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# Continuous $K$-Biframes in Hilbert Spaces

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**Abstract.** In this paper, we will introduce the concept of a continuous $K$-biframe for Hilbert spaces, and we will present various examples of continuous $K$-biframes. Furthermore, we investigate their characteristics from the perspective of operator theory by establishing various properties. These results are interesting and more general than what exists in the literature.

## 1. Introduction

The notion of frames in Hilbert spaces was introduced by Duffin and Schaffer [2] in 1952 to research certain difficult nonharmonic Fourier series problems. Following the fundamental paper [1] by Daubechies, Grossman, and Meyer, frame theory started to become popular, especially in the more specific context of Gabor frames and wavelet frames [6]. Currently, frames are frequently employed in distributed signal processing, image processing, operator theory, harmonic analysis, wireless communications, and many other fields. For more detailed information, readers are recommended to consult [4, 7–10, 12–16, 19–30].

The concept of $K$-frames was introduced by Laura Găvruţa and serves as a tool for investigating atomic systems with respect to a bounded linear operator $K$ in a separable Hilbert space. $K$-frames generalize ordinary frames by requiring that the lower frame bound is applicable only to elements within the range of $K$.

The idea of pair frames, which refers to a pair of sequences in a Hilbert space, was first presented in [5]. Parizi, Alijani, and Dehghan [17] studied Biframe, which is a generalization of controlled frame in Hilbert space. The concept of a frame is defined from a single sequence, but to define

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a biframe, we will need two sequences. In fact, the concept of biframe is a generalization of controlled frames and a special case of pair frames. Recently, Lfounoune, Karara, and Rossafi studied and developed the concept of continuous biframe for Hilbert $C^*$-modules [9].

In this paper, we will introduce the concept of continuous $K$-biframes in Hilbert space, which is a generalization of discrete $K$-biframes in Hilbert space [8]. Also, we will present some examples of this type of frame. Moreover, we investigate a characterization of continuous $K$-biframe by using the continuous biframe operator. Finally, in our exploration of continuous biframes, we investigate their characteristics from the perspective of operator theory by establishing various properties.

## 2. PRELIMINARIES

Throughout this paper, $\mathcal{H}$ represents a separable Hilbert space. The notation $\mathcal{B}(\mathcal{H},\mathcal{K})$ denotes the collection of all bounded linear operators from $\mathcal{H}$ to the Hilbert space $\mathcal{K}$. When $\mathcal{H} = \mathcal{K}$, this set is denoted simply as $\mathcal{B}(\mathcal{H})$. We will use $\mathcal{N}(\mathcal{T})$ and $\mathcal{R}(\mathcal{T})$ for the null and range space of an operator $\mathcal{T} \in \mathcal{B}(\mathcal{H})$. Also $\mathrm{GL}^+(\mathcal{H})$ is the collection of all invertible, positive bounded linear operators acting on $\mathcal{H}$.

In the following definition we define the notion of continuous frame in Hilbert spaces.

**Definition 2.1.** *[18] Let $\mathcal{H}$ be a complex Hilbert space and $(\Omega, \mu)$ be a measure space with positive measure $\mu$. A mapping $\mathcal{X} : \Omega \to \mathcal{H}$ is called a continuous frame with respect to $(\Omega, \mu)$ if*

*$(i)$ $\mathcal{X}$ is weakly-measurable, i.e., for all $f \in \mathcal{H}$, $\omega \mapsto \langle f, \mathcal{X}(\omega)\rangle$ is a measurable function on $\Omega$,*

*$(ii)$ there exist constants $0 < A \leq B < \infty$ such that*

$$A\|f\|^2 \leq \int_\Omega |\langle f, \mathcal{X}(\omega)\rangle|^2 d\mu \leq B\|f\|^2,$$

*for all $f \in \mathcal{H}$.*

The constants $A$ and $B$ are called continuous frame bounds. If $A = B$, then it is called a tight continuous frame. If the mapping $\mathcal{X}$ satisfies only the right inequality, then it is called continuous Bessel mapping with Bessel bound $B$.

Let $\mathcal{X} : \Omega \to \mathcal{H}$ be a continuous frame for $\mathcal{H}$. Then The synthesis operator $\mathcal{T}_\mathcal{X} : L^2(\Omega, \mu) \to \mathcal{H}$ weakly defined by

$$\langle \mathcal{T}_\mathcal{X}(\varphi), h\rangle = \int_\Omega \varphi(\omega)\langle \mathcal{X}(\omega), f\rangle d\mu$$

where $\varphi \in L^2(\Omega, \mu)$ and $f \in \mathcal{H}$ and its adjoint operator called the analysis operator $\mathcal{T}^*_\mathcal{X} : \mathcal{H} \to L^2(\Omega, \mu)$ is given by

$$\mathcal{T}^*_\mathcal{X}\mathcal{X}(\omega) = \langle f, \mathcal{X}(\omega)\rangle\ ,\ f \in \mathcal{H},\ \ \omega \in \Omega.$$

The frame operator $S_\mathcal{X} : \mathcal{H} \to \mathcal{H}$ is weakly defined by

$$\langle S_\mathcal{X} x, y\rangle = \int_\Omega \langle x, \mathcal{X}(\omega)\rangle\langle \mathcal{X}(\omega), y\rangle d\mu,\ \forall x, y \in \mathcal{H}.$$

The following definition and theorem are used to prove our results.

**Definition 2.2.** *[11] Let $\mathcal{H}$ be a Hilbert space, and suppose that $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ has a closed range. Then there exists an operator $\mathcal{T}^+ \in \mathcal{B}(\mathcal{H})$ for which*

$$\mathcal{N}\left(\mathcal{T}^+\right) = \mathcal{R}(\mathcal{T})^\perp, \quad \mathcal{R}\left(\mathcal{T}^+\right) = \mathcal{N}(\mathcal{T})^\perp, \quad \mathcal{T}\mathcal{T}^+ f = f, \quad f \in \mathcal{R}(\mathcal{T}).$$

*We call the operator $\mathcal{T}^+$ the pseudo-inverse of $\mathcal{T}$. This operator is uniquely determined by these properties. In fact, if $\mathcal{T}$ is invertible, then we have $\mathcal{T}^{-1} = \mathcal{T}^+$.*

Now we recall the famous Douglas majorization theorem.

