)$ being a constant.

Alternatively, the concept of an almost contact hyperbolic $(\phi, \xi, \eta, \mathfrak{g})$-structure was established by Upadhyay and Dube [15]. Das and Mandal [16] studied ρ-Yamabe solitons on 3-dimensional hyperbolic Kenmotsu manifolds. Moreover, the authors [17] studied hyperbolic Kenmotsu manifolds that admits a new type of semi-symmetric non-metric connection. Furthermore, Haseeb and Prasad [18] investigated certain results on Lorentzian para-Kenmotsu manifolds. Motivated by the above ideas, in this paper, we study the hyperbolic Kenmotsu manifolds with a Schouten-Van Kampen connection.

This article is structured in the following manner: Section 2 introduces the fundamental concepts of the hyperbolic Kenmotsu manifold. Section 3 focuses on defining the relationship between the curvature tensor of the Schouten-Van Kampen connection in the hyperbolic Kenmotsu manifold. Section 4 explores concircularly flat and ξ-concircularly flat hyperbolic Kenmotsu manifold concerning the Schouten-Van Kampen connection. Section 5 explores pseudo-concircularly flat and ϕ-concircularly flat hyperbolic Kenmotsu manifolds concerning the Schouten-Van Kampen connection. Section 6 discusses ϕ-concircularly semisymmetric hyperbolic Kenmotsu manifolds concerning the Schouten-Van Kampen connection. Section 7 examines Ricci semi-symmetric hyperbolic Kenmotsu manifolds with the Schouten-Van Kampen connection. Furthermore, sections 8 and 9 are dedicated to the study of Ricci solitons and η-Ricci solitons on hyperbolic Kenmotsu manifold associated with the Schouten-Van Kampen connection, respectively. In conclusion, the final section illustrates an example of a 3-dimensional hyperbolic Kenmotsu manifold with the Schouten-Van Kampen connection to validate our findings.

2. Preliminaries

Let $\mathcal{M}$ be a $(2n+1)$-dimensional contact metric manifold equipped with the structure $(\phi,\xi,\eta,\mathfrak{g})$ including a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric $\mathfrak{g}$ on $\mathcal{M}$ fulfilling

$$\phi^2(\mathcal{K}_1) = \mathcal{K}_1 + \eta(\mathcal{K}_1)\xi, \qquad \phi \circ \xi = 0, \qquad \eta \circ \phi = 0, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$\mathfrak{g}(\phi\mathcal{K}_1, \phi\mathcal{K}_2) = -\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2) - \eta(\mathcal{K}_1)\eta(\mathcal{K}_2), \tag{2.3}$$

$$\mathfrak{g}(\mathcal{K}_1, \phi\mathcal{K}_2) = -\mathfrak{g}(\phi\mathcal{K}_1, \mathcal{K}_2), \tag{2.4}$$

$$\mathfrak{g}(\mathcal{K}_1, \xi) = \eta(\mathcal{K}_1), \tag{2.5}$$

for any vector fields $\mathcal{K}_1, \mathcal{K}_2$ on $\mathcal{M}$. An $\mathcal{M}$ is said to be a hyperbolic Kenmotsu manifold [19,20], if it is satisfies

$$(\nabla_{\mathcal{K}_1}\phi)(\mathcal{K}_2) = -\eta(\mathcal{K}_2)\phi\mathcal{K}_1 + \mathfrak{g}(\phi\mathcal{K}_1, \mathcal{K}_2)\xi, \tag{2.6}$$

$$\nabla_{\mathcal{K}_1}\xi = -\mathcal{K}_1 - \eta(\mathcal{K}_1)\xi, \tag{2.7}$$

where ∇ is Levi-Civita connection on $\mathcal{M}$.

In an $\mathcal{M}$, the following relations hold [19,20]:

$$(\nabla_{\mathcal{K}_1}\eta)(\mathcal{K}_2) = \mathfrak{g}(\phi\mathcal{K}_1, \phi\mathcal{K}_2) = -\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2) - \eta(\mathcal{K}_1)\eta(\mathcal{K}_2), \tag{2.8}$$

$$\mathfrak{g}\Big(\mathcal{R}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3, \xi\Big) = \eta\Big(\mathcal{R}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3\Big) = \mathfrak{g}(\mathcal{K}_2, \mathcal{K}_3)\eta(\mathcal{K}_1) - \mathfrak{g}(\mathcal{K}_1, \mathcal{K}_3)\eta(\mathcal{K}_2), \tag{2.9}$$

$$\mathcal{R}(\mathcal{K}_1, \mathcal{K}_2)\xi = \eta(\mathcal{K}_2)\mathcal{K}_1 - \eta(\mathcal{K}_1)\mathcal{K}_2, \tag{2.10}$$

$$\mathcal{R}(\xi, \mathcal{K}_1)\mathcal{K}_2 = -\eta(\mathcal{K}_2)\mathcal{K}_1 + \mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2)\xi, \tag{2.11}$$

$$\mathcal{R}(\xi, \mathcal{K}_1)\xi = -\mathcal{K}_1 - \eta(\mathcal{K}_1)\xi, \tag{2.12}$$

$$\mathcal{S}(\mathcal{K}_1, \xi) = 2n\eta(\mathcal{K}_1), \tag{2.13}$$

$$Q\xi = 2n\xi, \tag{2.14}$$

$$\mathcal{S}(\xi, \xi) = -2n, \tag{2.15}$$

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2) = \mathfrak{g}(Q\mathcal{K}_1,\mathcal{K}_2), \tag{2.16}$$

$$\mathcal{S}(\phi\mathcal{K}_1,\phi\mathcal{K}_2) = -\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2) - 2n\eta(\mathcal{K}_1)\eta(\mathcal{K}_2), \tag{2.17}$$

for all $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3$ on $\mathcal{M}$, where $\mathcal{R}$ is the Riemannian curvature tensor and $\mathcal{S}$ is the Ricci tensor.

Consider the set $\{e_1, e_2, ...e_{2n}, e_{2n+1}\}$ as a frame of orthonormal vector fields in the manifold $\mathcal{M}$. Thus, the Ricci tensor $\mathcal{S}$ and the scalar curvature $\mathfrak{r}$ of the manifold are defined as follows:

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2) = \sum_{i=1}^{2n+1} \epsilon_i \mathfrak{g}\Big(\mathcal{R}(e_i,\mathcal{K}_1)\mathcal{K}_2, e_i\Big),$$

$$\mathfrak{r} = \sum_{i=1}^{2n+1} \epsilon_i \mathcal{S}(e_i, e_i).$$

Additionally, we have

$$\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_2) = \sum_{i=1}^{2n+1} \epsilon_i \mathfrak{g}(\mathcal{K}_1, e_i)\mathfrak{g}(\mathcal{K}_2, e_i),$$

where $\epsilon_i = \mathfrak{g}(e_i, e_i) = \pm 1$.

