Int. J. Anal. Appl. (2026), 24:231

International Journal of Analysis and Applications

# Exploring Fixed Point Results in Perturbed Controlled Metric Space

**Rajagopalan Ramaswamy**[1], **Neha Bhardwaj**[2], **Manoj Kumar**[2], **Gunaseelan Mani**[3,*], **Abdulkareem Saleh Hamarsheh**[1], **S. Padmaja**[4]

[1]*Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia*

[2]*Department of Mathematics, Maharishi Markandeshwar (Deemed to be university), Mullana, Ambala 133207, Haryana, India*

[3]*Centre for Excellence - Research, Rajalakshmi Engineering College, Chennai 602 105, India*

[4]*Department of Computer Science, College of Computer Engineering and Sciences, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia*

**Corresponding author:* mathsguna@yahoo.com

**Abstract.** In this manuscript, we shall introduce the notion of perturbed controlled metric spaces, where the control function is modified by an additional perturbation term. We establish fixed point theorems in this framework, demonstrating how classical results can be extended to perturbed settings. To prove the validity of our results some suitable examples are also provided. Some results from the literature are also deduced from our main theorems. Non trivial example and an application to solve integral equation have been provided to support the derived result.

## 1. Introduction

Banach's [1] fixed point theorem asserts that, if $(\Omega, d)$ is a complete metric space and $\mathcal{T} : \Omega \to \Omega$ is a contraction mapping, that is, $d(\mathcal{T}u, \mathcal{T}v) \leq \lambda d(u, v)\ \forall\ u, v \in \Omega$ for $\lambda \in [0, 1)$, then $\mathcal{T}$ admits one and only one fixed point. Since then, many generalisation on fixed point results have been reported in literature, see [2–6] as a generalisation of metric and metric like spaces. On the other hand, $\overline{d}(\mathcal{T}u, \mathcal{T}v) \leq \lambda \overline{d}(u, v)\ \forall\ u, v \in \Omega$, where $\overline{d}(u, v)$ is the experimental measurement of $d(u, v)$, Note that $\overline{d}$ is not necessarily a metric on $\Omega$. In this paper, we study the above question by introducing the concept of perturbed metric spaces.

Received: Feb. 28, 2026.

2020 *Mathematics Subject Classification.* 47H10; 54H25.

*Key words and phrases.* perturbed controlled metric space; fixed point; contraction mapping.

https://doi.org/10.28924/2291-8639-24-2026-231

ISSN: 2291-8639

Next, an interesting generalization of Banach's fixed point theorem is obtained in the setting of perturbed metric spaces and rest of the paper follows some topological concepts related to such spaces.

Throughout this paper, $\Omega$ denotes an arbitrary non-empty set. The set of non-negative integers is denoted by $\mathbb{N}$.

Mlaiki *et al*. [7] introduced the notion of controlled metric type space which is defined as follows

**Definition 1.1** ( [7]). *On a non empty set $\Omega$, define the mappings $\mu : \Omega \times \Omega \to [1, \infty)$ and $\overline{d} : \Omega \times \Omega \to [0, \infty)$ such that for all $u, e, f \in \Omega$ the following condition holds*

(i) $\overline{d}(u, e) = 0 \Longleftrightarrow u = e$,
(ii) $\overline{d}(u, e) = \overline{d}(e, u)$,
(iii) $\overline{d}(u, e) \leq \mu(u, f)\overline{d}(u, f) + \mu(f, e)\overline{d}(f, e)$.

*Then the pair $(\Omega, \overline{d})$ is called a controlled metric type space.*

Many authors established fixed point results in the setting of generalised form of controlled metric spaces with application to differential and integral equations, for example [8–14]. Recently, Jleli and Samet [15] proved fixed point theorems and gave some applications in complete perturbed metric spaces.

**Definition 1.2** ( [15]). *On a non empty set $\Omega$, define $\overline{d}, \psi :\ \Omega \times \Omega \to [0, \infty)$ be two mappings. Then we say that $\overline{d}$ is a perturbed metric on $\Omega$ with respect to $\psi$, if $\overline{d} - \psi : \Omega \times \Omega \to [0, \infty)$ for all $u, v, w \in \Omega$ is a metric on $\Omega$ if the following condition holds:*

(i) $(\overline{d} - \psi)(u, v) \geq 0$;
(ii) $(\overline{d} - \psi)(u, v) = 0$ *if and only if* $u = v$;
(iii) $(\overline{d} - \psi)(u, v) = (\overline{d} - \psi)(v, u)$;
(iv) $(\overline{d} - \psi)(u, v) \leq (\overline{d} - \psi)(u, w) + (\overline{d} - \psi)(w, v)$.

