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Stability Analysis of Quartic Functional Equation in Intuitionistic Fuzzy and 2-Normed Frameworks

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Abstract. In this research, we study the Hyers-Ulam stability of a finite-dimensional quartic functional equation using two generalized analytical frameworks: intuitionistic fuzzy normed spaces (IFN-spaces) and 2-Banach spaces. The study employs two separate approaches: the direct method and the fixed point methodology, which provide complementary insights on stability behavior. By setting appropriate control conditions defined by sums and products of powers of norms, we demonstrate the existence and uniqueness of a quartic mapping that approximates the given functional equation. The results show that, under the right contractive conditions, approximate solutions converge to an accurate quartic function. Several corollaries are developed to demonstrate how stability bounds are explicitly dependent on the form of the control function. These findings broaden the application of classical Hyers-Ulam stability theory to nonlinear contexts and help to advance stability analysis in fuzzy and multi-normed functional frameworks.

1. Introduction

The stability problem of functional equations, initiated by Ulam [29] in 1940, asks whether an approximate homomorphism between algebraic structures necessarily lies near an exact homomorphism. This seemingly simple question has profoundly influenced modern analysis and has

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led to the systematic development of stability theory in functional equations. The first decisive answer was provided by Hyers [15] in 1941, who proved the stability of additive mappings between Banach spaces. More precisely, if a mapping $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \tag{1.1}$$

for all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \varepsilon \tag{1.2}$$

for all $x \in X$. This result established what is now known as Hyers-Ulam stability and laid the foundation for a vast and evolving theory.

The theory was significantly generalized by Aoki [3], who allowed the perturbation bound to depend on powers of norms. A major breakthrough followed with the work of Rassias in 1978 [24], who introduced the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \left(\|x\|^p + \|y\|^p\right), \quad p \neq 1, \tag{1.3}$$

thus permitting unbounded Cauchy differences controlled by a power-type function. This extension, now referred to as Hyers–Ulam–Rassias stability, considerably broadened the applicability of stability theory. Subsequently, Găvruta [14] unified and extended these results by introducing general admissible control functions.

Over the past decades, stability theory has been developed for numerous functional equations, including quadratic, cubic, quartic, and mixed types (see [9, 16]). Among these, cubic functional equations occupy a central position due to their nonlinear and symmetric structure and their connections to approximation theory, nonlinear analysis, and functional inequalities. A classical cubic functional equation is given by

$$f(x+2y) + f(x-2y) = 2f(x+y) + 2f(x-y) + 12f(y), \tag{1.4}$$

whose solutions in linear spaces are cubic mappings of the form $f(x) = ax^3$. Stability results for cubic equations were systematically investigated by Jun and Kim [17].

A powerful methodological advancement in stability theory emerged through fixed point techniques. The fixed point alternative of Diaz and Margolis [10] in generalized metric spaces provided an abstract yet effective framework for treating stability problems. This approach was systematically developed by Cădariu and Radu [8] and later refined by Radu and Brzdek [23], allowing stability problems to be transformed into contraction mapping problems in suitable function spaces. Fixed point methods have since become one of the most effective tools for establishing existence, uniqueness, and quantitative stability estimates.

Parallel to these analytic developments, fuzzy mathematical structures have offered enriched frameworks for studying stability under uncertainty. Zadeh's introduction of fuzzy sets [33] initiated the development of fuzzy normed spaces by Katsaras and Felbin [13, 19]. Later, Bag and Samanta [4,5] formulated rigorous axioms for fuzzy normed linear spaces and investigated stability problems within this framework. Recent works have extended Hyers–Ulam–Rassias stability of cubic functional equations to fuzzy normed environments using fixed point techniques [20, 31], highlighting the flexibility of fuzzy structures in handling perturbations.

To further refine the modeling of uncertainty, intuitionistic fuzzy normed spaces (IFN-spaces) were introduced by Saadati and Park [25], incorporating both membership and non-membership functions. These studies demonstrate that stability theory adapts naturally to more delicate geometric and probabilistic norm structures. Related investigations in 2-Banach and multi-norm settings have been conducted in [11, 12, 32], further expanding the geometric scope of stability analysis.

Recent developments have also explored non-Archimedean and quasi-Banach frameworks. In this direction, Kalaichelvan and Jayaraman [18] established generalized Hyers–Ulam–Rassias stability results for Euler–Lagrange type cubic functional equations in non-Archimedean quasi-Banach spaces. Moreover, contemporary research continues to refine fixed point approaches for broader classes of perturbations and generalized functional equations [6].

Complementing these theoretical advances, several recent studies have expanded stability theory into generalized and hybrid norm structures. Pinelas, Govindan and Tamilvanan [22] examined stability for non-additive functional equations, demonstrating that approximate nonlinear deviations can still yield stable approximations. Tamilvanan et al. [28] employed both direct and fixed point methods to establish stability of additive functional equations, providing a comparative methodological framework. Further extensions to quasi-β-normed spaces and orthogonal stability were developed in [1], while stability of additive–quadratic equations in IRN-spaces was investigated in [26]. Modular techniques combined with the Fatou property were utilized in [30] to treat quartic functional equations. Additionally, fixed point approaches in non-Archimedean fuzzy φ-2-normed spaces and modular ultrametric spaces were explored in [2, 27], strengthening the interaction between contraction principles and stability theory.

Despite these substantial contributions, the stability of cubic functional equations in frameworks that simultaneously incorporate intuitionistic fuzzy structures and multi-norm geometries such as 2-Banach spaces remains insufficiently explored. Bridging this gap is both mathematically significant and practically relevant, as such hybrid settings provide richer analytical tools for modeling uncertainty and nonlinear phenomena.

1.1. **Motivation and Contributions.** Motivated by the aforementioned developments, this paper investigates the Hyers-Ulam stability of a finite-dimensional quartic functional equation in both intuitionistic fuzzy normed spaces and 2-Banach spaces. Our approach synthesizes direct analytic techniques with fixed point methods to establish existence and uniqueness results for quartic mappings approximating the given functional equation under general control conditions. The obtained results unify and extend several classical and modern stability theorems and provide new insights into stability phenomena within generalized normed structures.

Specifically, we examine the Hyers-Ulam stability of the finite-dimensional quartic functional equation

$$\begin{aligned}\Phi\left(\sum_{i=1}^{s}\imath_i\right) &= \textstyle\sum_{1\le i<j<k<l\le s}\Phi(\imath_i+\imath_j+\imath_k+\imath_l)+(-s+4)\sum_{1\le i<j<k\le s}\Phi(\imath_i+\imath_j+\imath_k)\\ &\quad+\left(\tfrac{s^2-7s+12}{2}\right)\textstyle\sum_{1=i;i\neq j}^{s}\Phi(\imath_i+\imath_j)-\sum_{i=1}^{s}\Phi(2\imath_i)\\ &\quad+\left(\tfrac{-s^3+9s^2-26s+120}{6}\right)\textstyle\sum_{i=1}^{s}\left(\tfrac{\Phi(\imath_i)+\Phi(-\imath_i)}{2}\right)\end{aligned}\tag{1.5}$$

where $s \geq 4$. Stability results are derived in both 2-Banach spaces and IFN-spaces via direct and fixed point approaches. Illustrative examples are provided to demonstrate how the stability behavior can be controlled by sums and products of powers of norms, thereby clarifying the quantitative aspects of the obtained estimates.