Definition 2.1. *An $\mathcal{M}$ is said to be an η-Einstein manifold if its Ricci tensor $\mathcal{S}(\neq 0)$ is of the form*

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2) = a\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_2) + b\eta(\mathcal{K}_1)\eta(\mathcal{K}_2), \tag{2.18}$$

for any arbitrary vector fields $\mathcal{K}_1,\mathcal{K}_2$; where a and b are scalar functions on $\mathcal{M}$. If $b = 0$ (resp., $a = 0$), then manifold M becomes an Einstein (resp., special type of η-Einstein) manifold.

Definition 2.2. *The concircular curvature tensor C of a $(2n+1)$-dimensional hyperbolic Kenmotsu manifold with respect to the connection ∇ is defined as [21,22]*

$$C(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 - \frac{\mathfrak{r}}{2n(2n+1)}\Big\{\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1 - \mathfrak{g}(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_2\Big\}, \tag{2.19}$$

for $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3$ on $\mathcal{M}$, $\mathcal{R}$ and $\mathfrak{r}$ are the curvature tensor and the scalar curvature with respect to ∇, respectively.

Definition 2.3. *The concircular curvature tensor $\bar{C}$ of a $(2n+1)$-dimensional hyperbolic Kenmotsu manifold with respect to the Schouten-Van Kampen connection $\bar{\nabla}$ is defined as*

$$\bar{C}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \bar{\mathcal{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 - \frac{\bar{\mathfrak{r}}}{2n(2n+1)}\Big\{\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1 - \mathfrak{g}(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_2\Big\}, \tag{2.20}$$

where $\bar{\mathcal{R}}$ and $\bar{\mathfrak{r}}$ are respectively the curvature tensor and the scalar curvature with respect to the connection $\bar{\nabla}$.

Definition 2.4. *A Ricci soliton $(\mathfrak{g}, V, \lambda)$ on a Riemannian manifold is expressed as*

$$\pounds_V\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_2) + 2\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2) + 2\lambda\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_2) = 0, \tag{2.21}$$

where $\pounds_V$ is the Lie derivative operator along the vector field V on $\mathcal{M}$, and λ denotes a real number. It is described as shrinking, steady, or expanding depending on whether $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$, respectively.

Definition 2.5. *An η-Ricci soliton consists of a quadruple $(\mathfrak{g}, \xi, \lambda, \mu)$, where ξ denotes a vector field on $\mathcal{M}$, and λ, μ are real constants, which fulfills the equation*

$$(\pounds_V \mathfrak{g})(\mathcal{K}_1, \mathcal{K}_2) + 2\mathcal{S}(\mathcal{K}_1, \mathcal{K}_2) + 2\lambda \mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2) + 2\mu\eta(\mathcal{K}_1)\eta(\mathcal{K}_2) = 0. \tag{2.22}$$

Specifically, when $\mu = 0$, the concept of an η-Ricci soliton $(\mathfrak{g}, \xi, \lambda, \mu)$ turns to the Ricci soliton $(\mathfrak{g}, \xi, \lambda)$. We recommend the papers [23–26] for more detailed studies on Ricci and η-Ricci solitons.

3. Hyperbolic Kenmotsu manifolds with respect to the connection $\bar{\nabla}$

The Schouten-Van Kampen connection $\bar{\nabla}$ on $\mathcal{M}$ can be expressed as follows [27,28]:

$$\bar{\nabla}_{\mathcal{K}_1}\mathcal{K}_2 = \nabla_{\mathcal{K}_1}\mathcal{K}_2 - \eta(\mathcal{K}_2)\nabla_{\mathcal{K}_1}\xi + (\nabla_{\mathcal{K}_1}\eta)(\mathcal{K}_2)\xi. \tag{3.1}$$

Using (2.7) and (2.8), (3.1) reduced to

$$\bar{\nabla}_{\mathcal{K}_1}\mathcal{K}_2 = \nabla_{\mathcal{K}_1}\mathcal{K}_2 + \eta(\mathcal{K}_2)\mathcal{K}_1 - \mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2)\xi. \tag{3.2}$$

Putting $\mathcal{K}_2 = \xi$ in (3.2) and using (2.1), (2.5) and (2.7) we get,

$$\bar{\nabla}_{\mathcal{K}_1}\xi = -2\Big(\mathcal{K}_1 + \eta(\mathcal{K}_1)\xi\Big). \tag{3.3}$$

Putting $\mathcal{K}_1 = \xi$ in (3.1), we obtain

$$\bar{\nabla}_{\xi}\mathcal{K}_2 = \nabla_{\xi}\mathcal{K}_2, \tag{3.4}$$

and

$$(\bar{\nabla}_{\xi}\mathfrak{g})(\mathcal{K}_1, \mathcal{K}_2) = 0. \tag{3.5}$$

The curvature tensor of $\mathcal{M}$ with respect to the connection $\bar{\nabla}$ is defined as follows:

$$\bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3 = \bar{\nabla}_{\mathcal{K}_1}\bar{\nabla}_{\mathcal{K}_2}\mathcal{K}_3 - \bar{\nabla}_{\mathcal{K}_2}\bar{\nabla}_{\mathcal{K}_1}\mathcal{K}_3 - \bar{\nabla}_{[\mathcal{K}_1, \mathcal{K}_2]}\mathcal{K}_3. \tag{3.6}$$

In view of (3.3), (3.6) takes the form

$$\begin{aligned} \bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3 &= \mathcal{R}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3 - 3\Big(\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_3)\mathcal{K}_2 - \mathfrak{g}(\mathcal{K}_2, \mathcal{K}_3)\mathcal{K}_1\Big) \\ &- 2\eta(\mathcal{K}_3)\Big(\eta(\mathcal{K}_1)\mathcal{K}_2 - \eta(\mathcal{K}_2)\mathcal{K}_1\Big) \\ &- 2\Big(\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_3)\eta(\mathcal{K}_2) - \mathfrak{g}(\mathcal{K}_2, \mathcal{K}_3)\eta(\mathcal{K}_1)\Big)\xi. \end{aligned} \tag{3.7}$$

Taking the inner product of (3.7) with $\mathcal{K}_4$, we get

$$\begin{aligned} \bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4) &= \mathcal{R}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4) \\ &- 3\Big(\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_3)\mathfrak{g}(\mathcal{K}_2, \mathcal{K}_4) - \mathfrak{g}(\mathcal{K}_2, \mathcal{K}_3)\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_4)\Big) \\ &- 2\eta(\mathcal{K}_3)\Big(\eta(\mathcal{K}_1)\mathfrak{g}(\mathcal{K}_2, \mathcal{K}_4) - \eta(\mathcal{K}_2)\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_4)\Big) \\ &- 2\Big(\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_3)\eta(\mathcal{K}_2)\eta(\mathcal{K}_4) - \mathfrak{g}(\mathcal{K}_2, \mathcal{K}_3)\eta(\mathcal{K}_1)\eta(\mathcal{K}_4)\Big), \end{aligned} \tag{3.8}$$

where $\bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4) = \mathfrak{g}\Big(\bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3, \mathcal{K}_4\Big)$.

