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Mathematical and Dynamical Analysis of Hepatitis B Model with Hybrid Operator Piecewise Classical and Fractional Derivative Insight of Different Stages of Vaccinations

Muhammad Farman[1,2,3], **Kottakkaran Sooppy Nisar**[4,*]

[1]*Faculty of Arts and Science, Department of Mathematics, Near East University, Northern Cyprus, Turkey*

[2]*Karadeniz Technical University, Faculty of Medicine, Department of Biostatistics and Medical Informatics, Trabzon, Turkey*

[3]*International Center for Interdisciplinary Research in Sciences, The University of Lahore, Lahore, Pakistan*

[4]*Department of Mathematics, College of Science and Humanities in Al Kharj, Prince Sattam bin Abdulaziz University, Al Kharj 11942, Saudi Arabia*

[*]*Corresponding author:* n.sooppy@psau.edu.sa, ksnisar1@gmail.com

Abstract. In order to observe the dynamics and impacts of vaccination at different phases, we suggested an epidemic model in this paper that includes classical and Atangana-Baleanu piecewise operators for the hybrid operator's memory effects. The created model's properties, which take into account linear growth and Lipschitz requirements relating the rate of effects within its many sub-compartments according to the equilibrium points, include positivity, unique solution, existence, and boundedness in the feasible domain. The reproductive number R_0 is derived for the biological feasibility of the model with sensitivity analysis with different parameters impacting the model for the piecewise fractional operator. Global stability employs Lyapunov function theory to gain insight into first derivative results. The chaotic behavior of the system was also determined through simulation, which shows the solution is bound to a steady state point and lies in a feasible region.For numerical solutions, classical and piecewise fractional hybrid operators at various fractional order values are solved using the Newton polynomial interpolation method.This research advances our knowledge of the dynamics of hepatitis B and offers a solid framework for assessing the effectiveness of interventions for disease control planning and decision-making.

1. Introduction

Mathematicians always strive to simulate the behaviours they see in everyday life. The notion of differentiation and integral has recently been used to successfully represent a number of significant

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biological issues. Hepatitis B is viral disease that effects the liver of a human body. It can be provoked by a variety of things, such as medicines, auto immune diseases, drinking excessive amounts of alcohol, germs, and viruses. It is a contagious disease that can result in serious complications and even death.About 1.4 million fatalities each year from prolonged infections, cirrhosis, and liver cancer linked to hepatitis are attributable to this condition. Around 47 percent of such fatalities belong to the hepatitis B virus [1]. Approximately 300 million people worldwide have chronic hepatitis B virus, making it the ninth most common cause of death in the list. It is a serious health problem throughout. This virus may cause long-lasting illnesses such liver cancer and cirrhosis, resulting to a high fatality rate. Hepatitis not only have a high sickness rate but also costs too much for medical supplies and could be harmful to national economies [2].

In light of its many applications in the mathematical modeling of dynamic systems over the past few decades, fractional derivatives have received a lot of attention.The memory and inheritance characteristics of fractional derivatives as well as their higher degree of freedom than traditional derivatives are the main factors of these applications. [3]. In [4], the generalised Mittag-Leffler fractal-fractional derivatives model of malaria illness in humans and mosquitoes was presented. According to the Caputo-Fabrizio derivative, the authors of [5] constructed a fractional order model of the co-dynamics of cholera and malaria. Nature has been shown to follow very straightforward laws, which results in the formation of complex phenomena. The research focuses on the features of respiratory lung tissue, whose natural solutions non-integer differ-integral solutions and non-integer parametric models appear in the middle of FC. In [6], the time-fractional HIV/AIDS model is explored using the modified Atangana-Baleanu in Caputo sense derivative. In order to better understand the dynamical spread of malaria, the research in [7] investigated a deterministic model that takes into account the transmission method through blood transfusion and saturation therapy function. With a constant proportional (CP) Caputo operator, Farman et al. [8] introduced a unique fractional order measles model. The Laplace Adomian decomposition (LAD) technique was used in [9] to examine the dynamics of Lassa fever transmission. Fractional Calculus has found practical applications across diverse scientific and engineering disciplines. Atangana-Baleanu (AB) fractional-order derivative proves to be better suited for describing numerous real phenomena due to its ability to retain essential information about the phenomenon as time elapses. In [10], authors developed the optimal control strategies for dengue fever by implementing ABC derivative operative. A model for smoking epidemic was examined by Farman et al. [11], utilising Caputo-Fabrizio fractional derivative. In [12], an application of (ABC) fractional derivative, predator-prey model of three species. The COVID-19 model was examined by Khan et al. [13] utilising Caputo and classical piecewise operators. Atangana et al. [14, 15] proposed a method based on piecewise differential and integral operators. In [16], researchers investigated the dynamics of a piecewise COVID-19 mathematical model with a vaccination and quarantine class. [17] proposed a deterministic compartmental model of hepatitis B disease with vaccination and treatment interventions. In order to illustrate the behaviour of the co-infection of the HIV-TB model, the study in reference [18] employed a piecewise operator in the traditional Caputo sense and some hybrid operator also studied in [19, 20, 23]. When just initial conditions are utilised to forecast future behaviours of the spread, the Hepatitis B model with classical differentiation may be used to illustrate dynamical systems of infectious illness. Yet in unforeseen situations, maybe as a result of ambiguity in real-world problems, classical differentiation and related integral operators fall short. Given all the unknowns, misunderstandings, and ambiguities surrounding the hepatitis B model, it is extremely challenging to develop a proper mathematical model using classical differentiation. In these situations, non-local operators are frequently more suitable since they may capture non-localities and specific memory effects depending on whether there are power law, fading memory, or crossover effects. On the other hand, if there are other more complex behaviours that cannot be replicated using power law, fading memory, and crossover [17],then the recently developed fractal fractional operators may be more suitable mathematical tools to manage such behaviours. Although it has been established that these new operators are adequate for simulating complicated issues, concerns have been raised since the operators cannot manage time at the origin.

2. Basic Concepts of Fractional Operator

In this part, we'll introduce hybrid fractal fractional operators, go over their characteristics, and show how to approximate them numerically using tried-and-true methods like Newton and Lagrange polynomials. In this paper, section 1, is the introduction. Section 2 describes the basic definitions, section 3, describes the model, qualitative analysis, pose ability and sensitivity analysis. In section 4, we verify the Lipschitz conditions and Lyapunov Stability analysis. In section 5, we examine the numerical scheme for the given model. In section 6 and 7 we discuss the results and conclusion, respectively.

Definition 2.1. *Let F(x) be a function that isn't always differentiable. Let η be a fractional order and $0 < \eta \leq 1$. According to [19, 20], a fractal fractional derivative (FFD) of order $0 < \eta \leq 1$ with classical and Mittag-Leffler kernel is defined as*

$$ {}_{0}^{PAB}\mathrm{D}_t^{\eta}F(t) = \begin{cases} F'(t),\ 0 < t < t_1, \\ {}_{0}^{PAB}\mathrm{D}_t^{\eta}F(t),\ t_1 < t < t_2. \end{cases} \tag{2.1}$$

Definition 2.2. *Let F(x) be a function that isn't always differentiable. Let η be a fractional order and $0 < \kappa \leq 1$. According to [18], a fractal fractional integral (FFI) of order $0 < \eta < 1$ with Mittag-Leffler kernel is defined as*

$$ {}_{0}^{PAB}\mathrm{J}_t^{\eta}F(t) = \begin{cases} \int -0^{t_1} F(\vartheta)d\vartheta,\ 0 < t < t_1, \\ \frac{1-\eta}{AB(\eta)} + \frac{\eta}{AB(\eta)\Gamma(\eta)} \int -t_1^{t_2} (\mathrm{t} - \#)^{\eta-1} F(\vartheta)d\vartheta,\ t_1 < t < t_2. \end{cases} \tag{2.2}$$

Where ${}_{0}^{PAB}\mathrm{J}_t^{\kappa}$ stands for the Mittag-Leffler kernel fractional integral on $t_1 < t < t_2$, and the classical integral on $0 < t < t_1$.

3. Fractional Order Hepatitis B Model

We constructed a compartmental mathematical model of the virus known as hepatitis B based on its routes of transmission which is proposed by Malede Atnaw Belay et al. [21] is motivation to developing modified form with fractional operator. The model will place precise emphasis on key factors such as the reproduction number, equilibrium points, and stability of Hepatitis B. Moreover the study will delve into the dynamical transmission and sensitivity analysis. Each compartment of the model will be broadly examined to gain a complete understanding of its behavior. The disease dynamics will be extensively explored through numerical simulations of the suggested approach, illustrating the steady progress of the solution under this approach. Furthermore, to enhance the understanding of Hepatitis B, the proposed model will undergo comprehensive numerical simulations using a piecewise Fractional Operator with $0 < \beta \leq 1$.

$$\begin{cases} {}_0^{PABC}\mathrm{D}_t^{\eta}(S(t)) = \bigwedge(1-\omega)(1-\tau C) + \theta_2 V_1 - (\lambda+\mu+\theta_1)S, \\ {}_0^{PABC}\mathrm{D}_t^{\eta}(V_1(t)) = \theta_1 S + \bigwedge\omega - (a+\mu+\theta_2)V_1, \\ {}_0^{PABC}\mathrm{D}_t^{\eta}(V_2(t)) = \alpha V_1 - (\epsilon+\mu)V_2 \\ {}_0^{PABC}\mathrm{D}_t^{\eta}(A(t)) = \lambda S - (\delta+\sigma+\mu)A, \\ {}_0^{PABC}\mathrm{D}_t^{\eta}(C(t)) = \delta A + \wedge(1-\omega)\tau C - (\vartheta+\mu+\psi)C, \\ {}_0^{PABC}\mathrm{D}_t^{\eta}(R(t)) = \delta A + \vartheta C - \mu R, \end{cases} \tag{3.1}$$

With the initial conditions

$$S(0) = S_0 > 0, V_1(0) = V_{10} > 0, V_2(0) = V_{20} > 0, A(0) = A_0 > 0, C(0) = C_0 > 0, R(0) = R_0 > 0. \tag{3.2}$$

They all are biologically positive. The suggested model divides the total population N into six divisions at a given period t. We created the fractional order mathematical version for the Hepatitis B virus model with vaccination in order to deal with the hepatitis B virus model's greatest potential $S(t)$ those who are susceptible to infection but have not yet contracted the disease , $V_1(t)$ are those who have received the first dose of vaccination,$V_2(t)$ are people who have taken second dose of vaccination after a month of the first dose, $A(t)$ are individuals who are at the acute stage of infection ,$C(t)$ are infected individuals who are at chronic stage of infection and $R(t)$ are people who recovered from the disease. Details of other parameters is given in table 1 [21].

Table 1. Parameter values and biological interpretation

Parameter	Description
Λ	Birth rate
μ	Natural mortality death
σ	Acute class recovery
θ	Chronic Class Recovery with Treatment
δ	Acute class switches to chronic class by symptoms growth
α	Second dose vaccine receivers
ϵ	Recovery after receiving second dose
ψ	Deaths due to chronic diseases
ϕ_1	Susceptible first dose of vaccine
ϕ_2	Waning rate of vaccine from vaccinated first dose class

3.1. **Qualitative Analysis of the Hepatitis B Model.** We are currently performing a qualitative analysis of our model of infectious disease with vaccination, which is represented by a set of nonlinear differential equations (3), to gain insights into its features and to better comprehend the variables that govern the dynamics of infectious disease transmission.

