Int. J. Anal. Appl. (2026), 24:139

International Journal of Analysis and Applications

Characterizations of Semisimple Semigroups by the Properties of Their Spherical Fuzzy Interior Ideals

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Abstract. The concept of spherical fuzzy sets was introduced by Ashraf et al. (2019) as a generalization of intuitionistic fuzzy sets, picture fuzzy sets, and Pythagorean fuzzy sets. Subsequently, in 2022, V. Chinnadurai et al. investigated spherical fuzzy ideals in semigroups. In this paper, we introduce the notion of spherical fuzzy interior ideals in semigroups and explore several of their fundamental properties. Furthermore, we provide a characterization of semisimple semigroups in terms of spherical fuzzy interior ideals.

1. Introduction

In real-world scenarios, many problems are characterized by uncertainty, imprecision, and incompleteness. To handle such complexities, Zadeh [11] proposed the concept of fuzzy set theory in 1965, which has since become a powerful and flexible mathematical framework for modeling and analyzing uncertain information.

In the early development of fuzzy algebraic structures, Kuroki [6] investigated fuzzy subsemigroups and several types of fuzzy ideals in semigroups, thus extending the classical semigroup theory into the fuzzy environment.

Received: Mar. 1, 2026.

2020 *Mathematics Subject Classification.* 20M10, 20N99.

Key words and phrases. regular; intra-regular; semisimple; spherical fuzzy interior ideals.

https://doi.org/10.28924/2291-8639-24-2026-139

ISSN: 2291-8639

As the study of fuzzy set extensions progressed, Yager [10] introduced the concept of Pythagorean fuzzy sets (PFSs), a generalization of intuitionistic fuzzy sets—in which each element is characterized by a pair of degrees $(\eta(x), \vartheta(x))$ satisfying

$$0 \leq (\eta(x))^2 + (\vartheta(x))^2 \leq 1.$$

This formulation provides a broader domain for representing uncertainty and inspired further generalizations.

Motivated by this idea, Gündoğdu *et al.* [4] introduced the concept of spherical fuzzy sets (SFSs) and examined their fundamental properties. The spherical fuzzy set framework generalizes intuitionistic, Pythagorean, and picture fuzzy sets by introducing a third parameter—the degree of refusal subject to the condition

$$0 \leq (\mu(x))^2 + (\nu(x))^2 + (\pi(x))^2 \leq 1,$$

where $\mu(x)$, $\nu(x)$, and $\pi(x)$ represent the degrees of membership, nonmembership, and refusal, respectively. This additional flexibility enables a more precise and balanced representation of uncertainty.

Subsequently, several researchers have extended spherical fuzzy sets to diverse algebraic structures. For instance, Veerappan and Venkatesan [9] studied spherical interval-valued fuzzy bi-ideals in Γ-near-rings, while Krailoet et al. [5] explored the interplay between spherical fuzzy sets and rough sets in ternary semigroups. More recently, in 2023, Nakkhasen and Chinram [8] investigated spherical fuzzy bi-ideals in ternary semigroups, and in the same year, Gaketem and Khamrot [3] introduced and examined spherical interval-valued fuzzy ideals in semigroups.

Inspired by these developments, the present paper introduces the notion of spherical fuzzy interior ideals in semigroups and investigates their algebraic properties. We also establish necessary and sufficient conditions for the equivalence between spherical fuzzy ideals and spherical fuzzy interior ideals in semigroups. Furthermore, we provide a characterization of semisimple semigroups in terms of spherical fuzzy interior ideals.

2. Preliminaries

To make this work self-sufficient, we briefly introduce a few definitions engaged in the remaining work.

A *subsemigroup* (SSG) of a semigroup $\mathfrak{G}$ is a non-empty set $\mathfrak{K}$ of $\mathfrak{G}$ such that $\mathfrak{K}^2 \subseteq \mathfrak{K}$. A *left (right) ideal* (LID [RID]) of a semigroup $\mathfrak{G}$ is a non-empty set $\mathfrak{K}$ of $\mathfrak{G}$ such that $\mathfrak{G}\mathfrak{K} \subseteq \mathfrak{K}$ ($\mathfrak{K}\mathfrak{G} \subseteq \mathfrak{K}$). By an *ideal* (ID) of a semigroup $\mathfrak{S}$, we mean a non-empty set of $\mathfrak{S}$ which is both a LID and a RID of $\mathfrak{G}$. A SSG $\mathfrak{K}$ of a semigroup $\mathfrak{G}$ is called a *interior ideal* (IID) of $\mathfrak{G}$ if $\mathfrak{G}\mathfrak{K}\mathfrak{G} \subseteq \mathfrak{K}$. A semigroup $\mathfrak{G}$ is said to be *regular* if for each element $u \in \mathfrak{G}$, there exists an element $x \in \mathfrak{G}$ such that $u = uxu$. A semigroup $\mathfrak{G}$ is called *intra-regular* if for every $u \in \mathfrak{G}$ there exist $x, y \in \mathfrak{G}$ such that $u = xu^2y$. A semigroup $\mathfrak{G}$ is said to be *left* (right) regular if for each element $u \in \mathfrak{S}$, there exists an element $x \in \mathfrak{G}$ such that $u = xu^2 (u = u^2x)$. A semigroup $\mathfrak{G}$ is called *semisimple* if for every $u \in \mathfrak{G}$, there exist $x, y, z \in \mathfrak{G}$ such that $u = xuyuz$. A semigroup $\mathfrak{G}$ is called *weakly regular* if for every $u \in \mathfrak{G}$ there exist $x, y \in \mathfrak{G}$ such that $u = uxuy$. A semigroup $\mathfrak{G}$ is a *left (right) quasi-regular* if for every $u \in \mathfrak{G}$, there exist $x, y \in \mathfrak{G}$ such that $u = xuyu$ ($u = uxuy$).

For any $\eta_i \in [0,1]$ where $i \in \mathcal{J}$ define

$$\underset{i\in\mathcal{J}}{\vee}\eta_i := \sup_{i\in\mathcal{J}}\{\eta_i\} \quad \text{and} \quad \underset{i\in\mathcal{J}}{\wedge}\eta_i := \inf_{i\in\mathcal{J}}\{\eta_i\}.$$

We see that for any $\eta_1, \eta_2 \in [0,1]$, we have

$$\eta_1 \vee \eta_2 = \max\{\eta_1, \eta_2\} \quad \text{and} \quad \eta_1 \wedge \eta_2 = \min\{\eta_1, \eta_2\}.$$

Definition 2.1. *[11] A fuzzy subset (fuzzy set) $\acute{\eta}$ of a non-empty set $\mathfrak{T}$ is a function from $\mathfrak{T}$ into the closed interval $[0,1]$, i.e, $\acute{\eta} : \mathfrak{T} \to [0,1]$.*

Definition 2.2. *[10] Let $\mathfrak{T}$ be a non-empty set. A* Pythagorean fuzzy set *(PFS) $P := \{u, \eta(u), \vartheta(u) \mid u \in \mathfrak{G}\}$ where $\eta : \mathfrak{T} \to [0,1]$ and $\vartheta : \mathfrak{T} \to [0,1]$ represent the degree of membership and non-membership of the object $z \in \mathfrak{T}$ to the set P subset to the condition $0 \le (\eta(u))^2 + (\vartheta(u))^2 \le 1$ for all $u \in \mathfrak{T}$. For the sake of simplicity, a PFS is denoted as $P = (\eta(u); \vartheta(u))$.*