*Here $\psi$ a perturbed mapping, $\overline{d}_p = \overline{d} - \psi$ an exact metric and $(\Omega, \overline{d}, \psi)$ a perturbed metric space. Note that a perturbed metric on $\Omega$ is not necessarily a metric on $\Omega$.*

Inspired, in our work, we establish fixed point results in the setting of perturbed controlled metric space. The rest of the paper is organised as follows: In section-2 we, present the notion of perturbed controlled metric space and prove fixed point theorem in the setting of the space. We also provide suitable example and an application to the derived result in Section 3 and Section 4. Finally in Section 5 we conclude the article with open problem for future work.

## 2. Main Results

In this section, first of all, we shall introduce a new notion of perturbed controlled metric space. In addition to this, some fixed point results are also proved for the same.

**Definition 2.1.** *On a non empty set $\Omega$, define $\overline{d}, \psi : \Omega \times \Omega \to [0, \infty)$ be two mappings and $\mu : \Omega \times \Omega \to [1, \infty)$. Then we say that $\overline{d}$ is a perturbed controlled metric on $\Omega$ with respect to $\psi$, if $\overline{d} - \psi : \Omega \times \Omega \to [0, \infty)$ for all $u, v, w \in \Omega$ is a controlled metric on $\Omega$ if the following condition holds:*

(i) $(\overline{d} - \psi)(u, v) \geq 0$;
(ii) $(\overline{d} - \psi)(u, v) = 0$ *if and only if* $u = v$;
(iii) $(\overline{d} - \psi)(u, v) = (\overline{d} - \psi)(v, u)$;
(iv) $(\overline{d} - \psi)(u, v) \leq \mu(u, w)(\overline{d} - \psi)(u, w) + \mu(w, v)(\overline{d} - \psi)(w, v)$.

*Here $\psi$ a perturbed mapping $\overline{d}_p = \overline{d} - \psi$ an exact controlled metric and $(\Omega, \overline{d}, \psi)$ a perturbed controlled metric space.*

**Definition 2.2.** *Let $(\Omega, \overline{d}, \psi)$ be a perturbed control metric space, $\{w_n\}$ a sequence in $\Omega$ and $\mathcal{T} : \Omega \to \Omega$.*

(1) *We say that $\{w_n\}$ is a perturbed convergent sequence in $(\Omega, \overline{d}, \psi)$, if $\{w_n\}$ is a convergent sequence in the controlled metric space $(\Omega, \overline{d}_p, \mu)$ where $\overline{d}_p$ is exact metric (i.e., $\overline{d}_p = \overline{d} - \psi$).*
(2) *We say that $\{w_n\}$ is a perturbed Cauchy sequence in $(\Omega, \overline{d}, \psi)$, if $\{w_n\}$ is a Cauchy sequence in the controlled metric space $(\Omega, \overline{d}_p, \mu)$.*
(3) *We say that $(\Omega, \overline{d}, \psi, \mu)$ is complete perturbed controlled metric space, if $(\Omega, \overline{d}_p)$ is a complete controlled metric space.*
(4) *We say that $\mathcal{T}$ is a perturbed continuous mapping, if $\mathcal{T}$ is continuous with respect to the exact controlled metric $\overline{d}$.*

**Example 2.1.** *Let $\Omega = \{0, 1, 2\}$ and $\mu : \Omega \times \Omega \to [1, \infty)$ by $\mu(u, v) = 8 \ \forall \ u, v \in \Omega$ and $\overline{d} : \Omega \times \Omega \to [0, \infty)$ defined by* $\overline{d}(u, v) = \begin{cases} 0, & u = v, \\ 1.5, & |u - v| = 1, \\ 4, & |u - v| = 2. \end{cases}$

