Int. J. Anal. Appl. (2026), 24:251

International Journal of Analysis and Applications

Sensitivity Analysis and Chaos Control of Toxoplasma Gondii Dynamics Model for the Cat Population with Early Identification

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Abstract. This research examines the ecological and domestic cat-to-cat dynamics of Toxoplasma gondii transmission explored in terms of vaccination and sanitation methods as major intervention steps. Considering the severe negative consequences of toxoplasmosis in the fetus and the implications of the mentioned issue on the health of the population, a justifiable conception of the parasites present in the cat circulation is necessary. Populations play a central role in the development of control policies. A compartmental mathematical structure is employed for this purpose designed by subdividing the cat population into four health-related subgroups, along with an environmental sub-compartment column responsible for maintaining oocyst pollution. The model that has been developed summarizes the dynamics between infectious cats and the environment in which they are located, both in direct and indirect patterns of transmission. Results of the analysis are provided define the conditions under which the disease can be eradicated and also the conditions when the parasite is not eradicated inevitable. It is demonstrated through a thorough sensitivity analysis

Received: Mar. 1, 2026.

2020 *Mathematics Subject Classification.* 92D30.

Key words and phrases. mathematical modeling; toxoplasma gondii; sensitivity analysis; human health; communicable disease; chaos control; early identification.

https://doi.org/10.28924/2291-8639-24-2026-251
ISSN: 2291-8639

that inadequate vaccination coverage and inadequate sanitation practices. Treatments also stimulate the survival of the parasites, resulting in more oocysts being shed and increasing the rate of infection in susceptible individuals cat populations. Moreover, the results highlight the importance of integrated control measures in mitigating environmental impacts contaminated mind and minimizing the exposure to zoonotic transmission, ongoing debates are presented to be discussed. How specific vaccination and better practices to control Toxoplasma gondii spread to other animals can be achieved vulnerable animals and possibly at-risk human populations. The experiences acquired in this model can be used as a reference in future epidemiological planning and in the development of practical guidelines that would decrease the burden of long-term conditions in the future toxoplasmosis.

1. Introduction

Toxoplasmosis is a zoonotic infection of the protozoan Toxoplasma gondii that leads to a disease worldwide. It is a considerable infectious agent, and since its initial discovery more than a century ago [1], it has been known to be contaminated by seroprevalence statistics showing up to a third of the world population is contaminated, and incidence has been reported at more than 60 percent in parts of the tropics [2]. It has a facultatively heteroxenous life cycle with the ultimate host being the fine, on which sexual reproduction takes place, resulting in oocyst that are environmentally resilient [3]. The wide range of homeothermic animals (e.g chickens, cattle, and avian species) can serve as intermediate hosts, developing cysts in tissues and becoming valuable reservoirs for the spread of infection [4]. Human infection predominantly occurs through unintentional ingestion of oocysts in contaminated soil, water, or food, or by eating undercooked meat infected with tissue cysts [5]. However, a less prevalent but severe type of the disease, congenital transmission, occurs when the first infection is acquired during pregnancy [6]. Although the immunologically sound individuals have no manifestations or a mild and self-limited illness, T. gondii may lead to a deadly disease in immunodeficient individuals, including persons with AIDS or undergoing immunosuppressive therapy, which can manifest itself as toxoplasmic encephalitis [7]. The congenital contamination may lead to various serious effects, such as spontaneous abortion, stillbirth, and post-effects of neurodevelopment, such as hydrocephalus, chorioretinitis, and cognitive impairments [8]. The clinical impact is significant as indicated by the long-term studies; in one study, 162 children with congenital toxoplasmosis were studied, and found that 66 out of them were in a very ill state when they were born, 50 children acquired eye lesions, and 38 children who had eye conditions had neurological damage [9].

Felines play a pivotal role in the epidemiology of toxoplasmosis. As the only hosts that give birth to oocysts, they are the major source of contamination in the environment. One of the critical biological peculiarities is the character of the infection in felines: once the initial exposure is provided, and then the period of excretion of oocysts is rather brief (1-3 weeks), the infection progresses into a chronic phase during which the organism forms latent tissue cysts, and the felines overcome immunity, preventing the subsequent excretion of oocysts but remaining infected throughout [10, 11]. This distinction between acute, infectious, and non-excreting chronic conditions is critical for accurate modeling of the parasite’s behavior. The complex dissemination behavior of T. gondii

has necessitated the use of mathematical constructs to unravel its complexities. Previous modeling studies have ranged from those that focused only on feline populations [12, 13] to those that included more complex systems involving human and animal hosts [14, 15]. A common weakness in many early models, however, is the presence of a so-called recovered or resistant compartment that allows the felines to return to a fully vulnerable state, a pattern that is not reflective of the permanent nature of infection and the development of non-sterilizing resistance, which does not allow re-shedding. Abdelrazec's scheme was a breakthrough because it included the non-excreting and chronic phase of felines. However, it did not determine the possible effects of management interventions. Two relevant ways that can assist in the prevention of the cycle of transmission are immunization and hygiene. Cat vaccination is undertaken to decrease the number of susceptible individuals, hence minimizing the rate at which new infections are shed [16]. Meanwhile, better hygiene might help in speeding up the process of degrading or eliminating oocysts in the environment, which will lower the severity of infection among felines and intermediate hosts [17]. These practical management methods ought to be integrated into a valid dynamic approach to lead to the health policy of the populace. The Caputo fractional derivative is widely used in fractional-order modeling because it allows the incorporation of classical initial conditions while effectively capturing memory and hereditary effects in complex dynamical systems [18,19].
We created an $(SEVI_1I_2R)$ model with a generalized incidence rate. We presented a new methodology to regulate it efficiently, with particular emphasis on both vaccinated and infected populations of cats, to improve the understanding of the reader. In Section 1, an introduction, as well as pertinent historical background and basic terminology, is provided. Section 2 presents a VS-ToxoCat mathematical model, which was developed directly in accordance with the goals of the research. Section 3 entails the analysis of the proposed model and finding out its correctness. In Section 4, an equilibrium-point analysis is provided, which comprises stability analysis. Section 5 then analysis the reproductive number to determine the possibility of spreading the disease and includes a sensitivity analysis. Chaos control, which was done in MATLAB to give a detailed physical understanding, is presented in Section 6. In Section 7, solutions of the developed model are obtained using a fractional operator. Lastly, there is the graphical representation and conclusion in Sections 8 and 9.

Definition 1.

Suppose U is a continuous mapping on L^1 ([0, T],$\mathbb{R}$). For any fractional order $\nu \in$ (0,1) the corresponding Riemann-Liouville fractional integral is given by:

$$I^{\nu}_{0+}U(t) = \frac{1}{\Gamma(\nu)}\left[\int_0^t (t-\varphi)^{\nu-1}U(\varphi)d\varphi\right], \tag{1.1}$$

where integral on right side is point wise defined on $(0, \infty)$.

Definition 2.

Let U be a continuous function on the interval $[0, T]$. The Caputo fractional derivative of U is expressed as:

$$ {}_{0}^{C}D_{t}^{\nu}U(t)=\frac{1}{\Gamma(x-\nu)}\left[\int_{0}^{t}(t-\varphi)^{x-\nu-1}\frac{d^{x}}{d\varphi^{x}}U(t)(\varphi)d\varphi\right], \tag{1.2} $$

where $\Gamma(\cdot)$ denotes the Gamma function. For the case $(x=1)$ equation (2) simplifies to:

$$ {}_{0}^{C}D_{t}^{\nu}U(t)=\frac{1}{\Gamma(x-\nu)}\left[\int_{0}^{t}(t-\varphi)^{-\nu}U'(t)(\varphi)d\varphi\right]. \tag{1.3} $$

Lemma 1.
The Caputo system is assumed to have an equilibrium at a specific fixed point denoted by κ.

$$ {}_{0}^{C}D_{t}^{\nu}U(t)=U(t,\kappa(t)),\nu\in(0,1) \tag{1.4} $$

if and only if $U(t,\kappa^{\bullet})=0$

2. VS-ToxoCat Model

The cat population framework is structured into six distinct groups: Susceptible (S(t)), Exposed (E(t)), Vaccinated (V(t)), Seriously Infectious $(I_1(t))$, Mildly Infectious $(I_2(t))$, and Recovered (R(t)). Their interactions are described by a set of differential equations employing fractional-order calculus. The Susceptible cat (S) group grows by the continuous addition of new members at a fixed pace α_1 and by individuals re-entering from the Recovered category at pace ξ_1R. This group shrinks due to direct disease transmission from contact with highly infectious cats I_1 at pace τ_1, indirect infection from a polluted environment at pace μ, and death from natural causes at pace ω. The differential equation is expressed as:

$$ D(S) = \alpha_1+\xi_1R-\tau_1SI_1-\omega S-\mu SE $$

The Exposed cat category (E) includes felines that have acquired the pathogen from their surroundings. Its size grows at a rate ÂţSE as susceptible cats contract the illness from the environment. It decreases when these felines transition into the Vaccinated group at a rate μ_1, advance into the Seriously Infectious group at a rate μ_2, or succumb to natural mortality at a rate ω. Therefore:

$$ D(E) = \mu SE-(\mu_1+\mu_2+\omega)E $$

The Vaccinated feline group (V) grows when exposed cats receive immunization at a rate μ_1 E. It declines because of diminishing protection from the vaccine at rate ξ_2 (transitionary to the Recovered group) and death from all natural causes at pace ω:

$$ D(V) = \mu_1E-(\omega+\xi_2)V $$

The Seriously Infectious feline group (I_1) expands via two routes: immediate transmission from infectious cats to vulnerable cats at pace τ_1SI_1, and progression of exposed cats to this severe state at pace μ_2E. This group contracts as a result of mortality from all natural causes at pace ω, transition to the persistent (mild) phase at pace τ_2, and healing at pace ψ_1:

$$ D(I_1) = \tau_1SI_1-(\omega+\tau_2+\psi_1)I_1+\mu_2E $$

The Mildly Infected feline group (I_2) grows when cats from the seriously infectious category enter this lasting phase at a pace $\tau_2 I_1$. It shrinks because of recuperation at a pace ψ_2 and death from non-disease causes at a pace ω:

$$D(I_2) = \tau_2 I_1 - (\omega + \psi_2) I_2$$

The final group, Recovered felines (R), arises from two sources: immunized cats whose protection declines and develop resistance at pace ξ_2V, and the convalescence of mildly ill cats at pace $\psi_2 I_2$. It declines because of fading protection (rejoining the Susceptible group) at pace ξ_1 and mortality from natural causes at pace ω:

$$D(R) = \xi_2 V + \psi_2 I_2 - (\omega + \xi_1) R$$

The system of fractional-order ODEs listed below models the dynamics of toxoplasmosis in the feline community.

$$\begin{aligned}
D(S) &= \alpha_1 + \xi_1 R - \tau_1 S I_1 - \omega S - \mu S E \\
D(E) &= \mu S E - (\mu_1 + \mu_2 + \omega) E \\
D(V) &= \mu_1 E - (\omega + \xi_2) V \\
D(I_1) &= \tau_1 S I_1 - (\omega + \tau_2 + \psi_1) I_1 + \mu_2 E \\
D(I_2) &= \tau_2 I_1 - (\omega + \psi_2) I_2 \\
D(R) &= \xi_2 V + \psi_2 I_2 - (\omega + \xi_1) R
\end{aligned} \tag{2.1}$$

The initial condition are

$$S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, V(0) = V_0 \geq 0, I(0) = S_0 \geq 0$$

Description of Variable.
Table 1 lists the variable used in the suggested model along with an explanation of each.

Variable	**Description**
S	Susceptible cats
E	Exposed cats
V	Vaccinated cats
I_1	Seriously infectious cats
I_2	mildly infected cats
R	Recovered cats

Table 1. Description of the Variable

Description of Parameter.
Table 2 lists the parameters used in the suggested model along with an explanation of each.

Parameter	Description	Value	Source
α_1	Recruitment rate of susceptible cats.	2.89	Assumed
ω	Natural death rate of cats.	0.53	Assumed
τ_1	Direct (cat-to-cat) transmission rate from acute infectious cats.	0.23	Assumed
μ	Transmission rate from exposed cats (E) to susceptible cats (S).	0.211	Assumed
μ_1	Exposed to vaccinated rate.	0.20	Assumed
μ_2	Exposed to infectious rate.	0.18	Assumed
τ_2	Acute to chronic progression rate.	0.34	Assumed
ξ_1	Immunity loss rate.	0.3	Assumed
ξ_2	Vaccine waning rate.	1.134	Assumed
ψ_1	Recovery rate from acute stage.	0.123	Assumed
ψ_2	Recovery rate from chronic stage.	1.23	Assumed

Table 2. Description of the Parameter

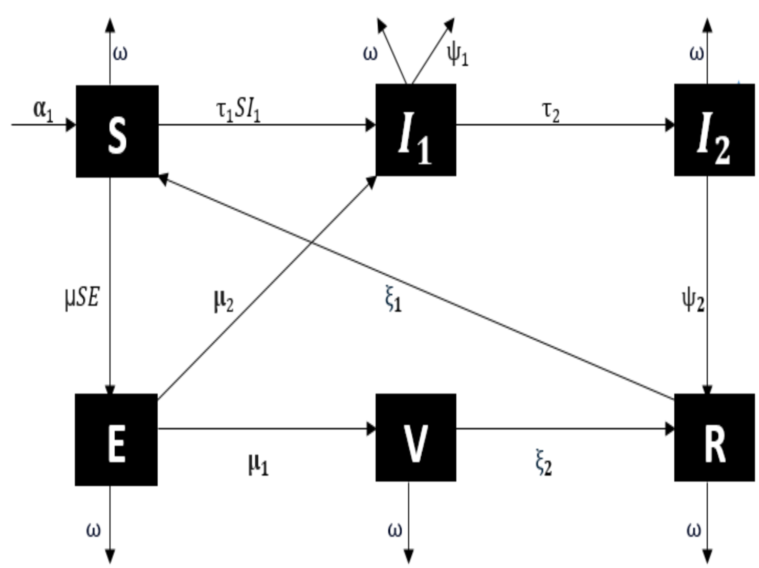


Figure 1. The flow diagram shows proposed model.