Definition 2.3. *[4] Let $\mathfrak{T}$ be a non-empty set. A* spherical fuzzy set *(SF-set) $\mathrm{SP} := \{u, \eta(u), \vartheta(u), \omega(u) \mid u \in \mathfrak{G}\}$ where $\eta : \mathfrak{T} \to [0,1]$, $\vartheta : \mathfrak{T} \to [0,1]$ and $\omega : \mathfrak{T} \to [0,1]$ represent the degree of membership, non-membership and hesitancy of the object $u \in \mathfrak{T}$ to the set $\mathrm{SP} = (\eta; \vartheta, \omega)$ subset to the condition $0 \le (\eta(u))^2 + (\vartheta(u))^2 + (\omega(u))^2 \le 1$ for all $u \in \mathfrak{T}$. For the sake of simplicity, an $\mathrm{SP} = (\eta(u); \vartheta(u), \omega(u))$ is denoted as $\mathrm{SP} = (\eta; \vartheta, \omega)$.*

Definition 2.4. *Let $\mathfrak{R}$ be a subset of a non-empty set $\mathfrak{T}$. A spherical characteristic function $\chi_{\mathfrak{R}}$ of $\mathfrak{R}$ is defined to be a function $\chi_{\mathfrak{R}} : \mathfrak{R} \to [0,1]$ by*

$$\eta_{\chi_{\mathfrak{R}}}(u) := \begin{cases} 1 & u \in \mathfrak{R}, \\ 0 & u \notin \mathfrak{R}, \end{cases}$$

and

$$\vartheta_{\chi_{\mathfrak{R}}}(u) := \begin{cases} 1 & u \in \mathfrak{R}, \\ 0 & u \notin \mathfrak{R}, \end{cases}$$

and

$$\omega_{\chi_{\mathfrak{R}}}(u) := \begin{cases} 0 & u \in \mathfrak{R}, \\ 1 & u \notin \mathfrak{R}, \end{cases}$$

for all $u \in \mathfrak{R}$.

Remark 2.1. *To simplify matters, we will employ the symbol $\chi_{\mathfrak{R}} = (\eta_{\chi_{\mathfrak{R}}}, \vartheta_{\chi_{\mathfrak{R}}}, \omega_{\chi_{\mathfrak{R}}})$ for the SF-set $\chi_{\mathfrak{R}} := \{(u, \eta_{\chi_{\mathfrak{R}}}(u), \vartheta_{\chi_{\mathfrak{R}}}(u), \omega_{\chi_{\mathfrak{R}}}(u),) \mid u \in \mathfrak{T}\}$.*

Definition 2.5. *Let $\mathrm{SP}_1 = (\eta, \vartheta, \omega)$ and $\mathrm{SP}_2 = (\tau, \nu, \alpha)$ be two SF-set of a semigroup $\mathfrak{G}$. Then the product $\mathrm{SP}_1 \,\bar{\circ}\, \mathrm{SP}_2$ is defined by*

$$\mathrm{SP}_1 \circ \mathrm{SP}_2 := \{u, (\eta \circ \tau)(u), (\vartheta \circ \nu)(u), (\omega \circ \alpha)(u) : u \in \mathfrak{G}\},$$

where

$$(\eta\circ\tau)(u)=\begin{cases}\bigvee_{(x,y)\in F_u}\{\eta(x)\wedge\tau(y)\} & \text{if } F_u\neq\emptyset,\\ 0 & \text{if } F_u=\emptyset,\end{cases}$$

$$(\vartheta\circ\nu)(u)=\begin{cases}\bigvee_{(x,y)\in F_u}\{\vartheta(x)\wedge\nu(y)\} & \text{if } F_u\neq\emptyset,\\ 0 & \text{if } F_u=\emptyset,\end{cases}$$

$$(\omega\circ\alpha)(u)=\begin{cases}\bigwedge_{(x,y)\in F_u}\{\omega(x)\vee\alpha(y)\} & \text{if } F_u\neq\emptyset,\\ 1 & \text{if } F_u=\emptyset,\end{cases}$$

for all $u\in\mathfrak{G}$.

Definition 2.6. *An SF-set* SP $=(\eta;\vartheta,\omega)$ *of a semigroup* $\mathfrak{G}$ *is called*

(1) *A* spherical fuzzy subsemigroup *(SFSSG) if* $\eta(uv)\geq\eta(u)\wedge\eta(v)$, $\vartheta(uv)\geq\vartheta(u)\wedge\vartheta(v)$ *and* $\omega(uv)\leq\omega(u)\vee\omega(v)$ *for all* $u,v\in\mathfrak{G}$.
(2) *A* spherical fuzzy left *(SFL) if* $\eta(uv)\geq\eta(v)$, $\vartheta(uv)\geq\vartheta(v)$ *and* $\omega(uv)\leq\omega(v)$ *for all* $u,v\in\mathfrak{G}$.
(3) *A* spherical fuzzy right *(SFR) if* $\eta(uv)\geq\eta(u)$, $\vartheta(uv)\geq\vartheta(u)$ *and* $\omega(uv)\leq\omega(u)$ *for all* $u,v\in\mathfrak{G}$.
(4) *A* spherical interval valued fuzzy ideal *(SFI) if it is both an SFL and SFR of* $\mathfrak{G}$.
(5) *A* spherical interval valued fuzzy bi-ideal *(SFBI) if* $\eta(uvw)\geq\eta(u)\wedge\eta(w)$, $\vartheta(uvw)\geq\vartheta(u)\wedge\vartheta(w)$ *and* $\omega(uvw)\leq\omega(u)\vee\omega(w)$ *for all* $u,v,w\in\mathfrak{G}$.

It is claerly every SFI of a semigroup $\mathfrak{G}$ is SFSSG of $\mathfrak{G}$ and every SFI of a semigroup $\mathfrak{G}$ is SFBI of $\mathfrak{G}$.

3. Spherical Fuzzy Interior Ideals in Semigroups

In this section, we will study concepts of spherical complex fuzzy sets in a semigroup, and we will study properties of those.

Definition 3.1. *An SFSSG* SP $=(\eta;\vartheta,\omega)$ *of a semigroup* $\mathfrak{G}$ *is called a* spherical fuzzy interior ideal *(SFII) if* $\eta(uvw)\geq\eta(v)$, $\vartheta(uvw)\geq\vartheta(v)$ *and* $\omega(uvw)\leq\omega(v)$ *for all* $u,v,w\in\mathfrak{G}$.

Example 3.1. *Let* $\mathfrak{G}=\{\Psi,\Omega,\Upsilon,\Pi\}$ *be semigroup with the following Cayley table:*

$\cdot$	Ψ	Ω	Υ	Π
Ψ	Ψ	Ψ	Ψ	Ψ
Ω	Ψ	Ψ	Ψ	Ψ
Υ	Ψ	Ψ	Ω	Ψ
Π	Ψ	Ψ	Ψ	Ω

Define SF-set $\eta:\mathfrak{G}\to[0,1]$ *by* $\eta(\Psi)=0.5$, $\eta(\Omega)=0.3$, $\eta(\Upsilon)=0.2$, $\eta(\Pi)=0.3$,; $\vartheta:\mathfrak{G}\to[0,1]$ *by* $\vartheta(\Psi)=0.4$, $\vartheta(\Omega)=0.2$, $\vartheta(\Upsilon)=0.1$, $\vartheta(\Pi)=0.2$; *and* $\omega:\mathfrak{G}\to[0,1]$ *by* $\omega(\Psi)=0.3$, $\omega(\Omega)=0.4$, $\omega(\Upsilon)=0.3$, $\omega(\Pi)=0.6$. *Then* SPC *is an SFI of* $\mathfrak{G}$.