*For $u = 0$, $v = 2$, $w = 1$, $\overline{d}(0, 2) = 4$, $\overline{d}(0, 1) + \overline{d}(1, 2) = 1.5 + 1.5 = 3$.*
*So the triangle inequality fails.*
*Therefore, $\overline{d}$ is not a metric space but it is a controlled metric space, since*

$$\begin{aligned} \overline{d}(0, 2) &\leq \mu(0, 1)\overline{d}(0, 1) + \mu((1, 2))\overline{d}(1, 2) \\ &\leq 8(1.5) + 8(1.5) \\ &\leq 24. \end{aligned}$$

*Now, let us define a Perturbed function $\psi : \ \Omega \times \Omega \to [0, \infty)$ by*

$$\psi(u, v) = \begin{cases} 0, & u = v, \\ 0.5, & u, v = (0, 1), (1, 0), \\ 1, & \textit{otherwise} \end{cases}$$

*and*

$$\overline{d}_p = \overline{d} - \psi : \Omega \times \Omega \to [0, \infty).$$

*Clearly, $\overline{d}_p$ is a Perturbed controlled metric space because*

$$\begin{aligned}\overline{d}_p(0,2) &= \overline{d}(0,2) - \psi(0,2) = 4 - 1 = 3,\\ \overline{d}_p(0,1) &= \overline{d}(0,1) - \psi(0,1) = 1.5 - 0.5 = 1,\\ \overline{d}_p(1,2) &= \overline{d}(1,2) - \psi(1,2) = 1.5 - 1 = 0.5.\end{aligned}$$

*So,*

$$\begin{aligned}\overline{d}_p(0,2) &\leq \mu(0,1)\overline{d}_p(0,1) + \mu((1,2))\overline{d}_p(1,2)\\ &\leq 8(1) + 8(.5)\\ &= 8 + 4\\ &= 12.\end{aligned}$$

**Example 2.2.** *Let $\mathbb{R} = \Omega$ and define a function $\overline{d} : \Omega \times \Omega \to [0, \infty)$ by*

$$\overline{d}(u,v) = \begin{cases} 0, & u = v,\\ |u - v| + 2, & u \neq v.\end{cases}$$

*also let the control function $\mu : \Omega \times \Omega \to [1, \infty)$ by $\mu(u, v) = 4\ \forall\ u, v \in \Omega$, define a perturbation function $\psi : \Omega \times \Omega \to [0, \infty)$ by $\psi(u, v) = 0.5|u - v|$.*
*Then the function $\overline{d}_p = \overline{d} - \psi : \Omega \times \Omega \to [0, \infty)$ by $\overline{d}_p(u, v) = \overline{d}(u, v) - \psi(u, v)$ i.e.*

$$\overline{d}_p(u,v) = \begin{cases} 0, & u = v,\\ 0.5|u - v| + 2, & u \neq v,\end{cases} \quad \forall\ u, v \in \Omega$$

*forms a Perturbed controlled metric on $\Omega$.*

We extend the Banach fixed point theorem into perturbed controlled metric spaces.

**Theorem 2.1.** *Let $(\Omega, \overline{d}, \psi)$ be a complete perturbed controlled metric space and $\mathcal{T} : \Omega \to \Omega$ be a given mapping. Assume that the following conditions hold:*

(i) *$\mathcal{T}$ is a perturbed continuous mapping;*
(ii) *There exists $\lambda \in [0, 1)$ such that*

$$\overline{d}(\mathcal{T}u, \mathcal{T}v) \leq \lambda \overline{d}(u, v), \quad \forall\ u, v \in \Omega. \tag{2.1}$$

*Then, $\mathcal{T}$ admits one and only one fixed point.*

*Proof.* Let $w_0 \in \Omega$ be fixed. Consider the Picard sequence $\{w_n\} \subset \Omega$ defined by

$$w_{n+1} = \mathcal{T}w_n, \quad n \in \mathbb{N}.$$

Taking $(u, v) = (w_0, w_1)$ in (2.1), we obtain

$$\overline{d}(\mathcal{T}w_0, \mathcal{T}w_1) \leq \lambda \overline{d}(w_0, w_1)$$

that is

$$\overline{d}(w_1, w_2) \leq \lambda \overline{d}(w_0, w_1). \tag{2.2}$$

Similarly, taking $(u, v) = (w_1, w_2)$ in (2.1), we obtain

$$\overline{d}(w_2, w_3) \leq \lambda \overline{d}(w_1, w_2)$$

which implies by (2.2) that

$$\overline{d}(w_2, w_3) \leq \lambda^2 \overline{d}(w_0, w_1).$$

Continuing in the same way, by induction, we get

$$\overline{d}(w_n, w_{n+1}) \leq \lambda^n \mathsf{r}, \quad n \in \mathbb{N}, \tag{2.3}$$

where $\mathsf{r} = \overline{d}(w_0, w_1)$.

