International Journal of Analysis and Applications

A Tangent Similarity-Based Pentapartitioned Neutrosophic Soft Topological Framework for Cancer Diagnosis

Shoaib Arbab[1], Muhammad Idrees[1], Aqeedat Hussain[2], Walid Abdelfattah[3,*], Arif Mehmood[2], Cris L. Armada[4,5]

[1]Department of Mathematics, Islamia College Peshawar, Peshawar 25120, Khyber Pakhtunkhwa, Pakistan

[2]Department of Mathematics, Institute of Numerical Sciences, Gomal University, Dera Ismail Khan 29050, KPK, Pakistan

[3]Humanities and Social Research Center, Northern Border University, Arar, Saudi Arabia

[4]Vietnam National University Ho Chi Minh City, Linh Trung Ward, Thu Duc City, Ho Chi Minh City, Vietnam

[5]Vietnam Department of Applied Mathematics, Faculty of Applied Science, Ho Chi Minh City University of Technology (HCMUT), 268 Ly Thuong Kiet, Ward 14, District 10, Ho Chi Minh City, Vietnam

**Corresponding author:* walid.abdelfattah@nbu.edu.sa

Abstract. This research paper introduces the concept of pentapartitioned neutrosophic soft topological spaces (PNSTS). In PNSTS, the indeterminacy is divided into three components: truth (ReT), hesitation (H), and falsehood (ReF). Fundamental topological concepts and tangent-based similarity measures have been discussed and applied to a cancer data set to ascertain the similarity between patients and potential treatments. To analyze results more effectively, we have employed Heatmaps, 3D surface plotting, PCA, t-SNE, and DBSCAN clustering. The experimental results demonstrated that the proposed framework was highly effective, as the maximum similarity value reached 0.7508. This indicates that the patient's traits and the proposed diagnosis clearly aligned. This demonstrates that the proposed method can be effectively employed for intelligent medical decision-making.

Received Feb. 24, 2026

2020 Mathematics Subject Classification. 54A40.

Key words and phrases. neutrosophic soft set; neutrosophic soft topological space; PNSS; PNSTSs; tangent similarity measure.

https://doi.org/10.28924/2291-8639-24-2026-270

ISSN: 2291-8639

1. **Introduction**

Various fields, including mathematics, artificial intelligence, information sciences, and decision support systems, have been focusing on the issue of uncertainty modeling. Information is modeled in a binary format by classical set theory, which lacks adequate tools for describing vague and uncertain information. Zadeh proposed the fuzzy sets theory to address the issue, which models information with fuzzy membership values, and established the fundamentals of the uncertainty modeling problem [1]. Turksen expanded on the concept by introducing interval-valued fuzzy sets, where membership values are represented by intervals [2]. Atanassov expanded the concept even more by presenting intuitionistic fuzzy sets, which utilize membership and non-membership functions to model information [3]. The framework mentioned above has been broadened to include interval-valued intuitionistic fuzzy sets, where the intervals can represent both the degree of membership and non-membership [4]. The capability of fuzzy systems to manage uncertainty in real-world applications was significantly enhanced by these developments. Smarandache proposed neutrosophic sets to accommodate indeterminate and inconsistent information, where an element is characterized independently by the degrees of truth, indeterminacy, and falsehood [5]. The theoretical foundation of neutrosophic systems was established by neutrosophic topology [6]. To facilitate implementation while maintaining the expressive power of neutrosophic sets, single valued neutrosophic sets were introduced [7].

Subsequently, interval neutrosophic sets were proposed by representing each element of the neutrosophic set with intervals [8]. A key area of investigation in neutrosophic theory is the development of similarity and correlation measures, which enable quantitative comparisons of uncertain data. The cosine similarity measure for interval-valued neutrosophic sets was defined by Broumi and Smarandache [9]. They also proposed the distance and similarity measurements for interval neutrosophic sets [10]. Ye developed a method for measuring the similarity of interval neutrosophic sets, which can be applied to multiple criteria decision-making [11]. Broumi and Smarandache proposed the correlation coefficients of interval neutrosophic sets [12]. Majumdar and Samanta [13] provided similarity and entropy measures for neutrosophic sets. In the past, Chen proposed an effective method for the fuzzy decision-making process [14], while Chen et al. conducted a comparison of various similarity measurements of fuzzy values [15]. Molodtsov [16] has accomplished the development of a parameterized uncertainty model utilizing soft set theory. Additionally, it has been developed using research on soft groups [17], soft relations and functions [18], parameter reduction methods [19], [20], soft set-based decision making [21], adjustable fuzzy soft decision making [22], fuzzy soft decision models [23], and fuzzy soft set theory [25]. Conversely, the concept of intuitionistic fuzzy soft sets has been proposed for use in group decision-making applications [26]. Maji et al. [27] developed some fundamental concepts of soft set theory, while Çağman and Deli [28], [29] discussed the FP-Soft set and its applications. Neutrosophic soft sets, resulting from the integration of neutrosophic and soft set theories, provide a robust means of addressing parameterized uncertainty. Deli proposed interval valued neutrosophic soft sets and their use in decision-making issues [30]. Majumdar and Samanta [31] examined similarity measures on soft sets, while Kharal [32] proposed distance and similarity measures on these sets. Grzegorzewski examined distances derived from the Hausdorff metric between intuitionistic and interval-valued fuzzy sets [33].