3.2. **Pose-ability and Biological Suitability of the Model.** This section examines the historical context in which the solution to our system makes sense. All solutions and proposed parameters have previously been demonstrated to be positive for all t. We are aware that $t > 0$ for everyone.

$$ {}_{0}^{PABC}\mathrm{D}_{t}^{\eta}(N(t)) = \wedge - N\mu - C\psi. \tag{3.3} $$

Hence, if there is no sickness,

$$ {}_{0}^{PABC}\mathrm{D}_{t}^{\eta}(N(t)) = \wedge - N\mu \tag{3.4} $$

The total population N(t) must be a positively growing function with the condition that $\frac{dN}{dt} > 0$.

$$ {}_{0}^{PABC}\mathrm{D}_{t}^{\eta}(N(t)) \geq N(t) < \frac{\wedge}{\psi}. \tag{3.5} $$

The threshold population level is described in the literature as the aforementioned inequality. Due to this, we conclude that the recognized set of solutions for the proposed model should only include

$$ \Omega = \{(S, V_1, V_2, A, C, R) \in R_{+}^{6} : S + V_1 + V_2 + A + C + R = N(t) < N(t) < \frac{\wedge}{\psi}\} \tag{3.6} $$

In this case, R_{+}^{6} represents the positive cone of R_{+}^{6}, which also comprises its faces in the lower dimensions. We rule out the scenario in which ${}_{0}^{PABC}\mathrm{D}_{t}^{\eta}(N(t)) \geq 0$ takes place, it might mean that the host group will eventually decline to its carrying capabilities.

Theorem 3.1. *The region $\Im \in R_{+}^{6}$,*

$$ \Im = S, V_1, V_2, A, C, R \in R_{+}^{6} : N \leq frac \wedge - C\psi\mu $$

attracts all solutions of proposed system and is positively invariant when initial restrictions are not negative.

The results are provided below, and shall demonstrate the system's positive solutions.

$$ \begin{cases} {}_{0}^{PABC}\mathrm{D}_{t}^{\eta}(S(t))|_{S=0} = \bigwedge (1-\omega)(1-\tau C) + \theta_2 V_1 \geq 0, \\ {}_{0}^{PABC}\mathrm{D}_{t}^{\eta}(V_1(t))|_{V_1=0} = \theta_1 S + \bigwedge \omega \geq 0, \\ {}_{0}^{PABC}\mathrm{D}_{t}^{\eta}(V_2(t))|_{V_2=0} = \alpha V_1 \geq 0, \\ {}_{0}^{PABC}\mathrm{D}_{t}^{\eta}(A(t))|_{A=0} = \lambda S, \geq 0, \\ {}_{0}^{PABC}\mathrm{D}_{t}^{\eta}(C(t))|_{C=0} = \delta A +, \geq 0, \\ {}_{0}^{PABC}\mathrm{D}_{t}^{\eta}(R(t))|_{R=0} = \delta A + \vartheta C \geq 0, \end{cases} \tag{3.7} $$

The vector field is said to be located in the region R_+^6 on each hyperplane encompassing the non-negative orthant with $t \geq 0$ according to the system (22). Model (3) component elements of the population are added together to yield the following total population.

$$ {}_0^{PABC}\mathrm{D}_t^\eta N(t) = {}_0^{PABC}\mathrm{D}_t^\eta(S(t)) + {}_0^{PABC}\mathrm{D}_t^\eta(V_1(t)) + {}_0^{PABC}\mathrm{D}_t^\eta(V_2(t)) + {}_0^{PABC}\mathrm{D}_t^\eta(A(t)) \tag{3.8}$$
$$ + {}_0^{PABC}\mathrm{D}_t^\eta(C(t) + {}_0^{PABC}\mathrm{D}_t^\eta(R(t)) \tag{3.9}$$

$$ {}_0^{PABC}\mathrm{D}_t^\eta(N(t)) = \wedge - N\mu - C\psi$$
$$ Supposethat N(0) \leq frac \wedge - C\psi\mu$$
$$ N(t) \leq frac \wedge - C\psi\mu$$

Therefore, a fractional model solution continues to exist in $\mathfrak{I}$ for all $t > 0$. In light of the fractional model, the closed set $\mathfrak{I}$ is a positively invariant. As a result we may test our model, in the realm of possibility;

$$\mathfrak{I} = S, V_1, V_2, A, C, R \in R_+^6 : N(t) \leq frac \wedge - C\psi\mu$$

3.3. **Disease-Free Equilibrium and analysis of reproductive number.** The disease-free equilibrium (DFE) of model only exists in the absence of infections. Now, solve for DFE by setting the left-hand side of system equal to zero. Since $A = C = 0$ at disease-free equilibrium then the system becomes

$$\begin{cases} \bigwedge (1-\omega)(1-\tau C) + \theta_2 V_1 - (\lambda + \mu + \theta_1) = 0, \\ \theta_1 S + \bigwedge \omega - (a + \mu + \theta_2) V_1 = 0, \\ \alpha V_1 - (\epsilon + \mu) V_2 = 0, \\ \lambda S - (\delta + \sigma + \mu) A = 0, \\ \delta A + \bigwedge (1-\omega)\tau C - (\vartheta + \mu + \psi) C = 0, \\ \delta A + \vartheta C - \mu R = 0, \end{cases} \tag{3.10}$$

Thus, from system (8), the DFE of the model is given by
$E_0 = S^0, V_1^0, V_2^0, A^0, C^0, R^0$

$$\left(\frac{\bigwedge(1-\omega\nu)}{\mu\iota + \theta_1\nu}, \frac{\bigwedge(\theta_1 + \omega\mu)}{\mu\iota + \theta_1\nu}, \frac{\bigwedge \alpha(\theta_1 + \omega\mu)}{(\epsilon+\mu)(\mu\iota + \theta_1\nu)}, 0, 0, \frac{\bigwedge \epsilon\alpha(\theta_1 + \mu\omega)}{\mu(\epsilon+\mu)(\mu\iota + \theta_1\nu)}\right) \tag{3.11}$$

where

$$\begin{cases} \iota = \alpha + \mu + \theta_2, \\ \nu = \alpha + \mu, \\ \alpha + \mu < \omega(\alpha + \mu). \end{cases} \tag{3.12}$$

Lets look at [22], reproductive numbers are

$$R_0 = \zeta S_0\left(\frac{\eta}{\delta + \sigma + \mu} + \delta\right)(\delta + \sigma + \mu)\left(\vartheta + \mu + \psi - \bigwedge(1-\omega)\tau\right), \tag{3.13}$$

$$R_0 = \frac{\zeta \wedge (\alpha + \mu + \theta_2 - \omega\alpha - \omega\mu)}{\mu(\alpha + \mu + \theta_2) + \theta_1(\alpha + \mu)} \cdot \left(\frac{k}{\delta + \sigma + \mu} + \frac{\delta}{(\delta + \sigma + \mu)(\vartheta + \mu + \psi - \wedge(1-\omega)\tau)}\right) \tag{3.14}$$

Here, we construct the analysis of reproductive number by taking partial derivative with respect to each parameters and range of impact of parameters also shown in figure 1.

$$\frac{\partial R_0}{\partial \wedge} = \zeta \left(\frac{k}{\delta + \mu + \sigma} + \frac{\delta}{(\delta + \mu + \sigma)(\psi + \mu + \vartheta + \tau \wedge (\omega - 1))} \right) (\alpha + \mu + \theta_2) > 0$$

Similarly for others parameters such as $\tau, \theta_2, \delta, \vartheta, \zeta$ is increasing and $\omega, \theta_1, \psi, \mu, \sigma$, is decreasing. The sensitivity indices of the above parameters, shown in the graph below:

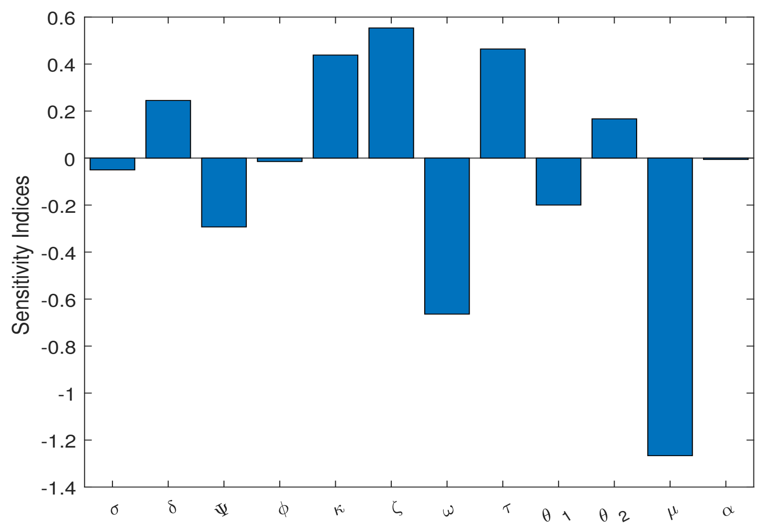

Figure 1. PRCC results for significance of parameters involved in R_0.

4. Piecewise Atangana-Baleanu hepatitis B Model

we can express the given equation as:

$$\begin{cases} {}_0^{PAB}\mathrm{D}_t^{\beta}(S(t)) = \begin{cases} \mathrm{D}_t^{\eta}(S(t)) = \mathfrak{I}_1(S, V_1, V_2, A, C, R), 0 < t < t_1, \\ {}_0^{AB}\mathrm{D}_t^{\eta}(S(t)) =^{AB} \mathfrak{I}_1(S, V_1, V_2, A, C, R), t_1 < t < T. \end{cases} \\ {}_0^{PAB}\mathrm{D}_t^{\beta}(V_1(t)) = \begin{cases} \mathrm{D}_t^{\eta}(V_1(t)) = \mathfrak{I}_2(S, V_1, V_2, A, C, R), 0 < t < t_1, \\ {}_0^{AB}\mathrm{D}_t^{\eta}(V_1(t)) =^{AB} \mathfrak{I}_2(S, V_1, V_2, A, C, R), t_1 < t < T. \end{cases} \\ {}_0^{PAB}\mathrm{D}_t^{\eta}(V_2(t)) = \begin{cases} \mathrm{D}_t^{\eta}(V_2(t)) = \mathfrak{I}_3(S, V_1, V_2, A, C, R), 0 < t < t_1, \\ {}_0^{AB}\mathrm{D}_t^{\eta}(V_2(t)) =^{AB} \mathfrak{I}_3(S, V_1, V_2, A, C, R), t_1 < t < T. \end{cases} \\ {}_0^{PAB}\mathrm{D}_t^{\beta}(A(t)) = \begin{cases} \mathrm{D}_t^{\eta}(A(t)) = \mathfrak{I}_4(S, V_1, V_2, A, C, R), 0 < t < t_1, \\ {}_0^{AB}\mathrm{D}_t^{\eta}(A(t)) =^{AB} \mathfrak{I}_4(S, V_1, V_2, A, C, R), t_1 < t < T. \end{cases} \\ {}_0^{PAB}\mathrm{D}_t^{\eta}(C(t)) = \begin{cases} \mathrm{D}_t^{\eta}(C(t)) = \mathfrak{I}_5(S, V_1, V_2, A, C, R), 0 < t < t_1, \\ {}_0^{AB}\mathrm{D}_t^{\eta}(C(t)) =^{AB} \mathfrak{I}_5(S, V_1, V_2, A, C, R), t_1 < t < T. \end{cases} \\ {}_0^{PAB}\mathrm{D}_t^{\eta}(R(t)) = \begin{cases} \mathrm{D}_t^{\beta}(R(t)) = \mathfrak{I}_6(S, V_1, V_2, A, C, R), 0 < t < t_1, \\ {}_0^{AB}\mathrm{D}_t^{\eta}(R(t)) =^{AB} \mathfrak{I}_6(S, V_1, V_2, A, C, R), t_1 < t < T. \end{cases} \end{cases} \tag{4.1}$$