Contracting (3.8) over $\mathcal{K}_1$ and $\mathcal{K}_4$, we obtain

$$\bar{\mathcal{S}}(\mathcal{K}_2, \mathcal{K}_3) = \mathcal{S}(\mathcal{K}_2, \mathcal{K}_3) + 2(3n-1)\mathfrak{g}(\mathcal{K}_2, \mathcal{K}_3) + 2(2n-1)\eta(\mathcal{K}_2)\eta(\mathcal{K}_3), \tag{3.9}$$

where $\bar{\mathcal{S}}$ is the Ricci tensor on $\mathcal{M}$ with respect to $\bar{\nabla}$. From (3.9), we derive

$$\bar{Q}\mathcal{K}_2 = Q\mathcal{K}_2 + 2(3n-1)\mathcal{K}_2 + 2(2n-1)\eta(\mathcal{K}_2)\xi, \tag{3.10}$$

where Q and $\bar{Q}$ are the Ricci operators on $\mathcal{M}$ with respect to ∇ and $\bar{\nabla}$, respectively. Also from (3.7), we get

$$\bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2)\xi = 2\{\eta(\mathcal{K}_2)\mathcal{K}_1 - \eta(\mathcal{K}_1)\mathcal{K}_2)\}. \tag{3.11}$$

From (3.9), we obtain

$$\bar{\mathcal{S}}(\mathcal{K}_2, \xi) = 4n\eta(\mathcal{K}_2). \tag{3.12}$$

And from (3.10), we find

$$\bar{Q}\xi = 4n\xi. \tag{3.13}$$

Once again, setting $\mathcal{K}_2 = \mathcal{K}_3 = e_i$ in (3.9) and taking the sum over i, we get

$$\bar{\mathfrak{r}} = \mathfrak{r} + 12n^2 - 2n, \tag{3.14}$$

here $\bar{\mathfrak{r}}$ and $\mathfrak{r}$ denote the scalar curvatures corresponding to the connections $\bar{\nabla}$ and ∇, respectively. Furthermore, from (3.5) and using Binachi's first identity $\mathcal{R}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3 + \mathcal{R}(\mathcal{K}_2, \mathcal{K}_3)\mathcal{K}_1 + \mathcal{R}(\mathcal{K}_3, \mathcal{K}_1)\mathcal{K}_2 = 0$, we get

$$\bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3 + \bar{\mathcal{R}}(\mathcal{K}_2, \mathcal{K}_3)\mathcal{K}_1 + \bar{\mathcal{R}}(\mathcal{K}_3, \mathcal{K}_1)\mathcal{K}_2 = 0.$$

Again from (3.8), we get

$$\bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4) = -\bar{\mathcal{R}}(\mathcal{K}_2, \mathcal{K}_1, \mathcal{K}_3, \mathcal{K}_4),$$

$$\bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4) = -\bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4, \mathcal{K}_3),$$

$$\bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4) = -\bar{\mathcal{R}}(\mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_1, \mathcal{K}_2).$$

Hence, we can state the following:

Proposition 3.1. *In an* $(\mathcal{M}, g, \bar{\nabla})$, *we have*
(i) *the curvature tensor* $\bar{\mathcal{R}}$ *of* $\bar{\nabla}$ *is given by* (3.7),
(ii) *the Ricci tesor* $\bar{\mathcal{S}}$ *of* $\bar{\nabla}$ *is given by* (3.9),
(iii) *the scalar curvature* $\bar{\mathfrak{r}}$ *of* $\bar{\nabla}$ *is given by* (3.14),
(iv) $\bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3 = -\bar{\mathcal{R}}(\mathcal{K}_2, \mathcal{K}_1)\mathcal{K}_3$,
(v) $\bar{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3 + \bar{\mathcal{R}}(\mathcal{K}_2, \mathcal{K}_3)\mathcal{K}_1 + \bar{\mathcal{R}}(\mathcal{K}_3, \mathcal{K}_1)\mathcal{K}_2 = 0.$

One notable invariant of a concircular transformation is the concircular curvature tensor. By interchanging $\mathcal{K}_1$ and $\mathcal{K}_2$ in (2.20), we get

$$\bar{\mathcal{C}}(\mathcal{K}_2, \mathcal{K}_1)\mathcal{K}_3 = \bar{\mathcal{R}}(\mathcal{K}_2, \mathcal{K}_1)\mathcal{K}_3 - \frac{\bar{\mathfrak{r}}}{2n(2n+1)}\Big\{\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_3)\mathcal{K}_2 - \mathfrak{g}(\mathcal{K}_2, \mathcal{K}_3)\mathcal{K}_1\Big\}. \tag{3.15}$$

By combining (2.20) and (3.15) and applying the fact that $\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3+\mathcal{R}(\mathcal{K}_2,\mathcal{K}_1)\mathcal{K}_3=0$, we get

$$\bar{C}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3+\bar{C}(\mathcal{K}_2,\mathcal{K}_1)\mathcal{K}_3=0. \tag{3.16}$$

From (2.20) and (3.7) and first Bianchi identity $\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3+\mathcal{R}(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1+\mathcal{R}(\mathcal{K}_3,\mathcal{K}_1)\mathcal{K}_2=0$ regarding ∇, we obtain

$$\bar{C}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3+\bar{C}(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1+\bar{C}(\mathcal{K}_3,\mathcal{K}_1)\mathcal{K}_2=0. \tag{3.17}$$

Therefore, the relations (3.16) and (3.17) respectively show that the concircular curvature tensor regarding the connection $\bar{\nabla}$ in an $\mathcal{M}$ is skew-symmetric and cyclic.