$$\begin{aligned}
\overline{d}(w_j, w_i) &\leq \mu(w_j, w_{j+1})\overline{d}(w_j, w_{j+1}) + \mu(w_{j+1}, w_i)\overline{d}(w_{j+1}, w_i) \\
&\leq \mu(w_j, w_{j+1})\overline{d}(w_j, w_{j+1}) + \mu(w_{j+1}, w_i)\mu(w_{j+1}, w_{j+2})\overline{d}(w_{j+1}, w_{j+2}) \\
&\quad + \mu(w_{j+1}, w_i)\mu(w_{j+2}, w_i)\overline{d}(w_{j+2}, w_i) \\
&\leq \mu(w_j, w_{j+1})\overline{d}(w_j, w_{j+1}) + \mu(w_{j+1}, w_i)\mu(w_{j+1}, w_{j+2})\overline{d}(w_{j+1}, w_{j+2}) \\
&\quad + \mu(w_{j+1}, w_i)\mu(w_{j+2}, w_i)\mu(w_{j+2}, w_{j+3})\overline{d}(w_{j+2}, w_{j+3}) \\
&\quad + \mu(w_{j+1}, w_i)\mu(w_{j+2}, w_i)\mu(w_{j+3}, w_i)\overline{d}(w_{j+3}, w_i) \\
&\leq \mu(w_j, w_{j+1})\lambda^j\overline{d}(w_0, w_1) + \mu(w_{j+1}, w_{j+2})\mu(w_{j+1}, w_i)\lambda^{j+1}\overline{d}(w_0, w_1) \\
&\quad + \mu(w_{j+1}, w_i)\mu(w_{j+2}, w_i)\mu(w_{j+2}, w_{j+3})\lambda^{j+2}\overline{d}(w_0, w_1) \\
&\quad + \mu(w_{j+1}, w_i)\mu(w_{j+2}, w_i)\mu(w_{j+3}, w_i)\lambda^{j+3}\overline{d}(w_0, w_1) + \dots \\
&\quad + \mu(w_{j+1}, w_i)\mu(w_{j+2}, w_i)\mu(w_{j+3}, w_i)\dots\mu(w_{i-1}, w_i)\lambda^{i-1}\overline{d}(w_0, w_1) \\
&= \mu(w_j, w_{j+1})\lambda^j\overline{d}(w_0, w_1) + \sum_{k=j+1}^{i-1} \lambda^k \overline{d}(w_0, w_1) \prod_{m=j+1}^{n} \mu(w_m, w_i)\mu(w_k, w_{k+1}) \\
&= \mu(w_j, w_{j+1})\lambda^j\mathsf{r} + \sum_{k=j+1}^{i-1} \lambda^k \mathsf{r} \prod_{m=j+1}^{n} \mu(w_m, w_i)\mu(w_k, w_{k+1}) \\
&= (\mu(w_j, w_{j+1})\lambda^j + \sum_{k=j+1}^{i-1} \lambda^k \prod_{m=j+1}^{n} \mu(w_m, w_i)\mu(w_k, w_{k+1}))\mathsf{r}.
\end{aligned}$$

Let $\overline{d}_p = \overline{d} - \psi$ be exact perturbed controlled metric.
From (2.3), we deduce that

$$\begin{aligned}
&\overline{d}_p(w_j, w_i) + \psi(w_j,\ w_i) \leq \lambda^j \mathsf{r}, \quad j \in \mathbb{N}, \\
&\overline{d}_p(w_j,\ w_i) \leq \overline{d}_p(w_j, w_i) + \psi(w_j, w_i),
\end{aligned}$$