We will now determine the existence and uniqueness of a solution for the hypothetical piecewise derivable function and its specific solution property. We can use system (22) for further clarification, we have to prove that $\forall[0, W_1]$ and $[W_1, W_2]\mathfrak{I}_i(S, V_1, V_2, A, C, R)$ satisfy where $i = 1, 2, 3, 4$.
A_1 is the condition for Linear Growth [22]

$$|\mathfrak{I}_i(x_i, t)|^2 = K_i(1 + |x_i|^2). \tag{4.2}$$

A_2 The Lipschitz condition [22]

$$|\mathfrak{I}_i(x_i^1, t) - \mathfrak{I}_i(x_i^2, t)|^2 \leq \hat{K}_i|x_i^1 - x_i^2|^2. \, fori = 1, 2, 3, 4. \tag{4.3}$$

We know determine the norm as, $\|\varphi\|_\infty = \sup_{t\in D\varphi}|\varphi(t)|$. Now, we are going to discuss piecewisely the existence and uniqueness of the solution for $[0, W2]$. For $[0, W2]$, there are four positive constants $M_1 < \infty$, $M_2 < \infty$, $M_3 < \infty$, and $M_4 < \infty$ such that
$\|S\|_\infty < M_1$, $\|V_1\|_\infty < M_2$, $\|V_2\|_\infty < M_3$, $\|A\|_\infty < M_4$, $\|C\|_\infty < M_5$, and $\|R\|_\infty < M_6$. Write system as below

$$\begin{cases} \dot{S} = \mathfrak{I}_1(S, V_1, V_2, A, C, R),\ 0 \leq t \leq W_2, \\ \dot{V}_1 = \mathfrak{I}_2(S, V_1, V_2, A, C, R),\ 0 \leq t \leq W_2, \\ \dot{V}_2 = \mathfrak{I}_3(S, V_1, V_2, A, C, R),\ 0 \leq t \leq W_2, \\ \dot{A} = \mathfrak{I}_4(S, V_1, V_2, A, C, R),\ 0 \leq t \leq W_2, \\ \dot{C} = \mathfrak{I}_5(S, V_1, V_2, A, C, R),\ 0 \leq t \leq W_2, \\ \dot{R} = \mathfrak{I}_6(S, V_1, V_2, A, C, R),\ 0 \leq t \leq W_2, \end{cases} \tag{4.4}$$

For proof, we consider the function

$$\begin{aligned} |\mathfrak{I}_1(S, V_1, V_2, A, C, R)|^2 &= |\wedge(1-\omega)(1-\tau C) + \theta_2 V_1 - (\lambda+\mu+\theta_1)S|^2, \\ &\leq 3\wedge^2(1-\omega)^2(1-\tau \mid C\mid)^2 + 3\theta_2^2 \mid V_1\mid^2 + 3(\lambda+\mu+\theta_1)^2 \mid S\mid^2, \\ &\leq 3\wedge^2(1-\omega)^2(1-\tau sup_{t\in D_\varphi}\mid C\mid)^2 + 3\theta_2^2 sup_{t\in D_\varphi}\mid V_1\mid^2 + 3(\lambda+\mu+\theta_1)^2\mid S\mid^2, \\ &\leq 3\wedge^2(1-\omega)^2(1-\tau \parallel C\parallel_\infty)^2 + 3\theta_2^2 \parallel V_1\parallel_\infty^2 + 3(\lambda+\mu+\theta_1)^2\mid S\mid^2, \\ &\leq 3\wedge^2(1-\omega)^2(1-\tau\parallel C\parallel_\infty)^2 + 3\theta_2^2\parallel V_1\parallel_\infty^2 \{1 + \frac{3(\lambda+\mu+\theta_1)^2\mid S\mid^2}{\leq 3\wedge^2(1-\omega)^2(1-\tau\parallel C\parallel_\infty)^2 + 3\theta_2^2\parallel V_1\parallel_\infty^2}\} \end{aligned} \tag{4.5}$$

under the condition that

$$\{1 + \frac{3(\lambda+\mu+\theta_1)^2}{\leq 3\wedge^2(1-\omega)^2(1-\tau\parallel C\parallel_\infty)^2 + 3\theta_2^2\parallel V_1\parallel_\infty^2}\mid S\mid^2\} < 1,$$

then we have

$$\mid \mathfrak{I}_2(S, V_1, V_2, A, C, R)\mid \leq K_1(1+\mid S\mid^2)$$

using the same routine

$$\begin{aligned} \mid \mathfrak{I}_2(S, V_1, V_2, A, C, R)\mid^2 &= \mid \theta_1 S + \wedge\omega - (\alpha+\mu+\theta_2)V_1\mid^2 \\ &\leq 2\theta_1^2\mid S\mid^2 + 2(\wedge\omega)^2 + 2(\alpha+\mu+\theta_2)^2\mid V_1\mid^2 \\ &\leq 2\theta_1^2 sup_{t\in D_\varphi} S\mid^2 + 2(\wedge\omega)^2 + 2(\alpha+\mu+\theta_2)^2\mid V_1\mid^2 \end{aligned}$$

$$\leq 2\theta_1^2 \parallel S \parallel_\infty^2 + 2(\wedge\omega)^2 + 2(\alpha+\mu+\theta_2)^2 \mid V_1 \mid^2$$

$$\leq \theta_1^2 \parallel S \parallel_\infty^2 + (\wedge\omega)^2\{1 + \frac{(\alpha+\mu+\theta_2)^2 \mid V_1 \mid^2}{\theta_1^2 \parallel S \parallel_\infty^2 + (\wedge\omega)^2}\}$$

under the condition that

$$\{\frac{(\alpha+\mu+\theta_2)^2 \mid^2}{{\theta_1}^2 \parallel S \parallel_\infty^2} + (\wedge\omega)^2 \mid V_1\} < 1,$$

then we have

$$\mid \Im_2(S, V_1, V_2, A, C, R) \mid \leq K_2(1+ \mid V_1 \mid^2)$$

For the function $\Im_3(S, V_1, V_2, A, C, R)$ so

$$\mid \Im_3(S, V_1, V_2, A, C, R) \mid^2 = \mid \alpha V_1 - (\epsilon+\mu)V_2 \mid^2$$

$$\leq 2\alpha^2 \mid V_1 \mid^2 + 2(\epsilon+\mu)^2 \mid V_2 \mid^2$$

$$\leq 2\alpha^2 sup_{t\in D_\varphi} \mid V_1 \mid^2 + 2(\epsilon+\mu)^2 \mid V_2 \mid^2$$

$$\leq 2\alpha^2 \parallel V_1 \parallel^2{}_\infty + 2(\epsilon+\mu)^2 \mid V_2 \mid^2$$

$$\leq \alpha^2 \parallel V_1 \parallel^2{}_\infty \leq \{1 + \frac{(\epsilon+\mu)^2}{\alpha^2 \parallel V_1 \parallel^2{}_\infty} \mid V_2 \mid^2\}$$

under the condition that $\frac{(\epsilon+\mu)^2}{\alpha^2 \|V_1\|^2{}_\infty} \mid V_2 \mid^2 < 1$,then we have

$$\mid \Im_3(S, V_1, V_2, A, C, R) \mid \leq K_3(1+ \mid V_2 \mid^2)$$

For the function $\Im_4(S, V_1, V_2, A, C, R)$ so,

$$\mid \Im_4(S, V_1, V_2, A, C, R) \mid^2 = \mid \lambda S - (\delta+\sigma+\mu)A \mid^2,$$

$$\leq 2\lambda^2 sup_{t\in D_\varphi} \mid S \mid^2 + 2(\delta+\sigma+\mu)^2 \mid A \mid^2$$

$$\leq 2\lambda^2 \parallel S \parallel^2{}_\infty + 2(\delta+\sigma+\mu)^2 \mid A \mid^2$$

$$\leq \lambda^2 \parallel S \parallel^2{}_\infty \{1 + \frac{(\delta+\sigma+\mu)^2}{\lambda^2 \parallel S \parallel^2{}_\infty} \mid A \mid^2\}$$

under the condition that $\frac{(\delta+\sigma+\mu)^2}{\lambda^2 \|S\|^2{}_\infty} < 1$, then we have

$$\mid \Im_4(S, V_1, V_2, A, C, R) \mid \leq K_4(1+ \mid A \mid^2)$$

for the function $\Im_5(S, V_1, V_2, A, C, R)$, So

$$\mid \Im_5(S, V_1, V_2, A, C, R) \mid^2 = \mid \delta A + \wedge(1-\omega)\tau C - (\vartheta+\mu+\psi) \mid^2$$

$$\leq 2\delta^2 \mid A \mid^2 + 2 \wedge^2 (1-\omega)^2 \tau^2 \mid C \mid^2 + 2(\vartheta+\mu+\psi)^2$$

$$\leq 2\delta^2 sup_{t\in D_\varphi} \mid A \mid^2 + 2\wedge^2(1-\omega)^2\tau^2 \mid C \mid^2 + 2(\vartheta+\mu+\psi)^2$$

$$\leq 2\delta^2 \parallel A \parallel^2{}_\infty + 2\wedge^2(1-\omega)^2\tau^2 \mid C \mid^2 + 2(\vartheta+\mu+\psi)^2$$

$$\leq \delta^2 \parallel A \parallel^2{}_\infty + (\vartheta+\mu+\psi)^2\{1 + \frac{\wedge^2(1-\omega)^2\tau^2}{\delta^2 \parallel A \parallel^2{}_\infty + (\vartheta+\mu+\psi)^2} \mid C \mid^2\}$$

under the condition that $\frac{\wedge^2(1-\omega)^2\tau^2}{\delta^2\|A\|^2{}_\infty+(\vartheta+\mu+\psi)^2} < 1$, then we have

$$| \Im_5(S, V_1, V_2, A, C, R) | \leq K_5(1+ | C |^2)$$

for the final function $\Im_6(S, V_1, V_2, A, C, R)$, So

$$\begin{aligned} | \Im_6(S, V_1, V_2, A, C, R) |^2 &= | \delta A + \vartheta C - \mu R |^2 \\ &\leq 3\delta^2 | A |^2 + 3\vartheta^2 | C |^2 + 3\mu^2 | R |^2 \\ &\leq 3\delta^2 sup_{t\in D_\varphi} | A |^2 + 3\vartheta^2 sup_{t\in D_\varphi} | C |^2 + 3\mu^2 | R |^2 \\ &\leq 3\delta^2 \| A \|_\infty^2 + 3\vartheta^2 \| C \|_\infty^2 + 3\mu^2 | R |^2 \\ &\leq \delta^2 \| A \|_\infty^2 + \vartheta^2 \| C \|_\infty^2 \{1 + \frac{\mu^2}{\delta^2 \| A \|_\infty^2 + \vartheta^2 \| C \|_\infty^2} | R |^2\} \end{aligned}$$

under the condition that $\frac{\mu^2}{\delta^2\|A\|_\infty^2+\vartheta^2\|C\|_\infty^2} < 1$, then we have

$$| \Im_6(S, V_1, V_2, A, C, R) | \leq K_5(1+ | R |^2)$$

The condition of linear growth is confirmed if

$$max\begin{cases} \frac{(\lambda+\mu+\theta_1)^2}{\leq\wedge^2(1-\omega)^2(1-\tau\|C\|_\infty)^2+\theta_2^2\|V_1\|_\infty^2}, \\ \frac{(\alpha+\mu+\theta_2)^2}{\theta_1^2\|S\|_\infty^2+(\wedge\omega)^2}, \\ \frac{(\delta+\sigma+\mu)^2}{\lambda^2\|S\|^2{}_\infty}, \\ \frac{(\delta+\sigma+\mu)^2}{\lambda^2\|S\|^2{}_\infty}, \\ \frac{\wedge^2(1-\omega)^2\tau^2}{\delta^2\|A\|^2{}_\infty+(\vartheta+\mu+\psi)^2}, \\ \frac{\mu^2}{\delta^2\|A\|_\infty^2+\vartheta^2\|C\|_\infty^2}, \end{cases} \tag{4.6}$$