4. CONCIRCULARLY FLAT AND ξ-CONCIRCULARLY FLAT HYPERBOLIC KENMOTSU MANIFOLDS WITH RESPECT TO THE CONNECTION $\bar{\nabla}$

First, we assume that the manifold $\mathcal{M}$ with the connection $\bar{\nabla}$ is concircularly flat, that is, $\bar{C}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3=0$. Then from (2.20) it follows that

$$\bar{\mathcal{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3=\frac{\bar{\mathfrak{r}}}{2n(2n+1)}\Big\{\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1-\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_2\Big\}. \tag{4.1}$$

Taking the inner product in both side of (4.1) with ξ, we have

$$\mathfrak{g}\Big(\bar{\mathcal{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3,\xi\Big)=\frac{\bar{\mathfrak{r}}}{2n(2n+1)}\Big\{\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)-\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)\Big\}. \tag{4.2}$$

Now, using (3.7), (3.14) and (2.9), we get

$$\frac{6n-\mathfrak{r}-4n^2}{2n(2n+1)}\Big\{\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)-\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)\Big\}=0.$$

Which suggests that either the scalar curvature of $\mathcal{M}$ is $\mathfrak{r}=-2n(2n-3)$, or

$$\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)-\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)=0.$$

Replacing $\mathcal{K}_2$ by ξ and $\mathcal{K}_1$ by $Q\mathcal{K}_1$ and using (2.13), we obtain

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_3)=-2n\eta(\mathcal{K}_1)\eta(\mathcal{K}_3).$$

Thus, we can state the following theorem:

Theorem 4.1. *For a concircularly flat hyperbolic Kenmotsu manifold concerning the connection $\bar{\nabla}$, either the scalar curvature is $-2n(2n-3)$, or the manifold is a special type of η-Einstein manifold.*

Next, we study ξ-concircularly flat hyperbolic Kenmotsu manifold admitting the Schouten-Van Kampen connection, i.e., $\bar{C}(\mathcal{K}_1,\mathcal{K}_2)\xi=0$. From (4.1) and (2.20), we find

$$\bar{\mathcal{R}}(\mathcal{K}_1,\mathcal{K}_2)\xi=\frac{\bar{\mathfrak{r}}}{2n(2n+1)}\Big\{\eta(\mathcal{K}_2)\mathcal{K}_1-\eta(\mathcal{K}_1)\mathcal{K}_2\Big\},$$

which by using (3.5) and (3.9) turns to

$$\frac{6n-\mathfrak{r}-4n^2}{2n(2n+1)}\Big\{\eta(\mathcal{K}_2)\mathcal{K}_1-\eta(\mathcal{K}_1)\mathcal{K}_2\Big\}=0. \tag{4.3}$$

Putting $\mathcal{K}_2 = \xi$ in (4.3) and using (2.2), we get

$$\frac{6n - \mathfrak{r} - 4n^2}{2n(2n+1)}\Big\{-\mathcal{K}_1 - \eta(\mathcal{K}_1)\xi\Big\} = 0. \tag{4.4}$$

Taking the inner product of (4.4) with $\mathcal{K}_3$, we have

$$\frac{6n - \mathfrak{r} - 4n^2}{2n(2n+1)}\Big\{\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_3) + \eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\Big\} = 0,$$

which suggests that either the scalar curvature of $\mathcal{M}$ is $\mathfrak{r} = -2n(2n-3)$, or

$$\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_3) = -\eta(\mathcal{K}_1)\eta(\mathcal{K}_3). \tag{4.5}$$

Replacing $\mathcal{K}_1$ by $Q\mathcal{K}_1$ in (4.5) and using (2.13), we obtain

$$\mathcal{S}(\mathcal{K}_1, \mathcal{K}_3) = -2n\eta(\mathcal{K}_1)\eta(\mathcal{K}_3). \tag{4.6}$$

Thus, we can state the following theorem:

Theorem 4.2. *For a ξ-concircularly flat hyperbolic Kenmotsu manifold with respect to the connection $\bar{\nabla}$, either the scalar curvature is $-2n(2n-3)$, or the manifold is a special type of η-Einstein manifold.*

5. PSEUDO-CONCIRCULARLY FLAT AND ϕ-CONCIRCULARLY FLAT HYPERBOLIC KENMOTSU MANIFOLDS WITH THE CONNECTION $\bar{\nabla}$

In this section, first we consider a pseudo-concircularly flat hyperbolic Kenmotsu manifold $\mathcal{M}$ with the connection $\bar{\nabla}$, i.e., $\phi^2\Big(\bar{C}(\phi\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3\Big) = 0$, which leads to

$$\mathfrak{g}\Big(\bar{C}(\phi\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3, \phi\mathcal{K}_4\Big) = 0, \tag{5.1}$$

for any vector fields $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$ on $\mathcal{M}$.

In view of (2.20), (5.1) turns to

$$\mathfrak{g}\Big(\bar{\mathcal{R}}(\phi\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3, \phi\mathcal{K}_4\Big) = \frac{\bar{\mathfrak{r}}}{2n(2n+1)}\Big\{\mathfrak{g}(\mathcal{K}_2, \mathcal{K}_3)\mathfrak{g}(\phi\mathcal{K}_1, \phi\mathcal{K}_4) + \mathfrak{g}(\phi\mathcal{K}_1, \mathcal{K}_3)\mathfrak{g}(\mathcal{K}_2, \phi\mathcal{K}_4)\Big\},$$

which by using (3.5) takes the form

$$\begin{aligned}
\mathfrak{g}\Big(\mathcal{R}(\phi\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3, \phi\mathcal{K}_4\Big) &= 3\{\mathfrak{g}(\phi\mathcal{K}_1, \mathcal{K}_3)\mathfrak{g}(\mathcal{K}_2, \phi\mathcal{K}_4) - \mathfrak{g}(\mathcal{K}_2, \mathcal{K}_3)\mathfrak{g}(\phi\mathcal{K}_1, \phi\mathcal{K}_4)\} \\
&- 2\eta(\mathcal{K}_3)\{\eta(\mathcal{K}_2)\mathfrak{g}(\phi\mathcal{K}_1, \phi\mathcal{K}_4)\} \\
&+ \frac{\bar{\mathfrak{r}}}{2n(2n+1)}\Big\{\mathfrak{g}(\mathcal{K}_2, \mathcal{K}_3)\mathfrak{g}(\phi\mathcal{K}_1, \phi\mathcal{K}_4) \\
&+ \mathfrak{g}(\phi\mathcal{K}_1, \mathcal{K}_3)\mathfrak{g}(\mathcal{K}_2, \phi\mathcal{K}_4)\Big\}.
\end{aligned} \tag{5.2}$$