**Theorem 2.1.** *[3] Let $\mathcal{H}$ be a Hilbert space and $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{B}(\mathcal{H})$. The following statements are equivalent:*

(1) $\mathcal{R}(\mathcal{T}_1) \subset \mathcal{R}(\mathcal{T}_2)$;
(2) $\mathcal{T}_1\mathcal{T}_1^* \leq \lambda^2 \mathcal{T}_2\mathcal{T}_2^*$ *for some* $\lambda \geq 0$;
(3) $\mathcal{T}_1 = \mathcal{T}_2 U$ *for some* $U \in \mathcal{B}(\mathcal{H})$.

## 3. Continuous $K$-biframe in Hilbert spaces

In this section, we begin by presenting the definition of a continuous $K$-biframe in a Hilbert spaces, followed by a discussion of some of its properties.

**Definition 3.1.** *A pair $(\mathcal{X}, \mathcal{Y}) = (\mathcal{X} : \Omega \to \mathcal{H},\ \mathcal{Y} : \Omega \to \mathcal{H})$ of mappings is called a continuous K-biframe for $\mathcal{H}$ with respect to $(\Omega, \mu)$ if:*

- $(i)$ *$\mathcal{X}, \mathcal{Y}$ are weakly-measurable, i.e., for all $f \in \mathcal{H}$, $\omega \mapsto \langle f, \mathcal{X}(\omega)\rangle$ and $\omega \mapsto \langle f, \mathcal{Y}(\omega)\rangle$ are measurable functions on $\Omega$,*
- $(ii)$ *there exist constants $0 < A \leq B < \infty$ such that for all $f \in \mathcal{H}$:*

$$A\|K^* f\|^2 \leq \int_\Omega \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu \leq B\|f\|^2. \tag{3.1}$$

*The constants A and B are called continuous K-biframe bounds. If $A = B$, then it is called a tight continuous K-biframe and if $A = B = 1$, then it is called Parseval continuous K-biframe .*

*If $(\mathcal{X}, \mathcal{Y})$ satisfies only the right inequality (3.1), then it is called continuous K-biframe Bessel mapping with Bessel bound B.*

**Remark 3.1.** *Let $\mathcal{X} : \Omega \to \mathcal{H}$ be a mapping. Consequently, in light of the Definition 3.1, we express that*

- $(i)$ *If $(\mathcal{X}, \mathcal{X})$ is a continuous K-biframe for $\mathcal{H}$, then $\mathcal{X}$ is a continuous frame for $\mathcal{H}$.*
- $(ii)$ *If $P \in GL^+(\mathcal{H})$, $(\mathcal{X}, P\mathcal{X})$ is a continuous K-biframe for $\mathcal{H}$, then $\mathcal{X}$ is a P–controlled continuous frame for $\mathcal{H}$,*
- $(iii)$ *If $P, Q \in GL^+(\mathcal{H})$, $(P\mathcal{X}, Q\mathcal{X})$ is a continuous K-biframe for $\mathcal{H}$, then $\mathcal{X}$ is a $(P, Q)$–controlled continuous frame for $\mathcal{H}$.*

We now provide some examples that verify the description given above.

**Example 3.1.** *Let $\{e_k\}_{i=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. We consider two sequences $\mathcal{X} = \{f_i\}_{i=1}^\infty$ and $\mathcal{Y} = \{g_i\}_{i=1}^\infty$ defined as follows:*

$$\mathcal{X} = \{e_1, e_1, e_1, e_2, e_3, \cdots\cdots\},$$
$$\mathcal{Y} = \{0, e_1, e_1, e_2, e_3, \cdots\cdots\}.$$

*We have* $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, *where* $\{\Omega_i\}_{i=1}^{\infty}$ *is a sequence of disjoint measurable subsets of* $\Omega$ *with* $\mu(\Omega_i) < \infty$. *For any* $\omega \in \Omega$, *we define the mappings* $\mathcal{X} : \Omega \to \mathcal{H}$ *by* $\mathcal{X}(\omega) = (\mu(\Omega_i))^{-1} f_i$ *and* $\mathcal{Y} : \Omega \to \mathcal{H}$ *by* $\mathcal{Y}(\omega) = (\mu(\Omega_i))^{-1} g_i$ *and* $K : \mathcal{H} \to \mathcal{H}$ *by*

$$Ke_1 = e_1,\ Ke_2 = e_1,\ Ke_3 = e_1,\ Ke_4 = e_2,\ Ke_5 = e_3, \cdots.$$

*Then for* $f \in \mathcal{H}$,

$$\begin{aligned}\int_{\Omega} \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{X}(\omega), f\rangle d\mu &= \sum_{i=1}^{\infty} \int_{\Omega_i} \langle f, f_i\rangle\langle f_i, f\rangle d\mu \\ &= 2\langle f, e_1\rangle\langle f, e_1\rangle + \sum_{i=1}^{\infty} \langle f, e_i\rangle\langle e_i, f\rangle \\ &= 2\langle f, e_1\rangle\langle f, e_1\rangle + \|f\|^2. \\ &\leq 3\|f\|^2\end{aligned}$$

*So,*

$$\|K^* f\|^2 \leq \int_{\Omega} \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{X}(\omega), f\rangle d\mu \leq 3\|f\|^2.$$

*Therefore,* $\mathcal{X}$ *is a continuous K-frame for* $\mathcal{H}$ *with bounds* 1 *and* 3. *Similarly, we have*

$$\begin{aligned}\int_{\Omega} \langle f, \mathcal{Y}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu &= \sum_{i=1}^{\infty} \int_{\Omega_i} \langle f, g_i\rangle\langle g_i, f\rangle d\mu \\ &= \langle f, e_1\rangle\langle f, e_1\rangle + \sum_{i=1}^{\infty} \langle f, e_i\rangle\langle e_i, f\rangle \\ &= \langle f, e_1\rangle\langle f, e_1\rangle + \|f\|^2. \\ &\leq 2\|f\|^2.\end{aligned}$$

*So,*

$$\|K^* f\|^2 \leq \int_{\Omega} \langle f, \mathcal{Y}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu \leq 2\|f\|^2$$

*Therefore* $\mathcal{Y}$ *is a continuous K-frame for* $\mathcal{H}$ *with bounds* 1 *and* 2.