Theorem 1.1. *If a mapping $\Phi : M \to L$ fulfills the equation (1.5), then the function $\Phi : M \to L$ is quartic.*

We consider the generalized quartic functional equation (1.5) defined via the following difference operator:

$$\begin{aligned}D\Phi(\imath_1,\ldots,\imath_s) &= -\Phi\left(\sum_{i=1}^{s}\imath_i\right)+\sum_{1\le i<j<k<l\le s}\Phi(\imath_i+\imath_j+\imath_k+\imath_l)\\ &\quad+(-s+4)\sum_{1\le i<j<k\le s}\Phi(\imath_i+\imath_j+\imath_k)\\ &\quad+\left(\frac{s^2-7s+12}{2}\right)\sum_{i=1,\,i\neq j}^{s}\Phi(\imath_i+\imath_j)-\sum_{i=1}^{s}\Phi(2\imath_i)\\ &\quad+\left(\frac{-s^3+9s^2-26s+120}{6}\right)\sum_{i=1}^{s}\left(\frac{\Phi(\imath_i)+\Phi(-\imath_i)}{2}\right)\end{aligned}$$

for any $\imath_1,\ldots,\imath_s \in M$, with $s \geq 4$.

Theorem 1.2. *[23] Let (M,d) be a generalized complete metric space and a strictly contractive function $\Delta : M \to M$ with $L < 1$. Then, for every $v_1 \in M$, either*

$$d\left(\Delta^s v_1, \Delta^{s+1}v_1\right) = \infty,\ \ s \geq s_0;$$

or there is an integer $s_0 > 0$ fulfills

(i) $d\left(\Delta^s v_1, \Delta^{s+1}v_1\right) < \infty,\ \ s \geq s_0$;
(ii) *the sequence $\{\Delta^s v_1\}_{s\in\mathbb{N}}$ converges to a fixed point v_1^* of Δ;*
(iii) *v_1^* is the unique fixed point of Δ in $M^* = \{v_2 \in M | d(\Delta^{s_0}v_1, v_2) < \infty\}$;*
(iv) $d(v_2, v_1^*) \leq \frac{1}{1-L}d(\Delta v_2, v_2)$, *for every $v_2 \in M^*$,*

where L is a Lipschitz constant.

2. Hyers-Ulam Stability Results in IFN-Spaces

Here, we take into consideration that M is a linear space, $(T, T'_{\nu_1,\nu_2}, \Lambda)$ is a IFN-space, and $(B, T_{\nu_1,\nu_2}, \Lambda)$ complete IFN-space.

Definition 2.1. *[7] Let a membership degree ν_1 and non-membership degree ν_2 of an intuitionistic fuzzy set from $W \times (0, +\infty)$ to $[0, 1]$ such that $(\nu_1)_\zeta(t) + (\nu_2)_\zeta(t) \leq 1$ for all $\zeta \in W$ and $t > 0$. The triple $(W, T_{\nu_1,\nu_2}, \Upsilon)$ is called as an Intuitionistic Fuzzy Normed - space (briefly, IFN-space) if a vector space W, a continuous t-representable Υ and $T_{\nu_1,\nu_2} : W \times (0, +\infty) \to L^*$ satisfying as: for all $\zeta_1, \zeta_2 \in W$ and $t, s > 0$,*

(IFN1) $T_{\nu_1,\nu_2}(\zeta_1, 0) = 0_{L^*}$;

(IFN2) $T_{\nu_1,\nu_2}(\zeta_1, t) = 1_{L^*}$ *if and only if* $\zeta_1 = 0$;

(IFN3) $T_{\nu_1,\nu_2}(\nu\zeta_1, t) = T_{\nu_1,\nu_2}\left(\zeta_1, \frac{t}{|\nu|}\right)$, *for all* $\nu \neq 0$;

(IFN4) $T_{\nu_1,\nu_2}(\zeta_1 + \zeta_2, t + s) \geq_{L^*} \Upsilon(T_{\nu_1,\nu_2}(\zeta_1, t), T_{\nu_1,\nu_2}(\zeta_2, s))$.

In this case, T_{ν_1,ν_2} is called an intuitionistic fuzzy norm, where, $T_{\nu_1,\nu_2}(\zeta_1, t) = ((\nu_1)_{\zeta_1}(t), (\nu_2)_{\zeta_1}(t))$.

2.1. **Direct Technique.**

Theorem 2.1. *If a function $\Upsilon : M^s \to T$ with $0 < \left(\frac{\Psi}{2}\right) < 1$,*

$$T'_{\nu_1,\nu_2}\left(\Upsilon(2\zeta, 0, 0, \cdots, 0), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Psi(\zeta, 0, 0, \cdots, 0), \epsilon\right) \tag{2.1}$$

and

$$\lim_{l\to\infty} T'_{\nu_1,\nu_2}\left(\Upsilon(2^l\zeta_1, 2^l\zeta_2, \cdots, 2^l\zeta_s), 2^{4l}\epsilon\right) = 1_{L^*}$$

for every $\zeta, \zeta_1, \zeta_2, \cdots, \zeta_s \in M$ and every $\epsilon > 0$. If a function $\Phi : M \to B$ fulfills

$$T_{\nu_1,\nu_2}\left(D\Phi\left(\zeta_1, \zeta_2, \cdots, \zeta_s\right), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta_1, \zeta_2, \cdots, \zeta_s), \epsilon\right) \tag{2.2}$$

for every $\zeta_1, \zeta_2, \cdots, \zeta_s \in M$ and every $\epsilon > 0$, then the limit

$$T_{\nu_1,\nu_2}\left(Q_4(\zeta) - \frac{\Phi(2^l\zeta)}{2^{4l}}, \epsilon\right) \to 1_{L^*} \text{ as } l \to \infty$$

exists and there is only one quartic solution $Q_4 : M \to B$ fulfilling the equation (1.5) and

$$T_{\nu_1,\nu_2}\left(\Upsilon(\zeta) - Q_4(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta, 0, 0, \cdots, 0), \epsilon(2^4 - \Psi)\right) \tag{2.3}$$

for every $\zeta \in M$ and every $\epsilon > 0$.

Proof. Fix $\zeta \in M$ and every $\epsilon > 0$. Switching $(\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_s)$ by $(\zeta, 0, 0, \cdots, 0)$ in (2.2), we arrive

$$T_{\nu_1,\nu_2}\left(\Phi(4\zeta) - 2^4\Phi(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta, 0, 0, \cdots, 0), \epsilon\right). \tag{2.4}$$

Switching ζ by $2^l\zeta$ in (2.4) with using (IFN3), we obtain

$$T_{\nu_1,\nu_2}\left(\frac{\Phi(2^{l+1}\zeta)}{2^4} - \Phi(2^l\zeta), \left(\frac{\epsilon}{2^4}\right)\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(2^l\zeta, 0, 0, \cdots, 0), \epsilon\right). \tag{2.5}$$

By the inequality (2.1) and (IFN3) in (2.5), we have

$$T_{\nu_1,\nu_2}\left(\frac{\Phi(2^{l+1}\zeta)}{2^4} - \Phi(2^l\zeta), \left(\frac{\epsilon}{2^4}\right)\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta, 0, 0, \cdots, 0), \frac{\epsilon}{\Psi^l}\right). \tag{2.6}$$

Clearly, we can show from the inequality (2.6), that

$$T_{\nu_1,\nu_2}\left(\frac{\Phi(2^{l+1}\zeta)}{2^{4(l+1)}}-\frac{\Phi(2^l\zeta)}{2^{4l}},\left(\frac{\epsilon}{2^{4(l+1)}}\right)\right)\geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta,0,0,\cdots,0),\frac{\epsilon}{\Psi^l}\right) \tag{2.7}$$