Let $\Big\{e_1, e_2, ...e_{2n}, \xi\Big\}$ be a local orthonormal basis of vector fields in $\mathcal{M}$. Using that $\Big\{\phi e_1, \phi e_2, ...\phi e_{2n}, \xi\Big\}$ is also local orthonormal basis in $\mathcal{M}$. If we take $\mathcal{K}_1 = \mathcal{K}_4 = e_i$ and taking sum over $i(1 \leq i \leq 2n)$, then we have

$$\begin{aligned}\sum_{i=1}^{2n}\epsilon_i\mathfrak{g}\Big(\mathcal{R}(\phi e_i,\mathcal{K}_2)\mathcal{K}_3,\phi e_i\Big) &= 3\sum_{i=1}^{2n}\epsilon_i\{\mathfrak{g}(\phi e_i,\mathcal{K}_3)\mathfrak{g}(\mathcal{K}_2,\phi e_i)-\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)\mathfrak{g}(\phi e_i,\phi e_i)\}\\ &- 2\eta(\mathcal{K}_3)\sum_{i=1}^{2n}\epsilon_i\{\eta(\mathcal{K}_2)\mathfrak{g}(\phi e_i,\phi e_i)\}\\ &+ \sum_{i=1}^{2n}\epsilon_i\frac{\bar{\mathfrak{r}}}{2n(2n+1)}\Big\{\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)\mathfrak{g}(\phi e_i,\phi e_i)\\ &+ \mathfrak{g}(\phi e_i,\mathcal{K}_3)\mathfrak{g}(\mathcal{K}_2,\phi e_i)\Big\}=0.\end{aligned}\tag{5.3}$$

It can be easily verified that

$$\sum_{i=1}^{2n}\epsilon_i\mathfrak{g}\Big(\mathcal{R}(\phi e_i,\mathcal{K}_2)\mathcal{K}_3,\phi e_i\Big)=\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3)-\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)-\eta(\mathcal{K}_2)\eta(\mathcal{K}_3),\tag{5.4}$$

and

$$\sum_{i=1}^{2n}\epsilon_i\mathfrak{g}(\phi e_i,\mathcal{K}_2)\mathfrak{g}(\phi e_i,\mathcal{K}_3)=\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3).\tag{5.5}$$

From the above equations, it follows that

$$\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3)=\frac{-12n^2+10n+r(2n-1)}{2n(2n+1)}\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)-(4n-1)\eta(\mathcal{K}_2)\eta(\mathcal{K}_3).\tag{5.6}$$

Thus, we can state the following theorem:

Theorem 5.1. *A hyperbolic Kenmotsu manifold admitting the connection $\bar{\nabla}$ is pseudo-concircularly flat if and only if the manifold is an η-Einstein manifold.*

Next, we consider a ϕ-concircularly flat hyperbolic Kenmotsu manifold with the connection $\bar{\nabla}$, i.e., $\phi^2\Big(\bar{C}(\phi\mathcal{K}_1,\phi\mathcal{K}_2)\phi\mathcal{K}_3\Big)=0$, which leads to

$$\mathfrak{g}\Big(\bar{C}(\phi\mathcal{K}_1,\phi\mathcal{K}_2)\phi\mathcal{K}_3,\phi\mathcal{K}_4\Big)=0.\tag{5.7}$$

In view of (2.20), (5.7) becomes

$$\begin{aligned}\mathfrak{g}\Big(\bar{\mathcal{R}}(\phi\mathcal{K}_1,\phi\mathcal{K}_2)\phi\mathcal{K}_3,\phi\mathcal{K}_4\Big) &= \frac{\bar{\mathfrak{r}}}{2n(2n+1)}\Big\{\mathfrak{g}(\phi\mathcal{K}_2,\phi\mathcal{K}_3)\mathfrak{g}(\phi\mathcal{K}_1,\phi\mathcal{K}_4)\\ &- \mathfrak{g}(\phi\mathcal{K}_1,\phi\mathcal{K}_3)\mathfrak{g}(\phi\mathcal{K}_2,\phi\mathcal{K}_4)\Big\}.\end{aligned}\tag{5.8}$$

Using (3.8) in (5.8), we have

$$\begin{aligned}\mathfrak{g}\Big(\mathcal{R}(\phi\mathcal{K}_1,\phi\mathcal{K}_2)\phi\mathcal{K}_3,\phi\mathcal{K}_4\Big) &= 3\{\mathfrak{g}(\phi\mathcal{K}_1,\phi\mathcal{K}_3)\mathfrak{g}(\phi\mathcal{K}_2,\phi\mathcal{K}_4)-\mathfrak{g}(\phi\mathcal{K}_2,\phi\mathcal{K}_3)\mathfrak{g}(\phi\mathcal{K}_1,\phi\mathcal{K}_4)\}\\ &+ \frac{\bar{\mathfrak{r}}}{2n(2n+1)}\Big\{\mathfrak{g}(\phi\mathcal{K}_2,\phi\mathcal{K}_3)\mathfrak{g}(\phi\mathcal{K}_1,\phi\mathcal{K}_4)\\ &+ \mathfrak{g}(\phi\mathcal{K}_1,\phi\mathcal{K}_3)\mathfrak{g}(\phi\mathcal{K}_2,\phi\mathcal{K}_4)\Big\}.\end{aligned}\tag{5.9}$$

Let $\left\{e_1, e_2, ...e_{2n}, \xi\right\}$ be a local orthonormal basis of vector fields in $\mathcal{M}$. Using that $\left\{\phi e_1, \phi e_2, ...\phi e_{2n}, \xi\right\}$ is also local orthonormal basis in $\mathcal{M}$. If we take $\mathcal{K}_1 = \mathcal{K}_4 = e_i$ and taking sum over $i(1 \leq i \leq 2n)$, then we get

$$\begin{aligned}\sum_{i=1}^{2n} \epsilon_i \mathfrak{g}\Big(\mathcal{R}(\phi e_i, \phi\mathcal{K}_2)\phi\mathcal{K}_3, \phi e_i\Big) &= 3\sum_{i=1}^{2n} \epsilon_i \{\mathfrak{g}(\phi e_i, \phi\mathcal{K}_3)\mathfrak{g}(\phi\mathcal{K}_2, \phi e_i) - \mathfrak{g}(\phi\mathcal{K}_2, \phi\mathcal{K}_3)\mathfrak{g}(\phi e_i, \phi e_i)\} \\ &+ \frac{\bar{\mathfrak{r}}}{2n(2n+1)}\sum_{i=1}^{2n} \epsilon_i \Big\{\mathfrak{g}(\phi\mathcal{K}_2, \phi\mathcal{K}_3)\mathfrak{g}(\phi e_i, \phi e_i) \\ &+ \mathfrak{g}(\phi e_i, \phi\mathcal{K}_3)\mathfrak{g}(\phi\mathcal{K}_2, \phi e_i\Big\} = 0. \end{aligned} \tag{5.10}$$