Broumi [46] subsequently introduced generalized neutrosophic soft sets. Maji [49] provided a rigorous definition of neutrosophic soft sets, while Deli and Broumi investigated neutrosophic soft relations and their mathematical characteristics [50]. Smarandache additionally demonstrated that neutrosophic sets naturally extend intuitionistic fuzzy sets [51]. Neutrosophic methods have achieved great success across various domains of application. Neutrosophic techniques have a wide range of applications, including relational database decomposition [34], natural language queries processing [35], intelligent control [36], image thresholding [37], image segmentation [38], watershed-based image segmentation [39], and medical artificial intelligence [40]. Theoretical contributions regarding operations on soft sets were provided by Ali et al. [41], Ali [42], Acar et al. [43], Jiang et al. [44], and Çağman [45]. The use of neutrosophic soft topologies has garnered significant attention recently due to their ability to integrate topology with uncertain models. In this context, significant contributions include generalized neutrosophic soft sets [46], similarity measures of neutrosophic sets [47], neutrosophic soft topological spaces [48], neutrosophic soft sets [49], neutrosophic soft relations [50], and generalized neutrosophic theory [51]. Recent studies have investigated additional applications of neutrosophic soft topological spaces [61]. Alongside these theoretical advancements, the methods of visualization and data analysis have demonstrated their importance as techniques for comprehending high-dimensional data with uncertainty. Efficient methods for managing extensive data volumes are provided by information retrieval techniques [52]. The grammar of graphics offers a robust theoretical foundation for statistical visualization [53]. Techniques of data mining have been significant in the realm of intelligent knowledge discovery [54]. Quantitative data analysis has been the subject of extensive study regarding visualization principles [55]. Principal component analysis is credited to Pearson [56], while spline-based numerical approximation techniques originate from de Boor [57]. MATLAB has become one of the most widely used computing tools for scientific visualization and numerical analysis [58]. The K-means clustering method proposed by Lloyd [59] is among the most widely used unsupervised algorithms, whereas Fisher's classical multivariate analysis [60] laid the groundwork for contemporary dimensionality reduction methods.

Modern research has significantly advanced neutrosophic mathematics through its applications across diverse areas of mathematics. Quadri-partitioned neutrosophic soft topological spaces represent recent advancements aimed at enhancing neutrosophic topology [61]. LPM polynomials have been used to derive numerical solutions for neutrosophic singular boundary value problems [62]. Irreversible threshold conversion problems have also been addressed using neutrosophic graph theory [63]. For (γ,τ)-rung fuzzy sets, tangent complex trigonometric operators have been established [64]. Neutrosophic topology has been further enriched by the n-plithogenic interval topological symbolic spaces [65]. Significant advancements have also been achieved in the areas of fractional differential equations [66], fixed point theory [67], fuzzy contractions [68], fractional soliton equations [69], correlation function of non-stationary fields [70], operator theory [71], [72], [73], splines for addressing boundary value problems [74], reaction-diffusion equations [75], numerical radius inequalities [76], and fuzzy n-refined neutrosophic rings [77].

1.1. Research gap

Although the conception of neutrosophic and soft set theory has reached a mature stage, certain restricting difficulties remain. A pentapartitioned paradigm, where indeterminacy is further broken down into discontinuous structures like relative truth, hesitation and relative falsehood is incompatible with the bulk of known quadripartitioned or TVNSS models. It limits the ability to simulate subtleties of uncertainty in fields like medical diagnosis, where many types of indeterminacy have varying practical significance. Second, the combined approach of topological operators (closure, interior and their connections) and similarity measures has been left out of most neutrosophic soft set type models, even if some have included mathematical ideas of topology in their frameworks. As a result, the architecture of relationships between data points and the borders of uncertainty have received less attention, particularly when it comes to high-dimensional decision spaces. Third, it is rare to combine sophisticated similarity metrics, such as tangent similarity which can match profiles with a high degree of accuracy with neutrosophic constructs in a way that makes visual analytics easier. Fourth, the models that are now available hardly ever provide complete visualization pipelines that would include density-based clustering (DBSCAN), Heatmaps, 3D surface visualization, PCA and t-SNE to identify both local and global similarity patterns. The actual performance of such models is unconfirmed in the case of mixed, incomplete and heterogeneous data sources because, despite the existence of many theoretical constructs, few of them apply to the kind of high-dimensional data that is characterized by non-stationarity in the real world (e.g., a wide range of cancer profiles or pathway featuring multiple choices of diagnosis methods).

1.2. Research Novelty

With a variety of theoretical, computational and artistic activities, the proposed research helps close those gaps by offering a simplified and hybrid analytical framework. In order to capture topological and relational uncertainty, it first defines a new mathematical structure known as the pentapartitioned neutrosophic soft topological spaces (PNSTS). This new structure decomposes indeterminacy into a number of clinically useful components using embedding operators like closure and interior. Second, it enables precise and comprehensible comparison of complicated profiles, such as a patient's records utilizing many diagnostic or treatment templates by combining pentapartitioned neutral similarity metrics with a tangent similarity metric. Third, to provide multi-scale responses on both global trends and local deviations in the data, the system combines similarity computation with a pipeline of visual analytics including Heatmaps, 3D surface plots, PCA and t-SNE with DBSCAN clustering. Fourth, the methodology is quantitatively good; the greatest distance-similarity value between patient R_2 and decision D_1 is 0.7508, indicating a strong diagnosis match between the two. This not only validates the framework's clinical decision-making capabilities but also its stability in the face of dimensionality reduction and clustering techniques. Finally yet importantly, the proposed system is not only domain-generic but also cancer-diagnostic, making it applicable to requirements in other domains like psychology decision support, health issue triage and military crisis management. Where there is a greater need for dependable decision-making and a computationally viable, mathematically modeled and pictorially interpretable capability that can be used to manage extreme uncertainty and contradiction.

2. Preliminaries and related concepts

This section is devoted to the basic concepts necessary for the upcoming sections.

Definition 2.1 [51]. On the universe of discourse Ú, a neutrosophic set A is defined as follows:
$A = \{(\S, T_A(\S), I_A(\S), Ƒ_A(\S)) : \S \in Ú\}$, where $T, I, Ƒ : Ú \to]^-0, 1^+[$ and $^-0 \leq T_A(\S) + I_A(\S) + Ƒ_A(\S) \leq 3^+$.
The neutrosophic set is philosophically viewed as deriving its values from the real non-standard or standard subsets of $]^-0, 1^+[$. However, when it comes to applying this concept to actual scientific and engineering problems, using a neutrosophic set with values from a real standard or non-standard subset of $]^-0, 1^+[$ is challenging Ú.

Definition 2.2 [16]. Let $P(Ú)$ be the power set of Ú, let Ě be a set of parameters and let Ú be a starting universe. When Ƒ is a mapping with the formula by $Ƒ : Ě \to P(Ú)$ the pair (Ƒ, Ě) is referred to as a soft set over Ú. To put it another way, the set is the set of e-approximate elements of the soft sets or $(Ƒ, Ě) = \{(e, Ƒ(e)) : e \in Ě, Ƒ : Ě \to P(Ú)\}$, or a parameterized family of subsets of the set Ú for e-elements of the soft set (Ƒ, Ě).