Lemma 3.1. *Every SFI of a semigroup $\mathfrak{G}$ is an SFII of $\mathfrak{G}$.*

Proof. Suppose that $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFI of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFI of $\mathfrak{G}$, we have that $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFR of $\mathfrak{G}$. Thus, $\eta(uv) \geq \eta(u),\quad \vartheta(uv) \geq \vartheta(u)$ and $\omega(uv) \leq \omega(u)$ and so $\eta(uv) \geq \eta(u) \geq \eta(u) \wedge \eta(v)$, $\vartheta(uv) \geq \vartheta(u) \geq \vartheta(u) \wedge \vartheta(v)$ and $\omega(uv) \leq \omega(u) \subseteq \omega(u) \vee \omega(v)$. Hence, $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFSSG of $\mathfrak{G}$. Let $a, u, v \in \mathfrak{G}$. Then,
$\eta(uav) = \eta(u(av)) \geq \eta(av) \geq \eta(a)$, $\vartheta(uav) = \vartheta(u(av)) \geq \vartheta(av) \geq \vartheta(a)$ and $\omega(uav) = \omega(u(av)) \leq \omega(av) \leq \omega(a)$ Thus, $\eta(uav) \geq \eta(a)$, $\vartheta(uav) \geq \vartheta(a)$ and $\omega(uav) \leq \omega(a)$. Hence, $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFII of $\mathfrak{G}$. □

In order to consider the converse of Lemma 3.1, we need to strengthen the condition of a semigroup $\mathfrak{G}$.

Lemma 3.2. *In a regular semigroup $\mathfrak{G}$, the SFIIs and the SFIs coincide.*

Proof. Suppose that $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is regular, there exists $x \in \mathfrak{G}$ such that $u = uxu$. Thus,

$$\eta(uv) = \eta((uxu)v) = \eta((ux)uv) \geq \eta(u),$$

$$\vartheta(uv) = \vartheta((uxu)v) = \vartheta((ux)uv) \geq \vartheta(u),$$

and

$$\omega(uv) = \omega((uxu)v) = \omega((ux)uv) \leq \omega(u).$$

Hence, $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFR of $\mathfrak{G}$. Similarly, we can show that $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFL of $\mathfrak{G}$. Thus, $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFI of $\mathfrak{G}$. □

Lemma 3.3. *In a left (right) regular semigroup $\mathfrak{G}$, the SFIIs and the SFIs coincide.*

Proof. Suppose that $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is left regular, there exists $k \in \mathfrak{G}$ such that $u = ku^2$. Thus,

$$\begin{aligned} \eta(uv) &= \eta((ku^2)v) = \eta(kuuv) \\ &= \eta((ku)uv) \geq \eta(u), \end{aligned}$$

$$\begin{aligned} \vartheta(uv) &= \vartheta((ku^2)v) = \vartheta(kuuv) \\ &= \vartheta((ku)uv) \geq \vartheta(u) \end{aligned}$$

and

$$\begin{aligned} \omega(uv) &= \omega((ku^2)v) = \omega(kuuv) \\ &= \omega((ku)uv) \leq \omega(u). \end{aligned}$$

Hence $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFR of $\mathfrak{G}$. Similarly, we can show that $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFL of $\mathfrak{G}$. Thus, $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFI of $\mathfrak{G}$. □

Lemma 3.4. *In an intra-regular semigroup $\mathfrak{G}$, the SFIIs and the SFIs coincide.*

Proof. Suppose that SP $= (\eta; \vartheta, \omega)$ is an SFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is intra-regular, there exist $x, y \in \mathfrak{G}$ such that $u = xu^2y$. Thus,

$$\begin{aligned} \eta(uv) &= \eta((xu^2y)v) = \eta((xuuy)v) \\ &= \eta((xu)uv) \geq \eta(u), \end{aligned}$$

$$\begin{aligned} \vartheta(uv) &= \vartheta((xu^2y)v) = \vartheta((xuuy)v) \\ &= \vartheta((xu)uv) \geq \vartheta(u) \end{aligned}$$

and

$$\begin{aligned} \omega(uv) &= \omega((xu^2)v) = \omega((xuuy)v) \\ &= \omega((xu)uv) \leq \omega(u). \end{aligned}$$

Hence, SP $= (\eta; \vartheta, \omega)$ is an SFR of $\mathfrak{G}$. Similarly, we can show that SP $= (\eta; \vartheta, \omega)$ is an SFL of $\mathfrak{G}$. Thus, SP $= (\eta; \vartheta, \omega)$ is an SFI of $\mathfrak{G}$. □

Lemma 3.5. *In a semisimple semigroup $\mathfrak{G}$, the SFIIs and the SFIs coincide.*

Proof. Suppose that SP $= (\eta; \vartheta, \omega)$ is an SFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is semisimple, there exist $x, y, z \in \mathfrak{G}$ such that $u = xuyuz$. Thus,

$$\eta(uv) = \eta((xuyuz)v) = \eta((xuy)u(zv)) \geq \eta(u),$$

$$\vartheta(uv) = \vartheta((xuyuz)v) = \vartheta((xuy)u(zv)) \geq \vartheta(u)$$

and

$$\omega(uv) = \omega((xuyuz)v) = \omega((xuy)u(zv)) \leq \omega(u).$$

Hence, SP $= (\eta; \vartheta, \omega)$ is an SFR of $\mathfrak{G}$. Similarly, we can show that SP $= (\eta; \vartheta, \omega)$ is an SFL of $\mathfrak{G}$. Thus, SP $= (\eta; \vartheta, \omega)$ is an SFI of $\mathfrak{G}$. □

Lemma 3.6. *In a let (right) quasi-regular semigroup $\mathfrak{G}$, the SFIIs and the SFIs coincide.*

Proof. Suppose that SP $= (\eta; \vartheta, \omega)$ is an SFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is left quasi-regular, there exist $x, y \in \mathfrak{G}$ such that $v = xvyv$. Thus,

$$\eta(uv) = \eta(u(xvyv)) = \eta((ux)v(yv)) \geq \eta(v),$$

$$\vartheta(uv) = \vartheta(u(xvyv)) = \vartheta((ux)v(yv)) \geq \vartheta(v)$$

and

$$\omega(uv) = \omega(u(xvyv)) = \omega((ux)v(yv)) \leq \omega(v).$$

Hence, SP $= (\eta; \vartheta, \omega)$ is an SFR of $\mathfrak{G}$. Similarly, we can show that SP $= (\eta; \vartheta, \omega)$ is an SFL of $\mathfrak{G}$. Thus, SP $= (\eta; \vartheta, \omega)$ is an SFI of $\mathfrak{G}$. □

Lemma 3.7. *In a weakly regular semigroup $\mathfrak{G}$, the SFIIs and the SFIs coincide.*

Proof. Suppose that SP $= (\eta; \vartheta, \omega)$is an SFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is weakly regular, there exist $p, q \in \mathfrak{G}$ such that $u = upuq$. Thus,

$$\eta(uv) = \eta((upuq)v) = \eta((up)u(qv)) \geq \eta(u),$$

$$\vartheta(uv) = \vartheta((upuq)v) = \vartheta((up)u(qv)) \geq \vartheta(u),$$

and

$$\omega(uv) = \omega((upuq)v) = \omega((up)u(qv)) \leq \omega(u),$$

Hence, SP $= (\eta; \vartheta, \omega)$ is an SFR of $\mathfrak{G}$. Similarly, we can show that SP $= (\eta; \vartheta, \omega)$ is an SFL of $\mathfrak{G}$. Thus, SP $= (\eta; \vartheta, \omega)$ is an SFI of $\mathfrak{G}$. □

By Lemma 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7, we have Theorem 3.1.

Theorem 3.1. *Let $\mathfrak{G}$ be a semigroup. If $\mathfrak{G}$ is regular, left (right) regular, intra-regular, semisimple, left (right) quasi-regular or weakly regular, then SFIIs and SFIs coincide.*

The following theorems study the spherical characteristic function of types of subsemigroups of semigroups.