it holds that

$$\overline{d}_p(w_j, w_i) \leq \lambda^j \mathsf{r}, \quad j \in \mathbb{N}.$$

From here, we can say that $\{w_j\}$ is a perturbed Cauchy sequence in the perturbed controlled metric space $(\Omega, \overline{d}, \psi)$.
By the completeness of the perturbed metric space $(\Omega, \overline{d}, \psi)$, we deduce that, there exists, $w^* \in \Omega$ such that

$$\lim_{j\to\infty} \overline{d}_p(w_j, w^*) = 0. \tag{2.4}$$

We now show that $w^*$ is a fixed point of $\mathcal{T}$. Since $\mathcal{T}$ is a perturbed continuous mapping, then by (2.4), we get

$$\lim_{j\to\infty} \overline{d}_p(\mathcal{T}w_j, \mathcal{T}w^*) = 0,$$

i.e.

$$\lim_{j\to\infty} \overline{d}_p(w_{j+1}, \mathcal{T}w^*) = 0. \tag{2.5}$$

Since $\overline{d}_p = \overline{d} - \psi$ is a controlled perturbed metric on $\Omega$, by the uniqueness of the limit, we get $\mathcal{T}w^* = w^*$ i.e., $w^*$ is a fixed point of $\mathcal{T}$.
We now show that $\mathcal{T}$ admits a unique fixed point.
On the contrary suppose that $u, v \in \Omega$ are two distinct fixed points of $\mathcal{T}$.
From (2.1), we have

$$\overline{d}(u, v) = \overline{d}(\mathcal{T}u, \mathcal{T}v) \leq \lambda \overline{d}(u, v),$$

which yields

$$\overline{d}_p(u, v) + \psi(u, v) \leq \lambda(\overline{d}_p(u, v) + \psi(u, v)).$$

Since $u \neq v$ then $\overline{d}_p(u, v) + \psi(u, v) \neq 0$ and the above inequality yields $\lambda \geq 1$, which contradicts the condition $\lambda \in [0, 1)$.
Consequently, $\mathcal{T}$ admits a unique fixed point. □

**Corollary 2.1.** *Let $(\Omega, \overline{d}, \psi)$ be a complete controlled metric space and be a given mapping. Assume that there exists $\lambda \in [0, 1)$ such that*

$$\overline{d}_p(\mathcal{T}u, \mathcal{T}v) \leq \lambda \overline{d}_p(u, v), \tag{2.6}$$

*for all $u, v \in \Omega$. Then $\mathcal{T}$ admits one and only one fixed point.*

*Proof.* Let $\overline{d} = \overline{d}_p$ and $\psi \equiv 0$ ($\psi(u, v) = 0$ for all $u, v \in \Omega$). Then $(\Omega, \overline{d}, \psi)$ is a perturbed controlled metric space. Furthermore, by (2.6), $\mathcal{T}$ is continuous with respect to the exact metric $\overline{d}_p$ and (2.1) holds. □

We provide an example to illustrate the Theorem 2.1.

**Example 2.3.** *Let $\Omega = [0, \infty)$, define $\overline{d} : \Omega \times \Omega \to [0, \infty)$ by*

$$\overline{d}(u, v) = \begin{cases} 0, & u = v, \\ |u - v| + (u - v)^2, & u \neq v. \end{cases}$$

*Define a control function $\mu : \Omega \times \Omega \to [1, \infty)$ by $\mu(u, v) = 1 + \max(u, v)$. So $(\Omega, \overline{d}, \mu)$ is a perturbed controlled metric space.*

***Solution:*** *Let $\overline{d}_p = |u - v|$ then*

$$\overline{d}_p(u, v) = \overline{d}(u, v) - \psi(u, v),$$

*where $\psi(u, v) = (u - v)^2 \geq 0$.*

*Thus $\overline{d}$ is a perturbation of the standard metric.*

*First we will prove $\overline{d}$ is not a metric space.*

*Let us take $u = 0, v = 1, w = 2$.*

*So by the definition of $\overline{d}$, we get*

$$\overline{d}(0, 2) = 2 + 4 = 6, \ \ \overline{d}(0, 1) = 2, \ \ \overline{d}(1, 2) = 2.$$