Definition 2.3 [49]. Let Ě be the collection of parameters and let Ú be the starting universe set. P(Ú) represent the set of all neutrosophic sets of Ú. A set valued function $ƒ$: Ě $\rightarrow P$(Ú) defines a neutrosophic soft set $(ƒ, Ě)$ over Ú. The approximate function of the NSS (ƒ, Ě) is denoted by the ƒ. A collection of ordered pairs can be used to represent it:

$$(ƒ, Ě) = \left\{\left(e, \left(ş, T_{ƒ(e)}(ş), I_{ƒ(e)}(ş), ƒ_{ƒ(e)}(ş) : ş \in Ú\right)\right) : e \in Ě\right\},$$

where $T_{ƒ(e)}(ş), I_{ƒ(e)}(ş), ƒ_{ƒ(e)}(ş) \in [0,1]$ are the truth-membership, indeterminacy-membership and falsity-membership functions of ƒ(e) respectively. Naturally, the inequality $^{-}0 \leq T_{ƒ(e)}(ş) + I_{ƒ(e)}(ş) + ƒ_{ƒ(e)}(ş) \leq 3^{+}$ is true.

Definition 2.4 [48]. Over the universe set Ú, let (ƒ, Ě) be NSS. Then, $(ƒ, Ě)^c$ which represents the complement of (ƒ, Ě) is defined as follows: $(ƒ, Ě)^c = \left\{\left(e, \left(ş, ƒ_{ƒ(e)}(ş), 1 - I_{ƒ(e)}(ş), T_{ƒ(e)}(ş) : ş \in Ú\right)\right) : e \in Ě\right\}$. The fact that $((ƒ, Ě)^c)^c = (ƒ, Ě)$ is evident.

Definition 2.5 [49]. Assume that two NSSs over set Ú are $(ƒ, Ě)$ *and* $(G, Ě)$. If for every ş in Ú and e in Ě, $T_{ƒ(e)}(ş) \leq T_{G(e)}(ş)$, $I_{ƒ(e)}(ş) \leq I_{G(e)}(ş)$, $ƒ(ş) \geq ƒ_{G(e)}(ş)$, then (ƒ, Ě) is NSSS of $(G, Ě)$ and the relationship is represented by $(ƒ, Ě) \subseteq (G, Ě)$.

Definition 2.6 [78]. Assume that two NSSs over set Ú are $(ƒ_1, Ě)$ *and* $(ƒ_2, Ě)$. The union denoted by $(ƒ_1, Ě) \cup (ƒ_2, Ě) = (ƒ_3, Ě)$ is defined as

$(ƒ_3, Ě) = \left\{\left(e, \left(ş, T_{ƒ_3(e)}(ş), I_{ƒ_3(e)}(ş), ƒ_{ƒ_3(e)}(ş)\right) : ş \in Ú\right) : e \in Ě\right\}$, where

$$T_{ƒ_3(e)}(ş) = \text{maş}\{T_{ƒ_1(e)}(ş), T_{ƒ_2(e)}(ş)\},$$
$$T_{ƒ_3(e)}(ş) = \text{maş}\{I_{ƒ_1(e)}(ş), I_{ƒ_2(e)}(ş)\},$$
$$T_{ƒ_3(e)}(ş) = \min\{ƒ_{ƒ_1(e)}(ş), ƒ_{ƒ_2(e)}(ş)\}.$$

Definition 2.7 [78]. Assume that two NSSs over set Ú are $(ƒ_1, Ě)$ *and* $(ƒ_2, Ě)$. The intersection denoted by $(ƒ_1, Ě) \cap (ƒ_2, Ě) = (ƒ_3, Ě)$ is defined as

$(ƒ_3, Ě) = \left\{\left(e, \left(ş, T_{ƒ_3(e)}(ş), I_{ƒ_3(e)}(ş), ƒ_{ƒ_3(e)}(ş)\right) : ş \in Ú\right) : e \in Ě\right\}$, where

$$T_{ƒ_3(e)}(ş) = \min\{T_{ƒ_1(e)}(ş), T_{ƒ_2(e)}(ş)\},$$
$$T_{ƒ_3(e)}(ş) = \min\{I_{ƒ_1(e)}(ş), I_{ƒ_2(e)}(ş)\},$$
$$T_{ƒ_3(e)}(ş) = \text{maş}\{ƒ_{ƒ_1(e)}(ş), ƒ_{ƒ_2(e)}(ş)\}.$$

Definition 2.8 [78]. Assume that two NSSs over set Ú are $(ƒ_1, Ě)$ *and* $(ƒ_2, Ě)$. Then difference operation between them denoted by $(ƒ_1, Ě) \setminus (ƒ_2, Ě) = (ƒ_3, Ě)$, defined as $(ƒ_3, Ě) = (ƒ_1, Ě) \cap (ƒ_2, Ě)^c$ as fallows:

$(ƒ_3, Ě) = \left\{\left(e, \left(ş, T_{ƒ_3(e)}(ş), I_{ƒ_3(e)}(ş), ƒ_{ƒ_3(e)}(ş)\right) : ş \in Ú\right) : e \in Ě\right\}$, where

$$T_{ƒ_3(e)}(ş) = \min\{T_{ƒ_1(e)}(ş), ƒ_{ƒ_2(e)}(ş)\},$$
$$T_{ƒ_3(e)}(ş) = \min\{I_{ƒ_1(e)}(ş), 1 - I_{ƒ_2(e)}(ş)\},$$

$$T_{Ƒ_3(e)}(§) = maş\{Ƒ_{Ƒ_1(e)}(§), T_{Ƒ_2(e)}(§)\}.$$

Definition 2.9 [78]. Let $\{(Ƒ_i, Ĕ) | i \in I\}$ be a family of NSSs over set Ú, then

$$\bigcup_{i\in I}(Ƒ_i, Ĕ) = \left\{\left(e, \left(§, \sup\left[T_{Ƒ_i(e)}(§)\right]_{i\in I}, \sup\left[I_{Ƒ_i(e)}(§)\right]_{i\in I}, \inf\left[Ƒ_{Ƒ_i(e)}(§)\right]_{i\in I}\right) : § \in Ú\right) : e \in Ĕ\right\},$$

$$\bigcap_{i\in I}(Ƒ_i, Ĕ) = \left\{\left(e, \left(§, \inf\left[T_{Ƒ_i(e)}(§)\right]_{i\in I}, \inf\left[I_{Ƒ_i(e)}(§)\right]_{i\in I}, \sup\left[Ƒ_{Ƒ_i(e)}(§)\right]_{i\in I}\right) : § \in Ú\right) : e \in Ĕ\right\}.$$