*Now, for* $f \in \mathcal{H}$, *we have*

$$\begin{aligned}&\int_{\Omega} \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu = \sum_{i=1}^{\infty} \int_{\Omega_i} \langle f, f_i\rangle \langle g_i, f\rangle d\mu \\ &= \langle f, e_1\rangle \langle e_1, f\rangle + \langle f, e_1\rangle \langle e_1, f\rangle + \langle f, e_2\rangle \langle e_2, f\rangle + \cdots \\ &= \langle f, e_1\rangle \langle e_1, f\rangle + \sum_{i=1}^{\infty} \langle f, e_i\rangle\langle e_i, f\rangle \\ &= \langle f, e_1\rangle \langle e_1, f\rangle + \|f\|^2\end{aligned}$$

$$\leq 2\|f\|^2.$$

*So,*

$$\|K^*f\|^2 \leq \int_\Omega \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu \leq 2\|f\|^2.$$

*Thus,* $(\mathcal{X},\mathcal{Y})$ *is a continuous K-biframe for* $\mathcal{H}$ *with bounds* 1 *and* 2*.*

**Example 3.2.** *Assume that* $\mathcal{S} = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}\right\}$*, with the inner product:*

$$\langle .,. \rangle : \mathcal{S}\times\mathcal{S} \to \mathbb{R}$$

$$(M,N) \longmapsto MN^t.$$

*for all* $M, N \in \mathcal{S}$*. It is then straightforward to verify that* $\langle\cdot,\cdot\rangle$ *forms a real inner product on* $\mathcal{S}$*. Now, let's consider a measure space* $(\Omega = [0,1], \mu)$ *where* $\mu$ *is the Lebesgue measure. Define* $\mathcal{X} : \Omega \to \mathcal{S}$ *by*

$$\mathcal{X}(\omega) = \begin{pmatrix} 2\omega & 0 \\ 0 & 1-\omega \end{pmatrix},\ \omega \in \Omega$$

*and* $\mathcal{Y} : \Omega \to \mathcal{S}$ *by*

$$\mathcal{Y}(\omega) = \begin{pmatrix} 3\omega & 0 \\ 0 & \omega+1 \end{pmatrix},\ \omega \in \Omega$$

*For all* $M \in \mathcal{S}$*, it's straightforward to verify that the functions* $\omega \mapsto \langle M, \mathcal{X}(\omega)\rangle$ *and* $\omega \mapsto \langle M, \mathcal{Y}(\omega)\rangle$ *are measurable on* $\Omega$*. Let's define*

$$K : \mathcal{S} \to \mathcal{S} \quad by \quad KM = \sqrt{2}M \quad for\ all \quad M \in \mathcal{S}.$$

*Then, it's easy to verify that* $\|K^*M\|^2 = 2\|M\|^2$*.*

*For every* $M_f = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathcal{S}$*, we have*

$$\begin{aligned}\int_\Omega \langle M_f, \mathcal{X}(\omega)\rangle\langle\mathcal{Y}(\omega), M_f\rangle d\mu(\omega) &= \int_{[0,1]} \left\langle \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 2\omega & 0 \\ 0 & 1-\omega \end{pmatrix} \right\rangle \\ &\qquad \left\langle \begin{pmatrix} 3\omega & 0 \\ 0 & \omega+1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\rangle d\mu(\omega) \\ &= \int_{[0,1]} \begin{pmatrix} 2\omega a & 0 \\ 0 & (1-\omega)b \end{pmatrix}\begin{pmatrix} 3\omega a & 0 \\ 0 & (\omega+1)b \end{pmatrix} d\mu(\omega) \\ &= \int_{[0,1]} \begin{pmatrix} 6\omega^2 & 0 \\ 0 & 1-\omega^2 \end{pmatrix}\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} d\mu(\omega) \\ &= \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\int_{[0,1]} \begin{pmatrix} 6\omega^2 & 0 \\ 0 & 1-\omega^2 \end{pmatrix} d\mu(\omega) \\ &= \begin{pmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix},\end{aligned}$$

*consequently*

$$\frac{1}{3}\|K^* f\|^2 \leq \int_\Omega \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu \leq 2\|f\|^2.$$

*Therefore,* $(\mathcal{X}, \mathcal{Y})$ *is a continuous K-biframe for* $\mathcal{H}$ *with bound* $\frac{1}{3}$ *and* 2.

Next, we introduce the continuous biframe operator and provide some of its properties.

**Definition 3.2.** *Let* $(\mathcal{X}, \mathcal{Y}) = (\mathcal{X} : \Omega \to \mathcal{H},\ \mathcal{Y} : \Omega \to \mathcal{H})$ *be a continuous biframe for* $\mathcal{H}$ *with respect to* $(\Omega, \mu)$*. Then the continuous biframe operator* $S_{\mathcal{X},\mathcal{Y}} : \mathcal{H} \to \mathcal{H}$ *is defined by*

$$S_{\mathcal{X},\mathcal{Y}} f = \int_\Omega \langle f, \mathcal{X}(\omega)\rangle \mathcal{Y}(\omega) d\mu,$$

*for all* $f \in \mathcal{H}$*.*

For every $f \in \mathcal{H}$, we have

$$\langle S_{\mathcal{X},\mathcal{Y}} f, f\rangle = \int_\Omega \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu. \tag{3.2}$$

This implies that, for each $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \langle S_{\mathcal{X},\mathcal{Y}} f, f\rangle \leq B\|f\|^2.$$

Hence $AI \leq S_{\mathcal{X},\mathcal{Y}} \leq BI$, where $I$ is the identity operator on $\mathcal{H}$. consequently, $S_{\mathcal{X},\mathcal{Y}}$ is positive and invertible.

