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Hardy–Rogers Type Contractions in Double Controlled G-Metric Type Spaces

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Abstract. In this study, we establish a series of novel fixed-point theorems for Hardy–Rogers type (β-$\mathcal{F}$)-contractions within the framework of double controlled G-metric type spaces (DCGMS). The presented results extend and unify several classical contraction principles, including those of Banach, Kannan, Chatterjea, and Reich, thereby offering a broader perspective on the existing fixed point theory. Moreover, we provide a detailed analysis of completeness and Cauchy sequence properties in DCGMS, laying a solid foundation for further theoretical developments. To illustrate the applicability and robustness of the proposed results, a nontrivial example is constructed. These findings contribute to the enrichment of the fixed point literature and open potential avenues for applications in nonlinear analysis and related fields.

1. INTRODUCTION

The theory of fixed points has become one of the most powerful tools in nonlinear analysis, with deep implications in mathematics and applied sciences. Many important results in this area have been established by extending or generalizing the concept of a metric space to accommodate broader types of mappings and convergence structures.

A significant milestone in this direction was achieved in 2006 when Mustafa and Sims [10] introduced the concept of a G-metric space, which generalizes the classical metric by employing a function of three variables instead of two. This framework preserves many essential properties of metric spaces while offering a richer geometric structure. Since its introduction, numerous fixed-point results for various classes of contraction mappings have been established within G-metric spaces ([6], [7]). Before this development, Bakhtin [4] proposed the notion of a b-metric space,

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another important generalization that inspired extensive research exploring the theoretical and practical implications of such metric extensions.

To introduce a class even broader than the G-metric space, Aghajani et al. [1] presented the concept of generalized G_b-metric spaces in 2014. Building on this, further generalizations were sought to incorporate additional flexibility through control functions. The idea of a controlled structure arises naturally from the need to relax the classical triangle-type inequalities, thereby enabling the development of more general analytical frameworks in which fixed-point results can be established. In particular, controlled and double-controlled frameworks enrich the theory by introducing one or two independent control functions, respectively, allowing greater adaptability in handling non-standard contractive conditions.

Motivated by these developments, a new extension of G-metric spaces was proposed in [12], termed the double controlled G-metric type space (DCGMS). This structure unifies the geometric framework of G-metric spaces with the flexibility of double-controlled approaches, providing a powerful and versatile setting for establishing novel fixed point theorems.

In this work, we establish new fixed point theorems for Hardy–Rogers type (β-$\mathcal{F}$)-contractions within the framework of double controlled G-metric type spaces (DCGMS). The obtained results unify and extend several classical contraction principles, including those of Banach, Kannan, Chatterjea, and Reich, thus providing a broader and more flexible perspective on fixed point theory. To demonstrate the validity and applicability of the proposed results, a nontrivial illustrative example is presented. These findings enrich the existing literature and offer promising directions for future research in nonlinear analysis and related fields.

Definition 1.1. *[10] Let $X \neq \emptyset$, and let $G : X^3 \to [0, \infty)$ be a mapping such that for all $x, y, z \in X$ and $a \in X$, it satisfies the following:*

(1) $G(x, y, z) = 0$ *if* $x = y = z$;
(2) $G(x, x, z) > 0$ *whenever* $x \neq z$;
(3) $G(x, x, y) \leq G(x, y, z)$ *whenever* $y \neq z$;
(4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ *(symmetry in all three variables);*
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$.

Then the function G is called a generalized metric, or, more specifically, a G- metric space on X, and the pair (X, G) is a G-metric space.

Definition 1.2. *[10]A G-metric space (X, G) is symmetric if*

$$G(x, y, y) = G(x, x, y), \text{ for all } x, y \in X.$$

There is a relationship between a G-metric space and a d_G metric. For any nonempty set X, it is known that one can construct a G-metric on X starting from any usual metric. Conversely, given a G- metric G on X, the expression

$$d_G(x, z) = G(x, z, z) + G(x, x, z)$$

readily defines a metric on X. This metric, often referred to as the metric associated with G, satisfies the required properties of a standard metric space.

Mustafa and Sims [10] introduced the notion of a G-metric space as a generalization of metric spaces and investigated the corresponding topological structure. In particular, they defined a G metric topology, denoted by $\tau(G)$, which is induced by the G-metric G and generated by G-balls. Mustafa and Sims [10] had shown that the $\tau(G)$ topology on X is equivalent to the d_G metric topology on X. Before proceeding further with generalized b-metric spaces or even our definition of double controlled G-metric type space, we recall a few important concepts regarding the convergence of sequences, as presented in [10], as these concepts will be utilized later in our article.

Definition 1.3. *[10] Let (X, G) be a G-metric space. A sequence $\{x_n\} \subset X$ is said to G-converge to w if $\{x_n\}$ converges to w with respect to the topology, $\tau(G)$ induced by G-metric.*

Proposition 1.1. *[10] Let (X, G) be a G-metric space. Consider $\{x_n\}$ any sequence in X and let $w \in X$. The following statements are all equivalent:*

1. *The sequence $\{x_n\}$ is G-convergent to $w \in X$.*
2. *$\{x_n\}$ converges to w relative to the metric d_G.*
3. *$G(x_n, x_n, w) \to 0$, as n tends to ∞.*
4. *$G(x_n, w, w) \to 0$, as n tends to ∞.*
5. *$G(x_m, x_n, w) \to 0$, as m, n tends to ∞.*

To introduce a class larger than a G-metric space, Aghajani et al. [1] presented the notion of generalized b-metric spaces in 2014.

Definition 1.4. *[1] Let $X \neq \emptyset$ and $b \geq 1$. Suppose the mapping $G : X^3 \to [0, \infty)$ fulfills the following conditions for every $x, y, z \in X$ and any $a \in X$:*

1. *$G(x, y, z) = 0$ if $x = y = z$;*
2. *$G(x, x, z) > 0$, whenever $x \neq z$;*
3. *$G(x, x, y) \leq G(x, y, z)$, whenever $y \neq z$;*
4. *$G(x, y, z) = G(y, z, x) = G(x, z, y) = ...$ (the symmetry in all variables);*
5. *$G(x, y, z) \leq b(G(x, a, a) + G(a, y, z))$.*

The pair (X, G) is called a generalized b-metric space or G_b-metric space.