Definition 2.10 [78]. Assume that two NSSs over set Ú are $(Ƒ_1, Ĕ)$ and $(Ƒ_2, Ĕ)$. The $''AND''$ operation on them is denoted by $(Ƒ_1, Ĕ) \wedge (Ƒ_2, Ĕ) = (Ƒ_3, Ĕ \times Ĕ)$ and is defined as:

$(Ƒ_3, Ĕ \times Ĕ) = \left\{\left((e_1, e_2), \left(§, T_{Ƒ_3(e_1,e_2)}(§), I_{Ƒ_3(e_1,e_2)}(§), Ƒ_{Ƒ_3(e_1,e_2)}(§)\right) : § \in Ú\right) : (e_1, e_2) \in Ĕ \times Ĕ\right\}$, where

$$T_{Ƒ_3(e_1,e_2)}(§) = \min\{T_{Ƒ_1(e_1)}(§), T_{Ƒ_2(e_2)}(§)\},$$

$$T_{Ƒ_3(e_1,e_2)}(§) = \min\{I_{Ƒ_1(e_1)}(§), I_{Ƒ_2(e_2)}(§)\},$$

$$T_{Ƒ_3(e_1,e_2)}(§) = maş\{Ƒ_{Ƒ_1(e_1)}(§), Ƒ_{Ƒ_2(e_2)}(§)\}.$$

Definition 2.11 [78]. Assume that two NSSs over set Ú are $(Ƒ_1, Ĕ)$ and $(Ƒ_2, Ĕ)$. The $''OR''$ operation on them is denoted by $(Ƒ_1, Ĕ) \vee (Ƒ_2, Ĕ) = (Ƒ_3, Ĕ \times Ĕ)$ and is defined as:

$(Ƒ_3, Ĕ \times Ĕ) = \left\{\left((e_1, e_2), \left(§, T_{Ƒ_3(e_1,e_2)}(§), I_{Ƒ_3(e_1,e_2)}(§), Ƒ_{Ƒ_3(e_1,e_2)}(§)\right) : § \in Ú\right) : (e_1, e_2) \in Ĕ \times Ĕ\right\}$, where

$$T_{Ƒ_3(e_1,e_2)}(§) = maş\{T_{Ƒ_1(e_1)}(§), T_{Ƒ_2(e_2)}(§)\},$$

$$T_{Ƒ_3(e_1,e_2)}(§) = maş\{I_{Ƒ_1(e_1)}(§), I_{Ƒ_2(e_2)}(§)\},$$

$$T_{Ƒ_3(e_1,e_2)}(§) = \min\{Ƒ_{Ƒ_1(e)}(§), Ƒ_{Ƒ_2(e_2)}(§)\}.$$

Definition 2.12 [78]. 1. Assume $(Ƒ, Ĕ)$ be NSS over set Ú. It is said to be null NSS if $T_{Ƒ(e)}(§) = 0$, $I_{Ƒ(e)}(§) = 0$, $Ƒ_{Ƒ(e)}(§) = 1$ $\forall\, e \in Ĕ$ and $§ \in Ú$, it is signified by $0_{(Ú,Ĕ)}$.

2. $(Ƒ, Ĕ)$ is said to be absolute NSS if $T_{Ƒ(e)}(§) = 1$, $I_{Ƒ(e)}(§) = 1$, $Ƒ_{Ƒ(e)}(§) = 0$ $\forall\, e \in Ĕ$, $§ \in Ú$, it is signified by $1_{(Ú,Ĕ)}$.

3. Pentapartitioned algebraic operations of neutrosophic soft sets

There are the concepts of a pentapartitioned neutrosophic set, a PNSS, a complement of PNSS, a subset of PNSSs, the equality of PNSSs, the union of a pair of PNSSs, the intersection of a pair of PNSSs and the difference of a pair of PNSSs.

Definition 3.1. A PNSS represented by A is defined as follows:

$A = \{(§, AbT_A(§), RelT_A(§), H_A(§), RelƑ_A(§), AbƑ_A(§)) : § \in Ú\}$, where $AbT, RelT, H, RelƑ, AbƑ :\ Ú \to\ ]^-0, 1^+[$ and $^-0 \leq AbT_A(§) + RelT_A(§) + H_A(§) + RelƑ_A(§) + AbƑ_A(§) \leq 5^+$.

Definition 3.2. Let Ĕ be a set of parameters and X be an initial universe set. A PNSS (Ƒ, Ĕ) over X is a set defined by a set valued function Ƒ, which represents a mapping: Ƒ : Ĕ →P(X), where Ƒ is an approximate function of the PNSS (Ƒ, Ĕ). Let P(X) represent the set of all PNS of X.

$\left(\digamma, \check{E}\right) = \left\{\left(e, \left(§, AbT_{\digamma(e)}(§), RelT_{\digamma(e)}(§), H_{\digamma(e)}(§), Rel\digamma_{\digamma(e)}(§), Ab\digamma_{\digamma(e)}(§)\right) : § \in Ú\right) : e \in \check{E}\right\}$,

where $AbT_{\digamma(e)}(§), RelT_{\digamma(e)}(§), H_{\digamma(e)}(§), Rel\digamma_{\digamma(e)}(§), Ab\digamma_{\digamma(e)}(§)$ are elements of [0,1], and respectively called the absolute truth-membership, relative truth-membership, hesitation-membership, relative false-membership and absolute false-membership function of $\digamma(e)$. The inequality $^{-}0 \leq AbT_{\digamma(e)}(§) + RelT_{\digamma(e)}(§) + H_{\digamma(e)}(§) + Rel\digamma_{\digamma(e)}(§) + Ab\digamma_{\digamma(e)}(§) \leq 5^{+}$ holds obviously.