2.1. **VS-ToxoCat Model with fractional derivative.** A wide range of epidemiological systems can be studied using fractional-order models, which generalize classical integer-order models by allowing derivatives of order between 0 and 1. In this work, we formulate a non-linear fractional-order system in the Caputo sense to describe the transmission dynamics of Toxoplasma gondii between the cat population and the environment. The model incorporates the effects of vaccination and sanitation to modify and extend the classical framework as follows:

$$\begin{cases} {}^{C}D_t^{\nu}S = \alpha_1 + \xi_1 R - \tau_1 S I_1 - \omega S - \mu S E, \\ {}^{C}D_t^{\nu}E = \mu S E - (\mu_1 + \mu_2 + \omega)E, \\ {}^{C}D_t^{\nu}V = \mu_1 E - (\omega + \xi_2)V, \\ {}^{C}D_t^{\nu}I_1 = \tau_1 S I_1 - (\omega + \tau_2 + \psi_1)I_1 + \mu_2 E, \\ {}^{C}D_t^{\nu}I_2 = \tau_2 I_1 - (\omega + \psi_2)I_2, \\ {}^{C}D_t^{\nu}R = \xi_2 V + \psi_2 I_2 - (\omega + \xi_1)R. \end{cases} \tag{2.2}$$

3. Analysis of Purposed Model

A comprehensive evaluation of the suggested framework entails looking at its accuracy, forecasting capability, and dependability. The mathematical model's structure, epidemiological parameters, and validation techniques all have a major role in how well it works.

3.1. **positivity and boundedness of Solutions.**

Given that its results are certain to be favorable, the suggested model may be utilized in real-world contexts with significant benefits. Regarding positive derivatives $\forall t = 0$, the following features apply:

$$\begin{cases} S(t) \geq S_o e^{\alpha_1 + \xi_1 R - \tau_1 S I_1 - \omega S - \mu S E}, \\ E(t) \geq E_o e^{\mu S E - (\mu_1 + \mu_2 + \omega)E}, \\ V(t) \geq V_o e^{\mu_1 E - (\omega + \xi_2)V}, \\ I_1(t) \geq I_o e^{\tau_1 S I_1 - (\omega + \tau_2 + \psi_1)I_1 + \mu_2 E}, \\ I_2(t) \geq I_o e^{\tau_2 I_1 - (\omega + \psi_2)I_2}, \\ R(t) \geq R_o e^{\xi_2 V + \psi_2 I_2 - (\omega + \xi_1)R}. \end{cases} \tag{3.1}$$

The norm has to be specified

$$\| x \|_{\infty} = \sup_{t \epsilon D_p} | x(t) | . \tag{3.2}$$

This norm is used to compute S(t).

$$\begin{aligned} {}^{C}D_t^{\nu}S(t) &= \alpha_1 + \xi_1 R - \tau_1 S I_1 - \omega S - \mu S E \\ {}^{C}D_t^{\nu}S(t) &\geq -(\tau_1 I_1 + \omega + \mu E)S \\ {}^{C}D_t^{\nu}S(t) &\geq -(\tau_1 \sup_{t\epsilon D_p} \mid I_1 \mid + \omega + \mu \sup_{t\epsilon D_p} \mid E \mid)S \\ {}^{C}D_t^{\nu}S(t) &\geq -(\tau_1 \parallel I_1 \parallel_\infty + \omega + \mu \parallel E \parallel_\infty)S, \forall t \geq 0 \end{aligned}$$

Therefore, the classical derivative's outcome is;

$$S(t) \geq S_o e^{-(\tau_1 \|I_1\|_\infty + \omega + \mu \|E\|_\infty)t}, \forall t \geq 0. \tag{3.3}$$

We calculate the function E(t).

$$\begin{aligned} {}^{C}D_t^{\nu}E(t) &= \mu S E - (\mu_1 + \mu_2 + \omega)E \\ {}^{C}D_t^{\nu}E(t) &\geq -(\mu_1 + \mu_2 + \omega)E \end{aligned}$$

Therefore, the classical derivative's outcome is;

$$E(t) \geq E_o e^{-(\mu_1 + \mu_2 + \omega)t}, \forall t \geq 0. \tag{3.4}$$

We calculate the function V(t).

$$\begin{aligned} {}^{C}D_t^{\nu}V(t) &= \mu_1 E - (\omega + \xi_2)V \\ {}^{C}D_t^{\eta}V(t) &\geq -(\omega + \xi_2)V. \end{aligned}$$

Therefore, the classical derivative's outcome is;

$$V(t) \geq V_o(t) e^{-(\omega + \xi_2)t} \tag{3.5}$$

We determine the function of $I_1(t)$.

$$\begin{aligned} {}^{C}D_t^{\nu}I_1(t) &= \tau_1 S I_1 - (\omega + \tau_2 + \psi_1)I_1 + \mu_2 E \\ {}^{C}D_t^{\nu}I_1(t) &\geq -(\omega + \tau_2 + \psi_1)I_1 \end{aligned}$$

Therefore, the classical derivative's outcome is;

$$I_1(t) \geq I_o e^{-(\omega + \tau_2 + \psi_1)t}, \forall t \geq 0. \tag{3.6}$$

We determine the function of $I_2(t)$.

$$\begin{aligned} {}^{C}D_t^{\nu}I_2(t) &= \tau_2 I_1 - (\omega + \psi_2)I_2 \\ {}^{C}D_t^{\nu}I_2(t) &\geq -(\omega + \psi_2)I_2 \end{aligned}$$

Therefore, the classical derivative's outcome is;

$$I_2(t) \geq I_o e^{-(\omega + \psi_2)t}, \forall t \geq 0. \tag{3.7}$$

We determine the function of $R(t)$.

$$ {}^{C}D_t^{\nu}R(t) = \xi_2 V + \psi_2 I_2 - (\omega - \xi_1)R $$

$$ {}^{C}D_t^{\nu}R(t) \quad \geq \quad -(\omega+\xi_1)R $$

Therefore, the classical derivative's outcome is;

$$ R(t) \quad \geq \quad R_0 e^{-(\omega+\xi_1)t}, \forall t \geq 0. \tag{3.8} $$

The system (6) solutions will undoubtedly remain positive once the Caputo fractional operator's initial requirements are satisfied $t > 0$ we get;

$$ \begin{cases} S(t) \geq S_o \Im\{-(\tau_1 \parallel I_1 \parallel_\infty +\omega + \mu \parallel E \parallel_\infty)t^{\nu}\}, \\ E(t) \geq E_o \Im\{-(\mu_1+\mu_2+\omega)t^{\nu}\}, \\ V(t) \geq V_o(t) \Im\{-(\omega+\xi_2)t^{\nu}\}, \\ I_1(t) \geq I_o \Im\{-(\omega+\tau_2+\psi_1)t^{\nu}\}, \\ I_2(t) \geq I_o \Im\{-(\omega+\psi_2)t^{\nu}\}, \\ R(t) \geq R_0 \Im\{-(\omega+\xi_1)t^{\nu}\}. \end{cases} \tag{3.9} $$

3.2. **Existence and uniqueness analysis.** In this section, the existence and uniqueness of the proposed framework are investigated using fixed-point theory. The existence of at least one solution is established via the Schauder fixed-point theorem, while the uniqueness of the solution is guaranteed by BanachÂŠs contraction principle under appropriate conditions. Once the existence of the solution is confirmed, numerical methods can be employed to approximate it. Furthermore, system (6) can be generalized by incorporating a Caputo fractional derivative of order $\omega \in$(0,1].

$$ \begin{cases} {}^{C}D_t^{\nu}S = \alpha_1 + \xi_1 R - \tau_1 S I_1 - \omega S - \mu S E, \\ {}^{C}D_t^{\nu}E = \mu S E - (\mu_1+\mu_2+\omega)E, \\ {}^{C}D_t^{\nu}V = \mu_1 E - (\omega+\xi_2)V, \\ {}^{C}D_t^{\nu}I_1 = \tau_1 S I_1 - (\omega+\tau_2+\psi_1)I_1 + \mu_2 E, \\ {}^{C}D_t^{\nu}I_2 = \tau_2 I_1 - (\omega+\psi_2)I_2, \\ {}^{C}D_t^{\nu}R = \xi_2 V + \psi_2 I_2 - (\omega+\xi_1)R. \end{cases} \tag{3.10} $$

Let

$$\begin{cases} \kappa_1(t,S) = \alpha_1 + \xi_1 R - \tau_1 S I_1 - \omega S - \mu S E, \\ \kappa_1(t,S) = \mu S E - (\mu_1 + \mu_2 + \omega) E, \\ \kappa_1(t,S) = \mu_1 E - (\omega + \xi_2) V, \\ \kappa_1(t,S) = \tau_1 S I_1 - (\omega + \tau_2 + \psi_1) I_1 + \mu_2 E, \\ \kappa_1(t,S) = \tau_2 I_1 - (\omega + \psi_2) I_2, \\ \kappa_1(t,S) = \xi_2 V + \psi_2 I_2 - (\omega + \xi_1) R. \end{cases} \tag{3.11}$$

Using the starting condition and fractional integral we obtain

$$\begin{aligned} S(t) &= S_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-\beta)^{\nu-1} \kappa_1(t,S) d\beta, \\ E(t) &= E_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-\beta)^{\nu-1} \kappa_2(t,E) d\beta, \\ V(t) &= V_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-\beta)^{\nu-1} \kappa_3(t,V) d\beta, \\ I_1(t) &= I_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-\beta)^{\nu-1} \kappa_4(t,I_1) d\beta, \\ I_2(t) &= I_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-\beta)^{\nu-1} \kappa_5(t,I_2) d\beta, \\ R(t) &= R_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-\beta)^{\nu-1} \kappa_6(t,R) d\beta. \end{aligned} \tag{3.12}$$

Suppose

$$\Psi(t) = \begin{cases} S(t) \\ E(t) \\ V(t) \\ I_1(t) \\ I_2(t) \\ R(t) \end{cases}, \Psi(t) = \begin{cases} S_0 \\ E_0 \\ V_0 \\ I_0 \\ I_0 \\ R_0 \end{cases}, \Omega(\beta, \Psi(\beta)) = \begin{cases} \kappa_1(t,S), \\ \kappa_2(t,E), \\ \kappa_3(t,V), \\ \kappa_4(t,I_1), \\ \kappa_5(t,I_2), \\ \kappa_6(t,R). \end{cases} \tag{3.13}$$

So the system (18) becomes

$$\Psi(t) = \Psi_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-\beta)^{\nu-1} \Omega(\beta, \Psi(\beta)) d\beta$$

Let us now examine a Banach space with a norm $\xi[0,T] = \mu$:

$$\|S,E,V,I_1,I_2,R\| = \sup_{t\epsilon[0,1]}[\mid S,E,V,I_1,I_2,R \mid]$$

The mapping $\diamond : \mu \to \mu$ should be considered.

$$\diamond\Psi(t) = \Psi_0 + \frac{1}{\Gamma(\nu)}\int_0^t (t-\beta)^{\nu-1}\Omega(\beta,\Psi(\beta))d\beta$$

Furthermore, we apply the subsequent hypothesis to a nonlinear function.

The $(F1)$ Constants χ_p and χ_p >0 exist such that

$$\mid \Omega(t,\Psi(t)) \mid \leq \chi_p \mid (t) \mid + \chi_p$$

A $(F2)$ constant called G_p exists 0 for every Ψ,$\bar{\Psi} \in \mu$.

$$\mid \Omega(t,\bar{\Psi}) - \Omega(t,\bar{\Psi}_1) \mid \leq G_p \mid \bar{\Psi} - \bar{\Psi}_1 \mid$$

Theorem 1. If assumption (F1) is true, there exists at least one solution of system (16).

Proof. Confirm that $\diamond$ remains within a bounded range.

Let

$$\Theta \quad = \quad \Psi \in \Theta | \chi \geq \|\Psi\|$$

Where;

$$\chi \quad \geq \quad \max_{t\in[0,T]} \frac{\Psi_0 + (\frac{\chi_p T^\nu}{\Gamma(\nu+1)})}{1 - (\frac{\chi_p T^\nu}{\Gamma(\nu+1)})}$$

A closed, convex subcollection of μ.

Now

$$\begin{aligned}
\|\diamond\Psi\| &= \max_{t\in[0,T]} \mid \Psi_0 + \frac{1}{\Gamma(\nu)}\int_0^t (t-\beta)^{\nu-1}\Omega(\beta,\Psi(\beta)))d\beta \mid \\
&\leq \mid \Psi_0 \mid + \frac{1}{\Gamma(\nu)}\int_0^t (t-\beta)^{\nu-1} \mid \Omega(\beta,\Psi(\beta))d\beta \mid \\
&\leq \mid \Psi_0 \mid + \frac{1}{\Gamma(\nu)}\int_0^t (t-\beta)^{\nu-1}[\chi_p \mid \Psi(t) \mid + \chi_p]d\beta \\
&\leq \mid \Psi_0 \mid + [\chi_p \parallel \Psi(t) \parallel + \chi_p]\frac{T^\nu}{\Gamma(\nu+1)} \leq \chi
\end{aligned}$$

Since $\Psi \in \Theta \Rightarrow \diamond(\Theta) \subseteq \Theta$ it shows that Θ is bounded. Let $t_1 < t_2 \in [0,T]$ to prove that $\in$ is completely continuous.