*But $6 \nleq 2 + 2$, so triangle inequality fails.*

*Now we will prove $\overline{d}$ is a controlled metric space.*

*Let $u, v, w \in \Omega$ and $u \neq w$ then, we have*

$$\begin{aligned} \overline{d}(u, w) &= |u - w| + (u - w)^2 \\ &= |u - v + v - w| + (u - v + v - w)^2 \\ &\leq |u - v| + |v - w| + 2(u - v)(v - w) + (u - v)^2 + (v - w)^2 \\ &\leq (1 + \max\{u, v\})(\overline{d}(u, v) + \overline{d}(v, w)). \end{aligned}$$

*Hence $(\Omega, \overline{d}, \mu)$ is a controlled metric space.*

*Let $\mathcal{T} : \Omega \to \Omega$ be $\mathcal{T}(u) = \frac{u}{3}$ then $\mathcal{T}$ is a contraction on the perturbed controlled metric space $(\Omega, \overline{d})$ as follows:*

$$\begin{aligned} \overline{d}(\mathcal{T}u, \mathcal{T}v) &= \left|\frac{u - v}{3}\right| + \left(\frac{u - v}{3}\right)^2 \\ &= \frac{1}{3}|u - v| + \frac{1}{9}(u - v)^2 \\ &\leq \frac{1}{3}(|u - v| + (u - v)^2) \\ &= \frac{1}{3}\overline{d}(u, v). \end{aligned}$$

*Thus,*

$$\overline{d}(\mathcal{T}u, \mathcal{T}v) \leq \lambda \overline{d}(u, v), \quad \lambda = \frac{1}{3} < 1.$$

*Clearly, 0 is the unique fixed point of $\mathcal{T}$.*

## 3. Application

To demonstrate the relevance of the acquired fixed point result within the context of perturbed controlled metric spaces, we provide the subsequent example:

Let $\Omega = [0,2]$ and define $\overline{d} : \Omega \times \Omega \to [0,\infty)$ such that $\overline{d}(u,v) = |u-v| + |u-v|^3$ is controlled metric with a control function $\mu : \Omega \times \Omega \to [1,\infty)$ by $\mu(u,v) = 4 + |u-v|^3$ and let us define a Perturbed function, $\psi : \Omega \times \Omega \to [0,\infty)$ by $\psi(u,v) = |u-v|$. Now let us take $\overline{d}_p = \overline{d} - \psi : \Omega \times \Omega \to [0,\infty)$ a mapping defined as $\overline{d}_p(u,v) = \overline{d}(u,v) - \psi(u,v) = |u-v|^3$, and we claim that $\overline{d}_p$ is a perturbed controlled metric space.

Firstly, we will show $\overline{d}$ is not a metric but a controlled metric.

For this, let $u = 0, v = 0.4, w = 1$.

Now $\overline{d}(u,w) = 1 + 1 = 2$,

$$\overline{d}(u,v) + \overline{d}(v,w) = (0.4 + 0.064) + (0.6 + 0.216) = 1.28.$$

Clearly,

$$\overline{d}(u,w) \nleq \overline{d}(u,v) + \overline{d}(v,w).$$

Now consider,

$$\begin{aligned} \mu(u,v)\overline{d}(u,v) + \mu(v,w)\overline{d}(v,w) &= 4.064(0.464) + 4.216(0.816) \\ &= 5.593216. \end{aligned}$$

Clearly,

$$\overline{d}(u,w) \leq \mu(u,v)\overline{d}(u,v) + \mu(v,w)\overline{d}(v,w).$$

Hence, $\overline{d}$ is not an exact metric but a controlled metric.

Since $\psi(u,v) = |u-v|$ and $\overline{d}_p(u,v) = \overline{d}(u,v) - \psi(u,v) = |u-v|^3$.

Now we will show $\overline{d}_p$ is not a metric but a perturbed controlled metric.

Let us take $u = 0, v = 1, w = 2$.

Here,

$$\begin{aligned} & \overline{d}_p(0,2) \nleq \overline{d}_p(0,1) + \overline{d}_p(1,2) \\ \Rightarrow \qquad & 8 \nleq 2 \end{aligned}$$

So $\overline{d}_p$ is not a metric.

Now let us consider

$$\begin{aligned} & \mu(0,1)\overline{d}_p(0,1) + \mu((1,2))\overline{d}_p(1,2) = 5(1) + 5(1) = 10 \\ \Rightarrow \qquad & \overline{d}_p(0,2) \leq \mu(0,1)\overline{d}_p(0,1) + \mu((1,2))\overline{d}_p(1,2). \end{aligned}$$

Hence $\overline{d}_p$ is not a metric but a perturbed controlled metric.