Definition 3.3. Let $(\digamma, \check{E})$ be PNSS over set Ú. The complement of $\left(\digamma, \check{E}\right)$ denoted by $(\digamma, \check{E})^{c}$, defined as;

$(\digamma, \check{E})^{c} = \left\{\left(e, \left(§, Ab\digamma_{\digamma(e)}(§), Rel\digamma_{\digamma(e)}(§), 1 - H_{\digamma(e)}(§), RelT_{\digamma(e)}(§), AbT_{\digamma(e)}(§)\right) : § \in Ú\right) : e \in \check{E}\right\}$.

Definition 3.4. Let $\left(\digamma, \check{E}\right)$ and $\left(G, \check{E}\right)$ be PNSS over set Ú, then $\left(\digamma, \check{E}\right)$ is said to be PNS subset of $\left(G, \check{E}\right)$ if $AbT_{\digamma(e)}(§) \geq AbT_{G(e)}(§)$, $RelT_{\digamma(e)}(§) \geq RelT_{G(e)}(§)$, $H_{\digamma(e)}(§) \geq H_{G(e)}(§)$, $Rel\digamma_{\digamma(e)}(§) \leq Rel\digamma_{G(e)}(§), Ab\digamma_{\digamma(e)}(§) \leq Ab\digamma_{G(e)}(§), \forall\, § \in Ú, e \in \check{E}$. It is symbolized by $\left(\digamma, \check{E}\right) \Subset \left(G, \check{E}\right)$.

Definition 3.5. Let $\left(\digamma, \check{E}\right)$ and $(G, \check{E})$ be PNSS over set Ú, then $(\digamma, \check{E})$ is said to be PNS equals $\left(G, \check{E}\right)$, if $\left(\digamma, \check{E}\right) \Subset \left(G, \check{E}\right)$ i.e.

$AbT_{\digamma(e)}(§) \geq AbT_{G(e)}(§),\ RelT_{\digamma(e)}(§) \geq RelT_{G(e)}(§),\ H_{\digamma(e)}(§) \geq H_{G(e)}(§),$

$Rel\digamma_{\digamma(e)}(§) \leq Rel\digamma_{G(e)}(§),\ Ab\digamma_{\digamma(e)}(§) \leq Ab\digamma_{G(e)}(§), \forall\, e \in \check{E}, § \in Ú$. If $\left(G, \check{E}\right) \Subset \left(\digamma, \check{E}\right)$ i.e.

$AbT_{G(e)}(§) \geq AbT_{\digamma(e)}(§),\ RelT_{G(e)}(§) \geq RelT_{\digamma(e)}(§),\ H_{G(e)}(§) \geq H_{\digamma(e)}(§),$

$Rel\digamma_{G(e)}(§) \leq Rel\digamma_{\digamma(e)}(§),\ Ab\digamma_{G(e)}(§) \leq Ab\digamma_{\digamma(e)}(§)$. It is symbolized by $\left(\digamma, \check{E}\right) = \left(G, \check{E}\right)$.

Definition 3.6. Let $\left(\digamma_1, \check{E}\right)$ and $\left(\digamma_2, \check{E}\right)$ be PNSS over set Ú. Then, their union is denoted by $\left(\digamma_3, \check{E}\right) = \left(\digamma_1, \check{E}\right) \cup \left(\digamma_2, \check{E}\right)$ and is defined as;

$\left(\digamma_3, \check{E}\right) = \left\{\left(e, \left(§, AbT_{\digamma_3(e)}(§), RelT_{\digamma_3(e)}(§), H_{\digamma_3(e)}(§), Rel\digamma_{\digamma_3(e)}(§), Ab\digamma_{\digamma_3(e)}(§)\right) : § \in Ú\right) : e \in \check{E}\right\}$, where

$AbT_{\digamma_3(e)}(§) = \text{ma§}\left\{AbT_{\digamma_1(e)}(§), AbT_{\digamma_2(e)}(§)\right\}$, $RelT_{\digamma_3(e)}(§) = \text{ma§}\left\{RelT_{\digamma_1(e)}(§), RelT_{\digamma_2(e)}(§)\right\}$,

$H_{\digamma_3(e)}(§) = \text{ma§}\left\{H_{\digamma_1(e)}(§), H_{\digamma_2(e)}(§)\right\}$, $Rel\digamma_{\digamma_3(e)}(§) = \min\left\{Rel\digamma_{\digamma_1(e)}(§), Rel\digamma_{\digamma_2(e)}(§)\right\}$,

$Ab\digamma_{\digamma_3(e)}(§) = \min\left\{Ab\digamma_{\digamma_1(e)}(§), Ab\digamma_{\digamma_2(e)}(§)\right\}$.

Definition 3.7. Let $\left(\digamma_1, \check{E}\right)$ and $\left(\digamma_2, \check{E}\right)$ be PNSS over set Ú. Then, their intersection is denoted by $\left(\digamma_3, \check{E}\right) = \left(\digamma_1, \check{E}\right) \cap \left(\digamma_2, \check{E}\right)$ and is defined as:

$\left(\digamma_3, \check{E}\right) = \left\{\left(e, \left(§, AbT_{\digamma_3(e)}(§), RelT_{\digamma_3(e)}(§), H_{\digamma_3(e)}(§), Rel\digamma_{\digamma_3(e)}(§), Ab\digamma_{\digamma_3(e)}(§)\right) : § \in Ú\right) : e \in \check{E}\right\}$.

Where, $AbT_{\digamma_3(e)}(§) = \min\left\{AbT_{\digamma_1(e)}(§), AbT_{\digamma_2(e)}(§)\right\}$, $RelT_{\digamma_3(e)}(§) = \min\left\{RelT_{\digamma_1(e)}(§), RelT_{\digamma_2(e)}(§)\right\}$,

$H_{\digamma_3(e)}(§) = \min\left\{H_{\digamma_1(e)}(§), H_{\digamma_2(e)}(§)\right\}$, $Rel\digamma_{\digamma_3(e)}(§) = \text{ma§}\left\{Rel\digamma_{\digamma_1(e)}(§), Rel\digamma_{\digamma_2(e)}(§)\right\}$,

$Ab\digamma_{\digamma_3(e)}(§) = \text{ma§}\left\{Ab\digamma_{\digamma_1(e)}(§), Ab\digamma_{\digamma_2(e)}(§)\right\}$.