Theorem 3.2. *Let $\mathfrak{K}$ be a nonempty subset of a semigroup $\mathfrak{G}$. Then $\mathfrak{K}$ is a subsemigroup (left ideal, right ideal) of $\mathfrak{G}$ if and only if the sphericalcharacteristic function $\chi_{\mathfrak{K}} = (\eta_{\chi_{\mathfrak{K}}}, \vartheta_{\chi_{\mathfrak{K}}}, \omega_{\chi_{\mathfrak{K}}})$ is an SFSSG (SFLI, SFRI) of $\mathfrak{G}$.*

Theorem 3.3. *Let $\mathfrak{K}$ be a nonempty subset of a semigroup $\mathfrak{G}$. Then $\mathfrak{K}$ is an interior ideal of $\mathfrak{G}$ if and only if the spherical characteristic function $\chi_{\mathfrak{K}} = (\eta_{\chi_{\mathfrak{K}}}, \vartheta_{\chi_{\mathfrak{K}}}, \omega_{\chi_{\mathfrak{K}}})$ is an SFII of $\mathfrak{G}$.*

Proof. Suppose that $\mathfrak{K}$ is an interior ideal of $\mathfrak{G}$. Then $\mathfrak{K}$ is a subsemigroup $\mathfrak{G}$. Thus, by Theorem 3.2, $\chi_{\mathfrak{K}} = (\eta_{\chi_{\mathfrak{K}}}, \vartheta_{\chi_{\mathfrak{K}}}, \omega_{\chi_{\mathfrak{K}}})$ is an SFSSG of $\mathfrak{G}$. Let $u, v, w \in \mathfrak{G}$. Then the following cases:

Case 1 If $v \in \mathfrak{K}$, then $uvw \in \mathfrak{K}$. Thus, $\eta_{\chi_{\mathfrak{K}}}(v) = 1 = \eta_{\chi_{\mathfrak{K}}}(uvw)$, $\vartheta_{\chi_{\mathfrak{K}}}(v) = 1 = \vartheta_{\chi_{\mathfrak{K}}}(uvw)$ and $\omega_{\chi_{\mathfrak{K}}}(v) = 0\omega_{\chi_{\mathfrak{K}}}(uvw)$. Hence, $\eta_{\chi_{\mathfrak{K}}}(uvw) \geq \eta_{\chi_{\mathfrak{K}}}(v)$, $\vartheta_{\chi_{\mathfrak{K}}}(uvw) \geq \vartheta_{\chi_{\mathfrak{K}}}(v)$ and $\omega_{\chi_{\mathfrak{K}}}(uvw) \leq \vartheta_{\chi_{\mathfrak{K}}}(v)$.

Case 2 If $v \notin \mathfrak{K}$, then $uvw \in \mathfrak{K}$. Thus, $\eta_{\chi_{\mathfrak{K}}}(uvw) \geq \eta_{\chi_{\mathfrak{K}}}(v)$, $\vartheta_{\chi_{\mathfrak{K}}}(uvw) \geq \vartheta_{\chi_{\mathfrak{K}}}(v)$ and $\omega_{\chi_{\mathfrak{K}}}(uvw) \leq \vartheta_{\chi_{\mathfrak{K}}}(v)$.

Therefore, $\chi_{\mathfrak{K}} = (\eta_{\chi_{\mathfrak{K}}}, \vartheta_{\chi_{\mathfrak{K}}}, \omega_{\chi_{\mathfrak{K}}})$ is an SFII of $\mathfrak{G}$.

Conversely suppose that $\chi_{\mathfrak{K}}^{RI}$ is an SFII of $\mathfrak{G}$ Then, $\chi_{\mathfrak{K}}^{RI}$ is an SFSSG of $\mathfrak{G}$. Thus, by Theorem 3.2, $\mathfrak{K}$ is a subsemigroup of $\mathfrak{G}$. Let $v \in \mathfrak{K}$. If $uvw \notin \mathfrak{K}$, then $\eta_{\chi_{\mathfrak{K}}}(uvw) \leq \eta_{\chi_{\mathfrak{K}}}(v)$, $\vartheta_{\chi_{\mathfrak{K}}}(uvw) \leq \vartheta_{\chi_{\mathfrak{K}}}(v)$ and $\omega_{\chi_{\mathfrak{K}}}(uvw) \geq \vartheta_{\chi_{\mathfrak{K}}}(v)$. Since $\chi_{\mathfrak{K}}$ is an SFII of $\mathfrak{G}$ we have $\eta_{\chi_{\mathfrak{K}}}(uvw) \geq \eta_{\chi_{\mathfrak{K}}}(v)$, $\vartheta_{\chi_{\mathfrak{K}}}(uvw) \geq \vartheta_{\chi_{\mathfrak{K}}}(v)$ and $\omega_{\chi_{\mathfrak{K}}}(uvw) \leq \vartheta_{\chi_{\mathfrak{K}}}(v)$, which is a contradiction. Thus, $uvw \in \mathfrak{K}$. Hence, $\mathfrak{K}$ is a bi-ideal of $\mathfrak{G}$. □

Definition 3.2. *Let* $\text{SP}_1 = (\eta, \vartheta, \omega)$ *and* $\text{SP}_2 = (\tau, \nu, \alpha)$ *be two SF-setof a non-empty set* $\mathfrak{T}$. *Then the*

(1) $\text{SP}_1 \subseteq \text{SP}_2$ *if* $\eta(u) \leq \tau(u)$ $\vartheta(u) \leq \nu(u)$ *and* $\omega(u) \geq \alpha(u)$,

(2) $\text{SP}_1 \sqcap \text{SP}_2$ *if* $\eta(u) \wedge \tau(u)$ $\vartheta(u) \wedge \nu(u)$ *and* $\omega(u) \vee \alpha(u)$,

for all $u \in \mathfrak{T}$.

Theorem 3.4. *Let* $\mathrm{SP} = (\eta; \vartheta, \omega)$ *be an SF-set of a semigroup* $\mathfrak{G}$. *Then* $\mathrm{SP} = (\eta; \vartheta, \omega)$ *is an SFSSG of* $\mathfrak{G}$ *if and only if* $\mathrm{SP} \circ \mathrm{SP} \subseteq \mathrm{SP}$.

Theorem 3.5. *Let* $\mathrm{SP} = (\eta; \vartheta, \omega)$ *be SF-set of a semigroup* $\mathfrak{G}$. *Then* $\mathrm{SP} = (\eta; \vartheta, \omega)$ *is an SFII of* $\mathfrak{G}$ *if and only if* $\mathrm{SP} \circ \mathrm{SP} \subseteq \mathrm{SP}$ *and* $\chi_{\mathfrak{G}} \circ \mathrm{SP} \circ \chi_{\mathfrak{G}} \subseteq \mathrm{SP}$, *where* $\chi_{\mathfrak{G}} = (\eta_{\mathfrak{G}}, \vartheta_{\mathfrak{G}}, \omega_{\mathfrak{G}})$.

Proof. Assume that $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFII of $\mathfrak{G}$ let $u \in \mathfrak{G}$. Then $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFSSG of $\mathfrak{G}$. Thus by Theorem 3.4, $\mathrm{SP} \circ \mathrm{SP} \subseteq \mathrm{SP}$. Next to show that $\chi_{\mathfrak{G}} \circ \mathrm{SP} \circ \chi_{\mathfrak{G}} \subseteq \mathrm{SP}$. Now,

If $F_u = \emptyset$. Then $(\eta_{\mathfrak{G}} \circ \eta \circ \eta_{\mathfrak{G}})(u) = 0 \leq \eta(u)$, $(\vartheta_{\mathfrak{G}} \circ \vartheta \circ \vartheta_{\mathfrak{G}})(u) = 0 \leq \vartheta(u)$ and $(\omega_{\mathfrak{G}} \circ \omega \circ \omega_{\mathfrak{G}})(u) = 1 \geq \omega(u)$.