We take

$$\begin{aligned}
\|\diamond\Psi(t_2) - \diamond\Psi(t_2)\| &= \mid \frac{1}{\Gamma(\nu)}\int_0^{t_2} (t_2-\beta)^{\nu-1}\Omega(\beta,\Psi(\beta))d\beta - \frac{1}{\Gamma(\nu)}\int_0^{t_1} (t_1-\beta)^{\nu-1}\Omega(\beta,\Psi(\beta))d\beta \mid \\
&\leq [\chi_p \parallel \Psi \parallel + \chi_p]\frac{[t_2^\nu - t_1^\nu]}{\Gamma(\nu+1)}
\end{aligned}$$

This suggests that $\|\diamond\Psi(t_2) - \diamond\Psi(t_2)\| \to 0$ as $t_2 \to t_1$. Operator $\diamond$ is continuous and compact according to the ArzelÃă-Ascoli theorem. Schauder's fixed-point principle states that the system (16) in question permits at least one solution. Additionally, the uniqueness of this solution (16) is established by applying Banach's contraction theorem.

Theorem 2. The existence of a unique solution to system (16) is ensured whenever condition (F2) holds.

Proof. Let $\bar{\Psi}, \bar{\Psi}_1 \in \mu$.

$$\begin{aligned}
\|\diamond(\bar{\Psi}) - \diamond(\bar{\Psi}_1)\| &= \max_{t\in[0,T]} \mid \frac{1}{\Gamma(\nu)}\int_0^{t_2}(t_2-\beta)^{\nu-1}\Omega(\beta,\gamma(\beta))d\beta - \frac{1}{\Gamma(\nu)}\int_0^{t_1}(t_1-\beta)^{\nu-1}\Omega(\beta,\bar{\Psi}(\beta))d\beta \mid \\
&\leq \frac{T^\nu}{\Gamma(\nu+1)}G_p \mid \bar{\Psi} - \Psi_1 \mid
\end{aligned}$$

The contraction property is satisfied by the operator $\diamond$. Therefore, the system (16) allows for a single solution, as stated by Banach's fixed-point theorem. The Caputo operator is used to examine stability and iterative behavior.

Theorem 3. LetÂŠs say that $\aleph$ is a self map defined as follows:

$$\begin{aligned}
\aleph[S_v(t)] &= S_{v+1}(t) = S_v(0) + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\alpha_1 + \xi_1 R_v - \tau_1 S_v I_{1v} - \omega S_v - \mu S_v E_v\}], \\
\aleph[E_v(t)] &= E_{v+1}(t) = E_v(0) + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\mu S_v E_v - (\mu_1 + \mu_2 + \omega)E_v\}], \\
\aleph[V_v(t)] &= V_{v+1}(t) = V_v(0) + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\mu_1 E_v - (\omega + \xi_2)V_v\}], \\
\aleph[I_{1v}(t)] &= I_{1(v+1)}(t) = I_{1v}(0) + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\tau_1 S_v I_{1v} - (\omega + \tau_2 + \psi_1)I_{1v} + \mu_2 E_v\}], \\
\aleph[I_{2v}(t)] &= I_{2(v+1)}(t) = I_{2v}(0) + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\tau_2 I_{1v} - (\omega + \psi_2)I_{2v}\}], \\
\aleph[R_v(t)] &= R_{v+1}(t) = R_v(0) + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\xi_2 V_v + \psi_2 I_{2v} - (\omega + \xi_1)R_v\}].
\end{aligned}$$

Iteration $\aleph$ Stable is G^1 $(v + \mu)$ if the following requirements are met.

$$\begin{cases}
1 + \xi_1\phi_1 - \tau_1(P_1 + P_4)\phi_2 - \omega\phi_3 - \mu(P_1 + P_2)\phi_4 < 1, \\
\mu(P_1 + P_2)\phi_4 - \mu_1\phi_5 - \mu_2\phi_6 - \omega\phi_7 < 1, \\
\mu_1\phi_5 - \omega\phi_8 - \xi_2\phi_9 < 1, \\
\tau_1(P_1 + P_4)\phi_2 - \omega\phi_{10} - \tau_2\phi_{11} - \psi_1\phi_{12} + \mu_2\phi_6 < 1, \\
\tau_2\phi_{11} - \omega\phi_{13} - \omega_2\phi_{14} < 1, \\
\xi_2\phi_9 - \psi_2\phi_{14} - \omega\phi_{15} + \xi_1\phi_1 < 1,
\end{cases}$$

Proof. To demonstrate the fixed point of $\aleph$, we calculated the following for $(v+\mu)\ \varepsilon\ PxP$.

$$\begin{aligned}
\aleph[S_\mu(t)-S_v(t)] &= S_\mu(t)-S_v(t)+\delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\alpha_1+\xi_1 R_\mu-\tau_1 S_\mu I_{1\mu}-\omega S_\mu-\mu S_\mu E_\mu\}] \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\alpha_1+\xi_1 R_v-\tau_1 S_v I_{1v}-\omega S_v-\mu S_v E_v\}], \\
\aleph[E_\mu(t)-E_v(t)] &= E_\mu(t)-E_v(t)+\delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\mu S_\mu E_\mu-(\mu_1+\mu_2+\omega)E_\mu\}] \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\mu S_v E_v-(\mu_1+\mu_2+\omega)E_v\}], \\
\aleph[V_\mu(t)-V_v(t)] &= V_\mu(t)-V_v(t)+\delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\mu_1 E_\mu-(\omega+\xi_2)V_\mu\}] \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\mu_1 E_v-(\omega+\xi_2)V_v\}], \\
\aleph[I_{1\mu}(t)-I_{1v}(t)] &= I_{1\mu}(t)-I_{1v}(t)+\delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\tau_1 S_\mu I_{1\mu}-(\omega+\tau_2+\psi_1)I_{1\mu}+\mu_2 E_\mu\}] \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\tau_1 S_v I_{1v}-(\omega+\tau_2+\psi_1)I_{1v}+\mu_2 E_v\}], \\
\aleph[I_{2\mu}(t)-I_{2v}(t)] &= I_{2\mu}(t)-I_{2v}(t)+\delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\tau_2 I_{1v}-(\omega+\psi_2)I_{2v}\}] \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\tau_2 I_{1v}-(\omega+\psi_2)I_{2v}\}], \\
\aleph[R_\mu(t)-R_v(t)] &= R_\mu(t)-R_v(t)+\delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\xi_2 V_\mu+\psi_2 I_{2\mu}-(\omega+\xi_1)R_\mu\}] \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\xi_2 V_v+\psi_2 I_{2v}-(\omega+\xi_1)R_v\}].
\end{aligned}$$

By computing the norm on both sides.

$$\begin{aligned}
\parallel \aleph[S_\mu(t)-S_v(t)] \parallel &= \parallel S_\mu(t)-S_v(t)+\delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\alpha_1+\xi_1 R_\mu-\tau_1 S_\mu I_{1\mu}-\omega S_\mu-\mu S_\mu E_\mu\}], \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\alpha_1+\xi_1 R_v-\tau_1 S_v I_{1v}-\omega S_v-\mu S_v E_v\}] \parallel, \\
\parallel \aleph[E_\mu(t)-E_v(t)] \parallel &= \parallel E_\mu(t)-E_v(t)+\delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\mu S_\mu E_\mu-(\mu_1+\mu_2+\omega)E_\mu\}] \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\mu S_v E_v-(\mu_1+\mu_2+\omega)E_v\}] \parallel \\
\parallel \aleph[V_\mu(t)-V_v(t)] \parallel &= \parallel V_\mu(t)-V_v(t)+\delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\mu_1 E_\mu-(\omega+\xi_2)V_\mu\}] \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\mu_1 E_v-(\omega+\xi_2)V_v\}] \parallel
\end{aligned}$$

$$\begin{aligned}
\parallel \aleph[I_{1\mu}(t)-I_{1v}(t)] \parallel &= \parallel I_{1\mu}(t)-I_{1v}(t)+\delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\tau_1 S_\mu I_{1\mu}-(\omega+\tau_2+\psi_1)I_{1\mu}+\mu_2 E_\mu\}] \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\tau_1 S_v I_{1v}-(\omega+\tau_2+\psi_1)I_{1v}+\mu_2 E_v\}] \parallel
\end{aligned}$$

$$
\begin{aligned}
\parallel \aleph[I_{2\mu}(t) - I_{2v}(t)] \parallel &= \parallel I_{2\mu}(t) - I_{2v}(t) + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\tau_2 I_{1\mu} - (\omega + \psi_2) I_{2\mu}\}] \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\tau_2 I_{1v} - (\omega + \psi_2) I_{2v}\}] \parallel \\
\parallel \aleph[R_\mu(t) - R_v(t)] \parallel &= \parallel R_\mu(t) - R_v(t) + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\xi_2 V_\mu + \psi_2 I_{2\mu} - (\omega + \xi_1) R_\mu\}] \\
&- \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\xi_2 V_v + \psi_2 I_{2v} - (\omega + \xi_1) R_v\}] \parallel
\end{aligned}
$$

We can make it simpler by using the triangle inequality, which gives;

$$
\begin{aligned}
\parallel \aleph[S_\mu(t) - S_v(t)] \parallel &= \parallel S_\mu(t) - S_v(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\parallel \xi_1(R_\mu - R_v) \parallel \\
&+ \parallel -\tau_1 S_\mu(I_{1\mu} - I_{1v}) \parallel + \parallel -\tau_1 I_{1v}(S_\mu - S_v) \parallel \\
&+ \parallel -\omega(S_\mu - S_v) \parallel + \parallel -\mu S_\mu(E_\mu - E_v) \parallel + \parallel -\mu E_v(S_\mu - S_v) \parallel\}], \\
\parallel \aleph[E_\mu(t) - E_v(t)] \parallel &= \parallel E_\mu(t) - E_v(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta \parallel \{\mu S_\mu(E_\mu - E_v) \parallel \\
&+ \parallel \{\mu E_v(S_\mu - S_v) \parallel + \parallel -(\mu_1 + \mu_2 + \omega)(E_\mu - E_v) \parallel\}] \\
\parallel \aleph[V_\mu(t) - V_v(t)] \parallel &= \parallel V_\mu(t) - V_v(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta \parallel \{\mu_1(E_\mu - E_v) \parallel + \parallel -(\omega + \xi_2)(V_\mu - V_v) \parallel\}] \\
\parallel \aleph[I_{1\mu}(t) - I_{1v}(t)] \parallel &= \parallel I_{1\mu}(t) - I_{1v}(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\parallel \tau_1 S_\mu(I_{1\mu} - I_{1v}) \parallel + \parallel \tau_1 I_{1v}(S_\mu - S_v) \parallel \\
&+ \parallel -(\omega + \tau_2 + \psi_1)(I_{1\mu} - I_{1v}) \parallel \\
&+ \parallel \mu_2(E_\mu - E_v) \parallel\}] \\
\parallel \aleph[I_{2\mu}(t) - I_{2v}(t)] \parallel &= \parallel I_{2\mu}(t) - I_{2v}(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\parallel \tau_2(I_{1\mu} - I_{1v}) \parallel + \parallel -(\omega + \psi_2)(I_{2\mu} - I_{2v}) \parallel\}] \\
\parallel \aleph[R_\mu(t) - R_v(t)] \parallel &= \parallel R_\mu(t) - R_v(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\parallel \xi_2(V_\mu - V_v) \parallel + \parallel \psi_2(I_{2\mu} - I_{2v}) \parallel \\
&+ \parallel -(\omega + \xi_1)(R_\mu - R_v) \parallel\}]
\end{aligned}
$$

Given the proportional significance of the two choices, we define

$$
\begin{aligned}
\parallel S_\mu(t) - S_v(t) \parallel &\cong \parallel E_\mu(t) - E_v(t) \parallel \cong \parallel V_\mu(t) - V_v(t) \parallel \cong \parallel I_{1\mu}(t) - I_{1v}(t) \parallel \cong \parallel I_{2\mu}(t) - I_{2v}(t) \parallel \\
&\cong \parallel R_\mu(t) - R_v(t) \parallel .
\end{aligned}
$$

As a result, we get the relationship below.

$$
\begin{aligned}
\parallel \aleph[S_\mu(t) - S_v(t)] \parallel &= \parallel S_\mu(t) - S_v(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\parallel \xi_1(S_\mu - S_v) \parallel \\
&+ \parallel -\tau_1 S_\mu(S_\mu - S_v) \parallel + \parallel -\tau_1 I_{1v}(S_\mu - S_v) \parallel \\
&+ \parallel -\omega(S_\mu - S_v) \parallel + \parallel -\mu S_\mu(S_\mu - S_v) \parallel + \parallel -\mu E_v(S_\mu - S_v) \parallel\}], \\
\parallel \aleph[E_\mu(t) - E_v(t)] \parallel &= \parallel E_\mu(t) - E_v(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta \parallel \{\mu S_\mu(E_\mu - E_v) \parallel \\
&+ \parallel \{\mu E_v(E_\mu - E_v) \parallel + \parallel -(\mu_1 + \mu_2 + \omega)(E_\mu - E_v) \parallel\}]
\end{aligned}
$$

$$\begin{aligned}
\parallel \aleph[V_\mu(t)-V_v(t)] \parallel &= \parallel V_\mu(t)-V_v(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta \parallel \{\mu_1(V_\mu - V_v) \parallel + \parallel -(\omega+\xi_2)(V_\mu - V_v) \parallel\}] \\
\parallel \aleph[I_{1\mu}(t)-I_{1v}(t)] \parallel &= \parallel I_{1\mu}(t)-I_{1v}(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\parallel \tau_1 S_\mu(I_{1\mu}-I_{1v}) \parallel + \parallel \tau_1 I_{1v}(I_{1\mu}-I_{1v}) \parallel \\
&+ \parallel -(\omega+\tau_2+\psi_1)(I_{1\mu}-I_{1v}) \parallel + \parallel \mu_2(I_{1\mu}-I_{1v}) \parallel\}] \\
\parallel \aleph[I_{2\mu}(t)-I_{2v}(t)] \parallel &= \parallel I_{2\mu}(t)-I_{2v}(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\parallel \tau_2(I_{2\mu}-I_{2v}) \parallel + \parallel -(\omega+\psi_2)(I_{2\mu}-I_{2v}) \parallel\}] \\
\parallel \aleph[R_\mu(t)-R_v(t)] \parallel &= \parallel R_\mu(t)-R_v(t) \parallel + \delta^{-1}[\frac{1}{\mathfrak{I}_\omega}\delta\{\parallel \xi_2(R_\mu - R_v) \parallel + \parallel \psi_2(R_\mu - R_v) \parallel \\
&+ \parallel -(\omega+\xi_1)(R_\mu - R_v) \parallel\}]
\end{aligned}$$