Motivated by the developments in controlled and double controlled metric type spaces [1,3,8,9], Swamy [12] introduced the double controlled G-metric type space as a natural extension of G-metric spaces. This framework merges G-metric geometry with the double controlled approach, providing a broader structure for nonlinear analysis and yielding Banach- and Kannan-type fixed point results.

To the best of our knowledge, [12] was the first to define double controlled G-metric type spaces and formulate their axioms. However, no concrete example was provided to verify these axioms in a detailed, step-by-step manner, nor was an explicit construction given to demonstrate the existence of genuinely non-comparable control functions.

In this paper, we address this deficiency in a decisive way. We construct the first fully worked and explicit model of a double controlled G-metric type space on $\mathbb{R}$ equipped with independent, genuinely non-comparable control functions $\alpha_1, \alpha_2 \geq 1$. We rigorously verify the axioms (DCG1) to (DCG5) individually, thereby establishing a concrete structural foundation for the theory. Our construction clarifies the precise role and interaction of the control functions and strengthens the mathematical framework of double-controlled G-metric spaces. Furthermore, we introduce a new class of contractions, namely the Hardy–Rogers type $(\beta - F_G)$-contraction mapping, within the setting of a double controlled G-metric spaces (DCGMS).

Definition 1.5. *[12] Let $X \neq \emptyset$, and let $G : X^3 \to [0, \infty)$ be a mapping and let $\alpha_1, \alpha_2 : X^3 \to [1, \infty)$ be two non-comparable functions, such that for all $x, y, z \in X$, it satisfies the following:*
(DCG1) $G(x, y, z) = 0$ *if* $x = z = y$;
(DCG2) $G(x, x, z) > 0$. *whenever* $x \neq z$;
(DCG3) $G(x, x, y) \leq G(x, y, z)$, *whenever* $y \neq z$;
(DCG4) $G(x, y, z) = G(y, z, x) = G(x, z, y) = \ldots$ *(the symmetry in all variables);*
(DCG5) $G(x, y, z) \leq \alpha_1(x, a, a)\, G(x, a, a) + \alpha_2(a, y, z) G(a, y, z)$, *for all* $x, y, z, a \in X$.

The pair (X, G) is called a double controlled G-metric type space, which will be denoted as DCGMS.

Example 1.1. *For $X = \mathbb{R}$. The G-metric is defined as;*

$$G_D(x, y, z) \ = \ \left[\frac{|x - y| + |y - z| + |x - z|}{3} \right]^2$$

and control functions

$$\alpha_1(x, w, w) = \begin{cases} 2, & \text{if } |x| < 1, \\ 4, & \text{if } |x| \geq 1, \end{cases} \qquad \alpha_2(w, y, z) = \begin{cases} 4, & \text{if } |z| < 1, \\ 2, & \text{if } |z| \geq 1. \end{cases}$$

Note that $\alpha_1, \alpha_2 \geq 2$, and they are non-comparable*: for instance, $\alpha_1(2, w, w) = 4 > \alpha_2(w, y, 3) = 2$, while $\alpha_1(0, w, w) = 2 < \alpha_2(w, y, 0) = 4$.*

We verify the axioms of a double-controlled G-metric type space.

(DCG1)

Clearly $[\frac{|x-y|+|y-z|+|x-z|}{3}]^2 \geq 0$. *Moreover,* $[\frac{|x-y|+|y-z|+|x-z|}{3}]^2 = 0$ *iff* $|x - y| = |y - z| = |x - z| = 0$, *i.e.,* $x = y = z$. *Hence* $G_D \geq 0$ *and* $G_D(x, y, z) = 0 \iff x = y = z$.

(DCG2) *If* $x \neq z$ *then*

$$G_D(x, x, z) = \left[\frac{|x - x| + |x - z| + |x - z|}{3} \right]^2 = \left[\frac{2|x - z|}{3} \right]^2 > 0$$

so $G_D(x, x, z) > 0$.

(DCG3) *For* $y \neq z$,

$$|x-y| \leq |x-z| + |z-y|$$

and if we add $|x-y|$ *to both sides of the inequality, we obtain*

$$|x-y| + |x-z| + |z-y| \ \geq \ |x-y| + |x-y| \ \geq \ 2|x-y|.$$

Dividing by two and squaring preserves order, so $G_D(x,x,y) \leq G_D(x,y,z)$.

(DCG4) *The expression* $|x-y| + |y-z| + |x-z|$ *is invariant under any permutation of* (x,y,z).

(DCG5) *For all* $x,y,z,w \in \mathbb{R}$,

$$\frac{2|x-w| + |w-y| + |y-z| + |w-z|}{3} \geq \frac{|x-y| + |x-z| + |y-z|}{3},$$

since $|x-y| \leq |x-w| + |w-y|$ *and* $|x-z| \leq |x-w| + |w-z|$.

Now square both sides and use $(a+b)^2 \leq 2a^2 + 2b^2$ *for* $a,b \geq 0$:

$$G_D(x,y,z) \ \leq \ 2\,G_D(x,w,w) + 2\,G_D(w,y,z).$$

Because by construction $\alpha_1(x,w,w) \geq 2$ *and* $\alpha_2(w,y,z) \geq 2$, *we obtain*

$$G_D(x,y,z) \ \leq \ \alpha_1(x,w,w)\,G_D(x,w,w) \ + \ \alpha_2(w,y,z)\,G_D(w,y,z),$$

which is exactly $(DCG5)$.

Therefore, $(\mathbb{R}, G_D)$ *is a double controlled G-metric type space with the above non-comparable control functions* α_1, α_2.

The following are some topological concepts of double-controlled G-metric type spaces that are useful for deriving related fixed-point results.