Now, The Lipschitz condition for equations has to be verified.
for the function $\Im_1(S, V_1, V_2, A, C, R)$

$$\begin{aligned} \Im_1(S^1, V_1, V_2, A, C, R) - \Im_1(S^2, V_1, V_2, A, C, R) &\leq (\frac{(\lambda+\mu+\theta_1)}{\wedge(1-\omega)(1-\tau C)+\theta_2 V_1}) | S^1 - S^2 |, \\ &\leq \widehat{K_1} | S^1 - S^2 | . \end{aligned}$$

for the function $\Im_2(S, V_1, V_2, A, C, R)$

$$\begin{aligned} \Im_2(S, {V_1}^1, V_2, A, C, R) - \Im_2(S, {V_1}^2, V_2, A, C, R) &\leq \frac{(\alpha+\mu+\theta_2)}{\theta_1 S + (\wedge\omega)} | V_1^1 - V_1^2 | \\ &\leq \widehat{K_2} | {V_1}^1 - {V_1}^2 | . \end{aligned}$$

for the function $\Im_3(S, V_1, V_2, A, C, R)$

$$\begin{aligned} \Im_3(S, V_1, V_2^1, A, C, R) - \Im_3(S, V_1, V_2^2, A, C, R) &\leq \frac{(\delta+\sigma+\mu)}{\lambda^2 S} | V_2^1 - V_2^2 |, \\ &\leq \widehat{K_3} | V_2^1 - V_2^2 | . \end{aligned}$$

for the function $\Im_4(S, V_1, V_2, A, C, R)$

$$\Im_4(S, V_1, V_2, A^1, C, R) - \Im_4(S, V_1, V_2, A^2, C, R) \leq \frac{(\delta + \sigma + \mu)}{\lambda S} | A^1 - A^2 |,$$

$$\leq \widehat{K_4} | A^1 - A^2 |.$$

for the function $\Im_5(S, V_1, V_2, A, C, R)$

$$\Im_5(S, V_1, V_2, A, C^1, R) - \Im_5(S, V_1, V_2, A, C^2, R) \leq \frac{\wedge(1 - \omega)\tau}{\delta A + (\vartheta + \mu + \psi)} | C^1 - C^2 |,$$

$$\leq \widehat{K_5} | C^1 - C^2 |.$$

for the function $\Im_4(S, V_1, V_2, A, C, R)$

$$\Im_5(S, V_1, V_2, A, C, R^1) - \Im_5(S, V_1, V_2, A, C, R^1) \leq \frac{\mu}{\delta A + \vartheta C} | R^1 - R^2 |,$$

$$\leq \widehat{K_6} | R^1 - R^2 |.$$

We completed the proof by confirming the Lipschitz condition.

Now we have to proof the last part of the piecewise equation. In this section,we are looking $\forall t \in [W_2, W]$. In the model, we assume for $\{S(t), V_1(t), V_2(t), A(t), C(t), R(t)\} \in [W_2, \Psi_e)$, where Ψ_e shows the timing of explosion. Now we need to show that the system solution is global. That's why we have to prove that $\Psi_e = \infty$.

Let us consider $l_0 \in \mathbb{F}_4^+$ has a positive value such that $\{S(W_2), V_1(W_2), V_2(W_2), A^(W_2), C^(W_2), R^(W_2)\}$ lies within $[\frac{1}{l_0}, l_0]$. The stopping time is as

$$\Psi_l = \{t \in [W_2, \Psi_e) : \frac{1}{l} \geq \min\{S(t), V_1(t), V_2(t), A(t), C(t), R(t)\} \text{ or } \max\{S(t), V_1(t), V_2(t), A(t),$$

$$C(t), R(t)\} \geq l\},$$

For each $l \geq l_0$, while as $l \to \infty$, Ψ_l is monotonically increasing. $\lim_{l\to\infty} \Psi_\infty$ with $\Psi_e \geq \Psi_\infty$ for all $t \geq 0$, if we show that $\Psi_\infty = 0$, then we can conclude that $\Psi_e = \infty$ and $\{(S(t), V_1(t), V_2(t), A(t), C(t), R(t)\} \in \mathbb{F}_6^+$ is a solution.

So, we need to prove that $\Psi_e = \infty$. If the conclusion is contradictory, then $\exists\, 0 < W$ and $\varepsilon \in (0, 1)$ such that $P\{(W) \geq \Psi_\infty) > \varepsilon$. A function has now been defined. $H(X) : \mathbb{F}_6^+ \to \mathbb{F}_6^+$ in $H \in L^2$ such that

$$\overline{H}(X) = dH(X) = \sum_{i=1}^{6} (1 - \frac{1}{x_i}) dx_i + \sum_{i=1}^{6} \iota_i (x_i - 1) dB_i(t),$$

$$= \sum_{i=1}^{4} (1 - \frac{1}{x_i}) x_i' + \sum_{i=1}^{4} \iota_i (x_i - 1) dB_i(t),$$

where

$y_1 = S(t), y_2 = V_1(t), y_3 = V_2(t), y_4 = A(t), y_5 = C(t), y_6 = R(t).$

$$\iota_i = (\iota_1, \iota_2, \iota_3, \iota_4, \iota_5, \iota_6)$$

$$B_i(t) = (B_1(t), B_2(t), B_3(t), B_4(t), B_5(t), B_6(t)).$$

In our model, we obtained $\overline{H}(X)$ by following equality

$$\overline{H}(X) = \sum_{i=1}^{6}(1-\frac{1}{x_i})x_i' = (1-\frac{1}{S})S' + (1-\frac{1}{V_1})V_1' + (1-\frac{1}{V_2})V_2' + (1-\frac{1}{A})A' + (1-\frac{1}{C})C'$$
$$+ (1-\frac{1}{R})R' + \sum_{i=1}^{6}\frac{\iota_i^2}{2},$$
$$< 3\xi + \pi + \varpi + \phi = L,$$

and

$$\overline{H}(X) = Ldt + \sum_{i=1}^{6}\iota_i(x_i-1)dB_i(t),$$

By taking integration from $0 to \Psi_l^W$, we have

$$\Phi[\overline{H}(\Psi_l^W)] \leq \overline{H}(X(W_2)) + \Phi\int_0^{\Psi_l^W} L,$$
$$\leq \overline{H}(X(W_2)) + \Gamma W,$$

setting $\Theta_l = \{W > \Psi_l\}$ for $l_1 \leq l$ and thus $P(\Theta_1) \geq \Gamma$.Nothing that for $\forall Y \in \Theta_l$, At least one $X(\Psi_l, W)$ must be exist there and it is equal to $\frac{1}{l}$ or l. Then $l - logl$ or $\frac{1}{l} + logl - 1$ as result

$$(\frac{1}{l} + logl - 1)^{l-logl-1} < \overline{H}(X(\Psi_l)).$$

We can write from above

$$\overline{H}(X(W_2)) + LW > \Phi(l_{Y_1}\overline{H}(X(\Psi))),$$
$$\geq \Psi(l - logl - 1)^{\frac{1}{l}+logl-1}.$$

Here l_{Y_1} is a function known as the indicator function of Y. Hence $lim_{l\to\infty}$ shows

$$\infty > \overline{H}(X(W_2)) + LW = 0.$$

It is contradictory. So, giving the earlier described conditions $\Psi_\infty = \infty$ which completes the proof.

4.1. **Lyapunov Stability Analysis.** In the most recent year, various analyses in numerous domains have made use of the Lyapunov function formulation. This function has been employed in epidemiology to evaluate the stability analysis of an epidemiological model. It has been suggested that the Lyaponuv might be regarded as energy; as a result, determining stability may benefit from knowing the sign of the function's first derivative. Even so, determining whether we have a local maximum or local minimum may not be as simple as looking at the sign of the first derivative of a function. It seems that in light of this, it would be wise to research the second derivative's sign. We will do such analysis to determine the sign of our models' first and second derivatives in this section.

4.2. **Lyapunov for Endemic Case.** The Lyapunov function is a endemic, $f(S, V_1, V_2, A, C, R) < 0$, is the endemic equilibrium E^*. When the reproductive number $R_0 > 1$, the proposed model's endemic equilibrium points E^* are globally asymptotically stable.
Proof. The Lyapunov function is used to establish the theorem in the following way:

$$L(S^*, V_1^*, V_{2,}^*, A^*, C^*, R^*) = (S - S^* - S^* \log \frac{S^*}{S}) + (V_1 - V_1^* - V_1^* \log \frac{V_1^*}{V_1}) \tag{4.7}$$

$$+ (V_2 - V_2^* - V_2^* \log \frac{V_2^*}{V_2}) + (A - A^* - A^* \log \frac{A^*}{A}) \tag{4.8}$$

$$+ (C - C^* - C^* \log \frac{C^*}{C}) + (R - R^* - R^* \log \frac{R^*}{R}) \tag{4.9}$$

As a result, when we use the derivative, we get the following.

$$\begin{gathered}{}^{PABC}_{0}\mathrm{D}^{\eta}_{t}L(t) \leq (\frac{S - S^*}{S}){}^{PABC}_{0}\mathrm{D}^{\beta}_{t}S(t) + (\frac{V_1 - V_1^*}{V_1}){}^{PABC}_{0}\mathrm{D}^{\beta}_{t}V_1(t) \\ (\frac{V_2 - V_2^*}{V_2}){}^{PABC}_{0}\mathrm{D}^{\eta}_{t}V_2(t) + (\frac{A - A^*}{A}){}^{PABC}_{0}\mathrm{D}^{\beta}_{t}A(t) \\ (\frac{C - C^*}{C}){}^{PABC}_{0}\mathrm{D}^{\eta}_{t}C(t) + (\frac{R - R^*}{R}){}^{PABC}_{0}\mathrm{D}^{\beta}_{t}R(t)\end{gathered} \tag{4.10}$$