Also the convergent sequences $S_\mu, E_\mu, V_\mu, I_{1\mu}, I_{2\mu}, R_\mu$ are bounded. We can then derive seven distinct positive constants $P_1, P_2, P_3, P_4, P_5, P_6$ for all t such that

$$\parallel S_\mu \parallel < P_1, \parallel E_\mu \parallel < P_2, \parallel V_\mu \parallel < P_3, \parallel I_{1\mu} \parallel < P_4, \parallel I_{2\mu} \parallel < P_5, \parallel R_\mu \parallel < P_6.$$

Further, considering equations we get

$$\begin{aligned}
\parallel \aleph[S_\mu(t)-S_v(t)] \parallel &\leq (1+\xi_1\phi_1 - \tau_1(P_1+P_4)\phi_2 - \omega\phi_3 - \mu(P_1+P_2)\phi_4) \parallel \aleph S_\mu(t)-S_v(t) \parallel, \\
\parallel \aleph[E_\mu(t)-E_v(t)] \parallel &\leq (\mu(P_1+P_2)\phi_4 - \mu_1\phi_5 - \mu_2\phi_6 - \omega\phi_7) \parallel \aleph E_\mu(t)-E_v(t) \parallel, \\
\parallel \aleph[V_\mu(t)-V_v(t)] \parallel &\leq (\mu_1\phi_5 - \omega\phi_8 - \xi_2\phi_9) \parallel \aleph V_\mu(t)-V_v(t) \parallel, \\
\parallel \aleph[I_{1\mu}(t)-I_{1v}(t)] \parallel &\leq (\tau_1(P_1+P_4)\phi_2 - \omega\phi_{10} - \tau_2\phi_{11} - \psi_1\phi_{12} + \mu_2\phi_6) \parallel \aleph I_{1\mu}(t)-I_{1v}(t) \parallel, \\
\parallel \aleph[I_{2\mu}(t)-I_{2v}(t)] \parallel &\leq (\tau_2\phi_{11} - \omega\phi_{13} - \omega_2\phi_{14}) \parallel \aleph I_{2\mu}(t)-I_{2v}(t) \parallel, \\
\parallel \aleph[R_\mu(t)-R_v(t)] \parallel &\leq (\xi_2\phi_9 - \psi_2\phi_{14} - \omega\phi_{15} + \xi_1\phi_1) \parallel \aleph R_\mu(t)-R_v(t) \parallel .
\end{aligned}$$

Where

$$\begin{cases}
1+\xi_1\phi_1 - \tau_1(P_1+P_4)\phi_2 - \omega\phi_3 - \mu(P_1+P_2)\phi_4 < 1, \\
\mu(P_1+P_2)\phi_4 - \mu_1\phi_5 - \mu_2\phi_6 - \omega\phi_7 < 1, \\
\mu_1\phi_5 - \omega\phi_8 - \xi_2\phi_9 < 1, \\
\tau_1(P_1+P_4)\phi_2 - \omega\phi_{10} - \tau_2\phi_{11} - \psi_1\phi_{12} + \mu_2\phi_6 < 1, \\
\tau_2\phi_{11} - \omega\phi_{13} - \omega_2\phi_{14} < 1, \\
\xi_2\phi_9 - \psi_2\phi_{14} - \omega\phi_{15} + \xi_1\phi_1 < 1.
\end{cases}$$

For this reason $\aleph$ has a fixed point. In relation to equations, we presume

$$\nu = (0,0,0,0,0)$$

$$\Upsilon = \begin{cases}
1+\xi_1\phi_1 - \tau_1(P_1+P_4)\phi_2 - \omega\phi_3 - \mu(P_1+P_2)\phi_4, \\
\mu(P_1+P_2)\phi_4 - \mu_1\phi_5 - \mu_2\phi_6 - \omega\phi_7, \\
\mu_1\phi_5 - \omega\phi_8 - \xi_2\phi_9, \\
\tau_1(P_1+P_4)\phi_2 - \omega\phi_{10} - \tau_2\phi_{11} - \psi_1\phi_{12} + \mu_2\phi_6, \\
\tau_2\phi_{11} - \omega\phi_{13} - \omega_2\phi_{14}, \\
\xi_2\phi_9 - \psi_2\phi_{14} - \omega\phi_{15} + \xi_1\phi_1.
\end{cases}$$

This complete the proof.

4. Equilibrium point analysis

A thorough analysis of the equilibrium points is given in this section. System (6) must be set to zero to find an equilibrium.

$$\begin{cases} \alpha_1 + \xi_1 R - \tau_1 S I_1 - \omega S - \mu S E = 0, \\ \mu S E - (\mu_1 + \mu_2 + \omega) E = 0, \\ \mu_1 E - (\omega + \xi_2) V = 0 \\ \tau_1 S I_1 - (\omega + \tau_2 + \psi_1) I_1 + \mu_2 E = 0, \\ \tau_2 I_1 - (\omega + \psi_2) I_2 = 0, \\ \xi_2 V + \psi_2 I_2 - (\omega + \xi_1) R = 0. \end{cases}$$

The following are equilibrium points devoid of disease:

$$E^0 = \begin{cases} S \to \frac{\alpha_1}{\omega}, \\ E \to 0, \\ V \to 0 \\ I_1 \to 0, \\ I_2 \to 0, \\ R \to 0. \end{cases}$$

Thus if $E^* = 0$, then we have $S^* = \frac{\alpha_1}{\omega}$, $I_1 = 0$,$I_2 = 0$,,$R = 0$, which are disease free equilibrium Points:

$$E^0 = (\frac{\alpha_1}{\omega}, 0, 0, 0, 0, 0)$$

5. Reproductive Number and Sensitivity Analysis

According to the disease-free level equilibrium point, the functions F and V are inferred by the next generation technique, respectively. This approach was previously used to explain how an illness spreads. Obtaining FV^{-1} for the reproduction R_0.

$$\mathbf{J}=\begin{pmatrix} -\omega & -\mu S & 0 & -\tau_1 S & 0 & \xi_1 \\ 0 & \mu S-(\mu_1+\mu_2+\omega) & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & -(\omega+\xi_2) & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & \tau_1 S-(\omega+\tau_2+\psi_1) & 0 & 0 \\ 0 & 0 & 0 & \tau_2 & -(\omega+\psi_2) & 0 \\ 0 & 0 & \xi_2 & 0 & \psi_2 & -(\omega+\xi_1) \end{pmatrix}$$

and

$$J_0 = F - V$$

Where;

$$F=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \xi_1 \\ 0 & \mu S & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & \tau_1 S & 0 & 0 \\ 0 & 0 & 0 & \tau_2 & 0 & 0 \\ 0 & 0 & \xi_2 & 0 & \psi_2 & 0 \end{pmatrix}$$

$$V=\begin{pmatrix} \omega & \mu S & 0 & \tau_1 S & 0 & 0 \\ 0 & (\mu_1+\mu_2+\omega) & 0 & 0 & 0 & 0 \\ 0 & 0 & (\omega+\xi_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & (\omega+\tau_2+\psi_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & (\omega+\psi_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & (\omega+\xi_1) \end{pmatrix}$$

and

$$V^{-1}=\begin{pmatrix} \frac{\omega+\mu_1+\mu_2}{(\omega^2+\omega\mu_1+\omega\mu_2)} & -\frac{\mu\alpha_1+\frac{\mu\alpha_1\xi_2}{\omega}}{(\omega^2+\omega\mu_1+\omega\mu_2)(\omega+\xi_2)} & 0 & -\frac{\alpha_1(\omega+\mu_1+\mu_2)\tau_1}{\omega(\omega^2+\omega\mu_1+\omega\mu_2)(\omega+\tau_2+\psi_1)} & 0 & \frac{\alpha_1(\omega+\mu_1+\mu_2)\tau_1}{(\omega+\xi_1)(\omega^2+\omega\mu_1+\omega\mu_2)(\omega+\tau_2+\psi_1)} \\ 0 & \frac{\omega^2+\omega\xi_2}{(\omega^2+\omega\mu_1+\omega\mu_2)(\omega+\xi_2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(\omega+\xi_2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{(\omega+\tau_2+\psi_1)} & 0 & -\frac{\omega}{(\omega+\xi_1)(\omega+\tau_2+\psi_1)} \\ 0 & 0 & 0 & 0 & \frac{1}{(\omega+\psi_2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{(\omega+\xi_1)} \end{pmatrix}$$

Where

$$Y = FV^{-1}$$

$$Y=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\xi_1}{(\omega+\xi_1)} \\ 0 & -\frac{\mu\alpha_1(\omega^2+\omega\xi_2)}{\omega(\omega^2+\omega\mu_1+\omega\mu_2)(\omega+\xi_2)} & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu_1(\omega^2+\omega\xi_2)}{(\omega^2+\omega\mu_1+\omega\mu_2)(\omega+\xi_2)} & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu_2(\omega^2+\omega\xi_2)}{(\omega^2+\omega\mu_1+\omega\mu_2)(\omega+\xi_2)} & 0 & \frac{\alpha_1\tau_1}{\omega(\omega+\tau_2+\psi_1)} & 0 & -\frac{\alpha_1\tau_1}{(\omega+\xi_1)(\omega+\tau_2+\psi_1)} \\ 0 & 0 & 0 & \frac{\tau_2}{(\omega+\tau_2+\psi_1)} & 0 & -\frac{\omega\tau_2}{(\omega+\xi_1)(\omega+\tau_2+\psi_1)} \\ 0 & 0 & \frac{\xi_2}{(\omega+\xi_2)} & 0 & \frac{\psi_2}{(\omega+\psi_2)} & 0 \end{pmatrix}$$

Thus

$$| Y - \lambda I |= 0$$

$$\begin{vmatrix} -\lambda & 0 & 0 & 0 & 0 & \frac{\xi_1}{(\omega+\xi_1)} \\ 0 & -\frac{\mu\alpha_1(\omega^2+\omega\xi_2)}{\omega(\omega^2+\omega\mu_1+\omega\mu_2)(\omega+\xi_2)} - \lambda & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu_1(\omega^2+\omega\xi_2)}{(\omega^2+\omega\mu_1+\omega\mu_2)(\omega+\xi_2)} & -\lambda & 0 & 0 & 0 \\ 0 & \frac{\mu_2(\omega^2+\omega\xi_2)}{(\omega^2+\omega\mu_1+\omega\mu_2)(\omega+\xi_2)} & 0 & \frac{\alpha_1\tau_1}{\omega(\omega+\tau_2+\psi_1)} - \lambda & 0 & -\frac{\alpha_1\tau_1}{(\omega+\xi_1)(\omega+\tau_2+\psi_1)} \\ 0 & 0 & 0 & \frac{\tau_2}{(\omega+\tau_2+\psi_1)} & -\lambda & -\frac{\omega\tau_2}{(\omega+\xi_1)(\omega+\tau_2+\psi_1)} \\ 0 & 0 & \frac{\xi_2}{(\omega+\xi_2)} & 0 & \frac{\psi_2}{(\omega+\psi_2)} & -\lambda \end{vmatrix} = 0$$

The characteristics equation is

$$R_0 = \frac{\mu\alpha_1}{\omega(\omega + \mu_1 + \mu_2)}$$

The reproductive no is

$$R_0 = \frac{\mu\alpha_1}{\omega(\omega + \mu_1 + \mu_2)}$$

5.1. **Senstivity.** A sensitivity analysis is performed to evaluate the impact of various factors on the proposed framework. The analytical examples that follow how each parameter effects R_0.

$$R_0 = \frac{\mu\alpha_1}{\omega(\omega + \mu_1 + \mu_2)}$$

By calculating the relevant threshold's partial derivative for the relative documents given below.