Definition 1.6. *[12] Consider a DCGMS* (X,G) *and let* $\{x_n\}$ *be a sequence in* X:

(1) The open ball with center $x_0 \in X$ *and radius* $\varepsilon > 0$ *has the form;*

$$B_G(x_0,\varepsilon) = \{x \in X : G(x,x_0,x_0) < \varepsilon\}.$$

(2) It is said to be a G-Cauchy sequence, if for every $\varepsilon > 0$, *one can find some* $N_0 \in \mathbb{N}$, *so* $G(x_m,x_n,x_l) < \varepsilon$ *for all* $n,m,l \geq N_0$.

(3) It is said to be a G-convergent sequence and converges to some $w \in X$, *if* $\lim_{n\to\infty} G(x_n,x_n,w) = 0$.

(4) The space (X, G) is said to be complete if every Cauchy sequence $\{x_n\}$ *is convergent.*

Following closely Proposition 1.1, we have the next Lemma.

Lemma 1.1. *Let* (X,G) *be a DCGMS. Then for any sequence* $\{x_n\} \subset X$. *The following are all equivalent:*

(1) *The sequence* $\{x_n\}$ *is G-convergent and converges to some* $w \in X$.

(2) $\lim_{n\to\infty} G(x_n,x_n,w) = 0$.

(3) $\lim_{n\to\infty} G(x_n,w,w) = 0$.

Recently, Wardowski [13] proposed the F-contraction framework and established several fixed-point results within it.

Definition 1.7. *Assume $\mathcal{F}$ represents the collection of all functions $F : (0, \infty) \to \mathbb{R}$ that fulfills the conditions stated below:*

(W1) F is a strictly increasing.

(W2) For every sequence $\{t_n\}$ of positive real numbers, the forthcoming equivalence holds:

$$\lim_{n\to\infty} t_n = 0 \text{ iff } \lim_{n\to\infty} F(t_n) = -\infty.$$

(W3) A constant $k \in (0,1)$ can be found such that $\lim_{t\to 0^+} t^k F(t) = 0$.

In 2019, Gupta and Kaurav [5] introduced F-contractions in G-metric spaces as follows:

Definition 1.8. *Let (X, G) be a G-metric space and consider a mapping $T : X \to X$. We call T an F-contraction if there exists a constant $\tau > 0$ such that*

$$G(Tx, Ty, Tz) > 0 \Rightarrow \tau + F\big(G(Tx, Ty, Tz)\big) \leq F\big(G(x, y, z)\big), \qquad \forall\, x, y, z \in X.$$

In 2024, G Sudhaamsh Mohan Reddy introduced an F-contractive mappings of Hardy–Rogers type in G-metric space as follows [11]:

Definition 1.9. *Let (X, G) be a G-metric space. A map $T : X \to X$ is called a Hardy–Rogers type G-F-contraction if there exists an $F \in \mathcal{F}$ and a constant $\tau > 0$ such that the inequality:*

$$\begin{aligned}\tau + F\big(G(Tx, Ty, Tz)\big) \leq\ & F\big(\alpha\, G(x, y, z) + \beta\, G(x, Tx, Tx) + \gamma\, G(y, Ty, Ty) \\ &+ h\, G(z, Tz, Tz) + \delta\, G(x, Ty, Ty) + \Delta\, G(y, Tz, Tz) + e\, G(z, Tx, Tx)\big),\end{aligned}$$

holds for every $x, y, z \in X$ satisfying $G(Tx, Ty, Tz) > 0$. The constants are required to satisfy:

$$\alpha + \beta + \gamma + h + \Delta + 2\delta = 1, \qquad h \neq 1, \qquad e \geq 0.$$

We recall the definition of an admissible mapping.

Definition 1.10. *[2] Let $T : X \longrightarrow X$ be a mapping, and let $\beta : X^3 \to [0, \infty)$ be a given function, where X is a nonempty set. The mapping T is called β-admissible, if for every $\hat{x}, \hat{y}, \hat{z} \in X$, we have*

$$\beta(\hat{x}, \hat{y}, \hat{z}) \geq 1 \Longrightarrow \beta(T\hat{x}, T\hat{y}, T\hat{z}) \geq 1. \tag{1.1}$$

Example 1.2. *[2] Assume $X = \mathbb{R}$ and the mapping $T : X \to X$ is defined by:*

$$T(a) = \begin{cases} 2\ln a, & \text{if } a \neq 0, \\ e, & \text{otherwise,} \end{cases}$$

and $\beta : X^3 \to [0, \infty)$ is given as

$$\beta(x, y, z) = \begin{cases} e, & \text{if } x \geq y \geq z, \\ 0, & \text{otherwise.} \end{cases}$$

Then the mapping T is β-admissible.

2. Main Results

Before presenting our main theorem, we introduce a novel concept the Hardy–Rogers type ($\beta-F_G$)-contraction mapping within the framework of double-controlled G-metric spaces ($DCGMS$). Under suitable coefficient constraints, this generalized form encompasses the classical Banach, Kannan, Chatterjea, and Reich contractions as special cases. Thus, the Hardy–Rogers type ($\beta-F_G$)-contraction serves as a unified framework for establishing fixed-point theorems in $DCGMS$.
First, we define the (β-$\mathcal{F}$)-contraction mapping on $DCGMS$ as follows.

Definition 2.1. *Assume* (X, G) *is a double controlled G-metric type space. Let* $X \neq \emptyset$*, a mapping* $T : X \to X$ *is called a* (β-$\mathcal{F}$)*-contraction mapping, if there exists a function* $\beta : X^3 \to [0, \infty)$*, an* $F \in \mathcal{F}$*, and a constant* $\tau > 0$ *such that the following condition is satisfied;*

$$\tau + \beta(x, y, z)F(G(Tx, Ty, Tz)) \leq F(G(x, y, z)), \tag{2.1}$$

for $x, y, z \in X$ *such that* $G(Tx, Ty, Tz) > 0$.

An illustrative example of a (β -F)-contraction on $DCGMS$ is presented below.