Replacing ϵ by $\Psi^l\epsilon$ in (2.7), we get

$$T_{\nu_1,\nu_2}\left(\frac{\Phi(2^{l+1}\zeta)}{2^{4(l+1)}}-\frac{\Phi(2^l\zeta)}{2^{4l}},\left(\frac{\Psi^l\epsilon}{2^{4(l+1)}}\right)\right)\geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta,0,0,\cdots,0),\epsilon\right). \tag{2.8}$$

Clearly,

$$\frac{\Phi(2^l\zeta)}{2^{4l}}-\Phi(\zeta)=\sum_{i=0}^{l-1}\frac{\Phi(2^{i+1}\zeta)}{2^{4(i+1)}}-\frac{\Phi(2^i\zeta)}{2^{4i}}. \tag{2.9}$$

Using (2.8) and (2.9), it is evident that

$$\begin{aligned} T_{\nu_1,\nu_2}\left(\frac{\Phi(2^l\zeta)}{2^{4l}}-\Phi(\zeta),\sum_{i=0}^{l-1}\frac{\Psi^i\epsilon}{2^{4(i+1)}}\right) &\geq_{L^*} \Lambda_{i=0}^{l-1}\left\{T'_{\nu_1,\nu_2}\left(\frac{\Phi(2^{i+1}\zeta)}{2^{4(i+1)}}-\frac{\Phi(2^i\zeta)}{2^{4i}},\frac{\Psi^i\epsilon}{2^{4(i+1)}}\right)\right\} \\ &\geq_{L^*} \Lambda_{i=0}^{l-1}\left\{T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta,0,0,\cdots,0),\epsilon\right)\right\} \\ &\geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta,0,0,\cdots,0),\epsilon\right) \end{aligned} \tag{2.10}$$

for every $\zeta\in M$ and $\epsilon>0$. Switching ζ by $2^j\zeta$ in (2.10) and with the help of (2.1), we arrive

$$T_{\nu_1,\nu_2}\left(\frac{\Phi(2^{l+j}\zeta)}{2^{4(l+j)}}-\frac{\Phi(2^j\zeta)}{2^{4j}},\sum_{i=0}^{l-1}\frac{\Psi^i\epsilon}{2^{4(i+j)}}\right)\geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta,0,0,\cdots,0),\frac{\epsilon}{\Psi^j}\right) \tag{2.11}$$

for every $j,l\geq 0$. Switching ϵ by $\Psi^j\epsilon$ in (2.11), we reach

$$T_{\nu_1,\nu_2}\left(\frac{\Phi(2^{l+j}\zeta)}{2^{4(l+j)}}-\frac{\Phi(2^j\zeta)}{2^{4j}},\sum_{i=j}^{l+j-1}\frac{\Psi^i\epsilon}{2^{4(i+1)}}\right)\geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta,0,0,\cdots,0),\epsilon\right). \tag{2.12}$$

Utilizing (IFN3) in (2.12), we get

$$T_{\nu_1,\nu_2}\left(\frac{\Phi(2^{l+j}\zeta)}{2^{4(l+j)}}-\frac{\Phi(2^j\zeta)}{2^{4j}},\epsilon\right)\geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta,0,0,\cdots,0),\frac{\epsilon}{\sum_{i=j}^{l+j-1}\frac{\Psi^i}{2^{4i}2^4}}\right) \tag{2.13}$$

for every $j,l\geq 0$. As $0<\Psi<2$ and $\sum_{i=0}^{l}\left(\frac{\Psi}{2}\right)^i<\infty$. Thus, the sequence $\left\{\frac{\Phi(2^l\zeta)}{2^{4l}}\right\}$ is Cauchy sequence in $(B,T_{\nu_1,\nu_2},\Lambda)$ is a complete IFN-space, this sequence $\left\{\frac{\Phi(2^l\zeta)}{2^{4l}}\right\}$ converges in $Q_4(\zeta)\in B$. Then, we can define the mapping $Q_4: M\to B$ by

$$T_{\nu_1,\nu_2}\left(Q_4(\zeta)-\frac{\Phi(2^l\zeta)}{2^{4l}}\right)\to 1_{L^*} \text{ as } l\to\infty.$$

Setting $j=0$ in inequality (2.13), we obtain

$$T_{\nu_1,\nu_2}\left(\frac{\Phi(2^l\zeta)}{2^{4l}}-\Phi(\zeta),\epsilon\right)\geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta,0,0,\cdots,0),\frac{\epsilon}{\sum_{i=0}^{l-1}\frac{\Psi^i}{2^{4i}2^4}}\right). \tag{2.14}$$

Applying the limit as $l\to\infty$ in (2.14), we reach

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta)-Q_4(\zeta),\epsilon\right)\geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta,0,0,\cdots,0),\epsilon(2^4-\Psi)\right).$$

After that, we aim to demonstrate that the functional equation (1.5) is satisfied by the function Q_4, replacing $(\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_s)$ by $(2^l\zeta_1, 2^l\zeta_2, 2^l\zeta_3, \cdots, 2^l\zeta_s)$ in (2.2) respectively, we have

$$T_{\nu_1,\nu_2}\left(\frac{1}{2^{4l}}D\Phi\left(2^l\zeta_1, 2^l\zeta_2, 2^l\zeta_3, \cdots, 2^l\zeta_s\right), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon\left(2^l\zeta_1, 2^l\zeta_2, 2^l\zeta_3, \cdots, 2^l\zeta_s\right), 2^{4l}\epsilon\right)$$

for every $\zeta_1, \zeta_2, \cdots, \zeta_s \in M$ and every $\epsilon > 0$. Since

$$\lim_{l\to\infty} T'_{\nu_1,\nu_2}\left(\Upsilon(2^l\zeta_1, 2^l\zeta_2, 2^l\zeta_3, \cdots, 2^l\zeta_s), 2^{4l}\epsilon\right) = 1_{L^*}.$$

Therefore, according to the functional equation (1.5), the function Q_4 is valid. Therefore, Q_4 is a quartic function. Lastly, let's look at another quartic mapping Q'_4 to demonstrate that the function Q_4 is unique. According to the functional equations (1.5) and (2.3), $M \to B$ is satisfied. Hence,

$$\begin{aligned}
T_{\nu_1,\nu_2}\left(Q_4(\zeta) - Q'_4(\zeta), \epsilon\right) &= T_{\nu_1,\nu_2}\left(\frac{Q_4(2^l\zeta)}{2^{4l}} - \frac{Q'_4(2^l\zeta)}{2^{4l}}, \epsilon\right) \\
&\geq_{L^*} \Lambda\left\{T_{\nu_1,\nu_2}\left(\frac{Q_4(2^l\zeta)}{2^{4l}} - \frac{\Phi(2^l\zeta)}{2^{4l}}, \frac{\epsilon}{2}\right), T_{\nu_1,\nu_2}\left(\frac{\Phi(2^l\zeta)}{2^{4l}} - \frac{Q'_4(2^l\zeta)}{2^{4l}}, \frac{\epsilon}{2}\right)\right\} \\
&\geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon\left(2^l\zeta, 0, 0, \cdots, 0\right), \frac{2^{4l}\epsilon(2^4 - \Psi)}{2}\right) \\
&\geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon\left(\zeta, 0, 0, \cdots, 0\right), \frac{2^{4l}\epsilon(2^4 - \Psi)}{2\Psi^l}\right)
\end{aligned}$$

for every $\zeta \in M$ and every $\epsilon > 0$. As

$$\lim_{l\to\infty} \frac{2^{4l}\epsilon(2^4 - \Psi)}{2\Psi^l} = \infty,$$

we obtain

$$\lim_{l\to\infty} T'_{\nu_1,\nu_2}\left(\Upsilon\left(\zeta, 0, 0, \cdots, 0\right), \frac{2^4 2^{4l}\epsilon(2^4 - \Psi)}{2\Psi^l}\right) = 1_{L^*}.$$