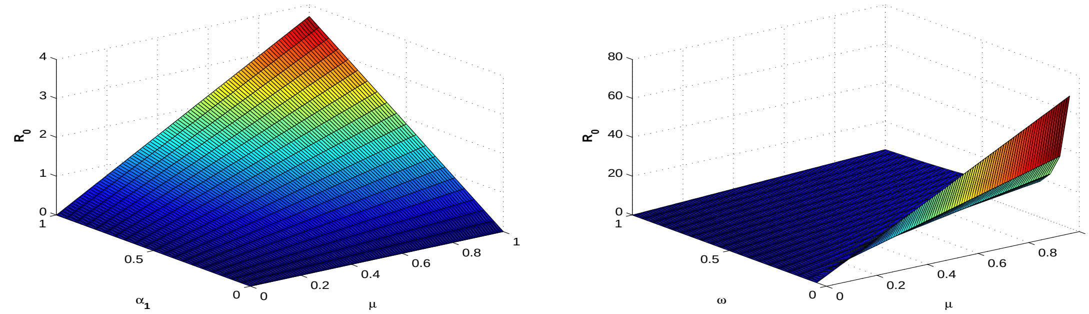
$$\frac{\partial R_0}{\partial \mu} = \frac{\alpha_1}{\omega(\omega + \mu_1 + \mu_2)} > 0,$$

$$\frac{\partial R_0}{\partial \alpha_1} = \frac{\mu}{\omega(\omega + \mu_1 + \mu_2)} > 0,$$

$$\frac{\partial R_0}{\partial \omega} = -\frac{\mu\alpha_1(2\omega + \mu_1 + \mu_2)}{\omega^2(\omega + \mu_1 + \mu_2)^2} < 0,$$

$$\frac{\partial R_0}{\partial \mu_1} = -\frac{\mu\alpha_1}{\omega(\omega + \mu_1 + \mu_2)^2} < 0,$$

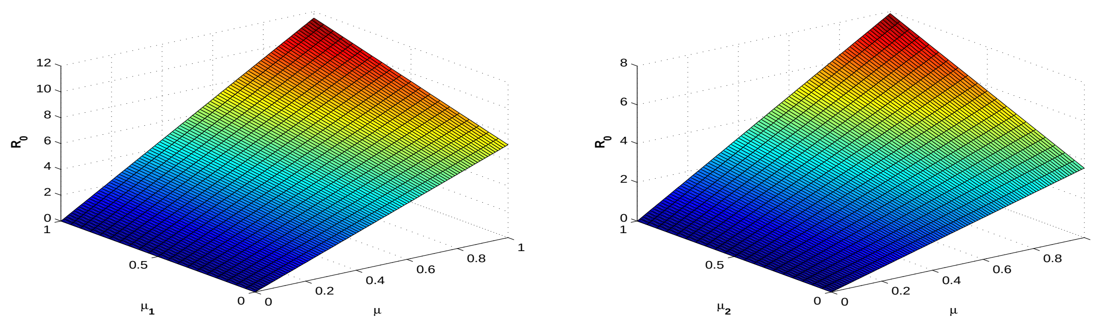
$$\frac{\partial R_0}{\partial \mu_2} = -\frac{\mu\alpha_1}{\omega(\omega + \mu_1 + \mu_2)^2} < 0.$$



(A) Sensitivity to the parameters α_1 and μ

(B) Sensitivity to the parameters ω and μ

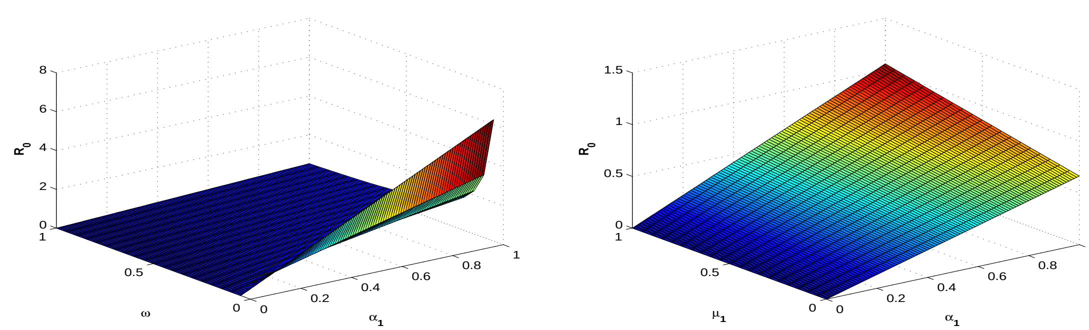
Figure 2. Multiple parameter sensitivity analysis



(A) Sensitivity to the parameters μ_1 and μ

(B) Sensitivity to the parameters μ_2 and μ

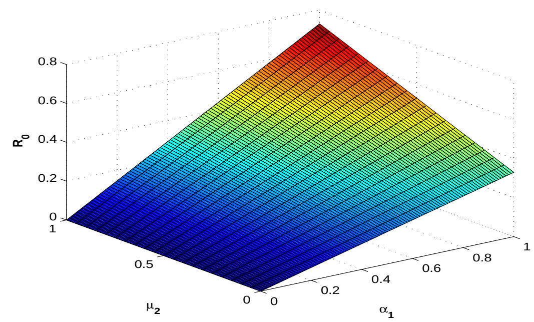
Figure 3. Multiple parameter sensitivity analysis



(A) Sensitivity to the parameters ω and α_1

(B) Sensitivity to the parameters μ_1 and α_1

Figure 4. Multiple parameter sensitivity analysis



(A) Sensitivity to the parameters μ_2 and α_1

Lemma 2. Suppose that for any t $\geq t_0$ U $\in R^+$ is a function that is continuous.
${}^{C}D_t^{\nu}\ (U - U^* - U^* log\frac{U}{U^*}) \leq (1 - \frac{U}{U^*})\ {}^{C}D_t^{\nu} U(t)$
$U^* \in R^+, \forall\ \alpha \in (0,1)$.

Theorem 4. For values of R_0 less than one, the disease-free equilibrium state E^0 exhibits global asymptotic stability.

Proof. The Lyapunov function is as follows:

$$G \quad = \quad (S - S^0 - S^0 log\frac{S}{S^0}) + E + V + I_1 + I_2 + R$$

Lemma gives us;

$${}^{C}D_t^{\nu}G \quad \leq \quad (1 - \frac{S^0}{S})^{C}D_t^{\nu}S +^{C} D_t^{\nu}E +^{C} D_t^{\nu}V +^{C} D_t^{\nu}I_1 +^{C} D_t^{\nu}I_2 +^{C} D_t^{\nu}R$$

Substituting the derivative values from;

$$\begin{aligned} {}^{C}D_t^{\nu}G \quad &\leq \quad (1 - \frac{S^0}{S})(\alpha_1 + \xi_1 R - \tau_1 S I_1 - \omega S - \mu S E) + (\mu S E - (\mu_1 + \mu_2 + \omega)E) + (\mu_1 E - (\omega + \xi_2)V) \\ &+ \quad (\tau_1 S I_1 - (\omega + \tau_2 + \psi_1)I_1 + \mu_2 E) + (\tau_2 I_1 - (\omega + \psi_2)I_2) + (\xi_2 V + \psi_2 I_2 - (\omega + \xi_1)R) \end{aligned}$$

Replacing $S = S - S^0, E = E - E^0, V = V - V^0, I_1 = I_1 - I_1^0, I_2 = I_2 - I_2^0, R = R - R^0$.

$$\begin{aligned} {}^{C}D_t^{\nu}G \quad &\leq \quad (1 - \frac{S^0}{S})(\alpha_1 + \xi_1(R - R^0) - \tau_1(S - S^0)(I_1 - I_1^0) - \omega(S - S^0) - \mu(S - S^0)(E - E^0)) \\ &+ \quad (\mu(S - S^0)(E - E^0) - (\mu_1 + \mu_2 + \omega)(E - E^0)) + (\mu_1(E - E^0) - (\omega + \xi_2)(V - V^0)) \\ &+ \quad (\tau_1(S - S^0)(I_1 - I_1^0) - (\omega + \tau_2 + \psi_1)(I_1 - I_1^0) + \mu_2(E - E^0)) \\ &+ \quad (\tau_2(I_1 - I_1^0) - (\omega + \psi_2)(I_2 - I_2^0)) + (\xi_2(V - V^0) + \psi_2(I_2 - I_2^0) - (\omega + \xi_1)(R - R^0)) \end{aligned}$$

As is evident, for $R_0 < 1 \Rightarrow {}^{C}D_t^{\nu}G \leq 0$
and ${}^{C}D_t^{\nu}G = 0$ only when;
$S = S - S^0, E = E - E^0, V = V - V^0, I_1 = I_1 - I_1^0, I_2 = I_2 - I_2^0, R = R - R^0$.
The existence of global asymptotically stable disease-free equilibrium states E^0 is the conclusion we draw from this.

At the endemic equilibrium E^0, the Lyapunov function defined on (S, E, V, I_1, I_2, R) satisfies G<0.

Theorem 5. The endemic equilibrium E^0 attains global asymptotic stability whenever R_0 is greater than one.

Proof. We analyze a Lyapunov function formulated using the Volterra approach.

$$\begin{aligned} G &= Q_1(S - S^0 - S^0 log\frac{S}{S^0}) + Q_2(E - E^0 - E^0 log\frac{E}{E^0}) + Q_3(V - V^0 - V^0 log\frac{V}{V^0}) \\ &+ Q_4(I_1 - I_1^0 - I_1^0 log\frac{I_1}{I_1^0}) + Q_5(I_2 - I_2^0 - I_2^0 log\frac{I_2}{I_2^0}) + Q_6(R - R^0 - R^0 log\frac{R}{R^0}) \end{aligned}$$

In contrast, the positive constants that were later chosen are $Q_i, i = 1, 2, 3, 4, 5$. Lemma (2) is then used after the equation has been placed in the main system.

$$\begin{aligned} {}^C D_t^\nu G &\leq Q_1(\frac{S - S^0}{S}){}^C D_t^\nu S + Q_2(\frac{E - E^0}{E}){}^C D_t^\nu E + Q_3(\frac{V - V^0}{V}){}^C D_t^\nu V + Q_4(\frac{I_1 - I_1^0}{I_1}){}^C D_t^\nu I_1 \\ &+ Q_5(\frac{I_2 - I_2^0}{I_2}){}^C D_t^\nu I_2 + Q_6(\frac{R - R^0}{R}){}^C D_t^\nu R \end{aligned}$$

In the equation above, the derivative values may be expressed as follows:

$$\begin{aligned} {}^C D_t^\nu G &= Q_1(\frac{S - S^0}{S})(\alpha_1 + \xi_1 R - \tau_1 S I_1 - \omega S - \mu S E) + Q_2(\frac{E - E^0}{E})(\mu S E - (\mu_1 + \mu_2 + \omega)E) \\ &+ Q_3(\frac{V - V^0}{V})(\mu_1 E - (\omega + \xi_2)V) + Q_4(\frac{I_1 - I_1^0}{I_1})(\tau_1 S I_1 - (\omega + \tau_2 + \psi_1)I_1 + \mu_2 E) \\ &+ Q_5(\frac{I_2 - I_2^0}{I_2})(\tau_2 I_1 - (\omega + \psi_2)I_2) + Q_6(\frac{R - R^0}{R})(\xi_2 V + \psi_2 I_2 - (\omega + \xi_1)R) \end{aligned}$$

Replacing $S = S - S^0, E = E - E^0, V = V - V^0, I_1 = I_1 - I_1^0, I_2 = I_2 - I_2^0, R = R - R^0$.

$$\begin{aligned} {}^C D_t^\nu G &= Q_1(\frac{S - S^0}{S})(\alpha_1 + \xi_1(R - R^0) - \tau_1(S - S^0)(I_1 - I_1^0) - \omega(S - S^0) - \mu(S - S^0)(E - E^0)) \\ &+ Q_2(\frac{E - E^0}{E})(\mu(S - S^0)(E - E^0) - (\mu_1 + \mu_2 + \omega)(E - E^0)) \\ &+ Q_3(\frac{V - V^0}{V})(\mu_1(E - E^0) - (\omega + \xi_2)(V - V^0)) \\ &+ Q_4(\frac{I_1 - I_1^0}{I_1})(\tau_1(S - S^0)(I_1 - I_1^0) - (\omega + \tau_2 + \psi_1)(I_1 - I_1^0) + \mu_2(E - E^0)) \\ &+ Q_5(\frac{I_2 - I_2^0}{I_2}) + (\tau_2(I_1 - I_1^0) - (\omega + \psi_2)(I_2 - I_2^0)) \\ &+ Q_6(\frac{R - R^0}{R})(\xi_2(V - V^0) + \psi_2(I_2 - I_2^0) - (\omega + \xi_1)(R - R^0)) \end{aligned}$$

Now let $Q_1 = Q_2 = Q_3 = Q_4 = Q_5 = Q_6 = 1$. The above can be arranged as follows:

$$\begin{aligned} {}^C D_t^\nu G &\leq \alpha_1 - \alpha_1\frac{S^0}{S} + \xi_1 R - \xi_1 R\frac{S^0}{S} - \xi_1 R^0 + \xi_1 R^0\frac{S^0}{S} - \tau_1 I_1\frac{(S - S^0)^2}{S} + \tau_1 I_1^0\frac{(S - S^0)^2}{S} \\ &- \omega\frac{(S - S^0)^2}{S} - \mu E\frac{(S - S^0)^2}{S} + \mu E^0\frac{(S - S^0)^2}{S} + \mu S\frac{(E - E^0)^2}{E} - \mu S^0\frac{(E - E^0)^2}{E} \end{aligned}$$

$$
\begin{aligned}
&- \mu_1\frac{(E-E^0)^2}{E} - \mu_2\frac{(E-E^0)^2}{E} - \omega\frac{(E-E^0)^2}{E} + \mu_1 E - \mu_1 E\frac{V^0}{V} - \mu_1 E^0 + \mu_1 E^0\frac{V^0}{V}\\
&- \omega\frac{(V-V^0)^2}{V} - \xi_2\frac{(V-V^0)^2}{V} + \tau S\frac{(I_1-I_1^0)^2}{I_1} - \tau S^0\frac{(I_1-I_1^0)^2}{I_1} - \omega\frac{(I_1-I_1^0)^2}{I_1} - \tau_2\frac{(I_1-I_1^0)^2}{I_1}\\
&- \psi_1\frac{(I_1-I_1^0)^2}{I_1} + \mu_2 E - \mu_2 E\frac{I_1^0}{I_1} - \mu_2 E^0 + \mu_2 E^0\frac{I_1^0}{I_1} + \tau_2 I_1 - \tau_2 I_1\frac{I_2^0}{I_2} + \tau_2 I_1^0\frac{I_2^0}{I_2} - \tau_2 I_1^0\\
&- \omega\frac{(I_2-I_2^0)^2}{I_2} - \psi_2\frac{(I_2-I_2^0)^2}{I_2} - \xi_2 V - \xi_2 V\frac{R^0}{R} - \xi_2 V^0 + \xi_2 V^0\frac{R^0}{R} + \psi_2 I_2\\
&- \psi_2 I_2\frac{R^0}{R} - \psi_2 I_2^0 + \psi_2 I_2^0\frac{R^0}{R} - \omega\frac{(R-R^0)^2}{R} - \xi_1\frac{(R-R^0)^2}{R}
\end{aligned}
$$