Now, let us consider a non linear Volterra-type integral equation

$$u(t) = \int_0^t \frac{t-s}{6+s^2}(u(s)^2 + 2)ds, \quad t \in [0,2],$$

where Kernel $K(t,s) = \frac{t-s}{6+s^2}$ is continuous.
Let us define a mapping $\mathcal{T} : C[0,2] \to C[0,2]$ by

$$\mathcal{T}u(t) = \int_0^t \frac{t-s}{6+s^2}(u(s)^2+2)ds, \quad t \in [0,2].$$

Now we will prove $\mathcal{T}$ is contractive.
Consider,

$$\begin{aligned} \overline{d}_p(\mathcal{T}u, \mathcal{T}v) &= |\mathcal{T}u - \mathcal{T}v|^3 \\ &\leq \int_0^t \left|\frac{t-s}{6+s^2}\right| |u(s)^2 - v(s)^2| ds \\ &\leq \int_0^t \left|\frac{t-s}{6+s^2}\right| |u(s) - v(s)|\,|u(s)+v(s)| ds. \end{aligned} \tag{3.1}$$

Let $|u(s)|, |v(s)| \leq \mathcal{M}$, $\forall\, s$.
Then $|u(s)+v(s)| \leq 2\mathcal{M}$.
So by equation (3.1), we get

$$\overline{d}_p(\mathcal{T}u, \mathcal{T}v) \leq 2\mathcal{M} \int_0^t \left|\frac{t-s}{6+s^2}\right| |u(s) - v(s)| ds\,. \tag{3.2}$$

Now since,

$$\left|\frac{t-s}{6+s^2}\right| \leq \frac{2}{6} = \frac{1}{3}, \quad \forall\, t,s \in [0,2].$$

Therefore by equation (3.2), we get

$$\overline{d}_p(\mathcal{T}u, \mathcal{T}v) \leq \frac{2}{3}\mathcal{M} \int_0^t |u(s) - v(s)| ds.$$

Let

$$\|u - v\| = \sup |u(s) - v(s)|, \quad \forall\, s \in [0,2].$$

Then

$$\begin{aligned} |\mathcal{T}u(t) - \mathcal{T}v(t)| &\leq \frac{2}{3}\mathcal{M}.2\|u-v\| \\ &\leq \frac{4}{3}\mathcal{M}\|u-v\|. \end{aligned}$$

If $\mathcal{M} < \frac{3}{4}$, for small bounded functions (for example during iterations then) $\frac{4}{3}\mathcal{M} < 1$ and we can write:

$$|\mathcal{T}u(t) - \mathcal{T}v(t)| \leq k|u(t) - v(t)|,$$

where $0 < k = \frac{3}{4}\mathcal{M} < 1$.
Thus, $\mathcal{T}$ is contractive self map. So the equation has a unique fixed point.

## 4. Numerical Illustration

Let us verify that sequence of iterates $u_0, u_1, u_2$ is converging to the unique fixed point at $u^*(t)$ at specific point $t = 2$.
Let $u_0(t) = 0$ and

$$\begin{aligned} & u_1 = \frac{2t}{\sqrt{6}} \tan^{-1} \frac{t}{\sqrt{6}} - \ln\left(\frac{6+t^2}{6}\right) \\ \Rightarrow \quad & u_1(2) \approx 0.61. \end{aligned}$$

Now,

$$\begin{aligned} & u_2(2) = \int_0^2 \frac{2-s}{6+s^2} (u(s)^2 + 2) ds \\ \Rightarrow \quad & u_2(2) \approx 0.645 \quad \text{(After taking several sample points between 0 to 2 of } u_1(s)\text{)}. \end{aligned}$$

The difference of values is small $|u_2(2) - u_1(2)| = 0.038$ showing quick convergence.
So, the unique fixed point value at $t = 2$ is approximately $u^*(2) \approx 0.65$.

## 5. Conclusion

In this study, we introduced the notion of Perturbed controlled metric space and established fixed point results in the setting of these spaces. Our result has generalised proven result in the past. The derived results have been supplemented with non trivial example. The result is applied to find the analytical solution to Integral equation. It will be an open problem to establish fixed point results by weakening the contractive conditions.