Example 2.1. *Let* $X = \mathbb{R}$ *and equip* X *with the double controlled G-metric type*

$$G_D(x, y, z) \ = \ \left(\frac{|x-y|+|y-z|+|x-z|}{3}\right)^2,$$

which is a standard DCGMS. Let $\beta : X^3 \to [0, \infty)$ *be the constant function* $\beta(x, y, z) = 1$*, and take* $F : (0, \infty) \to \mathbb{R}$ *as* $F(t) = \ln t$*. Fix any* $\tau > 0$ *and define* $T : X \to X$ *by* $T(x) = \rho\, x$ *with* $0 < \rho \leq e^{-\tau/2}$*. T is a* $(\beta-F)$*-contraction. Really, since* G_D *is homogeneous of degree* 2 *(because of the square), we have*

$$G_D(Tx, Ty, Tz) = G_D(\rho x, \rho y, \rho z) = \rho^2\, G_D(x, y, z).$$

Thus, for all x, y, z *with* $G_D(Tx, Ty, Tz) > 0$,

$$\begin{aligned}
\tau + \beta(x, y, z)F\Big(G_D(Tx, Ty, Tz)\Big) \ &= \ \tau + \ln\Big(\rho^2 G_D(x, y, z)\Big) \\
&= \ \underbrace{\Big(\tau + 2\ln\rho\Big)}_{\leq 0} + \ln G_D(x, y, z) \\
&\leq \ \ln G_D(x, y, z) \\
&= \ F\Big(G_D(x, y, z)
\end{aligned}$$

because $\rho \leq e^{-\tau/2}$ *implies* $\tau + 2\ln\rho \leq 0$*. Hence T satisfies* $\tau + \beta\, F(G_D(Tx, Ty, Tz)) \leq F(G_D(x, y, z))$*, i.e. T is a* ($\beta$-$F$)*-contraction.* □

Definition 2.2. *Let* (X, G) *be a double controlled G-metric type space. A mapping* $T : X \to X$ *is called a Hardy–Rogers type* $(\beta-F_G)$*-contraction, if there exists a function* $\beta : X^3 \to [0, \infty)$*, a function* $F \in \mathcal{F}$ *and* $\tau > 0$ *a constant, such that the condition below holds:*

$$\tau + \beta(x, y, z)F\Big(G(Tx, Ty, Tz)\Big) \leq F\Big(M(x, y, z)\Big),$$

where

$$\begin{aligned} M(x,y,z) = \gamma_1 G(x,y,z) + \gamma_2 G(x,Tx,Tx) \\ + \gamma_3 G(y,Ty,Ty) + \gamma_4 G(z,Tz,Tz) \\ + \gamma_5 G(x,Ty,Ty) + \gamma_6 G(y,Tz,Tz) \\ + \gamma_7 G(z,Tx,Tx) + \gamma_8 G(x,Tz,Tz) \\ + \gamma_9 G(y,Tx,Tx) + \gamma_{10} G(z,Ty,Ty). \end{aligned}$$

for $x,y,z \in X$ *such that* $G(Tx,Ty,Tz) > 0$*, here* $\gamma_i > 0$ *for* $i = 1,\ldots,10$ *and*

$$\sum_{i=1}^{10} \gamma_i < 1.$$

We now proceed to present our first main theorem.

Theorem 2.1. *Let* (X,G) *be a complete symmetric double controlled G-metric type space, with* $\alpha_1,\alpha_2 : X^3 \to [1,\infty)$ *being non-comparable functions. Assume the mapping* $T : X \to X$ *is a Hardy–Rogers type* $(\beta{-}F_G)$*-contraction, as in Definition 2.2. Suppose the following conditions hold:*

1. *You can find some* $x_0 \in X$ *so that* $\beta(x_0,Tx_0,Tx_0) \geq 1$.
2. T *is* β*-admissible.*
3. *Let* $x_0 \in X$ *be initial point defined in* (1) *and define the sequence* $\{x_n\}$ *by* $x_{n+1} = Tx_n$*, for all* $n \geq 0$*. In addition, the following properties are satisfied:*

$$\sup_{m\geq 1} \lim_{n\to\infty} \frac{\alpha_1(x_{n+1},x_{n+2},x_{n+2})\,\alpha_2(x_{n+1},x_m,x_m)}{\alpha_1(x_n,x_{n+1},x_{n+1})} < 1, \tag{2.2}$$

the limit

$$\lim_{n\to\infty} \alpha_1(x_n,x_{n+1},x_{n+1})$$

exists and is finite. Furthermore, for each $x \in X$*, the limit*

$$\lim_{n\to\infty} \alpha_2(x_n,x,x)$$

exists and is finite. Moreover, we assume $A_{n-1} + B_n < 1$*, where*

$$\begin{aligned} A_{n-1} &= (\gamma_1+\gamma_2) + (\gamma_5+\gamma_8)\,\alpha_1(x_{n-1},x_n,x_n), \\ B_n &= (\gamma_3+\gamma_4+\gamma_6+\gamma_{10}) + (\gamma_5+\gamma_8)\,\alpha_2(x_n,x_{n+1},x_{n+1}),\ n \geq 1. \end{aligned} \tag{2.3}$$

Consequently, T *has at least one fixed point. To show the fixed point is unique, suppose* u *and* v *are fixed points with* $\beta(u,v,v) \geq 1$*. Under this assumption,* T *admits a single fixed point in* X*.*

Proof. Choose $x_0 \in X$ so that $\beta(x_0,Tx_0,Tx_0) \geq 1$. The Picard sequence $\{x_n\}$ is formed by taking x_0 as initial point, and letting

$$x_n = T^n x_0 = Tx_{n-1}, \quad \forall n \geq 1.$$

In case $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, the sequence x_n becomes a fixed point of T. Thus, assume $x_n \neq x_{n-1}$ for each $n \geq 1$. As the mapping T is β-admissible, hence

$$\beta(x_0,x_1,x_1) = \beta(x_0,Tx_0,Tx_0) \geq 1 \;\Rightarrow\; \beta(Tx_0,Tx_1,Tx_1) = \beta_1(x_1,x_2,x_2) \geq 1.$$