We can write

$$
{}^C D_t^{\nu} G \quad \leq \quad \vartheta_1 - \vartheta_2
$$

$$
\begin{aligned}
\vartheta_1 \quad = \quad & \alpha_1 + \xi_1 R + \xi_1 R^0\frac{S^0}{S} + \tau_1 I_1^0\frac{(S-S^0)^2}{S} + \mu E^0\frac{(S-S^0)^2}{S} + \mu S\frac{(E-E^0)^2}{E}\\
& + \mu_1 E + \mu_1 E^0\frac{V^0}{V} + \tau S\frac{(I_1-I_1^0)^2}{I_1} + \mu_2 E + \mu_2 E^0\frac{I_1^0}{I_1} + \tau_2 I_1\\
& + \tau_2 I_1^0\frac{I_2^0}{I_2} + \xi_2 V^0\frac{R^0}{R} + \psi_2 I_2 + \psi_2 I_2^0\frac{R^0}{R}
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_2 \quad = \quad & \alpha_1\frac{S^0}{S} + \xi_1 R\frac{S^0}{S} + \xi_1 R^0 + \tau_1 I_1\frac{(S-S^0)^2}{S} + \omega\frac{(S-S^0)^2}{S} + \mu E\frac{(S-S^0)^2}{S}\\
& + \mu S^0\frac{(E-E^0)^2}{E} + \mu_1\frac{(E-E^0)^2}{E} + \mu_2\frac{(E-E^0)^2}{E} + \mu_1 E\frac{V^0}{V} + \omega\frac{(E-E^0)^2}{E}\\
& + \mu_1 E^0 + \omega\frac{(V-V^0)^2}{V} + \xi_2\frac{(V-V^0)^2}{V} + \tau S^0\frac{(I_1-I_1^0)^2}{I_1} + \omega\frac{(I_1-I_1^0)^2}{I_1}\\
& + \tau_2\frac{(I_1-I_1^0)^2}{I_1} + \psi_1\frac{(I_1-I_1^0)^2}{I_1} + \mu_2 E^0 + \tau_2 I_1\frac{I_2^0}{I_2} + \tau_2 I_{1^0}\\
& + \omega\frac{(I_2-I_2^0)^2}{I_2} + \psi_2\frac{(I_2-I_2^0)^2}{I_2} + \xi_2 V + \xi_2 V\frac{R^0}{R} + \xi_2 V^0 + \psi_2 I_2\frac{R^0}{R}\\
& + \psi_2 I_2^0 + \omega\frac{(R-R^0)^2}{R} + \xi_1\frac{(R-R^0)^2}{R} + \mu_2 E\frac{I_1^0}{I_1}
\end{aligned}
$$

We examine that if

$$
\vartheta_1 \leq \vartheta_2 \Rightarrow^C D_t^{\nu} G < 0.
$$

Still when $S = S^0, E = E^0, I_1 = I_1^0, I_2 = I_2^0, R = R^0$.
Then;

$$
\vartheta_1 - \vartheta_2 = 0, \Rightarrow^C D_t^{\nu} G = 0.
$$

From this we conclude that;

$$L^0 = S^0, E^0, V^0, I_1^0, I_2^0, R^0$$

This is the biggest set of compact invariant singletons in;

$$\{(S^0, E^0, V^0, I_1^0, I_2^0, R^0)\} \in \Phi :^C D_t^\nu G = 0$$

Consequently if $\vartheta_1 < \vartheta_2$, the locations of endemic equilibrium E^0 exist in the invariant zone and are globally asymptotically stable.

6. Chaos Control

Using a linear feedback regulation technique designed especially for a fractional-order system operating inside a regulated design framework, System (2) is stabilized around its equilibrium points.

$$\begin{cases} {}^C D_t^\nu S = \alpha_1 + \xi_1 R - \tau_1 S I_1 - \omega S - \mu S E - \rho_1 (S - S^0), \\ {}^C D_t^\nu E = \mu S E - (\mu_1 + \mu_2 + \omega) E - \rho_2 (E - E^0), \\ {}^C D_t^\nu V = \mu_1 E - (\omega + \xi_2) V - \rho_3 (V - V^0), \\ {}^C D_t^\nu I_1 = \tau_1 S I_1 - (\omega + \tau_2 + \psi_1) I_1 + \mu_2 E - \rho_4 (I_1 - I_1^0), \\ {}^C D_t^\nu I_2 = \tau_2 I_1 - (\omega + \psi_2) I_2 - \rho_5 (I_2 - I_2^0), \\ {}^C D_t^\nu R = \xi_2 V + \psi_2 I_2 - (\omega + \xi_1) R - \rho_6 (R - R^0). \end{cases}$$

where $S^0, E^0, V^0, I_1^0, I_2^0$ and R^0 indicate the system's point of equilibrium ρ_1, ρ_2, ρ_3, ρ_4, ρ_5 and ρ_6 represent the control parameters. We discover the corresponding Jacobian matrix:

$$\mathbf{J} = \begin{pmatrix} -\omega - \rho_1 & -\mu S & 0 & -\tau_1 S & 0 & \xi_1 \\ 0 & \mu S - (\mu_1 + \mu_2 + \omega) - \rho_2 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & -(\omega + \xi_2) - \rho_3 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & \tau_1 S - (\omega + \tau_2 + \psi_1) - \rho_4 & 0 & 0 \\ 0 & 0 & 0 & \tau_2 & -(\omega + \psi_2) - \rho_5 & 0 \\ 0 & 0 & \xi_2 & 0 & \psi_2 & -(\omega + \xi_1) - \rho_6 \end{pmatrix}$$

Selecting $\phi_1 = 1, \phi_2 = 2, \phi_3 = 3, \phi_4 = 4, \phi_5 = 5, \phi_6 = 6$ and given parameters $\alpha_1 = 2.89$, $\xi_1 = 0.3$, $\tau_1 = 0.23$, $\omega = 0.53$, $\mu = 0.211$, $\mu_1 = 0.20$, $\mu_2 = 0.18$, $\xi_2 = 1.134$, $\tau_2 = 0.34$, $\psi_1 = 0.123$, $\psi_2 = 1.23$ at the equilibrium point E_1, the corresponding characteristic equation is given by:

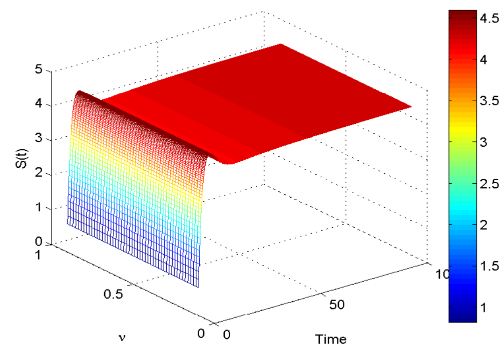
$$-0.43607 - 3.46232\lambda - 9.27597\lambda^2 - 10.4102\lambda^3 - 5.26444\lambda^4 - 1.\lambda^5 = 0$$

$$\{\lambda \to -1.66, \lambda \to -1.4535, \lambda \to -0.43, \lambda \to -0.267436, \lambda \to -1.4535\}$$

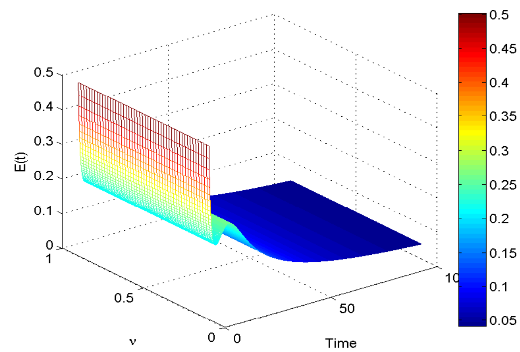
Selecting $\phi_1 = 1, \phi_2 = 2, \phi_3 = 3, \phi_4 = 4, \phi_5 = 5, \phi_6 = 6$ and given parameters $\alpha_1 = 2.89$, $\xi_1 = 0.3$, $\tau_1 = 0.23$, $\omega = 0.53$, $\mu = 0.211$, $\mu_1 = 0.20$, $\mu_2 = 0.18$, $\xi_2 = 1.134$, $\tau_2 = 0.34$, $\psi_1 = 0.123$, $\psi_2 = 1.23$ at the equilibrium point E_2, the corresponding characteristic equation is given by:

$$-0.394533 - 3.27948\lambda - 9.01423\lambda^2 - 10.2532\lambda^3 - 5.23039\lambda^4 - 1.0\lambda^5 = 0$$

$$\{\lambda \to -1.66, \lambda \to -1.4485, \lambda \to -0.43, \lambda \to -1.4485, \lambda \to -0.243391\}$$

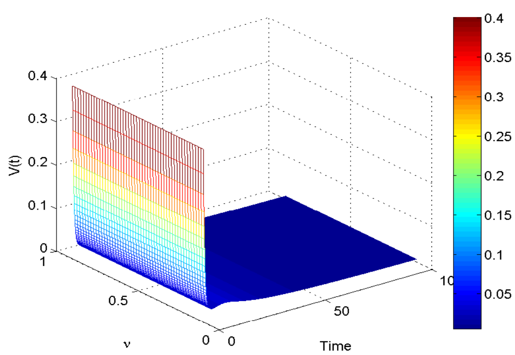


(B) Chaotic Behavior of $S(t)$

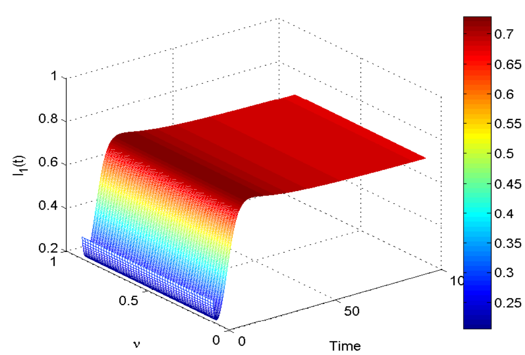


(C) Chaotic Behavior of $E(t)$

FIGURE 5. Graphs for chaos control

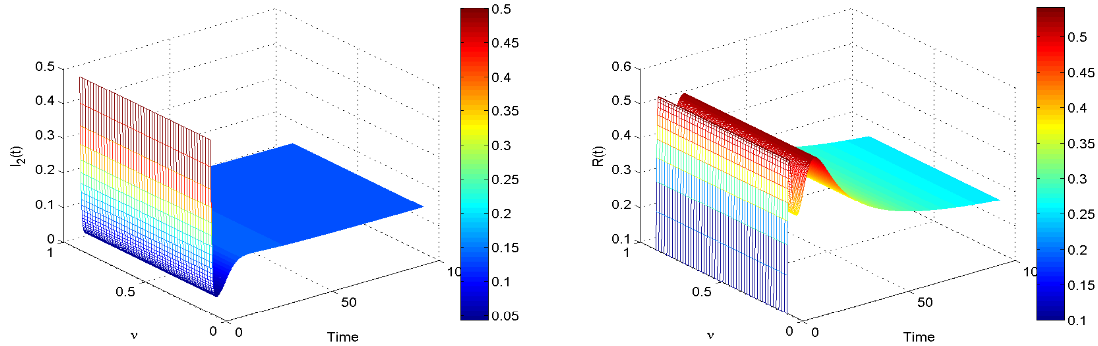


(A) Chaotic Behavior of $V(t)$



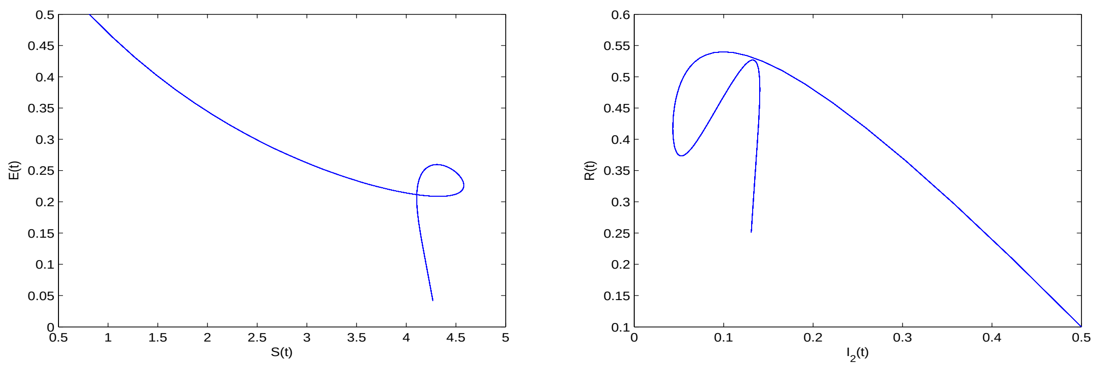
(B) Chaotic Behavior of $I_1(t)$

FIGURE 6. Graphs for chaos control



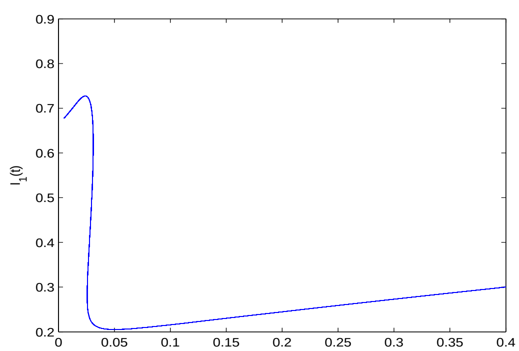
(A) Chaotic Behavior of $I_2(t)$ (B) Chaotic Behavior of $R(t)$

FIGURE 7. Graphs for chaos control



(A) Chaotic Behavior of $SE(t)$ (B) Chaotic Behavior of $I_2R(t)$

FIGURE 8. Graphs for chaos control



(A) Chaotic Behavior of $VI_1(t)$

FIGURE 9. Graphs for chaos control

7. Numerical scheme with Caputo

By adopting the Caputo fractional derivative in place of the standard time derivative and utilizing Newton polynomial interpolation, the fractional power law effect is captured. For system (6), we offer a numerical approach based on Newton polynomials. Forsimplicity:

$$\begin{cases} {}^{C}D_t^{\nu}S(t) = S(t,S), \\ \\ {}^{C}D_t^{\nu}E(t) = E(t,E), \\ \\ {}^{C}D_t^{\nu}V(t) = V(t,V), \\ \\ {}^{C}D_t^{\nu}I_1(t) = I_1(t,I_1), \\ \\ {}^{C}D_t^{\nu}I_2(t) = I_2(t,I_2), \\ \\ {}^{C}D_t^{\nu}R(t) = R(t,R). \end{cases}$$