The following is how their derivative values may now be written

$$\begin{cases} {}^{PABC}_{0}\mathrm{D}^{\eta}_{t}L(t) = (\frac{S-S^*}{S})(\wedge(1-\omega)(1-\tau C) + \theta_2 V_1 - (\lambda + \mu + \theta_1)S), \\ +(\frac{V_1 - V_1^*}{V_1})(\theta_1 S + \wedge\omega - (a + \mu + \theta_2)V_1), \\ +(\frac{V_2 - V_2^*}{V_2})(\alpha V_1 - (\epsilon + \mu)V_2), \\ +(\frac{A-A^*}{A})(\lambda S - (\delta + \sigma + \mu)A), \\ +(\frac{C-C^*}{C})(\delta A + \wedge(1-\omega)\tau C - (\vartheta + \mu + \psi)C), \\ +(\frac{R-R^*}{R})(\delta A + \vartheta C + \epsilon V_2 - \mu R), \end{cases} \tag{4.11}$$

Substituting values $S = S - S^*$, $V_1 = V_1 - V_1^*$, $V_2 = V_2 - V_2^*$,$A = A - A^*$,$C = C - C^*$, and $R = R - R^*$ leads to

$$\begin{cases} {}^{PABC}_{0}\mathrm{D}^{\eta}_{t}L(t) = (\frac{S-S^*}{S})(\wedge(1-\omega)(1-\tau(C-C^*)) + \theta_2(V_1 - V_1^*) - (\lambda + \mu + \theta_1)(S - S^*)), \\ +(\frac{V_1 - V_1^*}{V_1})(\theta_1(S - S^*)) + \wedge\omega - (a + \mu + \theta_2)(V_1 - V^*)), \\ +(\frac{A-A^*}{A})(\lambda(S - S^*) - (\delta + \sigma + \mu)(A - A^*)), \\ +(\frac{C-C^*}{C})(\delta(A - A^*) + \wedge(1-\omega)\tau(C - C^*) - (\vartheta + \mu + \psi)(C - C^*)), \\ +(\frac{R-R^*}{R})(\delta(A - A^*) + \vartheta(C - C^*) + \epsilon(V_2 - V_2^*) - \mu(R - R^*)), \end{cases} \tag{4.12}$$

The above can be organized as follows

$$\begin{aligned} {}^{PABC}_{0}\mathrm{D}^{\eta}_{t}L(t) \quad &= \quad \wedge - \wedge\tau C + \wedge\tau C^* - \wedge\omega + \wedge\omega\tau C - \wedge\omega\tau C^* \\ &+ \theta_2 V_1 - \theta_2 V_1^* - (\lambda + \mu + \theta_1)S + (\lambda + \mu + \theta_1)S^* + \theta_1 S - \theta_1 S^* \\ &- \frac{\wedge S^*}{S} + \frac{\wedge\tau C S^*}{S} + \frac{\wedge\omega S^*}{S} + \frac{\wedge\omega\tau C^* S^*}{S} - \frac{\wedge\omega\tau C S^*}{S} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\theta_2 V_1 S^*}{S}+\frac{(\lambda+\mu+\theta_1)}{S}-\frac{(\lambda+\mu+\theta_1)S^{*2}}{S}\\
&-(\alpha+\mu+\theta_2)V_1+(\alpha+\mu+\theta_2)V_1^*\\
&-\frac{\theta_1 S V_1^*}{V_1}+\frac{\theta_1 S^* V_1^*}{V_1}-\frac{V_1^*\wedge\omega}{V_1}+\frac{(\alpha+\mu+\theta_2)V_1V_1^*}{V_1}-\frac{(\alpha+\mu+\theta_2)V_1^{*2}}{V_1}\\
&+\alpha V_1-\alpha V_1^*-\epsilon V_2+\epsilon V_2^*-\mu V_2+\mu V_2^*\\
&-\frac{\alpha V_1 V_2^*}{V_2}+\frac{\alpha V_1^* V_2^*}{V_2}+\frac{\epsilon V_2 V_2^*}{V_2}-\frac{\epsilon V_2^*}{V_2}+\frac{\mu V_2 V_2^*}{V_2}+\frac{\mu V_2^*}{V_2}\\
&-\wedge\tau C-\wedge\tau C^*-\tau C\wedge\omega+\tau C^*\wedge\omega+\vartheta C-\mu C-\psi C\\
&+\vartheta C^*+\mu C^*+\vartheta C^*-\frac{\wedge\tau C C^*}{C}+\frac{\wedge\tau C^* C^*}{C}-\tau C^*\wedge\omega+\frac{\wedge\omega\tau C^{*2}}{C}\\
&-\vartheta C^*-\mu C^*-\psi C^*+\frac{\vartheta C^{*2}}{C}+\frac{\mu C^{*2}}{C}+\frac{\psi C^{*2}}{C}\\
&+\sigma A-\sigma A^*+\vartheta C-\vartheta C^*+\epsilon V_2-\epsilon V_2^*-\frac{\sigma A R^*}{R}+\frac{\sigma A^* R^*}{R}\\
&-\frac{\psi C R^*}{R}+\frac{\psi C^* R^*}{R}-\frac{\epsilon V_2 R^*}{R}+\frac{\epsilon V_2^* R^*}{R}
\end{aligned}
$$

For clarity, we may rewrite the above equivalency as follows

$$
{}_0^{PABC}\mathrm{D}_t^{\beta}(L(t))=\Omega_1-\Omega_2 \tag{4.13}
$$

where

$$
\begin{aligned}
\Omega_1 \;=\;& \wedge+\wedge\tau C^*+\wedge\omega\tau C+\theta_2 V_1+(\lambda+\mu+\theta_1)S^*+\theta_1 S\\
&+\frac{\wedge\tau C S^*}{S}+\frac{\wedge\omega S^*}{S}+\frac{\wedge\omega\tau C^* S^*}{S}+\frac{\theta_2 V_1 S^*}{S}+\frac{(\lambda+\mu+\theta_1)}{S}\\
&+(\alpha+\mu+\theta_2)V_1^*+\frac{\theta_1 S^* V_1^*}{V_1}+\alpha V_1+\epsilon V_2^*+\mu V_2^*\\
&+\frac{\alpha V_1^* V_2^*}{V_2}+\frac{\epsilon V_2 V_2^*}{V_2}+\frac{\mu V_2 V_2^*}{V_2}+\frac{\mu V_2^*}{V_2}+\frac{\sigma A A^*}{A}+\frac{\mu A A^*}{A}\\
&+\tau C^*\wedge\omega+\vartheta C+\vartheta C^*+\mu C^*+\vartheta C^*+\frac{\vartheta C^{*2}}{C}+\frac{\mu C^{*2}}{C}+\frac{\psi C^{*2}}{C}\\
&+\frac{\psi C^* R^*}{R}+\frac{\epsilon V_2^* R^*}{R}-\tau C^*\wedge\omega
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega_2 \;=\;& -\wedge\tau C-\wedge\omega-\wedge\omega\tau C^*-\theta_2 V_1^*-(\lambda+\mu+\theta_1)S-\theta_1 S^*\\
&-\frac{\wedge S^*}{S}-\frac{\wedge\omega\tau C S^*}{S}-\frac{(\lambda+\mu+\theta_1)S^{*2}}{S}\\
&-(\alpha+\mu+\theta_2)V_1-\frac{\theta_1 S V_1^*}{V_1}-\frac{V_1^*\wedge\omega}{V_1}-\frac{(\alpha+\mu+\theta_2)V_1^{*2}}{V_1}\\
&-\alpha V_1^*-\epsilon V_2-\mu V_2+\mu V_2^*-\frac{\alpha V_1 V_2^*}{V_2}-\frac{\delta A^{*2}}{A}-\frac{\sigma A^{*2}}{A}-\frac{\mu A^{*2}}{A}
\end{aligned}
$$

$$
\begin{aligned}
&- \wedge \tau C - \wedge \tau C^* - \tau C \wedge \omega - \mu C - \psi C - \frac{\wedge \tau C C^*}{C} - \tau C^* \wedge \omega \\
&- \vartheta C^* - \mu C^* - \psi C^* - \sigma A^* - \vartheta C^* - \epsilon V_2^* \\
&- \frac{\sigma A R^*}{R} - \frac{\psi C R^*}{R} - \tau C^* \wedge \omega
\end{aligned}
$$

It is achieved that if $\Omega_1 < \Omega_2$, this yields $\frac{dL}{dt} < 0$, however when $S = S^*$, $V_1 = V_1^*$, $V_2 = V_2^*$, $A = A^*$, $C = C^*$, and $R = R^*$.

$$0 = \Omega_1 - \Omega_2 \tag{4.14}$$

$${}_0^{PABC}\mathrm{D}_t^{\beta}(L(t)) = 0 \tag{4.15}$$

The recommended model model has the largest compact invariant set, as can be shown. The endemic equilibrium of the model under consideration is denoted by E^*. If $\Omega_1 < \Omega_2$ is globally asymptotically stable in G, then E^* is globally asymptotically stable in G, according to the LaSalle's invariance principle.The above derivation does not necessarily offer a clear constraint on the sign of the Lyapunov function's first derivative, despite its widespread use. While the derivation is extremely valuable, determining the sign of the Lyapunov function's first derivative necessitates further investigation. There is, however, another method for creating a clear condition that uses equilibrium points, which will be discussed below to explain how the two approaches differ.

4.3. **Analysis of Classical Piecewise with Atangana-Baleanu fractional Operator.** In this section, we construct the analysis of proposed of the system techniques given in [21], we can express the system in followings equation as :

$$
\begin{cases}
{}_0^{PABC}\mathrm{D}_t^{\eta}(S(t)) = [\mathrm{D}_t^{\eta}(S(t)) = H_1(S, V_1, V_2, A, C, R), 0 < t < t_1, \\
{}_0^{AB}\mathrm{D}_t^{\eta}(S(t)) =^{AB} H_1(S, V_1, V_2, A, C, R), t_1 < t < T.] \\
{}_0^{PABC}\mathrm{D}_t^{\eta}(V_1(t)) = [\mathrm{D}_t^{\eta}(V_1(t)) = H_2(S, V_1, V_2, A, C, R), 0 < t < t_1, \\
{}_0^{AB}\mathrm{D}_t^{\eta}(V_1(t)) =^{AB} H_2(S, V_1, V_2, A, C, R), t_1 < t < T.] \\
{}_0^{PABC}\mathrm{D}_t^{\eta}(V_2(t)) = [\mathrm{D}_t^{\eta}(V_2(t)) = H_3(S, V_1, V_2, A, C, R), 0 < t < t_1, \\
{}_0^{AB}\mathrm{D}_t^{\eta}(V_2(t)) =^{AB} H_3(S, V_1, V_2, A, C, R), t_1 < t < T.] \\
{}_0^{PABC}\mathrm{D}_t^{\eta}(A(t)) = [\mathrm{D}_t^{\eta}(A(t)) = H_4(S, V_1, V_2, A, C, R), 0 < t < t_1, \\
{}_0^{AB}\mathrm{D}_t^{\eta}(A(t)) =^{AB} H_4(S, V_1, V_2, A, C, R), t_1 < t < T.] \\
{}_0^{PABC}\mathrm{D}_t^{\eta}(C(t)) = [\mathrm{D}_t^{\eta}(C(t)) = H_5(S, V_1, V_2, A, C, R), 0 < t < t_1, \\
{}_0^{AB}\mathrm{D}_t^{\eta}(C(t)) =^{AB} H_5(S, V_1, V_2, A, C, R), t_1 < t < T.] \\
{}_0^{PABC}\mathrm{D}_t^{\eta}(R(t)) = [\mathrm{D}_t^{\eta}(R(t)) = H_6(S, V_1, V_2, A, C, R), 0 < t < t_1, \\
{}_0^{AB}\mathrm{D}_t^{\eta}(R(t)) =^{AB} H_6(S, V_1, V_2, A, C, R), t_1 < t < T.]
\end{cases}
\tag{4.16}
$$