Inductively, we obtain

$$\beta(x_n, x_{n+1}, x_{n+1}) \geq 1 \quad \text{for all } n = 0, 1, 2, \ldots$$

Since T is a Hardy–Rogers type $(\beta{-}F_G)$-contraction mapping, we obtain

$$\begin{aligned}
\tau + F\Big(G(x_n, x_{n+1}, x_{n+1})\Big) &= \tau + F\Big(G(Tx_{n-1}, Tx_n, Tx_n)\Big) \\
&\leq \tau + \beta(x_{n-1}, x_n, x_n)\, F\Big(G(Tx_{n-1}, Tx_n, Tx_n)\Big)
\end{aligned}$$

$$\begin{aligned}
\leq F\Big[&\gamma_1 G(x_{n-1}, x_n, x_n) + \gamma_2 G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + \gamma_3 G(x_n, Tx_n, Tx_n) + \gamma_4 G(x_n, Tx_n, Tx_n) \\
&+ \gamma_5 G(x_{n-1}, Tx_n, Tx_n) + \gamma_6 G(x_n, Tx_n, Tx_n) + \gamma_7 G(x_n, Tx_{n-1}, Tx_{n-1}) + \gamma_8 G(x_{n-1}, Tx_n, Tx_n) \\
&+ \gamma_9 G(x_n, Tx_{n-1}, Tx_{n-1}) + \gamma_{10} G(x_n, Tx_n, Tx_n)\Big]
\end{aligned}$$

$$\begin{aligned}
= \quad F\Big[&(\gamma_1 + \gamma_2) G(x_{n-1}, x_n, x_n) + (\gamma_3 + \gamma_4 + \gamma_6 + \gamma_{10}) G(x_n, x_{n+1}, x_{n+1}) \\
&+ (\gamma_5 + \gamma_8) G(x_{n-1}, x_{n+1}, x_{n+1})\Big]
\end{aligned}$$

This implies that

$$\begin{aligned}
0 \quad < \quad \tau \leq F\Big[&(\gamma_1 + \gamma_2) G(x_{n-1}, x_n, x_n) + (\gamma_3 + \gamma_4 + \gamma_6 + \gamma_{10}) G(x_n, x_{n+1}, x_{n+1}) \\
&+ (\gamma_5 + \gamma_8) G(x_{n-1}, x_{n+1}, x_{n+1})\Big] - F(G(x_n, x_{n+1}, x_{n+1}))
\end{aligned}$$

Since F is strictly increasing, we have

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) \leq{}& (\gamma_1 + \gamma_2) G(x_{n-1}, x_n, x_n) + (\gamma_3 + \gamma_4 + \gamma_6 + \gamma_{10}) G(x_n, x_{n+1}, x_{n+1}) \\
&+ (\gamma_5 + \gamma_8) G(x_{n-1}, x_{n+1}, x_{n+1}).
\end{aligned}$$

which leads to

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) \leq{}& (\gamma_1 + \gamma_2)\, G(x_{n-1}, x_n, x_n) + (\gamma_3 + \gamma_4 + \gamma_6 + \gamma_{10})\, G(x_n, x_{n+1}, x_{n+1}) \\
&+ (\gamma_5 + \gamma_8)\, \alpha_1(x_{n-1}, x_n, x_n)\, G(x_{n-1}, x_n, x_n) \\
&+ (\gamma_5 + \gamma_8)\, \alpha_2(x_n, x_{n+1}, x_{n+1})\, G(x_n, x_{n+1}, x_{n+1}).
\end{aligned}$$

Hence,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{\Big[(\gamma_1 + \gamma_2) + (\gamma_5 + \gamma_8)\alpha_1(x_{n-1}, x_n, x_n)\Big] G(x_{n-1}, x_n, x_n)}{1 - \Big[(\gamma_3 + \gamma_4 + \gamma_6 + \gamma_{10}) + (\gamma_5 + \gamma_8)\alpha_2(x_n, x_{n+1}, x_{n+1})\Big]}.$$

By equation 2.3, and the fact $A_{n-1} + B_n < 1$, we deduce that

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n), \quad \forall n \in \mathbb{N}.$$

Continuing in the same argument, we obtain

$$F(G(x_n, x_{n+1}, x_{n+1})) \leq F(G(x_0, x_1, x_1)) - n\tau.$$

Letting $n \to \infty$ and using $\tau > 0$ gives

$$\lim_{n\to\infty} F(G(x_n, x_{n+1}, x_{n+1})) = -\infty.$$

From property $(W2)$, we get

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

By $(W3)$ of Definition 1.7, there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1})^k F(G(x_n, x_{n+1}, x_{n+1})) = 0.$$

Thus for any n, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1})^k F(G(x_n, x_{n+1}, x_{n+1})) - G(x_n, x_{n+1}, x_{n+1})^k F(G(x_0, x_1, x_1)) \\ \leq -n\tau G(x_n, x_{n+1}, x_{n+1})^k < 0. \end{aligned}$$

Taking $n \to \infty$,

$$\lim_{n\to\infty} n\Big[G(x_n, x_{n+1}, x_{n+1})\Big]^k = 0.$$

This gives

$$\lim_{n\to\infty} n^{1/k} G(x_n, x_{n+1}, x_{n+1}) = 0,$$

Accordingly, there exists an integer n_0 satisfying

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{1}{n^{1/k}}, \quad \forall n \geq n_0. \tag{2.4}$$

We next demonstrate that the sequence $\{x_n\}$ is Cauchy in X. Let $n, m \in \mathbb{N}$ with $m > n$:

$$\begin{aligned} G(x_n, x_m, x_m) &\leq \alpha_1(x_n, x_{n+1}, x_{n+1})G(x_n, x_{n+1}, x_{n+1}) + \alpha_2(x_{n+1}, x_m, x_m)G(x_{n+1}, x_m, x_m) \\ &\leq \alpha_1(x_n, x_{n+1}, x_{n+1})G(x_n, x_{n+1}, x_{n+1}) \\ &\quad + \alpha_2(x_{n+1}, x_m, x_m)\Big[\alpha_1(x_{n+1}, x_{n+2}, x_{n+2})G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + \alpha_2(x_{n+2}, x_m, x_m)G(x_{n+2}, x_m, x_m)\Big] \\ &\leq \ldots \end{aligned}$$