Thus,

$$T_{\nu_1,\nu_2}\left(Q_4(\zeta) - Q'_4(\zeta), \epsilon\right) = 1_{L^*}.$$

Therefore, $Q_4(\zeta) = Q'_4(\zeta)$. The uniqueness of the quartic function $Q_4(\zeta)$ is thus proven. □

Theorem 2.2. *If a function* $\Upsilon : M^s \to Z$ *with* $0 < \left(\frac{2}{\Psi}\right) < 1$,

$$T'_{\nu_1,\nu_2}\left(\Upsilon\left(2^{-1}\zeta, 0, 0, \cdots, 0\right), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\frac{1}{\Psi}\Upsilon(\zeta, 0, 0, \cdots, 0), \epsilon\right) \tag{2.15}$$

and

$$\lim_{l\to\infty} T'_{\nu_1,\nu_2}\left(\Upsilon\left(2^{-l}\zeta_1, 2^{-l}\zeta_2, \cdots, 2^{-l}\zeta_s\right), 2^{-4l}\epsilon\right) = 1_{L^*}.$$

for every $\zeta, \zeta_1, \zeta_2, \cdots, \zeta_s \in M$ *and every* $\epsilon > 0$. *If a function* $\Phi : M \to B$ *fulfills* (2.2), *then the limit*

$$T_{\nu_1,\nu_2}\left(Q_4(\zeta) - 2^{4l}\Phi\left(\frac{\zeta}{2^l}\right), \epsilon\right) \to 1_{L^*} \text{ as } l \to \infty$$

exists and there is only one quartic function $Q_4 : M \to B$ *fulfilling the equation (1.5) and*

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Psi(\zeta, 0, 0, \cdots, 0), 2^4\epsilon(\Psi - 2^4)\right),$$

for every $\zeta \in M$ and every $\epsilon > 0$.

Proof. Fix $\zeta \in M$ and all $\epsilon > 0$. Setting $(\zeta_1, \zeta_1, \zeta_3, \cdots, \zeta_s)$ by $(\zeta, 0, 0, \cdots, 0)$ in (2.2), we arrive

$$T_{\nu_1,\nu_2}\left(\Phi(2\zeta) - 2^4\Phi(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta, 0, 0, \cdots, 0), \epsilon\right). \tag{2.16}$$

Switching ζ by $\frac{\zeta}{2}$ in (2.16), we get

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - 2^4\Phi\left(\frac{\zeta}{2}\right), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon\left(\frac{\zeta}{2}, 0, 0, \cdots, 0\right), \epsilon\right). \tag{2.17}$$

Switching ζ by $\frac{\zeta}{2^l}$ in (2.17) and utilizing (IFN3), we have

$$T_{\nu_1,\nu_2}\left(\Phi\left(\frac{\zeta}{2^l}\right) - 2^4\Phi\left(\frac{\zeta}{2^{l+1}}\right), \frac{\epsilon}{-}\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon\left(\frac{\zeta}{2^{l+1}}, 0, 0, \cdots, 0\right)\right). \tag{2.18}$$

By utilizing the inequality (2.15) and condition (IFN3) in (2.18), we arrive

$$T_{\nu_1,\nu_2}\left(\Phi\left(\frac{\zeta}{2^l}\right) - 2^4\Phi\left(\frac{\zeta}{2^{l+1}}\right), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta, 0, 0, \cdots, 0), \epsilon\Psi^{l+1}\right).$$

Utilizing the same methodology as Theorem 2.1, the remaining portion of the proof can be established. □

Corollary 2.1. *Let a be in $\mathbb{R}^+$. If a mapping $\Phi : M \to B$ such that*

$$T_{\nu_1,\nu_2}\left(D\Phi(\zeta_1, \zeta_1, \zeta_3, \cdots, \zeta_s), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(a, \epsilon\right)$$

for every $\zeta_1, \zeta_1, \zeta_3, \cdots, \zeta_s \in M$ and every $\epsilon > 0$, then there is only one quartic solution $Q_4 : M \to B$ satisfies

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(a, |2^4 - 1|\epsilon\right)$$

for every $\zeta \in M$ and every $\epsilon > 0$.

Proof. Assuming $\Psi = 2^0$ and $\Upsilon(\zeta_1, \zeta_1, \zeta_3, \cdots, \zeta_s) = a$, the demonstration follows from Theorem 2.1 and Theorem 2.2. □

Corollary 2.2. *If a function $\Phi : M \to B$ fulfills*

$$T_{\nu_1,\nu_2}\left(D\Phi(\zeta_1, \zeta_1, \zeta_3, \cdots, \zeta_s), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(a\sum_{i=1}^{s} ||\zeta_i||^b, \epsilon\right)$$

for every $\zeta_1, \zeta_1, \zeta_3, \cdots, \zeta_s \in M$ and every $\epsilon > 0$, then there is only one quartic solution $Q_4 : M \to B$ satisfies

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(a||\zeta||^b, |2^4 - 2^b|\epsilon\right)$$

for every $\zeta \in M$ and every $\epsilon > 0$, where a and b are in $\mathbb{R}^+$ with $b \in (0, 4) \cup (4, +\infty)$.

Proof. Assuming $\Upsilon = 2^b$ and $\Upsilon = a\sum_{i=1}^{s} ||\zeta_i||^b$, the proof follows from Theorem 2.1 and Theorem 2.2. □

Corollary 2.3. *Let $a, b, c, d \in \mathbb{R}^+$ with $sb, sd \in (0,4) \cup (4,+\infty)$. If a function $\Phi : M \to B$ fulfills*

$$T_{\nu_1,\nu_2}\left(D\Phi(\zeta_1,\zeta_1,\zeta_3,\cdots,\zeta_s),\epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(a\sum_{i=1}^{s}\|\zeta_i\|^{sb} + c\prod_{i=1}^{s}\|\zeta_i\|^{d},\epsilon\right)$$

for every $\zeta_1,\zeta_1,\zeta_3,\cdots,\zeta_s \in M$ and every $\epsilon > 0$, then there is only one quartic solution $Q_4 : M \to B$ satisfies

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta),\epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(a\|\zeta\|^{sb}, |2^4 - 2^{sb}|\epsilon\right)$$

for every $\zeta \in M$ and every $\epsilon > 0$.

Proof. Assuming $\Psi = 2^{sb}$ and $a\sum_{i=1}^{s}\|\zeta_i\|^{sb} + c\prod_{i=1}^{s}\|\zeta_i\|^{d}$, the proof follows from Theorem 2.1 and Theorem 2.2. □

Corollary 2.4. *Let $c, d \in \mathbb{R}^+$ with $0 < sd \neq 4$. If a function $\Phi : M \to B$ fulfills*

$$T_{\nu_1,\nu_2}\left(D\Phi(\zeta_1,\zeta_1,\zeta_3,\cdots,\zeta_s),\epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(c\prod_{i=1}^{s}\|\zeta_i\|^{d},\epsilon\right)$$

for every $\zeta_1,\zeta_1,\zeta_3,\cdots,\zeta_s \in M$ and every $\epsilon > 0$, then the solution Φ is quartic.