**Proposition 3.1.** *Let* $S_{\mathcal{X},\mathcal{Y}}$ *and* $S_{\mathcal{Y},\mathcal{X}}$ *be continuous biframe operators such that* $S_{\mathcal{X},\mathcal{Y}} = S_{\mathcal{Y},\mathcal{X}}$*. Then the pair* $(\mathcal{X}, \mathcal{Y})$ *is a continuous K-biframe for* $\mathcal{H}$ *with respect to* $(\Omega, \mu)$ *if and only if* $(\mathcal{Y}, \mathcal{X})$ *is a continuous K-biframe for* $\mathcal{H}$ *with respect to* $(\Omega, \mu)$

*Proof.* Let $(\mathcal{X}, \mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$ with bounds $A$ and $B$. Then for every $f \in \mathcal{H}$, we have

$$A\|K^* f\|^2 \leq \langle S_{\mathcal{X},\mathcal{Y}} f, f\rangle = \int_\Omega \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu \leq B\|f\|^2.$$

Since $S_{\mathcal{X},\mathcal{Y}} = S_{\mathcal{Y},\mathcal{X}}$ we have

$$\langle S_{\mathcal{Y},\mathcal{X}} f, f\rangle = \langle S_{\mathcal{X},\mathcal{Y}} f, f\rangle = \int_\Omega \langle f, \mathcal{Y}(\omega)\rangle\langle \mathcal{X}(\omega), f\rangle d\mu$$

Thus, for each $f \in \mathcal{H}$, we have

$$A\|K^* f\|^2 \leq \int_\Omega \langle f, \mathcal{Y}(\omega)\rangle\langle \mathcal{X}(\omega), f\rangle d\mu \leq B\|f\|^2.$$

Therefore, $(\mathcal{Y}, \mathcal{X})$ is a continuous $K$-biframe for $\mathcal{H}$.

Likewise, we can establish the converse part of this theorem. □

In the following theorem, we establish a characterization of a continuous $K$-biframe by utilizing its continuous biframe operator.

**Theorem 3.1.** *Let $(\mathcal{X}, \mathcal{Y})$ be a continuous biframe for $\mathcal{H}$ with respect to $(\Omega, \mu)$.Then $(\mathcal{X}, \mathcal{Y})$ is a continuous K-biframe with bounds A and B for $\mathcal{H}$ if and only if $S_{\mathcal{X},\mathcal{Y}} \geq AKK^*$, where $S_{\mathcal{X},\mathcal{Y}}$ is the continuous biframe operator for $(\mathcal{X}, \mathcal{Y})$.*

*Proof.* Let $(\mathcal{X}, \mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$ with bounds $A$ and $B$. Then using (3.1) and (3.2), for each $f \in \mathcal{H}$, we get

$$A\|K^* f\|^2 \leq \langle S_{\mathcal{X},\mathcal{Y}} f, f\rangle = \int_{\Omega} \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu \leq B\|f\|^2.$$

Thus,

$$A\langle K^* f, K^* f\rangle \leq \langle S_{\mathcal{X},\mathcal{Y}} f, f\rangle,$$

Hence,

$$S_{\mathcal{X},\mathcal{Y}} \geq AKK^*.$$

Conversely, assume that $S_{\mathcal{X},\mathcal{Y}} \geq AKK^*$. Then, for every $f \in \mathcal{H}$, we have

$$A\|K^* f\|^2 \leq \langle S_{\mathcal{X},\mathcal{Y}} f, f\rangle = \int_{\Omega} \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu.$$

Since $(\mathcal{X}, \mathcal{Y})$ is a continuous biframe for $\mathcal{H}$. Therefore, $(\mathcal{X}, \mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$. □

Furthermore, we provide a characterization of a continuous biframe with the assistance of an invertible operator on $\mathcal{H}$.

**Theorem 3.2.** *Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be invertible on $\mathcal{H}$. Then the following statements are equivalent:*

(1) *$(\mathcal{X}, \mathcal{Y})$ is a continuous K-biframe for $\mathcal{H}$ with respect to $(\Omega, \mu)$*

(2) *$(\mathcal{T}\mathcal{X}, \mathcal{T}\mathcal{Y})$ is a continuous K-biframe for $\mathcal{H}$ with respect to $(\Omega, \mu)$.*

*Proof.* (1)⇒(2) For each $f \in \mathcal{H}$,

$$\omega \mapsto \langle f, \mathcal{T}\mathcal{X}(\omega)\rangle$$

and

$$\omega \mapsto \langle f, \mathcal{T}\mathcal{Y}(\omega)\rangle$$

are measurable functions on $\Omega$. Let $(\mathcal{X}, \mathcal{Y})$ is a continuous biframe for $\mathcal{H}$ with bounds $A$ and $B$ and $\mathcal{T} \in \mathcal{B}(\mathcal{H})$. for $f \in \mathcal{H}$, we have

$$\begin{aligned}\int_{\Omega} \langle f, \mathcal{T}\mathcal{X}(\omega)\rangle\langle \mathcal{T}\mathcal{Y}(\omega), f\rangle d\mu &= \int_{\Omega} \langle \mathcal{T}^* f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), \mathcal{T}^* f\rangle d\mu \\ &\leq B\|\mathcal{T}^* f\|^2 \\ &\leq B\|\mathcal{T}\|^2\|f\|^2.\end{aligned}$$