Where;

$$\begin{aligned} S(t,S) &= \alpha_1 + \xi_1 R - \tau_1 S I_1 - \omega S - \mu S E, \\ E(t,E) &= \mu S E - (\mu_1 + \mu_2 + \omega)E, \\ V(t,V) &= \mu_1 E - (\omega + \xi_2)V, \\ I_1(t,I_1) &= \tau_1 S I_1 - (\omega + \tau_2 + \psi_1)I_1 + \mu_2 E, \\ I_2(t,I_2) &= \tau_2 I_1 - (\omega + \psi_2)I_2, \\ R(t,R) &= \xi_2 V + \psi_2 I_2 - (\omega + \xi_1)R. \end{aligned}$$

Using a fractional integration technique and a kernel based on a power-law function, we obtain:

$$(S)(t_\varsigma + 1) = S(0) + \frac{1}{\Gamma(\hbar)} \sum_{n=2}^{\varsigma} \int_{t_n}^{t_{n+1}} S_1(t,S)\beta^{1-\zeta}(t_{\varsigma+1} - \beta)^{\hbar-1} d\beta,$$

Similarly we can write for $(E)(t_\varsigma+1)$, $(V)(t_\varsigma+1)$, $(I_1)(t_\varsigma+1)$, $(I_2)(t_\varsigma+1)$, $(R)(t_\varsigma+1)$. Remember the Newton polynomial:

$$N(t,V) \simeq N(t_{\varsigma-2}, V_{\varsigma-2}) + \frac{1}{\Delta t}[N(t_{\varsigma-1}, V_{\varsigma-1}) - N(t_{\varsigma-2}, V_{\varsigma-2})](\beta - t_{\varsigma-2})$$

$$+ \frac{1}{2\Delta t^2}[N(t_\varsigma, V_\varsigma) - 2N(t_{\varsigma-1}, V_{\varsigma-1}) + N(t_{\varsigma-2}, V_{\varsigma-2})] \times (\beta - t_{\varsigma-2})(\beta - t_{\varsigma-1}).$$

Replacing the above formulae;

$$(S)(t_\varsigma + 1) = S(0) + \frac{1}{\Gamma(\hbar)} \sum_{n=2}^{\hbar} S_1(t_{(n-2)}, S^{(n-2)}) t_{n-2}^{1-\zeta} \int_{t_n}^{t_{n+1}} (t_{\varsigma+1} - \beta)^{\hbar-1} d\beta$$

$$
\begin{aligned}
&+ \quad \frac{1}{\Gamma(\hbar)} \sum_{n=2}^{\varsigma} \frac{1}{\Delta t}[t_{n-1}^{1-\zeta} S_1(t_{n-1}, S^{n-1}) - t_{n-2}^{1-\zeta} S_1(t_{n-2}, S^{n-2})] \times \int_{t_n}^{t_{n-1}} (\beta - t_{n-2})(t_{\varsigma+1} - \beta)^{\hbar-1} d\beta \\
&+ \quad \frac{1}{\Gamma(\hbar)} \sum_{n=2}^{\varsigma} \frac{1}{2\Delta t^2}[t_n^{1-\zeta} S_1(t_n, S^n) - 2t_{n-1}^{1-\zeta} S_1(t_{n-1}, S^{n-1}) \\
&+ \quad t_{n-2}^{1-\zeta} S_1(t_{n-2}, S^{n-2})] \int_{t_n}^{t_{n+1}} (\beta - t_{n-2})(\beta - t_{n-1})(t_{\varsigma+1} - \beta)^{\hbar-1} d\beta,
\end{aligned}
$$

Similarly we can find for $(E)(t_\varsigma + 1)$, $(V)(t_\varsigma + 1)$, $(I_1)(t_\varsigma + 1)$, $(I_2)(t_\varsigma + 1)$, $(R)(t_\varsigma + 1)$.
Calculations of the integral and get the equations as.

$$
\begin{aligned}
\int_{t_n}^{t_{n+1}} (t_{\varsigma+1} - \beta)^{\hbar-1} d\beta &= \frac{(\Delta t)^{\hbar}}{\hbar}(\varsigma - n + 1)^{\hbar} - (\varsigma - n)^{\hbar}, \\
\int_{t_n}^{t_{n+1}} (\beta - t_{n-2})(t_{\varsigma+1} - \beta)^{\hbar-1} d\beta &= \frac{(\Delta t)^{\hbar+1}}{\hbar(\hbar+1)}[(\varsigma - n + 1)^{\hbar}(\varsigma - n + 3 + 2\hbar) \\
&- (\varsigma - n)^{\hbar}(\varsigma - n + 3 + 3\hbar)], \\
\int_{t_n}^{t_{n+1}} (\beta - t_{n-2})(\beta - t_{n-1})(t_{\varsigma+1} - \beta)^{\hbar-1} d\beta &= \frac{(\Delta t)^{\hbar+2}}{\hbar(\hbar+1)(\hbar+2)}(\varsigma - n + 1)^{\hbar}(2(\varsigma - n)^2) \\
&+ (3\hbar + 10)(\varsigma - n) + (2\hbar^2 + 9\hbar + 12) \\
&- (\varsigma - n)^{\hbar} \times (2(\varsigma - n)^2 + (5\hbar + 10)(\varsigma - n) \qquad (7.\text{-}18) \\
&+ (6\hbar^2 + 18\hbar + 12)).
\end{aligned}
$$

Thus, we ultimately obtain

$$
\begin{aligned}
(S)(t_\varsigma + 1) &= S(0) + \frac{(\Delta t)^{\hbar}}{\Gamma(\hbar+1)} \sum_{n=2}^{\varsigma} t_{n-2}^{1-\zeta} S_1(t_{n-2}, S^{n-2}) F_1 \\
&+ \frac{(\Delta t)^{\hbar}}{\Gamma(\hbar+2)} \sum_{n=2}^{\varsigma} t_{n-1}^{1-\zeta} S_1(t_{n-1}, S^{n-1}) - t_{n-2}^{1-\zeta} S_1(t_{n-2}, S^{n-2}) F_2 \\
&+ \frac{(\Delta t)^{\hbar}}{\Gamma(\hbar+3)} \sum_{n=2}^{\varsigma} t_n^{1-\zeta} S_1(t_n, S^n) - 2t_{n-1}^{1-\zeta} S_1(t_{n-1}, S^{n-1}) + t_{n-2}^{1-\zeta} S_1(t_{n-2}, S^{n-2}) F_3,
\end{aligned}
$$

Similarly we can find for $(E)(t_\varsigma + 1)$, $(V)(t_\varsigma + 1)$, $(I_1)(t_\varsigma + 1)$, $(I_2)(t_\varsigma + 1)$, $(R)(t_\varsigma + 1)$.
where

$$
\begin{aligned}
F_1 &= (\varsigma - n + 1)^{\hbar} - (\varsigma - n)^{\hbar}, \\
F_2 &= (\varsigma - n + 1)^{\hbar}(\varsigma - n + 3 + 2\hbar) - (\varsigma - n)^{\hbar}(\varsigma - n + 3 + 3\hbar), \\
F_3 &= (\varsigma - n + 1)^{\hbar}(2(\varsigma - n)^2) + (3\hbar + 10)(\varsigma - n) + (2\hbar^2 + 9\hbar + 12) \\
&- (\varsigma - n)^{\hbar} \times (2(\varsigma - n)^2 + (5\hbar + 10)(\varsigma - n) + (6\hbar^2 + 18\hbar + 12)).
\end{aligned}
$$

7.1. **Stability of Numerical Scheme.** To analyze the stability of the numerical scheme property defined by the controlled behavior of small perturbations in the solution of $SEVI_1I_2R$ based model is expressed in the following general form;

$$^{C}D_t^{\nu}U(t) = U(t, \kappa(t)), \in \hbar(0,1), t \in [0,T]$$

where $^{C}D_t^{\nu}$ denotes the $SEVI_1I_2R$ fractional operator. Let U_{p+1} and $\bar{U}_{p+1}$ represent two numerical approximations obtained from identical initial data, $U_0 = \bar{U}_0$ evolving from the same initial value. If there exists a positive constant P_ν such that

$$| U_{p+1} - \bar{U}_{p+1} | \leq \bar{\Theta}_{\hbar,T} \| U_{p+1} - \bar{U}_{p+1} \|_\infty$$

This demonstrates that the proposed approach is stable.

Lemma 3. The weighting method used in this study is stated as follows;

$$\sum_{n=0}^{p+1} | j_n^{p+1} | \leq \bar{\Theta}_\hbar T^\hbar$$

where $\bar{\Theta}_\hbar$ is solely dependent upon $\hbar$.

t	**S**	**E**	**V**	I_1	I_2	**R**
0	0.83	0.7	0.5	0.55	0.2	0.25
5	3.91516972	0.01212008	0.000129707	0.089231161	0.017866804	0.018216112
10	4.07420081	0.010704961	0.001839337	0.081467487	0.01610602	0.016419457
15	4.129609725	0.009962081	0.001881952	0.078131586	0.015302338	0.015563847
20	4.153347876	0.009445353	0.00180113	0.076125757	0.014850969	0.015080541
25	4.165140107	0.009034818	0.001721934	0.074653228	0.014538222	0.014746293
30	4.171844431	0.008685907	0.001653251	0.0734391	0.014289793	0.014482
35	4.176182089	0.008378347	0.001593062	0.072371237	0.014075958	0.014255683
40	4.179333238	0.008101391	0.001539242	0.071397919	0.013883364	0.014052833
45	4.181845173	0.007848593	0.001490357	0.070492646	0.013705389	0.013866174
50	4.183986121	0.007615693	0.00144546	0.069640447	0.013538433	0.013691706

TABLE 3. Numerical results for variables S,E,V,I_1,I_2 and R over time t at $\nu = 1$

t	**S**	**E**	**V**	I_1	I_2	**R**
0	0.83	0.7	0.5	0.55	0.2	0.25
5	3.820534291	0.033944302	0.005330787	0.108969411	0.021674696	0.022302316
10	3.995447452	0.029218988	0.005618122	0.101868766	0.02002735	0.020513565
15	4.059575804	0.02664315	0.005182938	0.099401608	0.019365681	0.01976635
20	4.087979687	0.024829917	0.004812411	0.098179578	0.019061078	0.019411817
25	4.102387081	0.023390054	0.004517407	0.097339356	0.018875648	0.019193007
30	4.110688565	0.022173607	0.00427264	0.096604653	0.018728557	0.019021237
35	4.116126216	0.021110174	0.004061711	0.095880969	0.018590717	0.018863886
40	4.120142707	0.020161388	0.00387521	0.095138033	0.018451742	0.018708783
45	4.123417012	0.01930372	0.003707516	0.094369508	0.018308449	0.018551749
50	4.126282167	0.018521337	0.003555022	0.093577828	0.018160471	0.018391812

TABLE 4. Numerical results for variables S,E,V,I_1,I_2 and R over time t at$\nu = 0.98$

t	**S**	**E**	**V**	I_1	I_2	**R**
0	0.83	0.7	0.5	0.55	0.2	0.2
5	3.727339372	0.054603494	0.010915085	0.126934226	0.025249637	0.025249637
10	3.920566645	0.046033595	0.009278636	0.11926141	0.023446693	0.023446693
15	3.996147539	0.041311817	0.008201909	0.116690458	0.022727845	0.022727845
20	4.031552642	0.038011478	0.007470158	0.115469401	0.022409473	0.022409473
25	4.050445612	0.035423596	0.00692067	0.114619999	0.022220181	0.022220181
30	4.061820073	0.033266334	0.006476653	0.113831796	0.022065686	0.022065686
35	4.069524208	0.031404125	0.006100686	0.113005077	0.021912649	0.021912649
40	4.075334689	0.029761863	0.005772901	0.112114639	0.021750182	0.021750182
45	4.080116576	0.028293055	0.005481722	0.111162216	0.02157601	0.02157601
50	4.084307881	0.026966317	0.005219783	0.11015845	0.021391101	0.021391101

TABLE 5. Numerical results for variables S,E,V,I_1,I_2 and R over time t at $\nu = 0.96$

t	**S**	**E**	**V**	I_1	I_2	**R**
0	0.83	0.83	0.5	0.5	0.55	0.2
5	3.634340361	3.634340361	0.017061586	0.017061586	0.143581524	0.028693373
10	3.847071121	3.847071121	0.012996586	0.012996586	0.134525336	0.026523254
15	3.935884704	3.935884704	0.011103641	0.011103641	0.131266928	0.025617066
20	3.979851094	3.979851094	0.009944872	0.009944872	0.129631191	0.025191924
25	4.004424945	4.004424945	0.00911397	0.00911397	0.128478561	0.024932527
30	4.019734889	4.019734889	0.008460221	0.008460221	0.127431268	0.024723869
35	4.030303743	4.030303743	0.007916732	0.007916732	0.126360858	0.024523144
40	4.038307266	4.038307266	0.007449432	0.007449432	0.125230417	0.024315233
45	4.044848938	4.044848938	0.007038904	0.007038904	0.124037683	0.024096105
50	4.050511295	4.050511295	0.006672993	0.006672993	0.122792726	0.023866131

TABLE 6. Numerical results for variables S,E,V,I_1,I_2 and R over time t at$\nu = 0.90$

Tables 3 to 6 represents the numerical results of developed solutions for the newly developed model regarding Toxoplasma gondii between the cat population and the environment. Its observed that both Mildly and seriously infected start reducing with the passage of time by lowering the fractional values due to vaccination and early identification.

Theorem 6. Assume that the solutions to scheme denoted by U_n $(n = 1, 2, ..., P)$. Subse-quently, the proposed scheme is stable.