If $F_u \neq \emptyset$. Then

$$\begin{aligned}
(\eta_{\mathfrak{G}} \circ (\eta \circ \eta_{\mathfrak{G}}))(u) &= \bigvee_{(i,j) \in F_u} \eta_{\mathfrak{G}}(i) \wedge (\eta \circ \eta_{\mathfrak{G}})(j) \\
&= \bigvee_{(i,j) \in F_u} \eta_{\mathfrak{G}}(i) \wedge (\bigvee_{(k,r) \in F_j} \eta(k) \wedge \eta_{\mathfrak{G}}(r)) \\
&= \bigvee_{(i,j) \in F_u} 1 \wedge (\bigvee_{(k,r) \in F_j} \eta(k) \wedge 1) \\
&= \bigvee_{(i,k) \in F_u} \eta(k) \leq \eta(u),
\end{aligned}$$

$$\begin{aligned}
(\vartheta_{\mathfrak{G}} \circ (\vartheta \circ \vartheta_{\mathfrak{G}}))(u) &= \bigvee_{(i,j) \in F_u} \vartheta_{\mathfrak{G}}(i) \wedge (\vartheta \circ \vartheta_{\mathfrak{G}})(j) \\
&= \bigvee_{(i,j) \in F_u} \vartheta_{\mathfrak{G}}(i) \wedge (\bigvee_{(k,r) \in F_j} \vartheta(k) \wedge \vartheta_{\mathfrak{G}}(r)) \\
&= \bigvee_{(i,j) \in F_u} 1 \wedge (\bigvee_{(k,r) \in F_j} \vartheta(k) \wedge 1) \\
&= \bigvee_{(i,k) \in F_u} \vartheta(k) \leq \vartheta(u)
\end{aligned}$$

and

$$\begin{aligned}
(\omega_{\mathfrak{G}} \circ (\omega \circ \omega_{\mathfrak{G}}))(u) &= \bigwedge_{(i,j) \in F_u} \omega_{\mathfrak{G}}(i) \vee (\omega \circ \omega_{\mathfrak{G}})(j) \\
&= \bigwedge_{(i,j) \in F_u} \omega_{\mathfrak{G}}(i) \vee (\bigwedge_{(k,r) \in F_j} \omega(k) \vee \omega_{\mathfrak{G}}(r)) \\
&= \bigwedge_{(i,j) \in F_u} 0 \vee (\bigwedge_{(k,r) \in F_j} \omega(k) \vee 0) \\
&= \bigwedge_{(i,k) \in F_u} \omega(k) \geq \omega(u).
\end{aligned}$$

Hence, $\chi_{\mathfrak{G}} \circ \mathrm{SP} \circ \chi_{\mathfrak{G}} \subseteq \mathrm{SP}$.

Conversely, assume that $\mathrm{SP} \circ \mathrm{SP} \subseteq \mathrm{SP}$ and $\chi_{\mathfrak{G}} \circ \mathrm{SP} \circ \chi_{\mathfrak{G}} \subseteq \mathrm{SP}$. Let $u, v, w \in \mathfrak{G}$. Since $\mathrm{SP} \circ \mathrm{SP} \subseteq \mathrm{SP}$ we have $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFSSG of $\mathfrak{G}$ by Theorem 3.4. Thus, $\eta(uv) \geq \eta(u) \wedge \eta(v)$, $\vartheta(uv) \geq \vartheta(u) \wedge \vartheta(v)$ and $\omega(uv) \leq \omega(u) \vee \omega(v)$. Next to show that, $\eta(uvw) \geq \eta(v)$, $\vartheta(uvw) \geq \vartheta(v)$ and $\omega(uvw) \leq \omega(v)$. Now,

$$\begin{aligned}
\eta(uvw) &\geq (\eta_{\mathfrak{G}} \circ (\eta \circ \eta_{\mathfrak{G}}))(uvw) \\
&= \bigvee_{(i,j) \in F_{uvw}} \eta_{\mathfrak{G}}(i) \wedge (\eta \circ \eta_{\mathfrak{G}})(j)
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{(i,j)\in F_{uvw}} \eta_{\mathfrak{G}}(i) \wedge \Big(\bigvee_{(k,r)\in F_j} \eta(k) \wedge \eta_{\mathfrak{G}}(r) \Big) \\
&= \bigvee_{(i,j)\in F_{uvw}} 1 \wedge \Big(\bigvee_{(k,r)\in F_j} \eta(k) \wedge 1 \Big) \\
&\geq \eta(v),
\end{aligned}
$$

$$
\begin{aligned}
\vartheta(uvw) &\geq (\vartheta_{\mathfrak{G}} \circ (\vartheta \circ \vartheta_{\mathfrak{G}}))(uvw) \\
&= \bigvee_{(i,j)\in F_{uvw}} \vartheta_{\mathfrak{G}}(i) \wedge (\vartheta \circ \vartheta_{\mathfrak{G}})(j) \\
&= \bigvee_{(i,j)\in F_{uvw}} \vartheta_{\mathfrak{G}}(i) \wedge \Big(\bigvee_{(k,r)\in F_j} \vartheta(k) \wedge \vartheta_{\mathfrak{G}}(r) \Big) \\
&= \bigvee_{(i,j)\in F_{uvw}} 1 \wedge \Big(\bigvee_{(k,r)\in F_j} \vartheta(k) \wedge 1 \Big) \\
&\geq \vartheta(v)
\end{aligned}
$$

$$
\begin{aligned}
\omega(uvw) &\geq (\omega_{\mathfrak{G}} \circ (\omega \circ \omega_{\mathfrak{G}}))(uvw) \\
&= \bigwedge_{(i,j)\in F_{uvw}} \omega_{\mathfrak{G}}(i) \vee (\omega \circ \omega_{\mathfrak{G}})(j) \\
&= \bigwedge_{(i,j)\in F_{uvw}} \omega_{\mathfrak{G}}(i) \vee \Big(\bigwedge_{(k,r)\in F_j} \omega(k) \wedge \omega_{\mathfrak{G}}(r) \Big) \\
&= \bigwedge_{(i,j)\in F_{uvw}} 0 \vee \Big(\bigwedge_{(k,r)\in F_j} \omega(k) \vee 0 \Big) \\
&\leq \omega(v).
\end{aligned}
$$

Thus, $\eta(uvw) \geq \eta(v)$, $\vartheta(uvw) \geq \vartheta(v)$ and $\omega(uvw) \leq \omega(v)$. Hence, SP $= (\eta; \vartheta, \omega)$ is an SFII of $\mathfrak{G}$. □

The ensuing theorem is an essential property for an equivalent of an SFII of a left (right, intra-) regular semigroup.

Theorem 3.6. *Let* SP $= (\eta; \vartheta, \omega)$ *be an SF-set of a left (right, intra-) regular semigroup* $\mathfrak{G}$. *Then the following statements are equivalent*

(1) SP $= (\eta; \vartheta, \omega)$ *is an SFII of* $\mathfrak{G}$.

(2) SP $\circ$ SP $=$ SP *and* $\chi_{\mathfrak{G}} \circ$ SP $\circ\, \chi_{\mathfrak{G}} =$ SP.