We will now determine the existence of a solution for the hypothetical piecewise derivable function and its specific solution property.

$$ {}_{0}^{PABC}\mathrm{D}_{t}^{\eta}(f(t)) = H(t,f),\ 0 < \rho \leq 1. \tag{4.17}$$

is

$$F(t) = \begin{cases} F_0 + \int_0^t H(\rho, F(\rho))d\rho,\ 0 < t \leq t_1, \\ F(t_1) + \frac{1-\eta}{AB(\eta)}F(t) + \frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\eta-1}H(\rho, F(\rho)d\rho, \\ t_1 < t \leq t_2. \end{cases} \tag{4.18}$$

where

$$F =, \begin{cases} , S(t), \\ V_1(t), \\ V_2(t), \\ A(t), \\ C(t), \\ R(t), \end{cases} \quad F_0 = \begin{cases} S(0), \\ V_1(0), \\ V_2(0), \\ A(0), \\ C(0), \\ R(0), \end{cases} \quad F_{t_1} \begin{cases} S(t_1), \\ V_1(t_1), \\ V_2(t_1), \\ A(t_1), \\ C(t_1), \\ R(t_1), \end{cases} \tag{4.19}$$

$$H(t, F(t)) = \begin{cases} H_1 = \begin{cases} \frac{d}{dt}\Im_1(S, V_1, V_2, A, C, R), \\ {}^{C}\Im 1(S, V_1, V_2, A, C, R), \end{cases} \\ H_2 = \begin{cases} \frac{d}{dt}\Im_2(S, V_1, V_2, A, C, R), \\ {}^{C}\Im 2(S, V_1, V_2, A, C, R). \end{cases} \\ H_3 = \begin{cases} \frac{d}{dt}\Im_3(S, V_1, V_2, A, C, R), \\ {}^{C}\Im 3(S, V_1, V_2, A, C, R). \end{cases} \\ H_4 = \begin{cases} \frac{d}{dt}\Im_4(S, V_1, V_2, A, C, R), \\ {}^{C}\Im 4(S, V_1, V_2, A, C, R). \end{cases} \\ H_5 = \begin{cases} \frac{d}{dt}\Im_5(S, V_1, V_2, A, C, R), \\ {}^{C}\Im 5(S, V_1, V_2, A, C, R). \end{cases} \\ H_6 = \begin{cases} \frac{d}{dt}\Im_6(S, V_1, V_2, A, C, R), \\ {}^{C}\Im 6(S, V_1, V_2, A, C, R). \end{cases} \end{cases} \tag{4.20}$$

Taking $\infty > t_2 \geq t > t_1 > 0$ and the Banach space $B = C[0, T]$ with a norm

$$||F|| = \max_{t \in [0,T]} |F|. \tag{4.21}$$

We suppose that

A_1 Then there exist $K_F > 0$, $\forall > P$, and $F \in B$. We have

$$|H(t, F) - H(t, \bar{F})| = K_F|F - \bar{F}|. \tag{4.22}$$

A_2 Then $\exists C_H > 0$ $K_F > 0$,and $N_H > 0$, we have

$$|H(t, F(t))| \leq C_H|F| + N_H \tag{4.23}$$

Theorem 4.1. *It follows that equation (38) has at least one solution if H be piecewise continuous on $0 < t \leq t_1$ and $t_1 < t \leq t_2$ on $[0, T]$, also satisfying A_2. Proof:Assume that closed subset $\Re$ of B in both subintervals of$[0, \wp]$ as*

$$\Re = \{F \in B : ||F|| \leq P_{1,2}, P > 0\}. \tag{4.24}$$

Consider $T : \Re \to \Re$ and using (39) as

$$T(F) = \begin{cases} F_0 + \int_0^{t_1} H(\rho, F(\rho))d\rho, 0 < t \leq t_1, \\ F_{t_1} + \frac{1-\eta}{AB\eta}F(t) + \frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\rho-1}H(\rho, F(\rho))d\rho,\ t_1 < t \leq t_2 \end{cases} \tag{4.25}$$

Any $F \in B$, we have

$$|T(F)| \leq \begin{cases} |F_0| + \int_0^{t_1} |H(\rho, F(\rho))|d\rho, 0 < t \leq t_1, \\ |F(t_1)| + \frac{1-\eta}{AB(\eta)}|F(t)| + \frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\rho-1}H(\rho, F(\rho))d\rho,\ t_1 < t \leq t_2. \end{cases}$$

$$\leq \begin{cases} |F_0| + \int_0^{t_1}[C_H|F| + N_H]d\rho, 0 < t \leq t_1, \\ |F(t_1)| + \frac{1-\eta}{AB(\eta)}[C_H|F| + N_H] + \frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\rho-1}[C_H|F| + N_H]d\rho,\ t_1 < t \leq t_2. \end{cases}$$

$$\leq \begin{cases} P_1,\ 0 < t \leq t_1, \\ P_2,\ t_1 < t \leq t_2. \end{cases}$$

As established by the prior equation, as $F \in \Re$. Hence, $T(\Re) \subset \Re$. As a result, it proves that operator T is closed and completed. We can write as to further show that the recommended operator is fully continuous. Using $t_i > t_j \in [0, t_1]$ as the initial interval for the AB sense, consider

$$|T(F)(t_i) - T(F)(t_j)| = |\int_0^{t_i} H(\rho, F(\rho))d\rho - \int_0^{t_j} H(\rho, F(\rho))d\rho|,$$

$$\leq \int_0^{t_i} |H(\rho, F(\rho))|d\rho - \int_0^{t_j} |H(\rho, F(\rho))|d\rho,$$

$$\leq \int_0^{t_i} [C_H|F| + N_H]d\rho - \int_0^{t_j} [C_H|F| + N_H]d\rho,$$

When $t_j \to t_i$, then

$$|T(F)(t_i) - T(F)(t_j)|$$

As a result, the operator T equicontinuity is demonstrated in $[0, t_1]$. Think of $t_i, t_j \in [t1, T]$ in the sense of ABC as

$$l|T(F)(t_i) - T(F)(t_j)| =| \frac{1-\eta}{AB(\eta)}F(t) + \frac{\eta}{AB(\eta)\Gamma(\eta)}\int^{t_{i0}}(t_i - \rho)^{\eta-1}H(\eta, F(\eta))d\rho$$

$$-\frac{1-\kappa}{AB()}F(t) + \frac{\eta}{AB(\eta)\Gamma(\eta)}\int^{t_{j0}}(t_j - \rho)^{\eta-1}H(\eta, F(eta))d\rho\ |$$

$$\leq \frac{\kappa}{AB(\eta)\Gamma(eta)} \int^{t_{j_0}} [(t_j-\rho)^{eta-1} - (t_i-\rho)^{\eta-1}]|H(\rho, F(\rho))|d\rho.$$

$$\leq \frac{\eta}{AB(\eta)\Gamma(\eta)}[\int^{t_{j_0}} \{(t_j-\rho)^{\eta-1} - (t_i-\rho)^{\eta-1}\} + \int_j^i (t_i-\rho)^{\eta-1} d\rho](C_H|F| + N_H).$$

$$\leq \frac{\eta(C_H|F| + N_H)}{AB(\eta)\Gamma(\eta+1)}[t_i^\eta - t^\eta{}_j + 2(t_i - t_j)^\eta].$$

If $t_j \to t_i$, then

$$|T(F)(t_i) - T(F)(t_j)| \to 0$$

. So demonstrate the equi-continuity of T in $[t_1, t_2]$. Thus, T a map that is equicontinuous. The Arzel's Ascoli conclusion shows that T is the completely continuous operator, uniformly continuous, and bounded. Since the Schauder result demonstrates that the model under consideration has at least one solution in the sub intervals.

Theorem 4.2. *Assume that A1 holds, the suggested piecewise model has at most one root if T is a contraction operator. Proof.Since $T : \mathfrak{R} \to \mathfrak{R}$ is piecewise continuous, let F and $\overline{F} \in \mathfrak{R}$ on $[0, t_1]$ in classical form is*

$$\parallel T(F) - T(\widetilde{F}) \parallel = max_{t\in[0,t_1]} \mid \int_0^{t_i} H(\rho, F(\rho))d\rho - \int_0^{t_i} H(\rho, \overline{F}(\rho))d\rho \mid . \tag{4.26}$$

$$\leq t_1 I_H \parallel F - \widetilde{F} \parallel . \tag{4.27}$$

From equation(49),we have

$$\|T(F) - T(\widetilde{F})\| \leq t_1 I_H \|F - \widetilde{F}\|. \tag{4.28}$$

T is therefore a contraction. In view of the Banach result, the considered model thus has a singular solution in the given sub interval. Moreover, $t_2[t_1, t_2]$, we have

$$\|T(F) - T(\widetilde{F})\| = max_{t\in[t_1,t_2]}|\frac{1-\eta}{AB(\eta)}F(t) + \frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\eta-1}H(\rho, F(\rho))d\rho$$

$$\frac{1-\eta}{AB(\eta)}\widetilde{F}(t) + \frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\eta-1}H(\rho, \widetilde{F}(\rho))d\rho|$$

$$\leq \frac{1-\eta}{AB(\eta)} I_H\|F - \widetilde{F}\| + \frac{\eta(t_2-t_1)^\eta}{AB(\eta)\Gamma(\eta+1)} I_H\|F - \widetilde{F}\|. \tag{4.29}$$

Now from equation(41) we have

$$\|T(F) - T(\widetilde{F})\| \leq I_H[\frac{1-\eta}{AB(\eta)} + \frac{\eta(t_2-t_1)^\eta}{AB(\eta)\Gamma(\eta+1)} I_H\|F - \widetilde{F}\|$$

As a result, T is a contraction. Hence the suggested piecewise model has at most one solution.