Definition 3.8. Let $\left(\digamma_1, \check{E}\right)$ and $\left(\digamma_2, \check{E}\right)$ be PNSS. Then the difference of $\left(\digamma_1, \check{E}\right)$ and $\left(\digamma_2, \check{E}\right)$ operation on them is symbolized by $\left(\digamma_3, \check{E}\right) = \left(\digamma_1, \check{E}\right) \setminus \left(\digamma_2, \check{E}\right)$ and is defined as: $\left(\digamma_3, \check{E}\right) = \left(\digamma_1, \check{E}\right) \cap \left(\digamma_2, \check{E}\right)^{c}$

Definition 3.9. Let $\{(\digamma_i, \check{E}) | i \in I\}$ be a collection of PNSS . Then

$$\cup_{i\in I}(\digamma_i, \check{E}) = \left\{\left(e, \begin{pmatrix} \S, \mathrm{sp}[AbT_{\digamma_i(e)}(\S)]_{i\in I}, \mathrm{sp}[RelT_{\digamma_i(e)}(\S)]_{i\in I}, \mathrm{sp}[H_{\digamma_i(e)}(\S)]_{i\in I}, \\ \mathrm{if}[Rel\digamma_{\digamma_i(e)}(\S)]_{i\in I}, \mathrm{if}[Ab\digamma_{\digamma_i(e)}(\S)]_{i\in I} \end{pmatrix} : e \in \check{E}\right) : \S \in \acute{U}\right\},$$

$$\cap_{i\in I}(\digamma_i, \check{E}) = \left\{\left(e, \begin{pmatrix} \S, \mathrm{if}[AbT_{\digamma_i(e)}(\S)]_{i\in I}, \mathrm{if}[RelT_{\digamma_i(e)}(\S)]_{i\in I}, \mathrm{if}[H_{\digamma_i(e)}(\S)]_{i\in I}, \\ \mathrm{sp}[Rel\digamma_{\digamma_i(e)}(\S)]_{i\in I}, \mathrm{sp}[Ab\digamma_{\digamma_i(e)}(\S)]_{i\in I} \end{pmatrix} : e \in \check{E}\right) : \S \in \acute{U}\right\}.$$

Definition 3.10. Let $(\digamma_1, \check{E})$ and $(\digamma_2, \check{E})$ be PNSS. Then operation AND is symbolized by $(\digamma_3, \check{E} \times \check{E}) = (\digamma_1, \check{E}) \wedge (\digamma_2, \check{E})$ is given as:

$$(\digamma_3, \check{E} \times \check{E}) = \left\{\left((e_1, e_2), \begin{pmatrix} \S, AbT_{\digamma_3(e_1,e_2)}(\S), RelT_{\digamma_3(e_1,e_2)}(\S), H_{\digamma_3(e_1,e_2)}(\S), \\ Rel\digamma_{\digamma_3(e_1,e_2)}(\S), Ab\digamma_{\digamma_3(e_1,e_2)}(\S) \end{pmatrix} : (e_1, e_2) \in \check{E} \times \check{E}\right) : \S \in \acute{U}\right\},$$

where

$AbT_{\digamma_3(e_1,e_2)}(\S) = \min\{AbT_{\digamma_1(e_1)}(\S), AbT_{\digamma_2(e_2)}(\S)\}$,

$RelT_{\digamma_3(e_1,e_2)}(\S) = \min\{RelT_{\digamma_1(e_1)}(\S), RelT_{\digamma_2(e_2)}(\S)\}$,

$H_{\digamma_3(e_1,e_2)}(\S) = \min\{H_{\digamma_1(e_1)}(\S), H_{\digamma_2(e_2)}(\S)\}$,

$Rel\digamma_{\digamma_3(e_1,e_2)}(\S) = \mathrm{ma\S}\{Rel\digamma_{\digamma_1(e_1)}(\S), Rel\digamma_{\digamma_2(e_2)}(\S)\}$,

$Ab\digamma_{\digamma_3(e_1,e_2)}(\S) = \mathrm{ma\S}\{Ab\digamma_{\digamma_1(e_1)}(\S), Ab\digamma_{\digamma_2(e_2)}(\S)\}$.

Definition 3.11. Let $(\digamma_1, \check{E})$ and $(\digamma_2, \check{E})$ be PNSS. Then operation OR is given as $(\digamma_3, \check{E} \times \check{E}) = (\digamma_1, \check{E}) \vee (\digamma_2, \check{E})$ is given as:

$$(\digamma_3, \check{E} \times \check{E}) = \left\{\left((e_1, e_2), \begin{pmatrix} \S, AbT_{\digamma_3(e_1,e_2)}(\S), RelT_{\digamma_3(e_1,e_2)}(\S), H_{\digamma_3(e_1,e_2)}(\S), \\ Rel\digamma_{\digamma_3(e_1,e_2)}(\S), Ab\digamma_{\digamma_3(e_1,e_2)}(\S) \end{pmatrix} : (e_1, e_2) \in \check{E} \times \check{E}\right) : \S \in \acute{U}\right\},$$

where

$AbT_{\digamma_3(e_1,e_2)}(\S) = \mathrm{ma\S}\{AbT_{\digamma_1(e_1)}(\S), AbT_{\digamma_2(e_2)}(\S)\}$,

$RelT_{\digamma_3(e_1,e_2)}(\S) = \mathrm{ma\S}\{RelT_{\digamma_1(e_1)}(\S), RelT_{\digamma_2(e_2)}(\S)\}$,

$H_{\digamma_3(e_1,e_2)}(\S) = \mathrm{ma\S}\{H_{\digamma_1(e_1)}(\S), H_{\digamma_2(e_2)}(\S)\}$,

$Rel\digamma_{\digamma_3(e_1,e_2)}(\S) = \min\{Rel\digamma_{\digamma_1(e_1)}(\S), Rel\digamma_{\digamma_2(e_2)}(\S)\}$,

$Ab\digamma_{\digamma_3(e_1,e_2)}(\S) = \min\{Ab\digamma_{\digamma_1(e_1)}(\S), Ab\digamma_{\digamma_2(e_2)}(\S)\}$.

Definition 3.12. (i). Let $(\digamma, \check{E})$ is PNSS. It is said to be null PNSS if $AbT_{\digamma(e)}(\S) = 0, RelT_{\digamma(e)}(\S) = 0, H_{\digamma(e)}(\S) = 0, Rel\digamma_{\digamma(e)}(\S) = 1, Ab\digamma_{\digamma(e)}(\S) = 1;\ \forall\, e \in \check{E}, \S \in \acute{U}$. It is symbolized by $0_{(\acute{U},\check{E})}$.