Proof. Assume that SP $= (\eta; \vartheta, \omega)$ is an SFII of $\mathfrak{G}$ and let $u \in \mathfrak{G}$. Since $\mathfrak{G}$ is left regular, there exists $m \in \mathfrak{G}$ such that $u = mu^2 = m(uu) = (mu)u = (m(mu^2))u = (mmuu)u = ((m^2)uu)(mu^2) = (m^2uu)(muu)$. Thus,

$$
\begin{aligned}
(\eta \circ \eta)(u) &= \bigvee_{(k,o)\in F_u} \{\eta(k) \wedge \eta(o)\} = \bigvee_{(k,o)\in F_{(m^2uu)(muu)}} \{\eta(k) \wedge \eta(o)\} \\
&\geq \eta(m^2uu) \wedge \eta(muu) \geq \eta(u) \wedge \eta(u) = \eta(u),
\end{aligned}
$$

$$
\begin{aligned}
(\vartheta \circ \vartheta)(u) &= \bigvee_{(k,o)\in F_u} \{\vartheta(k) \wedge \vartheta(o)\} = \bigvee_{(k,o)\in F_{(m^2uu)(muu)}} \{\vartheta(k) \wedge \vartheta(o)\} \\
&\geq \vartheta(m^2uu) \wedge \vartheta(muu) \geq \vartheta(u) \wedge \vartheta(u) = \vartheta(u)
\end{aligned}
$$

and

$$\begin{aligned}(\omega \circ \omega)(u) &= \bigwedge_{(k,o)\in F_u} \{\omega(k) \vee \omega(o)\} = \bigwedge_{(k,o)\in F_{(m^2uu)(muu)}} \{\omega(k) \vee \omega(o)\} \\ &\leq \omega(m^2uu) \vee \omega(muu) \leq \omega(u) \vee \omega(u) = \omega(u).\end{aligned}$$

Hence, $(\eta \circ \eta)(u) \geq \eta(u)$, $(\vartheta \circ \vartheta)(u) \geq \vartheta(u)$ and $(\omega \circ \omega)(u) \leq \omega(u)$. Therefore, $\mathrm{SP} \subseteq \mathrm{SP} \circ \mathrm{SP}$.

Since $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFSSG of $\mathfrak{G}$, we have $\mathrm{SP} \circ \mathrm{SP} \subseteq \mathrm{SP}$, by Theorem 3.4. Thus, $\mathrm{SP} \circ \mathrm{SP} = \mathrm{SP}$. Since $\mathfrak{G}$ is left regular, there exists $m \in \mathfrak{G}$ such that $u = mu^2 = muu = m(muu)u$. Thus,

$$\begin{aligned}((\eta_{\mathfrak{G}} \circ \eta) \circ \mathfrak{G})(u) &= \bigvee_{(k,o)\in F_u} \{((\eta_{\mathfrak{G}} \circ \eta)(k)) \wedge \eta_{\mathfrak{G}}(o)\} \\ &= \bigvee_{(k,o)\in F_{m(muu)u}} \{(\eta_{\mathfrak{G}} \circ \eta)(k)) \wedge \eta_{\mathfrak{G}}(o)\} \\ &\geq (\eta_{\mathfrak{G}} \circ \eta)(mmuu)) \wedge \eta_{\mathfrak{G}}(u) = (\eta_{\mathfrak{G}} \circ \eta)(m(muu)) \wedge 1 = (\eta_{\mathfrak{G}} \circ \eta)(m(muu)) \\ &= \bigvee_{(p,q)\in F_{m(muu)}} \{(\eta_{\mathfrak{G}}(p) \wedge (\eta(q)\} \\ &\geq \eta_{\mathfrak{G}}(m) \wedge \eta(muu) = 1 \wedge \eta(muu) = \eta(muu) \geq \eta(u),\end{aligned}$$

$$\begin{aligned}((\vartheta_{\mathfrak{G}} \circ \vartheta) \circ \vartheta_{\mathfrak{G}})(u) &= \bigvee_{(k,o)\in F_u} \{((\vartheta_{\mathfrak{G}} \circ \vartheta)(k)) \wedge \vartheta_{\mathfrak{G}}(o)\} \\ &= \bigvee_{(k,o)\in F_{m(muu)u}} \{(\vartheta_{\mathfrak{G}} \circ \vartheta(k)) \wedge \vartheta_{\mathfrak{G}}(o)\} \\ &\geq (\vartheta_{\mathfrak{G}} \circ \vartheta)(mmuu)) \wedge \vartheta_{\mathfrak{G}}(u) = (\vartheta_{\mathfrak{G}} \circ \vartheta)(m(muu)) \wedge 1 = (\vartheta_{\mathfrak{G}} \circ \vartheta)(m(muu)) \\ &= \bigvee_{(p,q)\in F_{m(muu)}} \{(\vartheta_{\mathfrak{G}}(p) \wedge (\vartheta(q)\} \\ &\geq \vartheta_{\mathfrak{G}}(m) \wedge \vartheta(muu) = 1 \wedge \vartheta(muu) = \vartheta(muu) \geq \vartheta(u)\end{aligned}$$

and

$$\begin{aligned}((\mathfrak{G} \circ \omega) \circ \mathfrak{G})(u) &= \bigwedge_{(k,o)\in F_u} \{((\mathfrak{G} \circ \omega)(k) \vee \eta_{\mathfrak{G}}(o)\} \\ &= \bigwedge_{(k,o)\in F_{m(muu)u}} \{(\mathfrak{G} \circ \omega)(k) \vee \mathfrak{G}(o)\} \\ &\leq (\mathfrak{G} \circ \omega)(mmuu)) \vee \mathfrak{G}(u) = (\mathfrak{G} \circ \omega)(m(muu)) \vee 1 = (\mathfrak{G} \circ \omega)(m(muu)) \\ &= \bigwedge_{(p,q)\in F_{m(muu)}} \{(\mathfrak{G}(p) \vee (\omega(q)\} \\ &\leq \mathfrak{G}(m) \vee \omega(muu) = 0 \vee \omega(muu) = \omega(muu) \leq \omega(u).\end{aligned}$$

Hence, $((\eta_{\mathfrak{G}} \circ \eta) \circ \eta_{\mathfrak{G}})(u) \geq \eta(u)$, $((\vartheta_{\mathfrak{G}} \circ \vartheta) \circ \vartheta_{\mathfrak{G}})(u) \geq \vartheta(u)$ and $((\chi_{\mathfrak{G}} \circ \omega) \circ \chi_{\mathfrak{G}})(u) \leq \omega(u)$. Therefore, $\mathrm{SP} \subseteq \chi_{\mathfrak{G}} \circ \mathrm{SP} \circ \chi_{\mathfrak{G}}$. By Theorem 3.5 we have $\mathfrak{G} \circ \mathrm{SP} \circ \mathfrak{G} \subseteq \mathrm{SP}$. Thus, $\mathrm{SP} = \mathfrak{G} \circ \mathrm{SP} \circ \mathfrak{G}$.

For the converse, it follows from Theorem 3.5.

Similarly, we can prove the other cases also. □

Some equivalent conditions are essential properties for an SFII of semisimple and regular semigroups.

Theorem 3.7. *Let $\mathrm{SP} = (\eta; \vartheta, \omega)$ be an SF-set of a semisimple semigruop $\mathfrak{G}$. Then the following statements are equivalent*

(1) $\mathrm{SP} = (\eta; \vartheta, \omega)$ *is an SFII of $\mathfrak{G}$.*

(2) $\mathrm{SP} \circ \mathrm{SP} = \mathrm{SP}$

Proof. Assume that $\mathrm{SP} = (\eta; \vartheta, \omega)$ is an SFII of $\mathfrak{G}$ and let $u \in \mathfrak{G}$. By assumption, there exist $m, r, e, d \in \mathfrak{G}$ such that $u = mureud$. Thus,

$$\begin{aligned}(\eta \circ \eta)(u) &= \bigvee_{(k,o) \in F_u} \{\eta(k) \wedge \eta(o)\} \\ &= \bigvee_{(k,0) \in F_{(mur)(eud)}} \{\eta(k) \wedge \eta(o))\} \\ &\geq \eta(mur) \wedge \eta(eud) \geq \eta(u) \wedge \eta(u) = \eta(u),\end{aligned}$$

$$\begin{aligned}(\vartheta \circ \vartheta)(u) &= \bigvee_{(k,o) \in F_u} \{\vartheta(k) \wedge \vartheta(o)\} \\ &= \bigvee_{(k,0) \in F_{(mur)(eud)}} \{\vartheta(k) \wedge \vartheta(o))\} \\ &\geq \vartheta(mur) \wedge \vartheta(eud) \geq \vartheta(u) \wedge \vartheta(u) = \vartheta(u),\end{aligned}$$

and

$$\begin{aligned}(\omega \circ \omega)(u) &= \bigwedge_{(k,o) \in F_u} \{\omega(k) \vee \omega(o)\} \\ &= \bigwedge_{(k,0) \in F_{(mur)(eud)}} \{\omega(k) \vee \omega(o))\} \\ &\leq \omega(mur) \wedge \omega(eud) \leq \omega(u) \vee \omega(u) = \omega(u).\end{aligned}$$

Hence,$(\eta \circ \eta)(u) \geq \eta(u)$, $(\vartheta \circ \vartheta)(u) \geq \vartheta(u)$ and $(\omega \circ \omega)(u) \leq \omega(u)$. Therefore, $\mathrm{SP} \subseteq \mathrm{SP} \circ \mathrm{SP}$.