5. Numerical Scheme

In this section, we established the generalized scheme for numerical results with Mittag-Leffler kernel having property of nonlocal and nonsingular kernel with combination of classical operator. So, we have

$$\mathrm{S}(t)=\begin{cases}S_0+\int_0^{t_1}H_1(\rho,S)d\rho, 0<t\leq t_1,\\ S(t_1)+\frac{1-\eta}{AB(\eta)}f(t)+\frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\eta-1}H_1(\rho,S)d\rho, t_1<t\leq t_2,\end{cases} \tag{5.1}$$

$$\mathrm{V}_1(t)=\begin{cases}V_{1_0}+\int_0^{t_1}H_2(\rho,V_1)d\rho, 0<t\leq t_1,\\ V_1(t_1)+\frac{1-\eta}{AB(\eta)}f(t)+\frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\eta-1}H_2(\rho,V_1)d\rho, t_1<t\leq t_2,\end{cases} \tag{5.2}$$

$$\mathrm{V}_2(t)=\begin{cases}V_{2_0}+\int_0^{t_1}H_3(\rho,V_2)d\rho, 0<t\leq t_1,\\ V_2(t_1)+\frac{1-\eta}{AB(\eta)}f(t)+\frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\eta-1}H_3(\rho,V_2)d\rho, t_1<t\leq t_2,\end{cases} \tag{5.3}$$

$$\mathrm{A}(t)=\begin{cases}A_0+\int_0^{t_1}H_4(\rho,A)d\rho, 0<t\leq t_1,\\ A(t_1)+\frac{1-\eta}{AB(\eta)}f(t)+\frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\eta-1}H_4(\rho,A)d\rho, t_1<t\leq t_2,\end{cases} \tag{5.4}$$

$$\mathrm{C}(t)=\begin{cases}C_0+\int_0^{t_1}H_5(\rho,C)d\rho, 0<t\leq t_1,\\ C(t_1)+\frac{1-\eta}{AB(\eta)}f(t)+\frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\eta-1}H_5(\rho,C)d\rho, t_1<t\leq t_2,\end{cases} \tag{5.5}$$

$$\mathrm{R}(t)=\begin{cases}R_0+\int_0^{t_1}H_6(\rho,R)d\rho, 0<t\leq t_1,\\ R(t_1)+\frac{1-\eta}{AB(\eta)}f(t)+\frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\eta-1}H_6(\rho,R)d\rho, t_1<t\leq t_2,\end{cases} \tag{5.6}$$

We demonstrate the method for the first equation in equation (6) at and then apply it to the remaining equations.

$$\mathrm{S}(t)=\begin{cases}S_0+\int_0^{t_1}H_1(\rho,S)d\rho, 0<t\leq t_1,\\ S(t_1)+\frac{1-\eta}{AB(\eta)}f(t)+\frac{\eta}{AB(\eta)\Gamma(\eta)}\int_{t_1}^{t_2}(t-\rho)^{\eta-1}H_1(\rho,S)d\rho, t_1<t\leq t_2,\end{cases} \tag{5.7}$$

$$S(t_{n+1})=\begin{cases}S_0+\sum_{m=2}^{j}[\frac{5}{12}H_1(P_2,t_{m-2})\Lambda t-\\ \frac{4}{3}H_1(P_1,t_{m-1})\Lambda t+H_1(P,t_k)],\\ S(t_1)+\frac{1-\eta}{AB(\eta)}H_1(S^n,{V_1}^n,{V_2}^n,A^n,C^n,R^n)+\\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{\eta-1}}{\Gamma(\eta+1)}\sum_{m=j+3}^{n}[H_1(P_2,t_{m-2})]\gamma+\\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{\Gamma(\eta+2)}\sum_{m=j+3}^{n}[H_1(P_2,t_{m-1})-H_1(P_2,t_{m-2})]\phi+\\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{2\Gamma(\eta+3)}\sum_{m=j+3}^{n}[H_1(P,t_m)-2H_1(P_1,t_{m-1})+H_1(P_2,t_{m-2}]\Lambda\end{cases} \tag{5.8}$$

$$V_1(t_n+1)=\begin{cases} V_{1_0}+\sum_{m=2}^{j}[\frac{5}{12}H_2(P_2,t_{m-2})\Lambda t- \\ \frac{4}{3}H_2(P_1,t_{m-1})\Lambda t+H_2(P,t_k)], \\ V_1(t_1)+\frac{1-\eta}{AB(\eta)}H_2(S^n,{V_1}^n,{V_2}^n,A^n,C^n,R^n)+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{\eta-1}}{\Gamma(\eta+1)}\sum_{m=j+3}^{n}[H_2(P_2,t_{m-2})]\gamma+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{\Gamma(\eta+2)}\sum_{m=j+3}^{n}[H_2(P_2,t_{m-1})-H_2(P_2,t_{m-2})]\phi+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{2\Gamma(\eta+3)}\sum_{m=j+3}^{n}[H_2(P,t_m)-2H_2(P_1,t_{m-1})+H_2(P_2,t_{m-2}]\Lambda \end{cases} \tag{5.9}$$

$$V_2(t_n+1)=\begin{cases} V_{2_0}+\sum_{m=2}^{j}[\frac{5}{12}H_3(P_2,t_{m-2})\Lambda t- \\ \frac{4}{3}H_3(P_1,t_{m-1})\Lambda t+H_3(P,t_k)], \\ V_2(t_1)+\frac{1-\eta}{AB(\eta)}H_3(S^n,{V_1}^n,{V_2}^n,A^n,C^n,R^n)+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{\eta-1}}{\Gamma(\eta+1)}\sum_{m=j+3}^{n}[H_3(P_2,t_{m-2})]\gamma+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{\Gamma(\eta+2)}\sum_{m=j+3}^{n}[H_3(P_2,t_{m-1})-H_3(P_2,t_{m-2})]\phi+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{2\Gamma(\eta+3)}\sum_{m=j+3}^{n}[H_3(P,t_m)-2H_3(P_1,t_{m-1})+H_3(P_2,t_{m-2}]\Lambda \end{cases} \tag{5.10}$$

$$A(t_n+1)=\begin{cases} A_0+\sum_{m=2}^{j}[\frac{5}{12}H_4(P_2,t_{m-2})\Lambda t- \\ \frac{4}{3}H_4(P_1,t_{m-1})\Lambda t+H_4(P,t_k)], \\ A(t_1)+\frac{1-\eta}{AB(\eta)}H_4(S^n,{V_1}^n,{V_2}^n,A^n,C^n,R^n)+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{\eta-1}}{\Gamma(\eta+1)}\sum_{m=j+3}^{n}[H_4(P_2,t_{m-2})]\gamma+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{\Gamma(\eta+2)}\sum_{m=j+3}^{n}[H_4(P_2,t_{m-1})-H_4(P_2,t_{m-2})]\phi+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{2\Gamma(\eta+3)}\sum_{m=j+3}^{n}[H_4(P,t_m)-2H_4(P_1,t_{m-1})+H_4(P_2,t_{m-2}]\Lambda \end{cases} \tag{5.11}$$

$$C(t_{n+1})=\begin{cases} C_0+\sum_{m=2}^{j}[\frac{5}{12}H_5(P_2,t_{m-2})\Lambda t- \\ \frac{4}{3}H_5(P_1,t_{m-1})\Lambda t+H_5(P,t_k)], \\ C(t_1)+\frac{1-\eta}{AB(\eta)}H_5(S^n,{V_1}^n,{V_2}^n,A^n,C^n,R^n)+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{\eta-1}}{\Gamma(\eta+1)}\sum_{m=j+3}^{n}[H_5(P_2,t_{m-2})]\gamma+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{\Gamma(\eta+2)}\sum_{m=j+3}^{n}[H_5(P_2,t_{m-1})-H_5(P_2,t_{m-2})]\phi+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{2\Gamma(\eta+3)}\sum_{m=j+3}^{n}[H_5(P,t_m)-2H_5(P_1,t_{m-1})+H_5(P_2,t_{m-2}]\Lambda \end{cases} \tag{5.12}$$

$$R(t_{n+1})=\begin{cases} R_0+\sum_{m=2}^{j}[\frac{5}{12}H_6(P_2,t_{m-2})\Lambda t- \\ \frac{4}{3}H_6(P_1,t_{m-1})\Lambda t+H_6(P,t_k)], \\ R(t_1)+\frac{1-\eta}{AB(\eta)}H_6(S^n,{V_1}^n,{V_2}^n,A^n,C^n,R^n)+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{\eta-1}}{\Gamma(\eta+1)}\sum_{m=j+3}^{n}[H_6(P_2,t_{m-2})]\gamma+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{\Gamma(\eta+2)}\sum_{m=j+3}^{n}[H_6(P_2,t_{m-1})-H_6(P_2,t_{m-2})]\phi+ \\ \frac{\eta}{AB(\eta)}\frac{(\Lambda t)^{n-1}}{2\Gamma(\eta+3)}\sum_{m=j+3}^{n}[H_6(P,t_m)-2H_6(P_1,t_{m-1})+H_6(P_2,t_{m-2}]\Lambda \end{cases} \tag{5.13}$$

where

$$P = S^m, V_1^m, V_2^m, A^m, C^m, D^m$$
$$P_1 = S^{m-1}, V_1^{m-1}, V_2^{m-1}, A^{m-1}, C^{m-1}, D^{m-1}$$
$$P_2 = S^{m-2}, V_1^{m-2}, V_2^{m-2}, A^{m-2}, C^{m-2}, D^{m-2}$$

and

$$\Lambda = [(1+n-m)^v(2(n-m)^2 + (3v+10)(n-m) + 2v^2 + 9v + 12)-$$
$$(n-m)(2(n-m)^2 + (5v+10)(n-m) + 6v^2 + 18v + 12)],$$
$$\gamma = [(1+n-m)^v(3+2v-m+n) - (n-m)(3+3v-m+n)],$$
$$\phi = [(1+n-m)^v - (n-m)^v]$$

6. Results of Proposed Scheme

Mathematical investigation of Hepatitis B for fractional operator has been presented to see its real impact. Unexpected reactions resulting from the sections of the postulated fractional order model are obtained by utilizing non-integer values of parameters. Figure 2 shows the increasing behaviour of the susceptible population for decreasing value of η. Figure 3 and 4 shows the impact of vaccination to population who have taken first dose of vaccine and second dose of vaccine for decreasing value of η. Figure 5 shows the increasing behaviour of the population in acute stage for decreasing value of η. Figure 6 shows the increasing behaviour of the population at chronic stage for decreasing value of η. Figure 7 shows the increasing behaviour of the recovered population for increasing value of η. From the graphs we understand the increase and decrease in values of the given compartments by dropping the fractional values. Moreover, we find different patterns with respect to vaccinated infected and recovered people, as Figures 3,5 and 7 illustrate. The number of people who get vaccination rises steadily, but the number of people who recover drastically falls. This is explained by the concentration of germs in the dumpsite and the increasing numbers of humans in the water reservoir. These results demonstrate how significantly these parameters affect the transmission of infection. It is also clear that when fractional values drop, the number of infection units in the environment decreases and the number of recovered individuals rises. This study sheds light on how infection control is expected to develop in the future and how we may improve our efforts to reduce the spread of infection units in the environment. Compared to classical derivatives, fractal fractional analysis produces robust results for all compartments when examining steady states with non-integer order derivatives. It's also important to remember that fractional values reduction improves the accuracy and reliability of solutions for every compartment. Memory influence can be observed with greater degree of freedom and a solution restricted to a steady state point that lies in a feasible range by varying the fractal dimension.The chaotic behavior of the system is determined at different fractional order values through simulation in figures 8 to 13 which shows the solutions lie in feasible region. All compartments have positive and unique solutions also relation between compartments is positively invariant in figures 10,11,12 and 13. Figures 11 and 12 represent the impact of different stages of vaccine plays an important role in controlling the disease and recovering population increases.