(ii). Let $(\digamma, \check{E})$ is PNSS. It is said to be absolute PNSS if

$AbT_{\digamma(e)}(\S) = 1, RelT_{\digamma(e)}(\S) = 1, H_{\digamma(e)}(\S) = 1, Rel\digamma_{\digamma(e)}(\S) = 0, Ab\digamma_{\digamma(e)}(\S) = 0;\ \forall\, e \in \check{E}, \S \in \acute{U}$.

It is symbolized by $1_{(\acute{U},\check{E})}$ and clearly, $0^c_{(\acute{U},\check{E})} = 1_{(\acute{U},\check{E})}$ and $1^c_{(\acute{U},\check{E})} = 0_{(\acute{U},\check{E})}$.

Example 3.13. Let $\acute{U} = \{\S_1, \S_2, \S_3, \S_4\}$ and $\check{E} = \{e_1, e_2\}$, let the PNSSs $(\digamma_1, \check{E})$ and $(\digamma_2, \check{E})$ $be\ PNSSs\ over$ $\acute{U}$ $be\ defined\ as$ follows:

$$\left(\text{Ƒ}_1,\check{E}\right)=\begin{cases} e_1 = \left\{\begin{matrix}(\S_1,0.4,0.7,0.2,0.8,0.4),(\S_2,0.5,0.6,0.3,0.9,0.4),\\(\S_3,0.4,0.5,0.3,0.8,0.4),(\S_4,0.2,0.5,0.6,0.8,0.3)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(\S_1,0.7,0.8,0.4,0.5,0.3),(\S_2,0.4,0.9,0.2,0.6,0.8),\\(\S_3,0.6,0.7,0.2,0.8,0.9),(\S_4,0.3,0.4,0.7,0.9,0.4)\end{matrix}\right\}\end{cases},$$

$$\left(\text{Ƒ}_2,\check{E}\right)=\begin{cases} e_1 = \left\{\begin{matrix}(\S_1,0.5,0.6,0.9,0.2,0.5),(\S_2,0.6,0.8,0.3,0.4,0.5),\\(\S_3,0.7,0.3,0.1,0.2,0.8),(\S_4,0.3,0.4,0.9,0.4,0.6)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(\S_1,0.6,0.9,0.3,0.4,0.6),(\S_2,0.3,0.6,0.9,0.3,0.8),\\(\S_3,0.4,0.2,0.6,0.8,0.5),(\S_4,0.6,0.3,0.7,0.8,0.4)\end{matrix}\right\}\end{cases},$$

$$\left(\text{Ƒ}_1,\check{E}\right)\cup\left(\text{Ƒ}_2,\check{E}\right)=\begin{cases} e_1 = \left\{\begin{matrix}(\S_1,0.5,0.7,0.9,0.2,0.4),(\S_2,0.6,0.8,0.3,0.4,0.4),\\(\S_3,0.7,0.5,0.3,0.2,0.4),(\S_4,0.3,0.5,0.9,0.4,0.3)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(\S_1,0.7,0.9,0.4,0.4,0.3),(\S_2,0.4,0.9,0.9,0.3,0.8),\\(\S_3,0.6,0.7,0.6,0.8,0.5),(\S_4,0.6,0.4,0.7,0.8,0.4)\end{matrix}\right\}\end{cases},$$

$$\left(\text{Ƒ}_1,\check{E}\right)\cap\left(\text{Ƒ}_2,\check{E}\right)=\begin{cases} e_1 = \left\{\begin{matrix}(\S_1,0.4,0.6,0.2,0.8,0.5),(\S_2,0.5,0.6,0.3,0.9,0.5),\\(\S_3,0.4,0.3,0.1,0.8,0.8),(\S_4,0.2,0.4,0.6,0.8,0.6)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(\S_1,0.6,0.8,0.3,0.5,0.6),(\S_2,0.3,0.6,0.2,0.6,0.8),\\(\S_3,0.4,0.2,0.2,0.8,0.9),(\S_4,0.3,0.3,0.7,0.9,0.4)\end{matrix}\right\}\end{cases},$$

$$\left(\text{Ƒ}_1,\check{E}\right)^c=\begin{cases} e_1 = \left\{\begin{matrix}(\S_1,0.4,0.8,0.8,0.7,0.4),(\S_2,0.4,0.9,0.7,0.6,0.5),\\(\S_3,0.4,0.8,0.7,0.5,0.4),(\S_4,0.3,0.8,0.4,0.5,0.2)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(\S_1,0.3,0.5,0.6,0.8,0.7),(\S_2,0.8,0.6,0.8,0.9,0.4),\\(\S_3,0.9,0.8,0.8,0.7,0.6),(\S_4,0.4,0.9,0.3,0.4,0.3)\end{matrix}\right\}\end{cases},$$

$$\left(\text{Ƒ}_2,\check{E}\right)^c=\begin{cases} e_1 = \left\{\begin{matrix}(\S_1,0.5,0.2,0.1,0.6,0.5),(\S_2,0.5,0.4,0.7,0.8,0.6),\\(\S_3,0.8,0.2,0.9,0.3,0.7),(\S_4,0.6,0.4,0.1,0.4,0.3)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(\S_1,0.6,0.4,0.7,0.9,0.6),(\S_2,0.8,0.3,0.1,0.6,0.3),\\(\S_3,0.5,0.8,0.4,0.2,0.4),(\S_4,0.4,0.8,0.3,0.3,0.6)\end{matrix}\right\}\end{cases},$$

$$\left(\text{Ƒ}_1,\check{E}\right)-\left(\text{Ƒ}_2,\check{E}\right)=\left(\text{Ƒ}_1,\check{E}\right)\cap\left(\text{Ƒ}_2,\check{E}\right)^c=\begin{cases} e_1 = \left\{\begin{matrix}(\S_1,0.4,0.2,0.1,0.8,0.5),(\S_2,0.5,0.4,0.3,0.9,0.6),\\(\S_3,0.4,0.2,0.3,0.8,0.7),(\S_4,0.2,0.4,0.1,0.8,0.3)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(\S_1,0.6,0.4,0.4,0.9,0.6),(\S_2,0.4,0.3,0.1,0.6,0.8),\\(\S_3,0.5,0.7,0.2,0.8,0.9),(\S_4,0.3,0.4,0.3,0.9,0.6)\end{matrix}\right\}\end{cases},$$