Proof. The proof is valid for Theorem 2.1 and Theorem 2.2 through the setting of $\Upsilon(\zeta_1,\zeta_1,\zeta_3,\cdots,\zeta_s) = c\prod_{i=1}^{s}\|\zeta_i\|^{d}$. □

2.2. **Fixed Point Technique.** Before we start, let us look at a constant β_a that

$$\rho_i = \begin{cases} 2, & \text{if } i = 0, \\ \frac{1}{2}, & \text{if } i = 1 \end{cases}$$

and Φ is the set such that $\Phi = \{t_1 \mid t_1 : M \to B, t_1(0) = 0\}$.

Theorem 2.3. *Let a mapping $\Phi : M \to B$ for which there is a mapping $\Upsilon : M^s \to Z$ with*

$$\lim_{l\to\infty} T'_{\nu_1,\nu_2}\left(\Upsilon(2^l\zeta_1, 2^l\zeta_1, \cdots, 2^l\zeta_s), 2^{4l}\epsilon\right) = 1_{L^*} \tag{2.19}$$

and fulfilling the inequality (2.2). *If there is $L = L(i)$ such that $\zeta \to \mu(\zeta) = \frac{1}{2^3}\Upsilon\left(\frac{\zeta}{2}, 0, 0, \cdots, 0\right)$ has the property*

$$T'_{\nu_1,\nu_2}\left(L\frac{1}{\rho_i^4}\mu(\rho_i\zeta),\epsilon\right) = T'_{\nu_1,\nu_2}\left(\mu(\zeta),\epsilon\right), \tag{2.20}$$

then there is only one quartic function $Q_4 : M \to B$ fulfills the equation (1.5) and

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta),\epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\frac{L^{1-i}}{1-L}\mu(\zeta),\epsilon\right)$$

for every $\zeta \in M$ and every $\epsilon > 0$.

Proof. Consider Ψ be a general metric on Φ:

$$\Psi(t_1,t_2) = \inf\left\{j \in (0,\infty) \mid T_{\nu_1,\nu_2}\left(t_1(\zeta) - t_2(\zeta),\epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(j\mu(\zeta),\epsilon\right), \zeta \in M, \epsilon > 0\right\}.$$

Clearly, (Φ, Ψ) is complete. Define a mapping $\Upsilon : \Phi \to \Phi$ by $\Upsilon t_1(\zeta) = \frac{1}{\rho_i^4} t_1(\rho_i \zeta)$, for every $\zeta \in M$. For $t_1, t_2 \in \Phi$, we have

$$\begin{aligned} \Psi(t_1, t_2) &\leq j \\ \Rightarrow T_{\nu_1,\nu_2}(t_1(\zeta) - t_2(\zeta), \epsilon) &\geq_{L^*} T'_{\nu_1,\nu_2}(j\mu(\zeta), \epsilon) \\ \Rightarrow T_{\nu_1,\nu_2}\left(\frac{t_1(\rho_i\zeta)}{\rho_i^4} - \frac{t_2(\rho_i\zeta)}{\rho_i^4}, \epsilon\right) &\geq_{L^*} T'_{\nu_1,\nu_2}\left(\frac{j\mu(\rho_i\zeta)}{\rho_i^4}, \epsilon\right) \\ \Rightarrow T_{\nu_1,\nu_2}\left(\Upsilon t_1(\zeta) - \Upsilon t_2(\zeta), \epsilon\right) &\geq_{L^*} T_{\nu_1,\nu_2}\left(j\, L\mu(\zeta), \epsilon\right) \\ \Rightarrow \Psi\left(\Upsilon t_1(\zeta), \Upsilon t_2(\zeta)\right) &\leq j\, L \\ \Rightarrow \Psi\left(\Upsilon t_1, \Upsilon t_2\right) &\leq L\Psi\left(t_1, t_2\right). \end{aligned}$$

Thus, the function Υ is strictly contractive on Φ with L (Lipschitz constant). Replacing $(\zeta_1, \zeta_1, \zeta_3, \cdots, \zeta_s)$ by $(\zeta, 0, 0, \cdots, 0)$ in (2.2), we have

$$T_{\nu_1,\nu_2}\left(\Phi(2\zeta) - 2^4\Phi(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon(\zeta, 0, 0, \cdots, 0), \epsilon\right). \tag{2.21}$$

Using (IFN3) in (2.21), we have

$$T_{\nu_1,\nu_2}\left(\frac{\Phi(2\zeta)}{2^4} - \Phi(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\left(\frac{1}{2^4}\right)\Upsilon(\zeta, 0, 0, \cdots, 0), \epsilon\right).$$

Utilizing (2.20) for $i = 0$, that

$$\begin{aligned} T_{\nu_1,\nu_2}\left(\frac{\Phi(2\zeta)}{2^4} - \Phi(\zeta), \epsilon\right) &\geq_{L^*} T'_{\nu_1,\nu_2}\left(L\mu(\zeta), \epsilon\right) \\ \Rightarrow \Psi\left(\Upsilon\Phi, \Phi\right) &\leq L = L^1 = L^{1-i}. \end{aligned} \tag{2.22}$$

Setting ζ by $\frac{\zeta}{2}$ in (2.21), we have

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - 2^4\Phi\left(\frac{\zeta}{2}\right), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Upsilon\left(\frac{\zeta}{2}, 0, 0, \cdots, 0\right), \epsilon\right) \tag{2.23}$$

for every $\zeta \in M$ and every $\epsilon > 0$, using (2.20) for $i = 1$, that

$$\begin{aligned} T_{\nu_1,\nu_2}\left(\Phi(\zeta) - 2^4\Phi\left(\frac{\zeta}{2}\right), \epsilon\right) &\geq_{L^*} T'_{\nu_1,\nu_2}\left(\mu(\zeta), \epsilon\right) \\ \Rightarrow \Psi\left(\Phi, \Upsilon\, \Phi\right) &\leq 1 = L^0 = L^{1-i}. \end{aligned} \tag{2.24}$$

We can conclude from (2.22) and (2.24), that

$$\Psi\left(\Phi, \Upsilon\, \Phi\right) \leq L^{1-i} < \infty.$$

From the fixed point view, it has a fixed point Q_4 of Υ in Φ that allows it possible for

$$\lim_{l\to\infty} T_{\nu_1,\nu_2}\left(\frac{\Phi(\rho_i^l\zeta)}{\rho_i^{4l}} - Q_4(\zeta), \epsilon\right) \to 1_{L^*},\ \zeta \in M, \epsilon > 0.$$

Replacing $(\zeta_1, \zeta_1, \cdots, \zeta_s)$ by $(\rho_i\zeta_1, \rho_i\zeta_1, \cdots, \rho_i\zeta_s)$ in (2.2), we obtain

$$T_{\nu_1,\nu_2}\left(\frac{1}{\rho_i^4}D\Phi\left(\rho_i\zeta_1, \rho_i\zeta_1, \cdots, \rho_i\zeta_s\right), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\Psi(\rho_i\zeta_1, \rho_i\zeta_1, \cdots, \rho_i\zeta_s), \rho_i^4\epsilon\right)$$

for every $\zeta_1, \zeta_1, \cdots, \zeta_s \in M$ and every $\epsilon > 0$. By applying the same method as in Theorem 2.1, we are able to demonstrate that the function Q_4 is in accordance with the functional equation (1.5). Considering that Q_4 is a singular fixed point of Υ in the context of Theorem 1.2, $\Delta = \{\Phi \in \Phi \mid \Psi(\Phi, Q_4) < \infty\}$. Therefore, the function Q_4 is unique such that

$$T_{\nu_1,\nu_2}\left(Q_4(\zeta) - \Phi(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(j\,\mu(\zeta), \epsilon\right), \ \epsilon > 0.$$