Finally, we arrive at

$$\begin{aligned} G(x_n, x_m, x_m) \leq \alpha_1(x_n, x_{n+1}, x_{n+1})G(x_n, x_{n+1}, x_{n+1}) \\ + \sum_{i=n+1}^{m-2} \alpha_1(x_i, x_{i+1}, x_{i+1})G(x_i, x_{i+1}, x_{i+1}) \prod_{j=n+1}^{i} \alpha_2(x_j, x_m, x_m) \\ + \prod_{j=n+1}^{m-1} \alpha_2(x_j, x_m, x_m)G(x_{m-1}, x_m, x_m). \end{aligned}$$

Applying 2.4 gives

$$G(x_n, x_m, x_m) \leq \alpha_1(x_n, x_{n+1}, x_{n+1})\left(\frac{1}{n^{1/k}}\right) + \sum_{i=n+1}^{m-1} \alpha_1(x_i, x_{i+1}, x_{i+1})\left(\frac{1}{i^{1/k}}\right)\prod_{j=n+1}^{i} \alpha_2(x_j, x_m, x_m).$$

Let

$$L_p = \sum_{i=1}^{p-1}\left(\prod_{j=1}^{i} \alpha_2(x_j, x_m, x_m)\right)\alpha_1(x_i, x_{i+1}, x_{i+1})\frac{1}{i^{1/k}}.$$

Then

$$G(x_n, x_m, x_m) \leq \alpha_1(x_n, x_{n+1}, x_{n+1})\left(\frac{1}{n^{1/k}}\right) + L_m - L_{n+1}. \tag{2.5}$$

Applying the ratio test on equation 2.5, we obtain the existence of $\lim_{n\to\infty} L_n$, and hence $\{x_n\}$ is Cauchy. From the completeness (X, G), therefore the sequence will converge to some $u \in X$, that is

$$\lim_{n\to\infty} G(x_n, u, u) = 0. \tag{2.6}$$

We next demonstrate the uniqueness of the fixed point $u \in X$.

First, we show that

$$\lim_{n\to\infty} G(Tx_n, Tu, Tu) = 0.$$

Assume $G(Tx_n, Tu, Tu) > 0$ for all n. By Hardy–Rogers type $(\beta{-}F_G)$-contraction mapping,

$$\begin{aligned}\tau + F(G(Tx_n, Tu, Tu)) \leq F\Big(&\gamma_1 G(x_n, u, u) + \gamma_2 G(x_n, Tx_n, Tx_n) + \gamma_3 G(u, Tu, Tu)\\ &+ \gamma_4 G(u, Tx_n, Tx_n) + \gamma_5 G(x_n, Tu, Tu) + \gamma_6 G(u, Tu, Tu)\\ &+ \gamma_7 G(u, Tx_n, Tx_n) + \gamma_8 G(x_n, Tu, Tu) + \gamma_{10} G(u, Tu, Tu)\Big).\end{aligned}$$

Letting n tend to infinity and using 2.6, and $(W2)$ yields

$$\lim_{n\to\infty} F(G(Tx_n, Tu, Tu)) = -\infty \quad \Rightarrow \quad \lim_{n\to\infty} G(Tx_n, Tu, Tu) = 0.$$

Hence

$$G(Tu, u, u) \leq \alpha_1(Tu, x_{n+1}, x_{n+1})G(Tu, x_{n+1}, x_{n+1}) + \alpha_2(x_{n+1}, u, u)G(x_{n+1}, u, u).$$

Taking limit $n \to \infty$ gives $G(Tu, u, u) = 0 \Rightarrow Tu = u$.

Finally, suppose T has two fixed points u, v with $\beta(u, v, v) \geq 1$. Then $Tu = u \neq v = Tv$ implies $G(Tu, Tv, Tv) > 0$. Using Hardy–Rogers type $(\beta{-}F_G)$-contraction, we obtain

$$\begin{aligned}\tau + F(G(Tu, Tv, Tv)) \leq F\Big(&\gamma_1 G(u, v, v) + \gamma_2 G(u, Tu, Tu) + \gamma_3 G(v, Tv, Tv) + \gamma_4 G(v, Tv, Tv)\\ &+ \gamma_5 G(u, Tv, Tv) + \gamma_6 G(v, Tv, Tv) + \gamma_7 G(v, Tu, Tu) + \gamma_8 G(u, Tv, Tv)\\ &+ \gamma_9 G(v, Tu, Tu) + \gamma_{10} G(v, Tv, Tv)\Big)\\ = F\Big[&(\gamma_1 + \gamma_5 + \gamma_8)G(u, v, v) + (\gamma_7 + \gamma_9)G(v, u, u)\Big].\end{aligned}$$

Utilizing the symmetry property and by $(W1)$ of Definition 1.7, we have

$$\begin{aligned}\tau + F(G(Tu,Tv,Tv)) &= F\Big[(\gamma_1+\gamma_5+\gamma_8)G(u,v,v)+(\gamma_7+\gamma_9)G(v,u,u)\Big]\\ &= F\Big[(\gamma_1+\gamma_5+\gamma_8+\gamma_7+\gamma_9)G(u,v,v)\Big]\\ &\le F\Big[G(Tu,Tv,Tv)\Big].\end{aligned}$$

This gives a contradiction since $\tau > 0$. Thus, u is the unique fixed point. □

Our next main theorem is stated as follows.