It can be easily verified that

$$\sum_{i=1}^{2n} \epsilon_i \mathfrak{g}\Big(\mathcal{R}(\phi e_i, \phi\mathcal{K}_2)\phi\mathcal{K}_3, \phi e_i\Big) = \mathcal{S}(\phi\mathcal{K}_2, \phi\mathcal{K}_3) - \mathfrak{g}(\phi\mathcal{K}_2, \phi\mathcal{K}_3), \tag{5.11}$$

and

$$\sum_{i=1}^{2n} \epsilon_i \mathfrak{g}(\phi e_i, \phi\mathcal{K}_2)\mathfrak{g}(\phi e_i, \phi\mathcal{K}_3) = \mathfrak{g}(\phi\mathcal{K}_2, \phi\mathcal{K}_3). \tag{5.12}$$

From the above equations, it follows that

$$\mathcal{S}(\phi\mathcal{K}_2, \phi\mathcal{K}_3) = \frac{-12n^2 + 10n + r(2n-1)}{2n(2n+1)}\mathfrak{g}(\phi\mathcal{K}_2, \phi\mathcal{K}_3). \tag{5.13}$$

By using (2.3), (2.17) in (5.13), we get

$$\begin{aligned}\mathcal{S}(\mathcal{K}_2, \mathcal{K}_3) &= \frac{-12n^2 + 10n + r(2n-1)}{2n(2n+1)}\mathfrak{g}(\mathcal{K}_2, \mathcal{K}_3) \\ &+ \frac{-8n^3 - 16n^2 + 10n + r(2n-1)}{2n(2n+1)}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3). \end{aligned} \tag{5.14}$$

Thus, we can state the following theorem:

Theorem 5.2. *A hyperbolic Kenmotsu manifold admitting the connection $\bar{\nabla}$ is ϕ-concircularly flat if and only if the manifold is an η-Einstein manifold regarding the connection ∇.*

6. ϕ-CONCIRCULARLY SEMI-SYMMETRIC HYPERBOLIC KENMOTSU MANIFOLDS WITH RESPECT TO THE CONNECTION $\bar{\nabla}$

Definition 6.1. *An $\mathcal{M}$ is said to be ϕ-concircularly semi-symmetric with respect to the connection $\bar{\nabla}$, if it fulfills the condition $\bar{C}(\mathcal{K}_1, \mathcal{K}_2).\phi = 0$.*

Let the manifold be a ϕ-concircularly semi-symmetric with respect to the connection $\bar{\nabla}$. Then, we have

$$\Big(\bar{C}(\mathcal{K}_1, \mathcal{K}_2).\phi\Big)\mathcal{K}_3 = \bar{C}(\mathcal{K}_1, \mathcal{K}_2).\phi\mathcal{K}_3 - \phi\bar{C}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3. \tag{6.1}$$

Putting $\mathcal{K}_3 = \xi$ in (6.1), it follows that

$$\phi\Big(\bar{C}(\mathcal{K}_1,\mathcal{K}_2)\xi\Big) = 0. \tag{6.2}$$

Using (2.20), (2.10) and (3.9) in (6.2), we get

$$\frac{-4n^2+6n-\mathfrak{r}}{2n(2n+1)}\Big\{\eta(\mathcal{K}_2)\phi\mathcal{K}_1 - \eta(\mathcal{K}_1)\phi\mathcal{K}_2\Big\} = 0,$$

which by replacing $\mathcal{K}_2$ by ξ and $\mathcal{K}_1$ by $\phi\mathcal{K}_1$ turns to

$$\frac{-4n^2+6n-\mathfrak{r}}{2n(2n+1)}\Big\{-\mathcal{K}_1 - \eta(\mathcal{K}_1)\xi\Big\} = 0. \tag{6.3}$$

Taking the inner product of (6.3) with $\mathcal{K}_3$, we have

$$\frac{-4n^2+6n-\mathfrak{r}}{2n(2n+1)}\Big\{\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_3) + \eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\Big\} = 0. \tag{6.4}$$

This implies that either the scalar curvature of $\mathcal{M}$ is $\mathfrak{r} = -4n^2+6n$, or

$$\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_3) = -\eta(\mathcal{K}_1)\eta(\mathcal{K}_3). \tag{6.5}$$

Replacing $\mathcal{K}_1$ by $Q\mathcal{K}_1$ and using (2.13), we obtain

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_3) = -2n\eta(\mathcal{K}_1)\eta(\mathcal{K}_3). \tag{6.6}$$

Therefore, we can state the following theorem:

Theorem 6.1. *For a ϕ-concircularly semi-symmetric hyperbolic Kenmotsu manifold with respect to the connection $\bar{\nabla}$, either the scalar curvature is $-4n^2+6n$ or the manifold is a special type of η-Einstein manifold.*

7. Ricci semi-symmetric hyperbolic Kenmotsu manifolds with respect to the connection $\bar{\nabla}$

An $\mathcal{M}$ equipped with the connection $\bar{\nabla}$ is considered Ricci semi-symmetric if

$$\Big(\bar{\mathcal{R}}(\mathcal{K}_1,\mathcal{K}_2).\bar{\mathcal{S}}\Big)(\mathcal{K}_3,\mathcal{K}_4) = 0. \tag{7.1}$$

Therefore, it can be written as

$$\bar{\mathcal{S}}\Big(\bar{\mathcal{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4\Big) + \bar{\mathcal{S}}\Big(\mathcal{K}_3,\bar{\mathcal{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4\Big) = 0, \tag{7.2}$$

for any $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4$ on $\mathcal{M}$.

Putting $\mathcal{K}_1 = \xi$ in (7.2), we have

$$\bar{\mathcal{S}}\Big(\bar{\mathcal{R}}(\xi,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4\Big) + \bar{\mathcal{S}}\Big(\mathcal{K}_3,\bar{\mathcal{R}}(\xi,\mathcal{K}_2)\mathcal{K}_4\Big) = 0. \tag{7.3}$$

Using (3.3) in (7.3), we get

$$\eta(\mathcal{K}_3)\bar{\mathcal{S}}(\mathcal{K}_2,\mathcal{K}_4) - \mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)\bar{\mathcal{S}}(\xi,\mathcal{K}_4) + \eta(\mathcal{K}_4)\bar{\mathcal{S}}(\mathcal{K}_2,\mathcal{K}_3) - \mathfrak{g}(\mathcal{K}_2,\mathcal{K}_4)\bar{\mathcal{S}}(\mathcal{K}_3,\xi) = 0. \tag{7.4}$$

Replacing $\mathcal{K}_4 = \xi$ in (7.4) and using (2.2), (3.12) in (7.4), we obtain

$$\bar{\mathcal{S}}(\mathcal{K}_2,\mathcal{K}_4) = 4n\mathfrak{g}(\mathcal{K}_2,\mathcal{K}_4). \tag{7.5}$$