Since $\mathrm{SP} = (\eta; \vartheta, \omega)$ is a SFSSG of $\mathfrak{G}$, we have $\mathrm{SP} \circ \mathrm{SP} \subseteq \mathrm{SP}$, by Theorem 3.4. Thus, $\mathrm{SP} \circ \mathrm{SP} = \mathrm{SP}$.

For the converse, it follows from Theorem 3.5. □

4. Characterization of Semisimple Semigroups in Terms of Spherical Fuzzy Interior Ideals and Spherical Fuzzy Ideals

In this topic, we will characterize a semisimple semigroup in terms of SFIIs and SFIDs.

Lemma 4.1. *If* $\mathrm{SP}_1 = (\eta, \vartheta, \omega)$ *is an SFR and* $\mathrm{SP}_2 = (\tau, \nu, \alpha)$ *is an SFL of semigroup* $\mathfrak{G}$*, then* $\mathrm{SP}_1 \circ \mathrm{SP}_2 \subseteq \mathrm{SP}_1 \sqcap \mathrm{SP}_2$.

Proof. Assume that $\mathrm{SP}_1 = (\eta, \vartheta, \omega)$ is an SFR and $\mathrm{SP}_2 = (\tau, \nu, \alpha)$ is an SFL of $\mathfrak{G}$ and let $u \in \mathfrak{G}$.

If $F_u = \emptyset$, then it is easy to verify that, $(\eta \circ \tau)(u) \leq (\eta \wedge \tau)(u)$, $(\vartheta \circ \nu)(u) \leq (\vartheta \wedge \nu)(u)$ and $(\omega \circ \alpha)(u) \leq (\omega \vee \alpha)(u)$.

If $F_{\acute{h}} \neq \emptyset$, then

$$\begin{aligned}(\eta \circ \tau)(u) &= \bigvee_{(k,w) \in F_u} \{\eta(k) \wedge \tau(w)\} \leq \bigvee_{(k,w) \in F_u} \{\eta(kw) \wedge \tau(kw)\} \\ &= \eta(u) \wedge \tau(u) = (\eta \wedge \tau)(u),\end{aligned}$$

$$\begin{aligned}(\vartheta \circ \nu)(u) &= \bigvee_{(k,w) \in F_u} \{\vartheta(k) \wedge \nu(w)\} \leq \bigvee_{(k,w) \in F_u} \{\vartheta(kw) \wedge \nu(kw)\} \\ &= \vartheta(u) \wedge \nu(u) = (\vartheta \wedge \nu)(u)\end{aligned}$$

and

$$\begin{aligned}(\omega\circ\alpha)(u) &= \bigwedge_{(k,w)\in F_u}\{\omega(k)\vee\alpha(w)\}\geq\bigwedge_{(k,w)\in F_u}\{\omega(kw)\vee\alpha(kw)\}\\ &= \omega(u)\vee\alpha(u)=(\omega\vee\alpha)(u).\end{aligned}$$

Thus, $(\eta\circ\tau)(u)\leq(\eta\wedge\tau)(u)$, $(\vartheta\circ\nu)(u)\leq(\vartheta\wedge\nu)(u)$ and $(\omega\circ\alpha)(u)\leq(\omega\vee\alpha)(u)$. Hence, $\mathrm{SP}_1\circ\mathrm{SP}_2\subseteq\mathrm{SP}_1\sqcap\mathrm{SP}_2$. □

This theorem is a tool for characterizing semisimple in terms of SFIIs.

Theorem 4.1. *[7] Let $\mathfrak{K}$ and $\mathfrak{N}$ be non-empty subsets of a semifroup $\mathfrak{G}$. Then*

(1) $\chi_{\mathfrak{K}}\circ\chi_{\mathfrak{N}}=\chi_{\mathfrak{K}\mathfrak{N}}$,

(2) $\chi_{\mathfrak{K}}\sqcap\chi_{\mathfrak{N}}=\chi_{\mathfrak{K}\cap\mathfrak{N}}$

where $\chi_{\mathfrak{K}}=(\eta_{\chi_{\mathfrak{K}}},\vartheta_{\chi_{\mathfrak{K}}},\omega_{\chi_{\mathfrak{K}}})$ *and* $\chi_{\mathfrak{N}}=(\eta_{\chi_{\mathfrak{N}}},\vartheta_{\chi_{\mathfrak{N}}},\omega_{\chi_{\mathfrak{N}}})$.

Lemma 4.2. *[7] For $\mathfrak{G}$, the ensuing statements are equivalent.*

(1) *$\mathfrak{G}$ is semisimple,*

(2) *Every II $\mathfrak{K}$ of $\mathfrak{G}$ is idempotent,*

(3) *Every ID $\mathfrak{K}$ of $\mathfrak{G}$ is idempotent,*

(4) *For any II $\mathfrak{K}$ and $\mathfrak{N}$ of $\mathfrak{G}$, $\mathfrak{K}\cap\mathfrak{N}=\mathfrak{K}\mathfrak{N}$,*

(5) *For any IDs $\mathfrak{K}$ and $\mathfrak{N}$ of $\mathfrak{G}$, $\mathfrak{K}\cap\mathfrak{N}=\mathfrak{K}\mathfrak{N}$,*

(6) *For any II $\mathfrak{K}$ and any ID $\mathfrak{N}$ of $\mathfrak{G}$, $\mathfrak{K}\cap\mathfrak{N}=\mathfrak{K}\mathfrak{N}$,*

(7) *For any ID $\mathfrak{K}$ and any II $\mathfrak{N}$ of $\mathfrak{G}$, $\mathfrak{K}\cap\mathfrak{N}=\mathfrak{K}\mathfrak{N}$.*

The ensuing theorem presents an equivalent conditional statement for a semisimple OSG.

Theorem 4.2. *Let $\mathfrak{F}$ be a semigroup. Then the ensuing are equivalent:*

(1) *$\mathfrak{G}$ is semisimple,*

(2) $\mathrm{SP}_1\circ\mathrm{SP}_1=\mathrm{SP}_1$, *for every SFIIs* $\mathrm{SP}_1=(\eta,\vartheta,\omega)$ *of* $\mathfrak{G}$,

(3) $\mathrm{SP}_1\circ\mathrm{SP}_1=\mathrm{SP}_1$, *for every SFIDs* $\mathrm{SP}_1=(\eta,\vartheta,\omega)$ *of* $\mathfrak{G}$,

(4) $\mathrm{SP}_1\circ\mathrm{SP}_2=\mathrm{SP}_1\sqcap\mathrm{SP}_2$, *for every SFIIs* $\mathrm{SP}_1=(\eta,\vartheta,\omega)$ *and* $\mathrm{SP}_2=(\tau,\nu,\alpha)$ *of* $\mathfrak{G}$,

(5) $\mathrm{SP}_1\circ\mathrm{SP}_2=\mathrm{SP}_1\sqcap\mathrm{SP}_2$, *for every SFIDs* $\mathrm{SP}_1=(\eta,\vartheta,\omega)$ *and* $\mathrm{SP}_2=(\tau,\nu,\alpha)$ *of* $\mathfrak{G}$,

(6) $\mathrm{SP}_1\circ\mathrm{SP}_2=\mathrm{SP}_1\sqcap\mathrm{SP}_2$ *for every SFII* $\mathrm{SP}_1=(\eta,\vartheta,\omega)$ *of $\mathfrak{G}$ and every SFID* $\mathrm{SP}_2=(\tau,\nu,\alpha)$ *of* $\mathfrak{G}$,

(7) $\mathrm{SP}_1\circ\mathrm{SP}_2=\mathrm{SP}_1\sqcap\mathrm{SP}_2$, *for every SFID* $\mathrm{SP}_1=(\eta,\vartheta,\omega)$ *of $\mathfrak{G}$ and every SFII* $\mathrm{SP}_2=(\tau,\nu,\alpha)$ *of* $\mathfrak{G}$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathrm{SP}_1=(\eta,\vartheta,\omega)$ is an SFID of $\mathfrak{G}$. Then by assumption and by Theorem 3.7, $\mathrm{SP}_1\circ\mathrm{SP}_1=\mathrm{SP}_1$.