Theorem 2.2. *Let (X, G) be a complete double controlled G- metric type space, where $X \neq \emptyset$. Let $T: X \to X$ be $(\beta - F_G)$-contraction mapping, as in Definition 2.1. Assume the following holds:*

1. *There is $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \ge 1$.*
2. *T is β-admissible.*
3. *For a chosen $x_0 \in X$, generate the sequence $\{x_n\}$, by setting $x_n = T^n x_0$, furthermore these hold:*

$$\sup_{m\ge 1}\lim_{n\to\infty}\frac{\alpha_1(x_{n+1}, x_{n+2}, x_{n+2})\alpha_2(x_{n+1}, x_m, x_m)}{\alpha_1(x_n, x_{n+1}, x_{n+1})} < 1. \tag{2.7}$$

Furthermore, for each $x \in X$, the limits;

$$\lim_{n\to\infty}\alpha_1(x, x_n, x_n)\ ,\text{and}\ \lim_{n\to\infty}\alpha_2(x_n, x_n, x)\ \text{exists and finite.}$$

Thus, T admits a fixed point. To establish uniqueness, suppose u and v are fixed points with $\beta(u, v, v) \ge 1$. Under this condition, T has exactly one fixed point in X.

Proof. Choose $x_0 \in X$, with $\beta(x_0, x_0, Tx_0) \ge 1$. Generate $\{x_n\}$ by setting $Tx_0 = x_1, T^2x_0 = Tx_1 = x_2$. Thus, for any $n \in \mathbb{N}$, note

$$T^n x_0 = T^{n-1}x_1 = \cdots = Tx_{n-1} = x_n.$$

In addition, $T^n x_0 \neq T^{n+1}x_0$ is true for all $n \ge 0$.

Since T is β-admissible, we have

$$\begin{aligned}\tau + F(G(x_n, x_{n+1}, x_{n+1})) &= \tau + F(G(Tx_{n-1}, Tx_n, Tx_n)).\\ &\le \tau + \beta(x_{n-1}, x_n, x_n)F(G(Tx_{n-1}, Tx_n, Tx_n))\\ &\le F(G(x_{n-1}, x_n, x_n)).\end{aligned}$$

which gives

$$\begin{aligned}F(G(x_n, x_{n+1}, x_{n+1})) &\le F(G(x_{n-1}, x_n, x_n)) - \tau.\\ &\le F(G(x_{n-2}, x_{n-1}, x_{n-1})) - 2\tau.\\ &\le\\ &\ \ \vdots\end{aligned}$$

$$\leq \quad F(G(x_0,x_1,x_1)) - n\tau. \tag{2.8}$$

As $n \to \infty$ in (2.8) and with $\tau > 0$, we have

$$\lim_{n\to\infty} F(G(x_n,x_{n+1},x_{n+1})) = -\infty.$$

Because $F \in \mathcal{F}$, applying condition (W2) of Definition 1.7, yields $\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1}) = 0$. Moreover, by (W3), there is a constant $k \in (0,1)$ so that

$$\lim_{n\to\infty} (G(x_n,x_{n+1},x_{n+1}))^k F((G(x_n,x_{n+1},x_{n+1})) = 0.$$

Equation (2.8) yields

$$F(G(\, x_n,\, x_{n+1},\, x_{n+1})) - F(G(x_0,\, x_1,\, x_1)) \leq -n\tau.$$

Thus for any n, we have

$$(G(\, x_n,\, x_{n+1},\, x_{n+1}))^k F(G(x_n,\, x_{n+1},\, x_{n+1}) - (G(x_n,\, x_{n+1},\, x_{n+1})^k F(G(x_0,x_1,x_1))$$

$$\leq -n\tau(G(x_n,x_{n+1},x_{n+1}))^k \leq 0. \tag{2.9}$$

Making $n \to \infty$ in (2.9), we have

$$\lim_{n\to\infty} n(G(x_n,x_{n+1},x_{n+1}))^k = 0.$$

Implying, $\lim_{n\to\infty} n^{1/k}(G(x_n,x_{n+1},x_{n+1})) = 0$. Therefore, there exists some $n_0 \in \mathbb{N}$, such that

$$G(x_n,x_{n+1},x_{n+1}) \leq \frac{1}{n^{1/k}}, \text{ for all } n \geq n_0. \tag{2.10}$$

To illustrate $\{x_n\}$ is a Cauchy sequence, note for any natural numbers $n, m \in \mathbb{N}$ with condition $n < m$, we obtain

$$\begin{aligned}
G(x_n,\, x_m,\, x_m) &\leq \alpha_1(x_n,\, x_{n+1},\, x_{n+1})G(\, x_n,x_{n+1},\, x_{n+1}) + \alpha_2(x_{n+1},x_m,x_m)G(x_{n+1},x_m,x_m)\\
&\leq \alpha_1(\, x_n,\, x_{n+1},\, x_{n+1})G(x_n,\, x_{n+1},x_{n+1})\\
&+ \alpha_2(x_{n+1},x_m,x_m)[\alpha_1(x_{n+1},x_{n+2},x_{n+2})G(x_{n+1},x_{n+2},x_{n+2})\\
&+ \alpha_2(x_{n+2},x_m,x_m)G(x_{n+2},x_m,x_m)].\\
&\leq \alpha_1(x_n,x_{n+1},\, x_{n+1})G(x_n,x_{n+1},x_{n+1})\\
&+ \alpha_2(x_{n+1},x_m,x_m)\alpha_1(x_{n+1},x_{n+2},x_{n+2})G(x_{n+1},x_{n+2},x_{n+2})\\
&+ \alpha_2(x_{n+1},x_m,x_m)\alpha_2(x_{n+2},x_m,x_m)[\alpha_1(x_{n+2},x_{n+3},x_{n+3})G(x_{n+2},x_{n+3},x_{n+3})\\
&+ \alpha_2(x_{n+3},x_m,x_m)G(\, x_{n+3},\, x_m,\, x_m)].\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
G(\, x_n,\, x_m,x_m) \leq{} & \alpha_1(x_n,x_{n+1},\, x_{n+1})G(x_n,\, x_{n+1},\, x_{n+1})\\
&+ \sum_{i=n+1}^{m-2} [\alpha_1(x_i,x_{i+1},x_{i+1})]G(x_i,x_{i+1},x_{i+1})\left(\prod_{j=n+1}^{i} \alpha_2(x_j,x_m,x_m)\right)
\end{aligned}$$

$$+\left[\prod_{i=n+1}^{m-1}\alpha_2(x_i,x_m,x_m)\right]G(x_{m-1},x_m,x_m).$$

$$\begin{aligned}G(x_n,x_m,x_m)&\leq \alpha_1(x_n,x_{n+1},x_{n+1})G(x_n,x_{n+1},x_{n+1})\\&+\sum_{i=n+1}^{m-1}\alpha_1(x_i,x_{i+1},x_{i+1})G(x_i,x_{i+1},x_{i+1})\left(\prod_{j=n+1}^{i}\alpha_2(x_j,x_m,x_m)\right)\end{aligned}\tag{2.11}$$