**Funding:** The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2025/01/34316).

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References

[1] S. Banach, Sur les Opérations dans les Ensembles Abstraits et Leur Application aux Équations Intégrales, Fundam. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.
[2] D.W. Boyd, J.S.W. Wong, On Nonlinear Contractions, Proc. Amer. Math. Soc. 20 (1969), 458–458. https://doi.org/10.1090/s0002-9939-1969-0239559-9.
[3] S. Reich, Fixed Points of Contractive Functions, Boll. Unione Mat. Ital. 5 (1972), 26–42.
[4] L.B. Ciric, A Generalization of Banach's Contraction Principle, Proc. Amer. Math. Soc. 45 (1974), 267–273. https://doi.org/10.2307/2040075.
[5] S. Czerwik, Contraction Mappings in b-Metric Spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11.
[6] A. Branciari, A Fixed Point Theorem of Banach–Caccioppoli Type on a Class of Generalized Metric Spaces, Publ. Math. Debr. 57 (2000), 31–37. https://doi.org/10.5486/pmd.2000.2133.
[7] N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled Metric Type Spaces and the Related Contraction Principle, Mathematics 6 (2018), 194. https://doi.org/10.3390/math6100194.

[8] O. Popescu, Some Remarks on the Paper "Fixed Point Theorems for Generalized Contractive Mappings in Metric Spaces", J. Fixed Point Theory Appl. 23 (2021), 72. https://doi.org/10.1007/s11784-021-00908-7.

[9] H. Huang, X. Fu, Asymptotic Properties of Solutions for Impulsive Neutral Stochastic Functional Integro-Differential Equations, J. Math. Phys. 62 (2021), 013301. https://doi.org/10.1063/1.5139964.

[10] G. Mani, R. Ramaswamy, A.J. Gnanaprakasam, A. Elsonbaty, O.A. Abdelnaby, et al., Application of Fixed Points in Bipolar Controlled Metric Space to Solve Fractional Differential Equation, Fractal Fract. 7 (2023), 242. https://doi.org/10.3390/fractalfract7030242.

[11] R. Ramaswamy, G. Mani, A.J. Gnanaprakasam, O.A. Abdelnaby, S. Radenović, Solving an Integral Equation via Tricomplex-Valued Controlled Metric Spaces, Axioms 12 (2023), 56. https://doi.org/10.3390/axioms12010056.

[12] R. Ramadan, O. Ege, R. Ramaswamy, Novel Fixed Point Results in Rectangular $G_b$-Metric Spaces and Some Applications on Fractional Differential Equations, Fractal Fract. 9 (2025), 527. https://doi.org/10.3390/fractalfract9080527.

[13] M. Kumar, N. Bhardwaj, G. Mani, R. Ramaswamy, K. Hayat Khan, et al., Fixed Point Theorems for $(\beta, \phi)$-Expansive Mappings in Controlled Metric Space, Eur. J. Pure Appl. Math. 18 (2025), 7124. https://doi.org/10.29020/nybg.ejpam.v18i4.7124.

[14] M. Penumarthy Parvateesam, C.P. Dhuri, R. Ramaswamy, K.H. Khan, O.A. Abdelnaby, Common Fixed Point Theorems for Mappings Satisfying Implicit Relation in Bipolar Metric Space, Eur. J. Pure Appl. Math. 18 (2025), 5982. https://doi.org/10.29020/nybg.ejpam.v18i2.5982.

[15] M. Jleli, B. Samet, On Banach's Fixed Point Theorem in Perturbed Metric Spaces, J. Appl. Anal. Comput. 15 (2025), 993–1001. https://doi.org/10.11948/20240242.

[16] C. Li, J. Llibre, Uniqueness of Limit Cycles for Liénard Differential Equations of Degree Four, J. Differ. Equ. 252 (2012), 3142–3162. https://doi.org/10.1016/j.jde.2011.11.002.

[17] F. Vetro, Fixed Point for $\alpha$-$\Theta$-$\Phi$-Contractions and First-Order Periodic Differential Problem, RACSAM. Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 113 (2018), 1823–1837. https://doi.org/10.1007/s13398-018-0586-9.