On the other hand, Since $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ is invertible, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}\|K^* f\|^2 &= \|\left(\mathcal{T}\mathcal{T}^{-1}\right)^* K^* f\|^2 \\ &= \|\left(\mathcal{T}^{-1}\right)^* \mathcal{T}^* K^* f\|^2 \\ &\leq \left\|\left(\mathcal{T}^{-1}\right)^*\right\|^2 \|\mathcal{T}^* K^* f\|^2.\end{aligned}$$

Consequently, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}\int_{\Omega} \langle f, \mathcal{T}\mathcal{X}(\omega)\rangle \langle \mathcal{T}\mathcal{Y}(\omega), f\rangle d\mu &= \int_{\Omega} \langle \mathcal{T}^* f, \mathcal{X}(\omega)\rangle \langle \mathcal{Y}(\omega), \mathcal{T}^* f\rangle d\mu \\ &\geq A\|\mathcal{T}^* K^* f\|^2 \\ &\geq A \left\|\left(\mathcal{T}^{-1}\right)^*\right\|^{-2} \|K^* f\|^2.\end{aligned}$$

Hence, $(\mathcal{T}\mathcal{X}, \mathcal{T}\mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$ with bounds $A \left\|\left(\mathcal{T}^{-1}\right)^*\right\|^{-2}$ and $B\|\mathcal{T}\|^2$.

(2)⇒(1), Assume that $(\mathcal{T}\mathcal{X}, \mathcal{T}\mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$ with bounds $A$ and $B$. Now, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}A\|\mathcal{T}^*\|^{-2}\|K^* f\|^2 &= A\|\mathcal{T}^*\|^{-2}\| \left(\mathcal{T}\mathcal{T}^{-1}\right)^* K^* f\|^2 \\ &\leq A\| \left(\mathcal{T}^{-1}\right)^* K^* f\|^2 \\ &\leq \int_{\Omega} \langle \left(\mathcal{T}^{-1}\right)^* f, \mathcal{T}\mathcal{X}(\omega)\rangle \langle \mathcal{T}\mathcal{Y}(\omega), \left(\mathcal{T}^{-1}\right)^* f\rangle d\mu \\ &= \int_{\Omega} \langle \mathcal{T}^* \left(\mathcal{T}^{-1}\right)^* f, \mathcal{X}(\omega)\rangle \langle \mathcal{Y}(\omega), \mathcal{T}^* \left(\mathcal{T}^{-1}\right)^* f\rangle d\mu \\ &= \int_{\Omega} \langle f, \mathcal{X}(\omega)\rangle \langle \mathcal{Y}(\omega), f\rangle d\mu.\end{aligned}$$

On the other hand, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}&\int_{\Omega} \langle f, \mathcal{X}(\omega)\rangle \langle \mathcal{Y}(\omega), f\rangle d\mu \\ &= \int_{\Omega} \langle \mathcal{T}^* \left(\mathcal{T}^{-1}\right)^* f, \mathcal{X}(\omega)\rangle \langle \mathcal{Y}(\omega), \mathcal{T}^* \left(\mathcal{T}^{-1}\right)^* f\rangle d\mu \\ &= \int_{\Omega} \langle \left(\mathcal{T}^{-1}\right)^* f, \mathcal{T}\mathcal{X}(\omega)\rangle \langle \mathcal{T}\mathcal{Y}(\omega), \left(\mathcal{T}^{-1}\right)^* f\rangle d\mu \\ &\leq B\| \left(\mathcal{T}^{-1}\right)^* f\|^2 \\ &\leq B \left\|\left(\mathcal{T}^{-1}\right)^*\right\|^2 \|f\|^2.\end{aligned}$$

Therefore, $(\mathcal{X}, \mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$ with bounds $A\|\mathcal{T}^*\|^{-2}$ and $B \left\|\left(\mathcal{T}^{-1}\right)^*\right\|^2$. □

In the following proposition we will require a necessary condition for the operator $\mathcal{T}$ for which $(\mathcal{X}, \mathcal{Y})$ will be $\mathcal{T}$-biframe for $\mathcal{H}$.

**Proposition 3.2.** *Let $(\mathcal{X}, \mathcal{Y})$ be a continuous $K$-biframe for $\mathcal{H}$. Assume that $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ with $\mathcal{R}(\mathcal{T}) \subseteq \mathcal{R}(K)$. Then $(\mathcal{X}, \mathcal{Y})$ is a continuous $\mathcal{T}$-biframe for $\mathcal{H}$.*

*Proof.* Suppose that $(\mathcal{X}, \mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$. Then there are positive constants $0 < A \leq B < \infty$ such that

$$A \left\|K^* f\right\|^2 \leq \int_{\Omega} \langle f, \mathcal{X}(\omega)\rangle \langle \mathcal{Y}(\omega), f\rangle d\mu \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}.$$

Since $\mathcal{R}(\mathcal{T}) \subseteq \mathcal{R}(K)$, by Theorem 2.1, there exists $\alpha > 0$ such that $\mathcal{T}\mathcal{T}^* \leq \alpha^2 KK^*$.

Hence,

$$\frac{A}{\alpha^2}\left\|\mathcal{T}^* f\right\|^2 \leq A\left\|K^* f\right\|^2 \leq \int_\Omega \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}.$$

Hence, $(\mathcal{X}, \mathcal{Y})$ is a continuous $\mathcal{T}$-biframe for $\mathcal{H}$. □

**Theorem 3.3.** *Suppose $K \in \mathcal{B}(\mathcal{H})$ has a closed range. The continuous biframe operator of a continuous $K$-biframe is invertible on the subspace $\mathcal{R}(K)$ of $\mathcal{H}$.*

*Proof.* Assume that $(\mathcal{X}, \mathcal{Y})$ be a continuous $K$-biframe for $\mathcal{H}$. Then there are positive constants $0 < A \leq B < \infty$ such that

$$A\left\|K^* f\right\|^2 \leq \int_\Omega \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}.$$