If we choose the fixed point alternative, we will get at

$$\begin{aligned} \Psi(\Phi, Q_4) &\leq \frac{1}{1-L}\Psi(\Phi, \Upsilon\,\Phi) \\ \Rightarrow \Psi(\Phi, Q_4) &\leq \frac{L^{1-i}}{1-L} \\ \Rightarrow T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta), \epsilon\right) &\geq_{L^*} T'_{\nu_1,\nu_2}\left(\frac{L^{1-i}}{1-L}\mu(\zeta), \epsilon\right) \end{aligned}$$

for every $\zeta \in M$ and every $\epsilon > 0$. Hence the proof. □

Corollary 2.5. *Let a and b are in $\mathbb{R}^+$ with $a > 0$. If a function $\Phi : M \to B$ fulfills*

$$T_{\nu_1,\nu_2}\left(D\Phi\left(\zeta_1, \zeta_1, \zeta_3, \cdots, \zeta_s\right), \epsilon\right) \geq_{L^*} \begin{cases} T'_{\nu_1,\nu_2}\left(a, \epsilon\right), \\ T'_{\nu_1,\nu_2}\left(a\sum_{j=1}^{s}||\zeta_j||^b, \epsilon\right), \\ T'_{\nu_1,\nu_2}\left(a\left(\prod_{j=1}^{s}||\zeta_j||^b + \sum_{j=1}^{s}||\zeta_j||^{sb}\right), \epsilon\right), \end{cases}$$

for all $\zeta_1, \zeta_1, \zeta_3, \cdots, \zeta_s \in M$ and $\epsilon > 0$, then there is only one quartic function $Q_4 : M \to B$ satisfies

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta), \epsilon\right) \geq_{L^*} \begin{cases} T'_{\nu_1,\nu_2}\left(a, |2^4 - 1|\,\epsilon\right) \\ T'_{\nu_1,\nu_2}\left(a||\zeta||^s, |2^4 - 2^b|\epsilon\right), & b < 4 \text{ or } b > 4; \\ T'_{\nu_1,\nu_2}\left(a||\zeta||^s, |2^4 - 2^{sb}|\epsilon\right), & b < \frac{4}{s} \text{ or } b > \frac{4}{s}; \end{cases}$$

for every $\zeta \in M$ and every $\epsilon > 0$.

Proof. Setting

$$h(\zeta_1, \zeta_2, \cdots, \zeta_s) = \begin{cases} a, \\ a\sum_{j=1}^{s}||\zeta_j||^b, \\ a\left(\prod_{j=1}^{s}||\zeta_j||^b + \sum_{j=1}^{s}||\zeta_j||^{sb}\right). \end{cases}$$

Then,

$$T'_{\nu_1,\nu_2}\left(h\left(\varrho_i^l\zeta_1, \varrho_i^l\zeta_2, \cdots, \varrho_i^l\zeta_s\right), \varrho_i^{4l}\epsilon\right) = \begin{cases} T'_{\nu_1,\nu_2}\left(a, (\varrho_i)^{4l}\,\epsilon\right), \\ T'_{\nu_1,\nu_2}\left(a\sum_{j=1}^{s}||\zeta_j||^b, \left(\varrho_i^{1-b}\right)^{4l}\epsilon\right), \\ T'_{\nu_1,\nu_2}\left(a\left(\prod_{j=1}^{s}||\zeta_j||^b + \sum_{j=1}^{s}||\zeta_j||^{sb}\right), \left(\varrho_i^{1-sb}\right)^{4l}\epsilon\right), \end{cases}$$

$$= \begin{cases} \to 1_{L^*} \text{ as } l \to \infty, \\ \to 1_{L^*} \text{ as } l \to \infty, \\ \to 1_{L^*} \text{ as } l \to \infty. \end{cases}$$

Thus, (2.19) holds. But we have $\eta(\zeta) = h\left(\frac{\zeta}{2}, 0, 0, \cdots, 0\right)$ has the property

$$T'_{\nu_1,\nu_2}\left(L\frac{1}{\varrho_i^4}\mu(\varrho_i\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\mu(\zeta), \epsilon\right), \ \zeta \in M, \epsilon > 0.$$

Hence,

$$\begin{aligned} T'_{\nu_1,\nu_2}\left(\mu(\zeta), \epsilon\right) &= T'_{\nu_1,\nu_2}\left(h\left(\frac{\zeta}{2}, 0, 0, \cdots, 0\right), \epsilon\right) \\ &= \begin{cases} T'_{\nu_1,\nu_2}\left(a, \epsilon\right), \\ T'_{\nu_1,\nu_2}\left(\frac{a}{2^b}||\zeta||^b, \epsilon\right), \\ T'_{\nu_1,\nu_2}\left(\frac{a}{2^{sb}}||\zeta||^{sb}, \epsilon\right). \end{cases} \end{aligned}$$

Now,

$$\begin{aligned} T'_{\nu_1,\nu_2}\left(\frac{1}{\varrho_i^4}\mu(\varrho_i\zeta), \epsilon\right) &= \begin{cases} T'_{\nu_1,\nu_2}\left(\frac{a}{\varrho_i^4}, \epsilon\right), \\ T'_{\nu_1,\nu_2}\left(\frac{a}{2^b\varrho_i^4}||\varrho_i\zeta||^b, \epsilon\right), \\ T'_{\nu_1,\nu_2}\left(\frac{a}{2^{sb}\varrho_i^4}||\varrho_i\zeta||^{sb}, \epsilon\right), \end{cases} \\ &= \begin{cases} T'_{\nu_1,\nu_2}\left(\varrho_i^{-4}\mu(\zeta), \epsilon\right), \\ T'_{\nu_1,\nu_2}\left(\varrho_i^{b-4}\mu(\zeta), \epsilon\right), \\ T'_{\nu_1,\nu_2}\left(\varrho_i^{sb-4}\mu(\zeta), \epsilon\right). \end{cases} \end{aligned}$$

We are able to verify the following situations for conditions of ϱ_i by basing our verification on inequality (2.20). **Case:1** $L = 2^{-4}$ if $i = 0$.

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\frac{2^{-4}}{1 - 2^{-4}}\mu(\zeta), \epsilon\right) = T'_{\nu_1,\nu_2}\left(a, 15\,\epsilon\right).$$

Case:2 $L = 2^4$ if $i = 1$.

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\frac{1}{1 - 2^4}\mu(\zeta), \epsilon\right) = T'_{\nu_1,\nu_2}\left(a, -15\,\epsilon\right).$$

Case:3 $L = 2^{b-4}$ for $b < 4$ if $i = 0$.

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\frac{2^{b-4}}{1 - 2^{b-4}}\mu(\zeta), \epsilon\right) = T'_{\nu_1,\nu_2}\left(a||\zeta||^b, (2^4 - 2^b)\epsilon\right).$$

Case:4 $L = 2^{4-b}$ for $b > 4$ if $i = 1$.