Applying equation (2.10) into inequality (2.11), it becomes

$$\begin{aligned}G(x_n,x_m,x_m)&\leq \alpha_1(x_n,x_{n+1},x_{n+1})(\frac{1}{n^{1/k}})\\&+\sum_{i=n+1}^{m-1}\alpha_1(x_i,x_{i+1},x_{i+1})(\frac{1}{i^{1/k}})\left(\prod_{j=n+1}^{i}\alpha_2(x_j,x_m,x_m)\right)\\&\leq \alpha_1(x_n,x_{n+1},x_{n+1})(\frac{1}{n^{1/k}})\\&+\sum_{i=1}^{m-1}\alpha_1(x_i,x_{i+1},x_{i+1})(\frac{1}{i^{1/k}})\left(\prod_{j=1}^{i}\alpha_2(x_j,x_m,x_m)\right)\end{aligned}\tag{2.12}$$

Let $L_p=\sum_{i=1}^{p-1}\alpha_1(x_i,x_{i+1},x_{i+1})(\frac{1}{i^{1/k}})\left(\prod_{j=1}^{i}\alpha_2(x_j,x_m,x_m)\right)$.

Hence, inequality (2.12) can be written as

$$G(x_n,x_m,x_m)\leq \alpha_1(x_n,x_{n+1},x_{n+1})(\frac{1}{n^{1/k}})+(L_m-L_{n+1})\tag{2.13}$$

Applying the ratio test on inequality (2.13) and then applying that limit as both n and m tends to infinity, we obtain $\lim_{n,m\to\infty}[L_{m-1}-L_n]=0$. Moreover, employing (2.2), gives $\lim_{n\to\infty}\alpha_1(x_n,x_{n+1},x_{n+1})(\frac{1}{n^{1/k}})=0$.

Hence, we have shown that

$$\lim_{n,m\to\infty}G(x_n,x_m,x_m)=0.$$

We conclude $\{x_n\}$ is a Cauchy sequence. A the space (X,G) is complete, therefore the sequence will converge to some $u\in X$, i.e.

$$\lim_{n\to\infty}G(x_n,u,u)=0.\tag{2.14}$$

To show u is a fixed point of the mapping T. First, we demonstrate that $\lim_{n\to\infty}G(Tx_n,Tu,Tu)=0$.

Let $G(Tx_n,Tu,Tu)>0$ for all n. By Definition 2.1, we obtain

$$\tau+\beta(x_n,u,u)F(G(Tx_n,Tu,Tu))\leq F(G(x_n,u,u))\tag{2.15}$$

Taking the limit as $n\to\infty$ in(2.15), and using equation (2.14), and (W2), we obtain $\lim_{n\to\infty}F(G(Tx_n,Tu,Tu))=-\infty$. Again, by Definition 1.7 this implies

$$\lim_{n\to\infty} G(Tx_n, Tu, Tu) = 0.$$

To show point u is fixed by T, note

$$\begin{aligned} G(Tu, u, u) &\le \alpha_1(Tu, x_{n+1}, x_{n+1})G(Tu, x_{n+1}, x_{n+1}) + \alpha_2(x_{n+1}, u, u)G(x_{n+1}, u, u). \\ &\le \alpha_1(Tu, x_{n+1}, x_{n+1})G(Tu, Tx_n, Tx_n) + \alpha_2(x_{n+1}, u, u)G(x_{n+1}, u, u). \end{aligned} \tag{2.16}$$

As n tends to ∞ in (2.16), and by utilizing Lemma 1.1, we deduce that $G(Tu, u, u) = 0$, thus $Tu = u$.

Next, we show that the fixed point is unique. Assume, for contradiction, that u and v are two distinct points and $\beta(u, v, v) \ge 1$. Since $Tu = u \neq v = Tv$, we have $G(Tu, Tv, Tv) > 0$.

Because T satisfies the $(\beta{-}F_G)$-contraction condition, applying relation (2.1), yields

$$\begin{aligned} \tau + F(G(Tu, Tv, Tv)) &\le \tau + \beta(u, v, v)F(G(Tu, Tv, Tv)). \\ &\le F(G(u, v, v)) = F(G(Tu, Tv, Tv)) \end{aligned}$$

This leads to the inequality $\tau \le 0$, which is impossible, hence a contradiction arises. Consequently, the fixed point must be unique. □

3. CONCLUSIONS

In this work, we developed a new fixed point results for the Hardy–Rogers type $(\beta\text{-}\mathcal{F})$-contractions within the framework of double controlled G-metric type spaces (DCGMS). These findings broaden and unify various well-known contraction principles"including those of Banach, Kannan, Chatterjea, and Reich"offering a more general and cohesive perspective in fixed point theory.

To demonstrate the applicability and strength of the proposed theorems, we presented a nontrivial example that satisfies the required conditions and illustrates the effectiveness of the approach. The results not only enrich the existing literature on fixed point theory but also broaden the scope for further investigation within nonlinear analysis and related disciplines.

Future research may focus on exploring additional applications of the developed framework, particularly in integral equations, iterative algorithms, and dynamical systems, where double controlled G-metric structures naturally arise.

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