Since $\mathcal{R}(K)$ is closed, then $KK^+ f = f$, for all $f \in \mathcal{R}(K)$. That is,

$$KK^+\big|_{\mathcal{R}(K)} = I_{\mathcal{R}(K)},$$

we have $I^*_{\mathcal{R}(K)} = \left(K^+\big|_{\mathcal{R}(K)}\right)^* K^*$. For any $f \in \mathcal{R}(K)$, we obtain

$$\|f\|^2 = \left\|\left(KK^+\big|_{\mathcal{R}(K)}\right)^* f\right\|^2 = \left\|\left(K^+\big|_{\mathcal{R}(K)}\right)^* K^* f\right\| \leq \left\|K^+\right\|^2 \cdot \left\|K^* f\right\|^2.$$

Thus,

$$\left\|K^* f\right\|^2 \geq \left\|K^+\right\|^{-2}\|f\|^2.$$

So, we have

$$\int_\Omega \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu \geq A\left\|K^* f\right\|^2 \geq A\left\|K^+\right\|^{-2}\|f\|^2, \text{ for all } f \in \mathcal{R}(K).$$

Therefore, based on the definition of a continuous $K$-biframe, we have

$$A\left\|K^+\right\|^{-2}\|f\|^2 \leq \int_\Omega \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu \leq B\|f\|^2, \text{ for all } f \in \mathcal{R}(K).$$

Hence,

$$A\left\|K^+\right\|^{-2}\|f\| \leq \|\, S_{\mathcal{X},\mathcal{Y}}\big|_{\mathcal{R}(K)} f\| \leq B\|f\|, \text{ for all } f \in \mathcal{R}(K).$$

Consequently, $S_{\mathcal{X},\mathcal{Y}}\big|_{\mathcal{R}(K)} : \mathcal{R}(K) \to R(S)$ is a bounded linear operator and invertible on $\mathcal{R}(K)$. □

**Theorem 3.4.** *Let $K \in \mathcal{B}(\mathcal{H})$ be with a dense range. Suppose that $(\mathcal{X}, \mathcal{Y}) = (\mathcal{X} : \Omega \to \mathcal{H},\ \mathcal{Y} : \Omega \to \mathcal{H})$ be a continuous $K$-biframe and $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ have a closed range.*
*If $(\mathcal{T}\mathcal{X}, \mathcal{T}\mathcal{Y}) = (\mathcal{T}\mathcal{X} : \Omega \to \mathcal{H},\ \mathcal{T}\mathcal{Y} : \Omega \to \mathcal{H})$ is a continuous $K$-biframe for $\mathcal{H}$, then $\mathcal{T}$ is surjective.*

*Proof.* Suppose That $(\mathcal{T}\mathcal{X}, \mathcal{T}\mathcal{Y}) = (\mathcal{T}\mathcal{X} : \Omega \to \mathcal{H},\ \mathcal{T}\mathcal{Y} : \Omega \to \mathcal{H})$ is a continuous $K$-biframe for $\mathcal{H}$ with frame bounds $A$ and $B$. Then for each $f \in \mathcal{H}$,

$$A\left\|K^* f\right\|^2 \leq \int_\Omega \langle f, \mathcal{T}\mathcal{X}(\omega)\rangle\langle \mathcal{T}\mathcal{Y}(\omega), f\rangle d\mu \leq B\|f\|^2.$$

Hence,

$$A\left\|K^*f\right\|^2 \leq \int_\Omega \langle \mathcal{T}^*f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), \mathcal{T}^*f\rangle d\mu \leq B\|f\|^2. \tag{3.3}$$

Since $K$ has a dense range, $\mathcal{H} = \overline{\mathcal{R}(K)}$, so $K^*$ is injective. From (3.3), $\mathcal{T}^*$ is injective since $\mathcal{N}(\mathcal{T}^*) \subseteq \mathcal{N}(K^*)$. Moreover,

$$\mathcal{R}(\mathcal{T}) = \mathcal{N}(\mathcal{T}^*)^\perp = \mathcal{H}.$$

Therefore $\mathcal{T}$ is surjective. □

**Theorem 3.5.** *Let $K \in \mathcal{B}(\mathcal{H})$ and let $(\mathcal{X}, \mathcal{Y})$ be a continuous K-biframe for $\mathcal{H}$. If $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ has a closed range with $\mathcal{T}K = K\mathcal{T}$, then $(\mathcal{T}\mathcal{X}, \mathcal{T}\mathcal{Y}) = (\mathcal{T}\mathcal{X} : \Omega \to \mathcal{H},\ \mathcal{T}\mathcal{Y} : \Omega \to \mathcal{H})$ is a continuous K-biframe for $\mathcal{R}(\mathcal{T})$.*

*Proof.* Since $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ has a closed range. Then for each $f \in \mathcal{R}(\mathcal{T})$,

$$K^*f = \left(\mathcal{T}^+\right)^* \mathcal{T}^*K^*f,$$

so we have

$$\left\|K^*f\right\| = \left\|\left(\mathcal{T}^+\right)^* \mathcal{T}^*K^*f\right\| \leq \left\|\left(\mathcal{T}^+\right)^*\right\| \left\|\mathcal{T}^*K^*f\right\|.$$

Hence

$$\left\|\left(\mathcal{T}^+\right)^*\right\|^{-1} \left\|K^*f\right\| \leq \left\|\mathcal{T}^*K^*f\right\|.$$

Since $(\mathcal{X}, \mathcal{Y})$ is a continuous $K$-biframe with frame bounds $A, B$, then for each $f \in \mathcal{R}(\mathcal{T})$, we have

$$\begin{aligned}\int_\Omega \langle f, \mathcal{T}\mathcal{X}(\omega)\rangle\langle \mathcal{T}\mathcal{Y}(\omega), f\rangle d\mu &= \int_\Omega \langle \mathcal{T}^*f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), \mathcal{T}^*f\rangle d\mu \\ &\geq A\left\|K^*\mathcal{T}^*f\right\|^2 \\ &= A\left\|\mathcal{T}^*K^*f\right\|^2 \\ &\geq A\left\|\left(\mathcal{T}^+\right)^*\right\|^{-2}\left\|K^*f\right\|^2.\end{aligned}$$