(2) $\Rightarrow$ (1) Let $\mathfrak{K}$ be an IID of $\mathfrak{G}$. Then by Theorem 3.3, $\chi_{\mathfrak{K}}=(\eta_{\chi_{\mathfrak{K}}},\vartheta_{\chi_{\mathfrak{K}}},\omega_{\chi_{\mathfrak{K}}})$ is a SFII of $\mathfrak{G}$. By supposition and Lemma 4.1, we have

$$\chi_{\mathfrak{K}}^2=\chi_{\mathfrak{K}}\circ\chi_{\mathfrak{K}}=\chi_{\mathfrak{K}}.$$

Thus, $\mathfrak{K}^2=\mathfrak{K}$. By Lemma 4.2, we have $\mathfrak{G}$ is semisimple.

(1) ⇒ (4) Let $SP_1 = (\eta, \vartheta, \omega)$ and $SP_2 = (\tau, \nu, \alpha)$ be SFIIs of $\mathfrak{G}$. Then by Theorem 3.5, $SP_1 = (\eta, \vartheta, \omega)$ and $SP_2 = (\tau, \nu, \alpha)$ are SFIDs of $\mathfrak{G}$. Thus by Lemma 4.1, $SP_1 \circ SP_2 \subseteq SP_1 \sqcap SP_2$.
Let $u \in \mathfrak{G}$. Then there exist $r, t, e, d \in \mathfrak{G}$ such that $u = ruteud$. Thus,

$$\begin{aligned}(\eta \circ \tau)(u) &= \bigvee_{(k,w)\in F_u} \{\eta(k) \wedge \tau(w)\} = \bigvee_{(k,w)\in F_{ruteud}} \{\eta(k) \wedge \tau(w)\} \\ &\geq \eta(rut) \wedge \tau(eud) \geq \eta(u) \wedge \tau(u) = (\eta \wedge \tau)(u),\end{aligned}$$

$$\begin{aligned}(\vartheta \circ \nu)(u) &= \bigvee_{(k,w)\in F_u} \{\vartheta(k) \wedge \nu(w)\} = \bigvee_{(k,w)\in F_{ruteud}} \{\vartheta(k) \wedge \nu(w)\} \\ &\geq \vartheta(rut) \wedge \nu(eud) \geq \vartheta(u) \wedge \nu(u) = (\vartheta \wedge \nu)(u),\end{aligned}$$

and

$$\begin{aligned}(\omega \circ \alpha)(u) &= \bigwedge_{(k,w)\in F_u} \{\omega(k) \vee \alpha(w)\} = \bigwedge_{(k,w)\in F_{ruteud}} \{\omega(k) \vee \alpha(w)\} \\ &\leq \omega(rut) \vee \alpha(eud) \leq \Omega(u) \vee \alpha(u) = (\omega \vee \alpha)(u).\end{aligned}$$

Hence, $(\eta \circ \tau)(u) \geq (\eta \wedge \tau)(u)$, $(\vartheta \circ \nu)(u) \geq (\vartheta \wedge \nu)(u)$ and $(\omega \circ \alpha)(u) \leq (\omega \vee \alpha)(u)$ and so, $SP_1 \sqcap SP_2 \subseteq SP_1 \circ SP_2$ Therefore, $SP_1 \circ SP_2 = SP_1 \sqcap SP_2$.

(4) ⇒ (1) Let $\mathfrak{K}$ and $\mathfrak{N}$ be IIs of $\mathfrak{G}$. Then by Theorem 3.3, $\chi_{\mathfrak{K}} = (\eta_{\chi_{\mathfrak{K}}}, \vartheta_{\chi_{\mathfrak{K}}}, \omega_{\chi_{\mathfrak{K}}})$ and $\chi_{\mathfrak{N}} = (\eta_{\chi_{\mathfrak{N}}}, \vartheta_{\chi_{\mathfrak{N}}}, \omega_{\chi_{\mathfrak{N}}})$ are SFIIs of $\mathfrak{G}$. By supposition and Lemma 4.1, we have

$$\chi_{\mathfrak{K}\mathfrak{N}} = \chi_{\mathfrak{K}} \circ \chi_{\mathfrak{N}} = \chi_{\mathfrak{K}} \sqcap \chi_{\mathfrak{N}} = \chi_{\mathfrak{K}\cap\mathfrak{N}}.$$

Thus, $u \in \mathfrak{K}\mathfrak{N}$. Hence, $\mathfrak{K} \cap \mathfrak{N} = \mathfrak{K}\mathfrak{N}$ By Lemma 4.2, $\mathfrak{G}$ is semisimple.

(1) ⇒ (6) Let $SP_1 = (\eta, \vartheta, \omega)$ and $SP_2 = (\tau, \nu, \alpha)$ be an SFII and an SFID of $\mathfrak{G}$ respectively. Then $SP_2 = (\tau, \nu, \alpha)$ is an SFII of $\mathfrak{G}$. Thus by (4), $SP_1 \circ SP_2 = SP_1 \sqcap SP_2$.

(6) ⇒ (1) Let $\mathfrak{K}$ and $\mathfrak{N}$ be an II and ID of $\mathfrak{G}$ respectively. Then by Theorem 3.3 and 3.2, $\chi_{\mathfrak{K}} = (\eta_{\chi_{\mathfrak{K}}}, \vartheta_{\chi_{\mathfrak{K}}}, \omega_{\chi_{\mathfrak{K}}})$ and $\chi_{\mathfrak{N}} = (\eta_{\chi_{\mathfrak{N}}}, \vartheta_{\chi_{\mathfrak{N}}}, \omega_{\chi_{\mathfrak{N}}})$ is an SFII and is an SFID of $\mathfrak{G}$ respectively. Then by Lemma3.5, $\chi_{\mathfrak{N}} = (\eta_{\chi_{\mathfrak{N}}}, \vartheta_{\chi_{\mathfrak{N}}}, \omega_{\chi_{\mathfrak{N}}})$ is an SFII of $\mathfrak{G}$. Similarly from (4) ⇒ (1), we have $\mathfrak{G}$ is semisimple.

So, (1) ⇔ (3), (1) ⇔ (5) and (1) ⇔ (7) are Straightforward. □

5. Conclusion

In this paper, we introduce the notion of spherical fuzzy interior ideals in semigroups. We desire properties of spherical fuzzy interior ideals. We characterized the necessary and sufficient conditions of coincidence spherical fuzzy ideals and spherical fuzzy interior ideals in semigroups. Furthermore, we characterized semisimple semigroups in terms of spherical fuzzy interior ideals in continuity of this paper, we will investigate will be carried out about the spherical fuzzy interior ideals of a ternary semigroup and their algebraic properties.

Acknowledgments: This research was supported by the Pibulsongkram Rajabhat University, Phitsanulok, Thailand (RDI-1-69-16).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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