Proof. Assume that U_0 and $\bar{U}_{p+1}$ are the equation's initial conditions, and that U_{p+1} and $\bar{U}_{p+1}$ are the numerical answers to the previous equation. We're going to use mathematical induction. Suppose that

$$| U_n - \bar{U}_n | \leq \bar{\Theta}_{\hbar,T} \| U_0 - \bar{U}_0 \|_\infty$$

concerns $(n = 0, 1, 2, ..., p)$. We must show that for n = p + 1, this is the case. Keep be mind that the above initial circumstances assume the induction basis (n = 0).

$$\begin{aligned} \mid U_n - \bar{U}_n \mid &\leq \sum_{q=0}^{\hbar-1} \frac{t^q_{(p+1)}}{q!} \mid U_0^q - \bar{U}_0^q \mid + \frac{1}{\Gamma\hbar}[\sum_{n=0}^{p} \mid j_n^{p+1} \| g(t_n, U_n), g(t_n, \bar{U}_n) \mid \\ &+ \mid j_{p+1}^{p+1} \| g(t_{p+1}, U_{p+1}) - g(t_{p+1}, \bar{U}_{p+1}) \mid] \\ &\leq \bar{\Theta} \| U_0 - \bar{U}_0 \|_\infty + \frac{\mid \kappa \mid}{\Gamma(\hbar)}[\sum_{n=0}^{p} \mid j_n^{p+1} \| U_n - \bar{U}_0 \mid + \mid j_{p+1}^{p+1} \| U_{p+1} - \bar{U}_{p+1} \mid] \end{aligned}$$

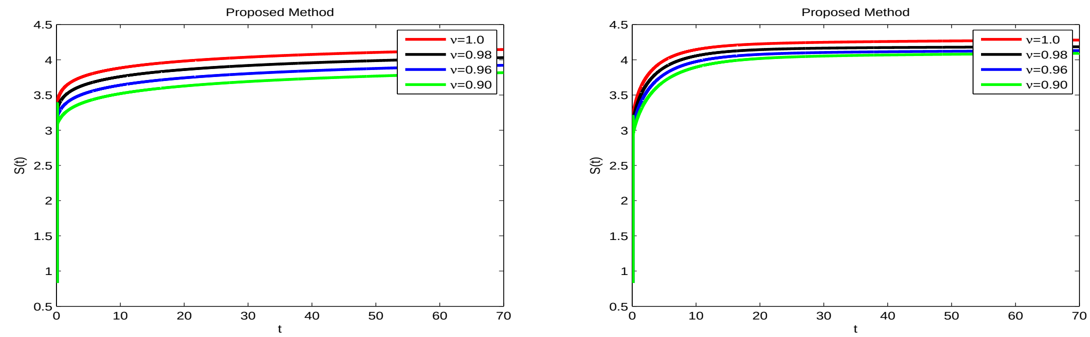
From Lemma (3) one can find

$$\begin{aligned} \mid U_{p+1} - \bar{U}_{p+1} \mid &\leq \bar{\Theta} \| U_0 - \bar{U}_0 \|_\infty + \frac{\mid \kappa \mid}{\Gamma(\hbar)}[\bar{\Theta}_{1,\hbar} T^{\hbar} \max_{0 \leq n \leq p} \mid U_n - \bar{U}_0 \mid + \bar{\Theta}_{p+1,\hbar} T^{\hbar} \mid U_{p+1} - \bar{U}_{p+1} \mid] \\ &\leq \frac{1}{1 - \frac{(|\kappa| \bar{\Theta}_{p+1,\hbar} T^{\hbar}}{\Gamma(\hbar)}}[\bar{\Theta} \| U_0 - \bar{U}_0 \|_\infty + \frac{\mid \kappa \mid}{\Gamma(\hbar)} + \bar{\Theta}_{1,\hbar} \max_{0 \leq n \leq p} T^{\hbar} \mid U_n - \bar{U}_n \mid] \end{aligned}$$

It is now possible to conclude the proof for sufficiently U by using mathematical induction and a sufficiently large constant $\bar{\Theta}_{\hbar,T}$.

8. Graphical Representation

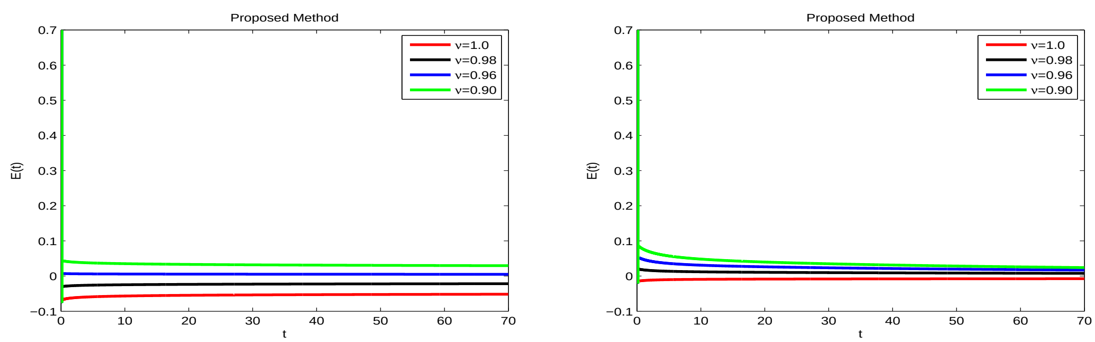
The ecology of Toxoplasma gondii between the natural habitat and cat hosts studied by applying the mathematical model that was proposed, which includes vaccination and sanitation interventions. The theoretical results were also confirmed by numerical modeling based on MATLAB. It was possible to simulate this in detail with these simulations. Simulation of the dynamics between the susceptible, exposed, and infected, vaccinated cats and the environmental oocyst compartments possessed by different parameter settings. Figures 10-15 show how the time changes of the model compartments vary with time. In particular, the impact of the vaccination and sanitation of the vulnerable, exposed, and infected populations of cats are exhibited, the environmental oocyst load. These findings indicate that the higher vaccination rates, the greater decrease in the susceptible and infected cats with time, and the more oocysts are removed with enhanced sanitation efforts environment therefore, decreasing the total transmission risk. Other issues that the simulations bring into focus are the sensitivity of the system to the key parameters, such as the effective contact rate, the rate of oocyst shedding, cat recruitment, vaccination, and environmental sanitation rate. In the favorable regimes of the parameters, all sub-compartments converge slowly, which is evidence of a possible change to graphical evidence supports that incorporating vaccination and frequent environmental sanitation produces better results. A synergistic effect which will effectively reduce T. gondii concentration in cat populations and reduce environmental contamination to a minimum. In general, the graphical representation provides visual confirmation and validation of the theoretical predictions and proves the critical role of the conceptual measures. Through compartmental trajectories a policymaker and a veterinarian can determine the best levels of intervention to control the spread of T. gondii in domestic and feral cats, thus limiting the risk of zoonotic transmission infection to other animals and humans.



(A) By taking 0.3 as a dimension

(B) By taking 0.7 as a dimension

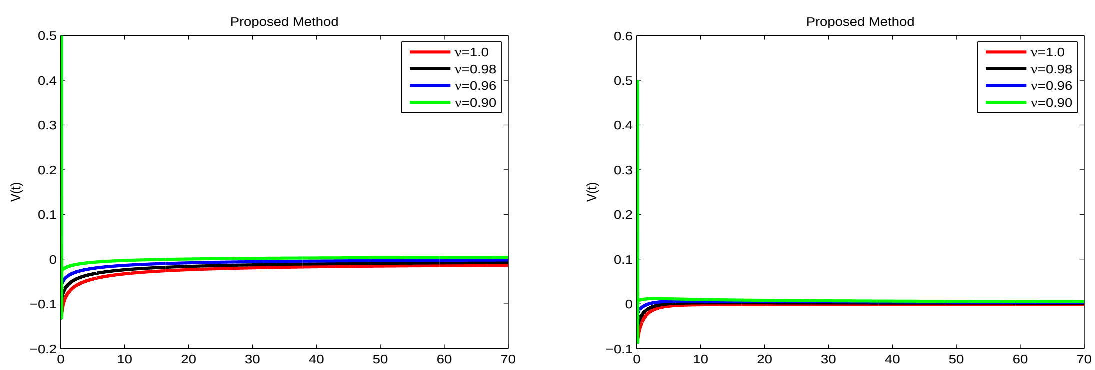
Figure 10. $S(t)$ under different dimensions



(A) By taking 0.3 as a dimension

(B) By taking 0.7 as a dimension

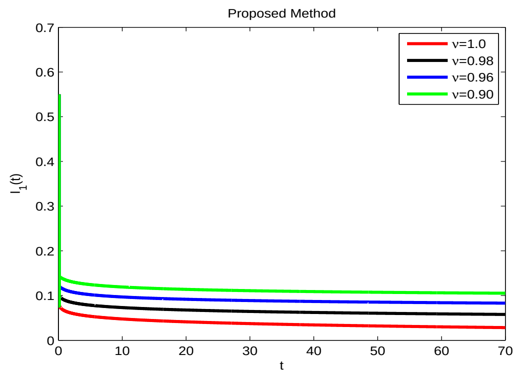
Figure 11. $E(t)$ under different dimensions



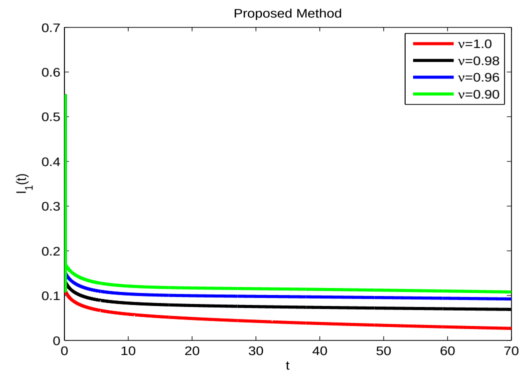
(A) By taking 0.3 as a dimension

(B) By taking 0.7 as a dimension

Figure 12. $V(t)$ under different dimensions

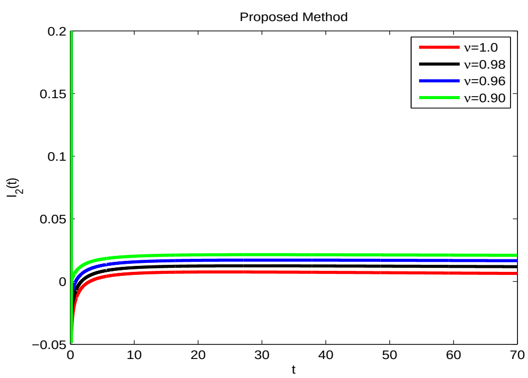


(A) By taking 0.3 as a dimension

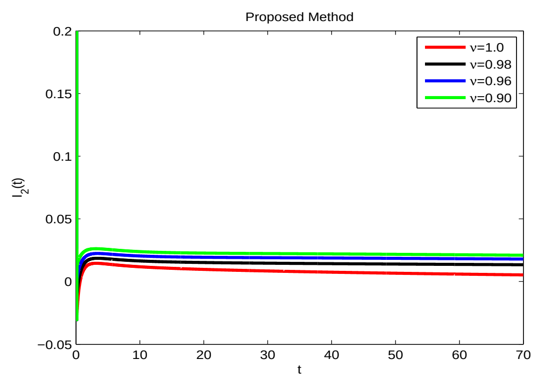


(B) By taking 0.7 as a dimension

FIGURE 13. $I_1(t)$ under different dimensions

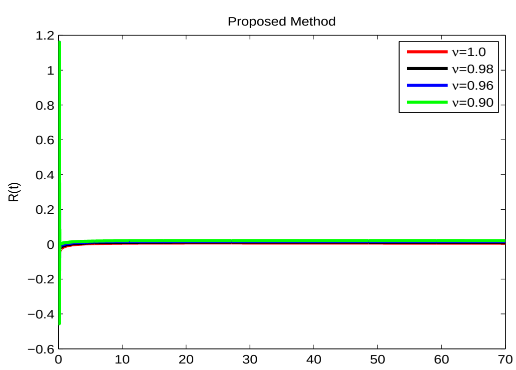


(A) By taking 0.3 as a dimension

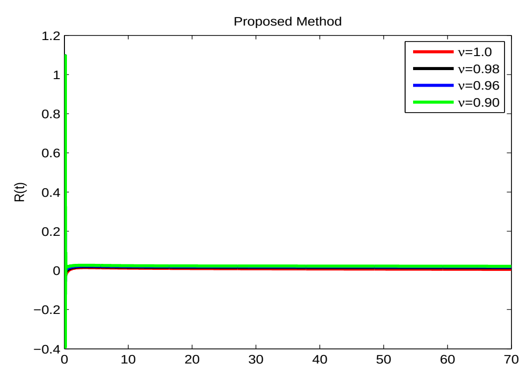


(B) By taking 0.7 as a dimension

FIGURE 14. $I_2(t)$ under different dimensions



(A) By taking 0.3 as a dimension



(B) By taking 0.7 as a dimension

FIGURE 15. $R(t)$ under different dimensions

9. Conclusion

This paper has presented a mathematical model of the dynamics of the transmission of Toxoplasma gondii between the cat population and the environment by including a separate vaccinated group and a sanitation factor in the environmental model. By using this framework, an in-depth study of the joint effect of vaccination and environmental hygiene measures on the propagation of T. gondii was possible. The model was numerically simulated, showing some complex dynamic behaviors such as oscillatory and chaotic behavior in the cat and environmental compartments of the model in some parameter regimes. The findings suggest that the moderate level of vaccination and sanitation can stabilize the system and decrease the prevalence of infections, but the lack of intervention can cause irregular changes in the density of oocysts and the population of infected cats, which may result in chaotic outbreaks. The implications of these findings are that uniform and powerful enough control measures should be applied to achieve stability and prevent sudden outbreaks of infection. In general, the paper proves that systematic measures, which should include routine cat vaccination along with strict environmental decontamination measures, would be crucial in decreasing the number of T. gondii and reducing environmental pollution. The model has given useful information by considering both stable and chaotic dynamics with an aim of designing effective intervention programs that can help to control toxoplasmosis among cats and reduce the chances of spilling the toxic to other vulnerable hosts. Findings demonstrate the importance of combined management efforts in the attainment of the long-term control of transmission of T. gondii.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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