On the other hand, we have

$$\int_\Omega \langle f, \mathcal{T}\mathcal{X}(\omega)\rangle\langle \mathcal{T}\mathcal{Y}(\omega), f\rangle d\mu = \int_\Omega \langle \mathcal{T}^*f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), \mathcal{T}^*f\rangle d\mu \leq \mu\left\|\mathcal{T}^*f\right\|^2 \leq \mu\|T\|^2\|f\|^2.$$

Hence,

$$A\left\|\left(\mathcal{T}^+\right)^*\right\|^{-2}\left\|K^*f\right\|^2 \leq \int_\Omega \langle f, \mathcal{T}\mathcal{X}(\omega)\rangle\langle \mathcal{T}\mathcal{Y}(\omega), f\rangle d\mu \leq B\|\mathcal{T}\|^2\|f\|^2.$$

Therefore $(\mathcal{T}\mathcal{X}, \mathcal{T}\mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{R}(\mathcal{T})$. □

**Theorem 3.6.** *Let $K \in \mathcal{B}(\mathcal{H})$ be with a dense range. Let $(\mathcal{X}, \mathcal{Y})$ be a continuous K-biframe and suppose $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ have a closed range. If $(\mathcal{T}\mathcal{X}, \mathcal{T}\mathcal{Y})$ and $(\mathcal{T}^*\mathcal{X}, \mathcal{T}^*\mathcal{Y})$ are continuous K-biframes then $\mathcal{T}$ is invertible.*

*Proof.* Assume that $(\mathcal{T}\mathcal{X}, \mathcal{T}\mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$ with frame bounds $A_1$ and $B_1$. Then for every $f \in \mathcal{H}$,

$$A_1 \left\|K^* f\right\|^2 \leq \int_\Omega \langle f, \mathcal{T}\mathcal{X}(\omega)\rangle\langle \mathcal{T}\mathcal{Y}(\omega), f\rangle d\mu \leq B_1\|f\|^2. \tag{3.4}$$

Since $K$ has a dense range, $K^*$ is injective. Then from (3.4), it follows that $\mathcal{T}^*$ is injective as $\mathcal{N}(\mathcal{T}^*) \subseteq \mathcal{N}(K^*)$. Moreover

$$\mathcal{R}(\mathcal{T}) = \mathcal{N}(\mathcal{T}^*)^\perp = \mathcal{H}.$$

Then $\mathcal{T}$ is surjective.

Suppose that $(\mathcal{T}^*\mathcal{X}, \mathcal{T}^*\mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$ with frame bounds $A_2$ and $B_2$. Then for every $f \in \mathcal{H}$,

$$A_2 \left\|K^* f\right\|^2 \leq \int_\Omega \langle f, \mathcal{T}^*\mathcal{X}(\omega)\rangle\langle \mathcal{T}^*\mathcal{Y}(\omega), f\rangle d\mu \leq B_2\|f\|^2. \tag{3.5}$$

Since $K$ has a dense range, then $K^*$ is injective. From (3.5), $\mathcal{T}$ is injective since $\mathcal{N}(\mathcal{T}) \subseteq \mathcal{N}(K^*)$. we can conclude that $\mathcal{T}$ is bijective. Therefore $\mathcal{T}$ is invertible. □

**Theorem 3.7.** *Let $K \in \mathcal{B}(\mathcal{H})$ be with a dense range. Let $(\mathcal{X}, \mathcal{Y})$ be a continuous $K$-biframe for $\mathcal{H}$ and let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be co-isometry with $\mathcal{T}K = K\mathcal{T}$. Then $(\mathcal{T}\mathcal{X}, \mathcal{T}\mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$.*

*Proof.* Assume that $(\mathcal{X}, \mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$. Then there are positive constants $0 < A \leq B < \infty$ such that

$$A \left\|K^* f\right\|^2 \leq \int_\Omega \langle f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), f\rangle d\mu \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}.$$

Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be co-isometry with $\mathcal{T}K = K\mathcal{T}$. Then for each $f \in \mathcal{H}$, we have

$$\begin{aligned}\int_\Omega \langle f, \mathcal{T}\mathcal{X}(\omega)\rangle\langle \mathcal{T}\mathcal{Y}(\omega), f\rangle d\mu &= \int_\Omega \langle \mathcal{T}^* f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), \mathcal{T}^* f\rangle d\mu \\ &\geq A \left\|K^*\mathcal{T}^* f\right\|^2 \\ &= A \left\|\mathcal{T}^* K^* f\right\|^2 \\ &= A \left\|K^* f\right\|^2.\end{aligned}$$

On the other hand, for $f \in \mathcal{H}$ all we have

$$\int_\Omega \langle f, \mathcal{T}\mathcal{X}(\omega)\rangle\langle \mathcal{T}\mathcal{Y}(\omega), f\rangle d\mu = \int_\Omega \langle \mathcal{T}^* f, \mathcal{X}(\omega)\rangle\langle \mathcal{Y}(\omega), \mathcal{T}^* f\rangle d\mu \leq B\|\mathcal{T}^* f\|^2 \leq B\|\mathcal{T}\|^2\|f\|^2.$$

Hence $(\mathcal{T}\mathcal{X}, \mathcal{T}\mathcal{Y})$ is a continuous $K$-biframe for $\mathcal{H}$. □

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## References

[1] I. Daubechies, A. Grossmann, Y. Meyer, Painless Nonorthogonal Expansions, J. Math. Phys. 27 (1986), 1271–1283. https://doi.org/10.1063/1.527388.

[2] R.J. Duffin, A.C. Schaeffer, A Class of Nonharmonic Fourier Series, Trans. Am. Math. Soc. 72 (1952), 341–366. https://doi.org/10.2307/1990760.