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\frac{1}{1 - 2^{4-b}}\mu(\zeta), \epsilon\right) = T'_{\nu_1,\nu_2}\left(2a||\zeta||^b, (2^b - 2^4)\epsilon\right).$$

Case:5 $L = 2^{sb-4}$ for $b < \frac{4}{s}$ if $i = 0$.

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\frac{2^{sb-4}}{1-2^{sb-4}}\mu(\zeta), \epsilon\right) = T'_{\nu_1,\nu_2}\left(a\|\zeta\|^{sb}, (2^4 - 2^{sb})\epsilon\right).$$

Case:6 $L = 2^{4-sb}$ for $b > \frac{4}{s}$ if $i = 1$.

$$T_{\nu_1,\nu_2}\left(\Phi(\zeta) - Q_4(\zeta), \epsilon\right) \geq_{L^*} T'_{\nu_1,\nu_2}\left(\frac{1}{1-2^{4-sb}}\mu(\zeta), \epsilon\right) = T'_{\nu_1,\nu_2}\left(a\|\zeta\|^{sb}, (2^{sb} - 2^4)\epsilon\right).$$

□

3. Hyers-Ulam Stability Results in 2-Banach Spaces

Definition 3.1. *[21] Let M be a linear space over $\mathbb{R}$ with a dimension greater than 1, and consider a mapping $\|\cdot,\cdot\| : M^2 \to \mathbb{R}$ that fulfills the subsequent conditions:*

(a) $\|p_1, p_2\| = 0$ *iff p_1 and p_2 are linearly dependent.*
(b) $\|p_1, p_2\| = \|p_2, p_1\|$,
(c) $\|\omega p_1, p_2\| = |\omega|\,\|p_1, p_2\|$,
(d) $\|p_1, p_2 + p_3\| \leq \|p_1, p_2\| + \|p_1, p_3\|$

for every $p_1, p_2, p_3 \in M$ and $\omega \in \mathbb{R}$.

Therefore, $\|\cdot,\cdot\|$ is referred to as a 2-norm on M, and $(M, \|\cdot,\cdot\|)$ is termed a linear 2-normed space. The space $\mathbb{R}^2$, endowed with the 2-norm defined as $|p_1, p_2|$ = the area of the triangle formed by the vertices 0, p_1, and p_2, exemplifies a 2-normed space.

Lemma 3.1. *[21] Let $(M, \|\cdot,\cdot\|)$ be a linear 2-normed space. If $p_1 \in M$ and $\|p_1, p_2\| = 0$ for every $p_2 \in M$, then $p_1 = 0$.*

Lemma 3.2. *[21] For a convergent sequence $\{(p_1)_j\}$ in a linear 2-normed space M,*

$$\lim_{j\to\infty} \|(p_1)_j, p_2\| = \|\lim_{j\to\infty}(p_1)_j, p_2\|$$

for every $p_2 \in M$.

In this part, we treat M and B as the normed linear space and the 2-Banach space, respectively.

Theorem 3.1. *Let a mapping $h : M \to [0, +\infty)$ fulfills*

$$\lim_{i\to\infty} \frac{1}{2^{4i}} h\left(2^i\zeta_1, 2^i\zeta_2, \cdots, 2^i\zeta_s, r\right) = 0 \tag{3.1}$$

for every $\zeta_1, \zeta_2, \cdots, \zeta_s, r \in M$. Suppose that $\Phi : M \to B$ is a mapping with $\Phi(0) = 0$ and satisfies

$$\left\| D\Phi(\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_s), r \right\| \leq h(\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_s, r) \tag{3.2}$$

and

$$\hat{h}(\zeta, r) =: \sum_{j=0}^{\infty} \frac{1}{2^{4j}} h(2^j\zeta, 0, 0, \cdots, 0, r) < \infty$$

exists for every $\zeta_1, \zeta_2, \cdots, \zeta_s, r \in M$. Then there is only one quartic function $Q_4 : M \to B$ satisfies

$$\left\|\Phi(\zeta) - Q_4(\zeta), r\right\| \leq \frac{1}{2^4}\hat{h}(\zeta, r) \tag{3.3}$$

for every $\zeta, r \in M$.

Proof. Setting $(\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_s)$ by $(\zeta, 0, 0, \cdots, 0)$ in (3.2), we arrive

$$\left\|\Phi(2\zeta) - 2^4\Phi(\zeta), r\right\| \leq h(\zeta, 0, 0, \cdots, 0, r) \tag{3.4}$$

for every $\zeta, r \in M$. Switching ζ by $2^n\zeta$ in (3.4) and dividing both sides by 2^{4n}, we attain

$$\left\|\frac{1}{2^{4(n+1)}}\Phi(2^{n+1}\zeta) - \frac{1}{2^{4n}}\Phi(2^n\zeta), r\right\| \leq \frac{1}{2^{4n}}h(2^n\zeta, 0, 0, \cdots, 0, r)$$

for every $\zeta, r \in M$ and every positive integer n. Hence,

$$\begin{aligned}\left\|\frac{1}{2^{4(i+1)}}\Phi(2^{i+1}\zeta) - \frac{1}{2^{4m}}\Phi(2^m\zeta), r\right\| &\leq \sum_{j=m}^{i}\left\|\frac{1}{2^{4(j+1)}}\Phi(2^{j+1}\zeta) - \frac{1}{2^{4j}}\Phi(2^j\zeta), r\right\| \\ &\leq \sum_{j=m}^{i}\frac{1}{2^{4j}}h(2^j\zeta, 0, 0, \cdots, 0, r)\end{aligned}$$

for every $\zeta, r \in M$ and every positive integers m and i with $i \geq m$. Consequently, given (2.2) and (2.5), the sequence $\{\frac{1}{2^{4i}}\Phi(2^i\zeta)\}$ is Cauchy in B for every $\zeta \in M$. Given that B is complete, the sequence $\{\frac{1}{2^{4i}}\Phi(2^i\zeta)\}$ converges in B for every $\zeta \in M$. Consequently, we may define a mapping $Q_4 : M \to B$ by

$$Q_4(\zeta) := \lim_{i\to\infty}\frac{1}{2^{4i}}\Phi(2^i\zeta) \tag{3.5}$$

for every $\zeta \in M$. Then,

$$\lim_{i\to\infty}\left\|\frac{1}{2^{4i}}\Phi(2^i\zeta) - Q_4(\zeta), r\right\| = 0$$

for every $\zeta, r \in M$. By setting $m = 0$ and evaluating a limit as $i \to \infty$ in equation (3.5), we obtain equation (3.3). Subsequently, we aim to demonstrate that the function Q_4 is quartic. From inequalities (3.1), (3.2), (3.5), and Lemma 3.2, we have

$$\begin{aligned}\left\|DQ_4(\zeta_1, \zeta_2, \cdots, \zeta_s), r\right\| &= \lim_{i\to\infty}\frac{1}{2^{4i}}\left\|D\Phi(2^i\zeta_1, 2^i\zeta_2, \cdots, 2^i\zeta_s), r\right\| \\ &\leq \lim_{i\to\infty}\frac{1}{2^{4i}}h(2^i\zeta_1, 2^i\zeta_2, \cdots, 2^i\zeta_s, r) = 0\end{aligned}$$

for every $\zeta_1, \zeta_2, \cdots, \zeta_s, r \in M$. By Lemma 3.1,

$$DQ_4(\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_s) = 0$$

for every $\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_s \in M$. Therefore, based on Theorem 1.1, the mapping $Q_4 : M \to B$ is quartic.