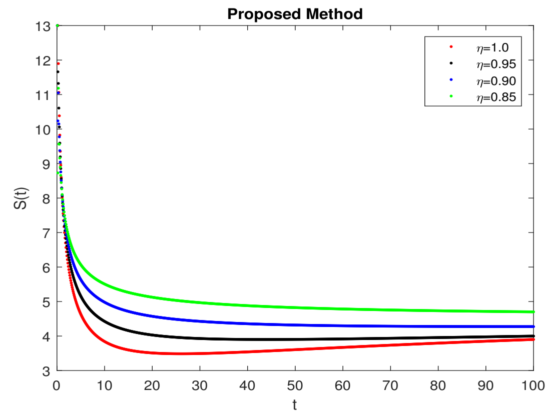

FIGURE 2. Dynamics of S(t) at different fractional order by using hybrid operator

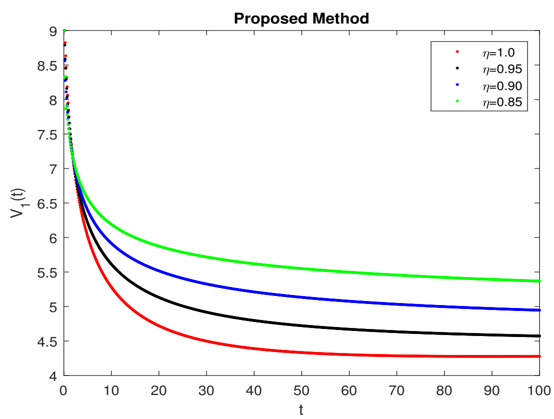

FIGURE 3. Dynamics of $V_1(t)$ at different fractional order by using hybrid operator

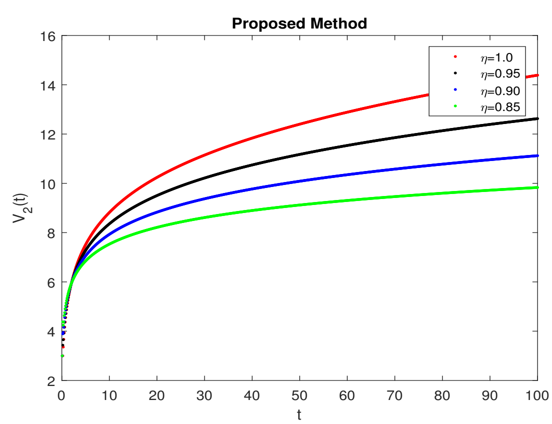

FIGURE 4. Dynamics of $V_2(t)$ at different fractional order by using hybrid operator

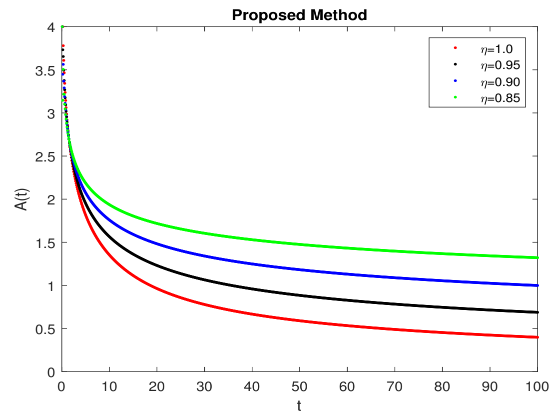

FIGURE 5. Dynamics of $A(t)$ at different fractional order by using hybrid operator

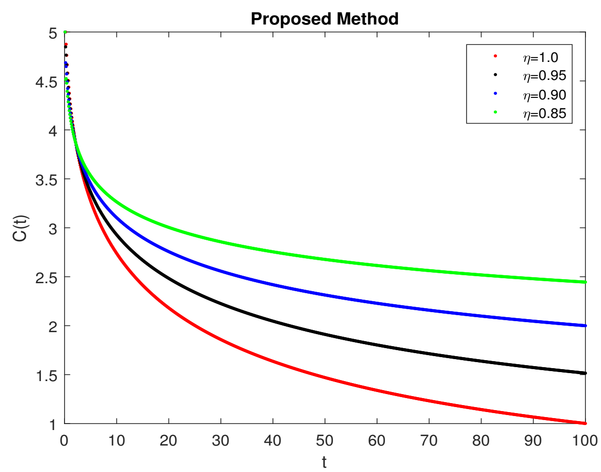

FIGURE 6. Dynamics of $C(t)$ at different fractional order by using hybrid operator

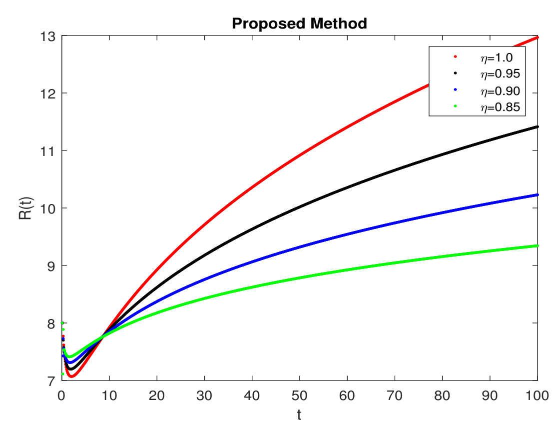

FIGURE 7. Dynamics of $R(t)$ at different fractional order by using hybrid operator

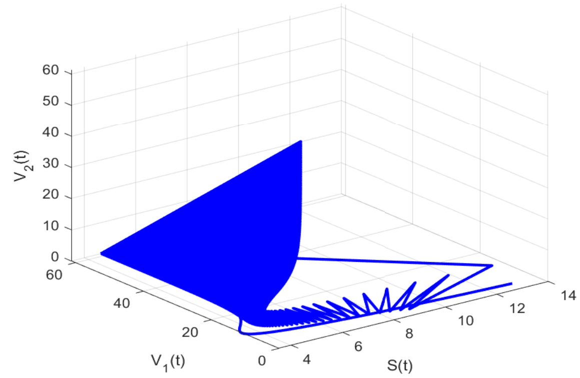

Figure 8. Chaotic Behavior of the different different compartments bounded to feasible region

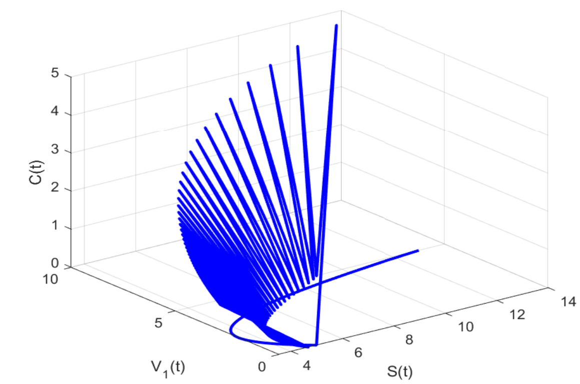

Figure 9. Chaotic Behavior of the different different compartments bounded to feasible region

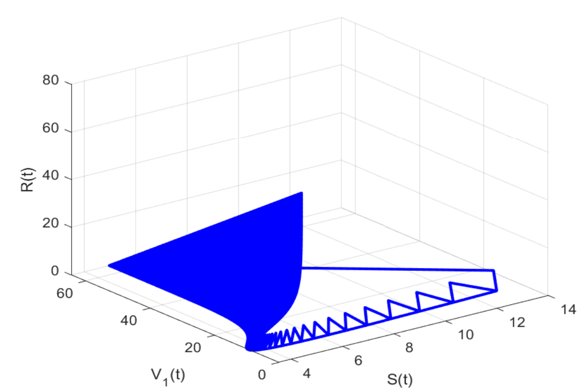

Figure 10. Chaotic Behavior of the different different compartments bounded to feasible region

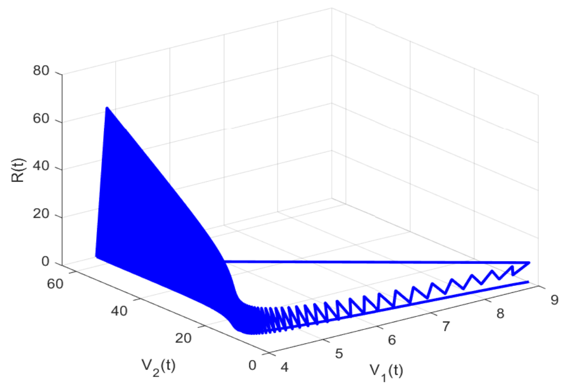

FIGURE 11. Chaotic Behavior of the different different compartments bounded to feasible region

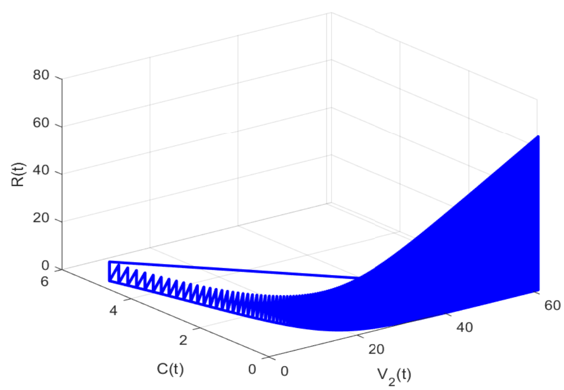

FIGURE 12. Chaotic Behavior of the different different compartments bounded to feasible region

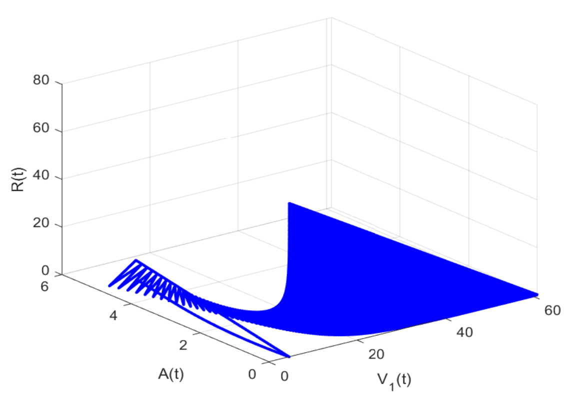

FIGURE 13. Chaotic Behavior of the different different compartments bounded to feasible region

7. Conclusion

In this study, we examined the behavior of the Hepatitis B model using piecewise operator in the classical and ABC Caputo sense. The existence and distinctiveness of a solution bearing a piecewise derivative are investigated for the disease model given earlier. To approximate the solution to the stated issue, the piecewise Newton polynomial method is utilized. The simulations for the proposed model have been given for various fractional orders. The figures show that the piecewise operators have better dynamics when compared separately to the fractional and classical operators. The significance of the fractional piecewise operator is clearly shown in Figures 2-7. All curves ultimately approach the integer order with rising derivative order. To stabilise the instability in a range of global phenomena involving sudden changes in the dynamics of distinct densities, a time interval can be split into two or more intervals. The control technique performs better at smaller fractional orders. Therefore, the proposed method is essential to the investigation of epidemiological models. Nevertheless, some dynamics of illness models show stochastic behaviour. Results in our figures are more dependable when the dimensions are changed. This research has significant implications for public health strategies in managing and controlling hepatitis B and other infectious diseases.

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