[3] R.G. Douglas, On Majorization, Factorization, and Range Inclusion of Operators on Hilbert Space, Proc. Am. Math. Soc. 17 (1966), 413–415. https://doi.org/10.1090/s0002-9939-1966-0203464-1.

[4] R. El Jazzar, R. Mohamed, On Frames in Hilbert Modules Over Locally $C^*$-Algebras, Int. J. Anal. Appl. 21 (2023), 130. https://doi.org/10.28924/2291-8639-21-2023-130.

[5] A. Fereydooni, A. Safapour, Pair Frames, Results Math. 66 (2014), 247–263.

[6] D. Gabor, Theory of Communications, J. Inst. Electr. Eng. 93 (1946) 429–457.

[7] A. Karara, M. Rossafi, M. Klilou, S. Kabbaj, Construction of Continuous K-g-Frames in Hilbert $C^*$-Modules, Palest. J. Math. 13 (2024), 198–209.

[8] A. Karara, M. Rossafi, A. Touri, K-Biframes in Hilbert Spaces, J. Anal. 33 (2024), 235–251. https://doi.org/10.1007/s41478-024-00831-3.

[9] A. Lfounoune, A. Karara, M. Rossafi, Continuous Biframes in Hilbert $C^*$-Modules, Int. J. Anal. Appl. 23 (2025), 104. https://doi.org/10.28924/2291-8639-23-2025-104.

[10] A. Lfounoune, R. El Jazzar, K-Frames in Super Hilbert $C^*-$Modules, Int. J. Anal. Appl., 23 (2025), 19. https://doi.org/10.28924/2291-8639-23-2025-19

[11] B.V. Limaye, Functional Analysis, New Age International Publishers Limited, New Delhi, (1996).

[12] H. Massit, M. Rossafi, C. Park, Some Relations Between Continuous Generalized Frames, Afr. Mat. 35 (2023), 12. https://doi.org/10.1007/s13370-023-01157-2.

[13] H. Massit, R. Eljazzar, M. Rossafi, Continuous Biframes in Hilbert Spaces, Afr. Mat. 37 (2026), 58. https://doi.org/10.1007/s13370-026-01453-7.

[14] F.D. Nhari, R. Echarghaoui, M. Rossafi, K-G-Fusion Frames in Hilbert $C^*$-Modules, Int. J. Anal. Appl. 19 (2021), 836–857. https://doi.org/10.28924/2291-8639-19-2021-836.

[15] F.D. Nhari, K. Mabrouk, M. Rossafi, On Woven K-g-Frame in Hilbert $C^*$-Modules, J. Anal. (2026). https://doi.org/10.1007/s41478-026-01064-2.

[16] E.H. Ouahidi, M. Rossafi, Woven Continuous Generalized Frames in Hilbert $C^*$-Modules, Int. J. Anal. Appl. 23 (2025), 84. https://doi.org/10.28924/2291-8639-23-2025-84.

[17] M.F. Parizi, A. Alijani, M.A. Dehghan, Biframes and Some of Their Properties, J. Inequal. Appl. 2022 (2022), 104. https://doi.org/10.1186/s13660-022-02844-7.

[18] A. Rahimi, A. Najati, Y.N. Dehghan, Continuous Frames in Hilbert Spaces, Methods Funct. Anal. Topol. 12 (2006), 170–182.

[19] M. Rossafi, F.D. Nhari, Controlled K-g-Fusion Frames in Hilbert $C^*$-Modules, Int. J. Anal. Appl. 20 (2022), 1. https://doi.org/10.28924/2291-8639-20-2022-1.

[20] M. Rossafi, F.D. Nhari, K-G-Duals in Hilbert $C^*$-Modules, Int. J. Anal. Appl. 20 (2022), 24. https://doi.org//10.28924/2291-8639-20-2022-24.

[21] M. Rossafi, F.D. Nhari, C. Park, S. Kabbaj, Continuous g-Frames with $C^*$-Valued Bounds and Their Properties, Complex Anal. Oper. Theory 16 (2022), 44. https://doi.org/10.1007/s11785-022-01229-4.

[22] M. Rossafi, S. Kabbaj, Generalized Frames for $B(H,K)$, Iran. J. Math. Sci. Inform. 17 (2022), 1–9. https://doi.org/10.52547/ijmsi.17.1.1.

[23] M. Rossafi, S. Kabbaj, $*$-K-Operator Frame for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, Asian-Eur. J. Math. 13 (2020), 2050060. https://doi.org/10.1142/S1793557120500606.

[24] M. Rossafi, S. Kabbaj, Operator Frame for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, J. Linear Topol. Algebra 8 (2019), 85–95.

[25] M. Rossafi, S. Kabbaj, $*$-K-$g$-Frames in Hilbert $\mathcal{A}$-Modules, J. Linear Topol. Algebra, 7 (2018), 63–71.

[26] M. Rossafi, S. Kabbaj, $*$-$g$-Frames in Tensor Products of Hilbert $C^*$-Modules, Ann. Univ. Paedagog. Crac. Stud. Math. 17 (2018), 17–25.

[27] M. Rossafi, A. Karara, R. El Jazzar, Biframes in Hilbert $C^*$-Modules, Montes Taurus J. Pure Appl. Math. 7 (2025), 69–80.

[28] M. Rossafi, S. Kabbaj, Frames and Operator Frames for $B(H)$, Asia Math. 2 (2018), 19–23.

[29] S. Touaiher, R. El Jazzar, M. Rossafi, Properties and Characterizations of Controlled K-g-Fusion Frames Within Hilbert $C^*$-Modules, Int. J. Anal. Appl. 23 (2025), 111. https://doi.org/10.28924/2291-8639-23-2025-111.

[30] S. Touaiher, M. Rossafi, Construction of New Continuous K-g-Frames Within Hilbert $C^*$-Modules, Int. J. Anal. Appl. 23 (2025), 232. https://doi.org/10.28924/2291-8639-23-2025-232.