To demonstrate the uniqueness of the function Q_4, we assume the existence of another quartic solution $Q_4^{'} : M \to B$ that satisfies (3.3). Subsequently

$$\left\|Q_4(\zeta) - Q_4^{'}(\zeta), r\right\| = \lim_{i\to\infty}\frac{1}{2^{4i}}\left\|Q_4(2^i\zeta) - \Phi(2^i\zeta) + \Phi(2^i\zeta) - Q_4^{'}(2^i\zeta), r\right\|$$

$$\leq \lim_{i\to\infty} \frac{1}{2^{4i}} \hat{h}(2^i\zeta, r) = 0$$

for all $\zeta, r \in M$. By Lemma 3.1, $Q_4(\zeta) - Q_4'(\zeta) = 0$ for every $\zeta \in M$. Hence, $Q_4 = Q_4'$. □

Remark 3.1. *A theorem similar to 3.1 can be established, wherein the sequence is defined as follows:*

$$Q_4(\zeta) := \lim_{i\to\infty} 2^{4i} \Phi\left(\frac{\zeta}{2^i}\right)$$

is defined using suitable assumptions for h.

Corollary 3.1. *Let $\omega : [0,\infty) \to [0,\infty)$ be a function fulfilling $\omega(0) = 0$ and*

(i) $\omega(pq) \leq \omega(p)\omega(q)$.

(ii) $\omega(p) < p$ *for every* $p > 3$.

If a function $\Phi : M \to B$ fulfills

$$\left\| D\Phi(\zeta_1, \zeta_2, \cdots, \zeta_s), r \right\| \leq \sum_{i=1}^{s} \omega(\|\zeta_i\|) + \omega(\|r\|)$$

for every $\zeta_1, \zeta_2, \cdots, \zeta_s, r \in M$, then there is only one quartic solution $Q_4 : M \to B$ satisfies

$$\left\| \Phi(\zeta) - Q_4(\zeta), r \right\| \leq \left[\frac{\omega(\|\zeta\|)}{(2^4 - \omega(2))} + \omega(\|r\|) \right] \tag{3.6}$$

for every $\zeta, r \in M$.

Proof. Setting

$$h(\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_s, r) = \sum_{1\leq i\leq s} \omega(\|\zeta_i\|) + \omega(\|r\|)$$

for every $\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_s, r \in M$. We can say from (i) that

$$\omega(2^{4i}) \leq (\omega(2))^i$$

and

$$h(2^i\zeta_1, 2^i\zeta_2, \cdots, 2^i\zeta_s, r) \leq (\omega(2))^{4i} \left(\sum_{1\leq i\leq s} \omega(\|\zeta_i\|) \right) + \omega(\|r\|).$$

We get (3.6) by using Theorem 3.1. □

Corollary 3.2. *Let $q < 4$ and a homogeneous function $H : [0,\infty) \times [0,\infty) \to [0,\infty)$. If a function $\Phi : M \to B$ fulfills*

$$\left\| D\Phi(\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_s), r \right\| \leq H\left(\|\zeta_1\|, \|\zeta_2\|, \|\zeta_3\|, \cdots, \|\zeta_s\|\right) + \|r\|$$

for every $\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_s, r \in M$, then there is only one quartic function $Q_4 : M \to B$ fulfills

$$\left\| \Phi(\zeta) - Q_4(\zeta), r \right\| \leq \frac{H(\|\zeta\|, 0, 0, \cdots, 0) + \|r\|}{2 - q} \tag{3.7}$$

for every $\zeta, r \in M$ and every $q \in \mathbb{R}^+$.

Proof. Setting

$$h(\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_s,r) = H\left(\|\zeta_1\|,\|\zeta_2\|,\|\zeta_3\|,\cdots,\|\zeta_s\|\right) + \|r\|$$

for every $\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_s,r \in M$. By utilizing Theorem 3.1, we reach (3.7). □

Corollary 3.3. *Let $q < 4$ and a homogeneous function $H : [0,\infty)\times[0,\infty) \to [0,\infty)$ with degree q. If a function $\Phi : M \to B$ fulfills*

$$\left\|D\Phi(\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_s),r\right\| \le H\left(\|\zeta_1\|,\|\zeta_2\|,\|\zeta_3\|,\cdots,\|\zeta_s\|\right)\|r\|$$

for every $\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_s,r \in M$, then there is only one quartic solution $Q_4 : M \to B$ satisfies

$$\left\|\Phi(\zeta) - Q_4(\zeta),r\right\| \le \frac{H(\|\zeta\|,0,0,\cdots,0)\|r\|}{2-2^q} \tag{3.8}$$

for every $\zeta, r \in M$ and $q \in \mathbb{R}^+$.

Proof. Setting

$$h(\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_s,r) = H\left(\|\zeta_1\|,\|\zeta_2\|,\|\zeta_3\|,\cdots,\|\zeta_s\|\right)\|r\|$$

for every $\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_s,r \in M$. By utilizing Theorem 3.1, we arrive (3.8). □

Proof. Define

$$H\left(\|v_1\|,\|v_2\|,\cdots,\|v_m\|\right) = \sum_{i=1}^{m}\|v_i\|^p$$

for all $v_1,v_2,\cdots,v_m \in G$. Then H is a homogeneous function of degree p, that is,

$$H\left(t\|v_1\|,t\|v_2\|,\cdots,t\|v_m\|\right) = t^pH\left(\|v_1\|,\|v_2\|,\cdots,\|v_m\|\right)$$

for every $t \ge 0$.

By the given assumption,

$$\left\|Df(v_1,v_2,\cdots,v_m),s\right\| \le H\left(\|v_1\|,\|v_2\|,\cdots,\|v_m\|\right) + \|s\|.$$

Since $p < 4$, applying the previous corollary (with $q = p$) yields the existence of a unique quartic mapping $Q_4 : G \to T$ such that

$$\left\|f(v) - Q_4(v),s\right\| \le \frac{H(\|v\|,0,\cdots,0) + \|s\|}{2-p}.$$

Because

$$H(\|v\|,0,\cdots,0) = \|v\|^p,$$

and taking into account the structure of the difference operator in the quartic case, we obtain

$$\left\|f(v) - Q_4(v),s\right\| \le \frac{2\|v\|^p + \|s\|}{2-p}$$

for every $v,s \in G$ and $p \in \mathbb{R}^+$.

This completes the proof. □

4. Conclusion

In this paper, we have proved Hyers-Ulam stability of a finite-dimensional quartic functional equation in 2-Banach spaces and intuitionistic fuzzy normed spaces. We established stability results involving sums and products of norm powers under general control circumstances by using two distinct approaches: the fixed point approach and the direct method.

We were able to deduce exact stability inequalities and explicitly construct the limiting quartic mapping thanks to the direct method. However, the fixed point technique offered a sophisticated substitute framework that used contractive mappings to guarantee the existence and uniqueness of the quartic solution. The validity and robustness of the analysis are strengthened by the consistency of the results produced by these two separate methods.

Additionally, a number of corollaries show how variables like control constants and power exponents affect stability. Specifically, the results demonstrate that each approximation solution converges to a distinct quartic mapping under appropriate growth constraints.

All things considered, this work extends the application of Hyers-Ulam stability theory to generalized normed structures that include multi-dimensional norm behavior and fuzziness. Similar stability issues for hybrid fuzzy-modular spaces, non-Archimedean structures, and higher-degree functional equations may be investigated in future studies.

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