Therefore, we can state the following theorem:

Theorem 7.1. *A hyperbolic Kenmotsu manifold admitting the connection $\bar{\nabla}$ is Ricci semi-symmetric if and only if the manifold is an Einstein manifold.*

8. Ricci solitons on hyperbolic Kenmotsu manifolds with the connection $\bar{\nabla}$

The Ricci soliton characterized by the data $(\mathfrak{g}, V, \lambda)$ is defined by Equation (2.21). Here, two cases arise corresponding to the vector field V: V$\in$ span(ξ) and $V \perp$ span(ξ). We focus exclusively on the case $V = \xi$. The Ricci soliton $(\mathfrak{g}, V, \lambda)$ on a hyperbolic Kenmotsu manifold admitting the Schouten-Van Kampen connection with $\bar{\nabla}$ is represented as

$$(\bar{\pounds}_V \mathfrak{g})(\mathcal{K}_1, \mathcal{K}_2) + 2\bar{\mathcal{S}}(\mathcal{K}_1, \mathcal{K}_2) + 2\lambda \mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2) = 0, \tag{8.1}$$

for any $\mathcal{K}_1, \mathcal{K}_2$ on $\mathcal{M}$. Here

$$(\bar{\pounds}_V \mathfrak{g})(\mathcal{K}_1, \mathcal{K}_2) = \mathfrak{g}(\bar{\nabla}_{\mathcal{K}_1}\xi, \mathcal{K}_2) + \mathfrak{g}(\mathcal{K}_1, \bar{\nabla}_{\mathcal{K}_2}\xi), \tag{8.2}$$

which signifies that

$$\mathfrak{g}(\bar{\nabla}_{\mathcal{K}_1}\xi, \mathcal{K}_2) + \mathfrak{g}(\mathcal{K}_1, \bar{\nabla}_{\mathcal{K}_2}\xi) + 2\bar{\mathcal{S}}(\mathcal{K}_1, \mathcal{K}_2) + 2\lambda \mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2) = 0. \tag{8.3}$$

Using (3.3) in (8.3), we obtain

$$\bar{\mathcal{S}}(\mathcal{K}_1, \mathcal{K}_2) - 2\eta(\mathcal{K}_1)\eta(\mathcal{K}_2) + (\lambda - 2)\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2) = 0. \tag{8.4}$$

Putting $\mathcal{K}_2 = \xi$ in (8.4), we obtain

$$\lambda = -4n < 0 \tag{8.5}$$

Putting this value of λ in (8.4), we get

$$\bar{\mathcal{S}}(\mathcal{K}_1, \mathcal{K}_2) = (4n + 2)\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2) + 2\eta(\mathcal{K}_1)\eta(\mathcal{K}_2). \tag{8.6}$$

Therefore, we can state the following theorem:

Theorem 8.1. *If a hyperbolic Kenmotsu manifold admitting the connection $\bar{\nabla}$ admits a Ricci soliton then the manifold is an η-Einstein manifold taking the form (8.6) and the Ricci soliton is always shrinking.*

9. η-Ricci solitons on hyperbolic Kenmotsu manifolds with the connection $\bar{\nabla}$

Consider a hyperbolic Kenmotsu manifold with a connection $\bar{\nabla}$ that possesses an η-Ricci soliton $(\mathfrak{g}, \xi, \lambda, \mu)$. Hence, (2.22) holds and we have

$$(\bar{\pounds}_V \mathfrak{g})(\mathcal{K}_1, \mathcal{K}_2) + 2\bar{\mathcal{S}}(\mathcal{K}_1, \mathcal{K}_2) + 2\lambda \mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2) + 2\mu\eta(\mathcal{K}_1)\eta(\mathcal{K}_2) = 0. \tag{9.1}$$

Using (8.2) in (9.1), we obtain

$$\mathfrak{g}(\bar{\nabla}_{\mathcal{K}_1}\xi, \mathcal{K}_2) + \mathfrak{g}(\mathcal{K}_1, \bar{\nabla}_{\mathcal{K}_2}\xi) + 2\bar{\mathcal{S}}(\mathcal{K}_1, \mathcal{K}_2) + 2\lambda \mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2) + 2\mu\eta(\mathcal{K}_1)\eta(\mathcal{K}_2) = 0. \tag{9.2}$$

Using (3.3) in (9.2), we get

$$\bar{\mathcal{S}}(\mathcal{K}_1, \mathcal{K}_2) = -(\lambda - 2)\mathfrak{g}(\mathcal{K}_1, \mathcal{K}_2) - (\mu - 2)\eta(\mathcal{K}_1)\eta(\mathcal{K}_2). \tag{9.3}$$

Putting $\mathcal{K}_1 = \mathcal{K}_2 = \xi$ in (9.3), we obtain

$$\lambda - \mu = -4n \tag{9.4}$$

Therefore, we can state the following theorem:

Theorem 9.1. *If $(\mathfrak{g}, \xi, \lambda, \mu)$ is an η-Ricci soliton on a hyperbolic Kenmotsu manifold admitting the connection $\bar{\nabla}$ is an η-Einstein manifold and the scalars λ and μ are related by $\lambda - \mu = -4n$.*

10. EXAMPLE

Let us consider the 3-dimensional manifold $\mathcal{M}=\{(k_1,k_2,k_3) \in R^3 : k_3 \neq 0\}$, with (k_1,k_2,k_3), representing the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = e^{k_3}\frac{\partial}{\partial k_1},\ \ e_2 = e^{k_3}\frac{\partial}{\partial k_2},\ \ e_3 = \frac{\partial}{\partial k_3} = \xi.$$

Let $\mathfrak{g}$ represents the Riemannian metric specified by

$$g(e_i, e_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let ϕ represents the (1,1)-tensor field expressed by

$$\phi(e_1) = e_2,\ \ \phi(e_2) = e_1,\ \ \phi(e_3) = 0.$$

Let η represents the 1-form expressed by $\eta(\mathcal{K}_1) = \mathfrak{g}(\mathcal{K}_1, e_3)$ for any $\mathcal{K}_1$ on $\mathcal{M}$.

Then using the linearity of g and ϕ, we get

$$\eta(e_3) = -1, \phi^2(\mathcal{K}_1) = \mathcal{K}_1 + \eta(\mathcal{K}_1)e_3, g(\phi\mathcal{K}_1, \phi\mathcal{K}_1) = -g(\mathcal{K}_1, \mathcal{K}_1) - \eta(\mathcal{K}_1)\eta(\mathcal{K}_1),$$

where $\mathcal{K}_1 \in \chi\ (M)$. Then, for $e_3 = \xi$, the structure $(\phi,\xi,\eta,\mathfrak{g})$ introduces an almost contact metric structure on $\mathcal{M}$.

Let ∇ represents the Levi-Civita connection related to the metric tensor $\mathfrak{g}$. The Lie bracket can be calculated by using the definition $[\mathcal{K}_1, \mathcal{K}_2]f = \mathcal{K}_1(\mathcal{K}_2 f) - \mathcal{K}_2(\mathcal{K}_1 f)$. Thus, we obtain

$$\begin{aligned} &[e_1, e_1] = 0, && [e_1, e_2] = 0, && [e_1, e_3] = -e_1, \\ &[e_2, e_1] = 0, && [e_2, e_2] = 0, && [e_2, e_3] = -e_2. \\ &[e_3, e_1] = e_1, && [e_3, e_2] = e_2, && [e_3, e_3] = 0. \end{aligned}$$

The Levi-Civita connection ∇ associated with the metric $\mathfrak{g}$ is expressed by the Koszul's formula

$$\begin{aligned} 2g(\nabla_{\mathcal{K}_1}\mathcal{K}_2, \mathcal{K}_3) &= \mathcal{K}_1 g(\mathcal{K}_2, \mathcal{K}_3) + \mathcal{K}_2 g(\mathcal{K}_3, \mathcal{K}_1) - \mathcal{K}_3 g(\mathcal{K}_1, \mathcal{K}_2) \\ &+ g([\mathcal{K}_1, \mathcal{K}_2], \mathcal{K}_3) - g([\mathcal{K}_2, \mathcal{K}_3], \mathcal{K}_1) + g([\mathcal{K}_3, \mathcal{K}_1], \mathcal{K}_2). \end{aligned}$$

By using the above formula, we obtain

$$\nabla_{e_1}e_1 = -e_3 \quad ,\nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = -e_1,$$
$$\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_2}e_3 = -e_2,$$
$$\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0 \ \ \nabla_{e_3}e_3 = 0.$$

From the above properties the manifold satisfies $\nabla_{\mathcal{K}_1}\xi = -\mathcal{K}_1 - \eta(\mathcal{K}_1)\xi$, for $e_3 = \xi$. Therefore, the manifold qualifies as a Kenmotsu manifold. Using the formula

$$\mathcal{R}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3 = \nabla_{\mathcal{K}_1}\nabla_{\mathcal{K}_2}\mathcal{K}_3 - \nabla_{\mathcal{K}_2}\nabla_{\mathcal{K}_1}\mathcal{K}_3 - \nabla_{[\mathcal{K}_1,\mathcal{K}_2]}\mathcal{K}_3,$$

we calculate the following expressions:

$$\mathcal{R}(e_1, e_2)e_1 = -e_2 = -\mathcal{R}(e_2, e_1)e_1, \qquad \mathcal{R}(e_1, e_3)e_1 = -e_3 = -\mathcal{R}(e_3, e_1)e_1,$$
$$\mathcal{R}(e_1, e_2)e_2 = -e_1 = -\mathcal{R}(e_2, e_1)e_2, \qquad \mathcal{R}(e_2, e_3)e_2 = e_3 = -\mathcal{R}(e_3, e_2)e_2,$$
$$\mathcal{R}(e_1, e_3)e_3 = -e_1 = -\mathcal{R}(e_3, e_1)e_3, \qquad \mathcal{R}(e_2, e_3)e_3 = -e_2 = -\mathcal{R}(e_3, e_2)e_3.$$

Hence we have the following values:

$$\mathcal{S}(e_1, e_1) = 2, \qquad \mathcal{S}(e_2, e_2) = -2, \qquad \mathcal{S}(e_3, e_3) = -2.$$

Consequently, $\sum_{i=1}^{3} \epsilon_i \mathcal{S}(e_i, e_i) = \mathfrak{r} = 6$.

Using (3.2) we calculate the following values:

$$\bar{\nabla}_{e_1}e_1 = -2e_3, \qquad \bar{\nabla}_{e_1}e_2 = 0, \qquad \bar{\nabla}_{e_1}e_3 = -2e_1$$
$$\bar{\nabla}_{e_2}e_1 = 0, \qquad \bar{\nabla}_{e_2}e_2 = 2e_3, \qquad \bar{\nabla}_{e_2}e_3 = -2e_2$$
$$\bar{\nabla}_{e_3}e_1 = 0, \qquad \bar{\nabla}_{e_3}e_2 = 0, \qquad \bar{\nabla}_{e_3}e_3 = 0.$$

Further, using (3.7) we can calculate

$$\bar{\mathcal{R}}(e_2, e_1)e_1 = 4e_2, \qquad \bar{\mathcal{R}}(e_3, e_1)e_1 = 2e_3, \qquad \bar{\mathcal{R}}(e_1, e_2)e_2 = -4e_1,$$
$$\bar{\mathcal{R}}(e_3, e_2)e_2 = -2e_3, \qquad \bar{\mathcal{R}}(e_1, e_3)e_3 = -2e_1, \qquad \bar{\mathcal{R}}(e_2, e_3)e_3 = -2e_2.$$

Hence we have the following values:

$$\bar{\mathcal{S}}(e_1, e_1) = 6, \qquad \bar{\mathcal{S}}(e_2, e_2) = -6, \qquad \bar{\mathcal{S}}(e_3, e_3) = -4 \implies \bar{\mathfrak{r}} = 16.$$

Which can also be verified from (3.14). Hence, from the above discussion, it is clear that theorems (8.1) and (9.1) are satisfied by the given example.

11. CONCLUSIONS

In this paper, we examined concircularly flat and ξ-concircularly flat hyperbolic Kenmotsu manifolds and concluded that the manifolds are special type of η-Einstein manifolds. Again, we discussed that pseudo-concircularly flat hyperbolic Kenmotsu manifold and it comes out to be an η-Einstein manifold. Next, we explored the ϕ-concircularly flat hyperbolic Kenmotsu manifold with the Schouten-Van Kampen connection and established that it is an η-Einstein manifold. Also, we explored a ϕ-concircularly semi-symmetric hyperbolic Kenmotsu manifold with the Schouten-Van Kampen connection and verified that it is a special type of η-Einstein manifold. Furthermore, we investigated a Ricci semi-symmetric hyperbolic Kenmotsu manifold with Schouten-Van Kampen connection that it is an Einstein manifold and η-Einstein manifold. In the last two sections, we discussed that the hyperbolic Kenmotsu manifold with the Schouten-Van Kampen connection admitting Ricci solitons and η-Ricci solitons are η-Einstein manifolds.

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