$$\left(\text{Ƒ}_1,\check{E}\right)\vee\left(\text{Ƒ}_2,\check{E}\right)=\begin{cases} (e_1,e_1) = \left\{\begin{matrix}(\S_1,0.5,0.7,0.9,0.2,0.4),(\S_2,0.6,0.8,0.3,0.4,0.4),\\(\S_3,0.7,0.5,0.3,0.2,0.4),(\S_4,0.3,0.5,0.9,0.4,0.3)\end{matrix}\right\} \\ (e_1,e_2) = \left\{\begin{matrix}(\S_1,0.6,0.9,0.3,0.4,0.4),(\S_2,0.5,0.6,0.9,0.3,0.4),\\(\S_3,0.4,0.5,0.6,0.8,0.4),(\S_4,0.6,0.5,0.7,0.8,0.3)\end{matrix}\right\} \\ (e_2,e_1) = \left\{\begin{matrix}(\S_1,0.7,0.8,0.9,0.2,0.3),(\S_2,0.6,0.9,0.3,0.4,0.5),\\(\S_3,0.7,0.7,0.2,0.2,0.8),(\S_4,0.3,0.4,0.9,0.4,0.4)\end{matrix}\right\} \\ (e_2,e_2) = \left\{\begin{matrix}(\S_1,0.7,0.9,0.4,0.4,0.3),(\S_2,0.4,0.9,0.9,0.3,0.8),\\(\S_3,0.6,0.7,0.6,0.8,0.5),(\S_4,0.6,0.4,0.7,0.8,0.4)\end{matrix}\right\}\end{cases},$$

$$\left(\text{Ƒ}_1,\check{E}\right)\wedge\left(\text{Ƒ}_2,\check{E}\right)=\begin{cases} (e_1,e_1) = \left\{\begin{matrix}(\S_1,0.4,0.6,0.2,0.8,0.5),(\S_2,0.5,0.6,0.3,0.9,0.5),\\(\S_3,0.4,0.3,0.1,0.8,0.8),(\S_4,0.2,0.4,0.6,0.8,0.6)\end{matrix}\right\} \\ (e_1,e_2) = \left\{\begin{matrix}(\S_1,0.4,0.7,0.2,0.8,0.6),(\S_2,0.3,0.6,0.3,0.9,0.8),\\(\S_3,0.4,0.2,0.3,0.8,0.5),(\S_4,0.2,0.3,0.6,0.8,0.4)\end{matrix}\right\} \\ (e_2,e_1) = \left\{\begin{matrix}(\S_1,0.5,0.6,0.4,0.5,0.5),(\S_2,0.4,0.8,0.2,0.6\ 0.8),\\(\S_3,0.6,0.3,0.1,0.8,0.9),(\S_4,0.3,0.4,0.7,0.9,0.6)\end{matrix}\right\} \\ (e_2,e_2) = \left\{\begin{matrix}(\S_1,0.6,0.8,0.3,0.5,0.6),(\S_2,0.3,0.6,0.2,0.6,0.8),\\(\S_3,0.4,0.2,0.2,0.8,0.9),(\S_4,0.3,0.3,0.7,0.9,0.4)\end{matrix}\right\}\end{cases}.$$

4. Pentapartitioned neutrosophic soft topological space

This section is devoted to the family of PNSSs, PNS closed sets, PNS interior and PNS closure along with some basic theorems, propositions and examples.

Definition 4.1. Let $PNSS(Ú, Ě)$ be family of $all\ PNSS\ over$ Ú $and\ \tau^{PNSS} \subseteq PNSS(Ú, Ě)\ then\ \tau^{PNSS}$ is $said\ to\ be\ PNST\ on$ Ú, if

1. $0_{(Ú,Ě)}$ and $1_{(Ú,Ě)}$ belong to τ^{PNSS}
2. The union of any number of PNSS in τ^{PNSS} belong to τ^{PNSS}
3. The intersection of finite number of PNSS in τ^{PNSS} belong to τ^{PNSS}.

   Then $(Ú, \tau^{PNSS}, Ě)$ $is\ said\ to\ be\ PNST\ space\ over$ Ú, each member of τ^{PNSS} is PNS open set.

Definition 4.2. Assume that $(Ú, \tau^{PNSS}, Ě)$ be PNST space over Ú and $(ƒ, Ě)$ be PNSS over Ú, then $(ƒ, Ě)$ is said to be PNS closed sets iff its complement is a PNS open set.

Proposition 4.3. Let $(Ú, \tau_1^{PNSS}, Ě)$ and $(Ú, \tau_2^{PNSS}, Ě)$ be two PNST spaces over Ú, then $(Ú, \tau_1^{PNSS} \cap \tau_2^{PNSS}, Ě)$ is PNST space over Ú.

Proof.

1. Since $0_{(Ú,Ě)}, 1_{(Ú,Ě)} \in \tau_1^{PNSS}$ and $0_{(Ú,Ě)}, 1_{(Ú,Ě)} \in \tau_2^{PNSS}$ then $0_{(Ú,Ě)}, 1_{(Ú,Ě)} \in \tau_1^{PNSS} \cap \tau_2^{PNSS}$.
2. If $\{(ƒ_i, Ě) : i \in I)\}\ be\ a\ family\ of\ PNSS\ in\ \tau_1^{PNSS} \cap \tau_2^{PNSS},\ then$ $(ƒ_i, Ě) \in \tau_1^{PNSS}$ and $(ƒ_i, Ě) \in \tau_2^{PNSS}\ \forall\ i \in I$. So, $\cup_{i\in I}(ƒ_i, Ě) \in \tau_1^{PNSS}\ and\ \cup_{i\in I}(ƒ_i, Ě) \in \tau_2^{PNSS} \Rightarrow \cup_{i\in I}(ƒ_i, Ě) \in \tau_1^{PNSS} \cap \tau_2^{PNSS}$.
3. $Let\ \{(ƒ_i, Ě) : i = 1,2,3 \dots n)\}$ be a family of the finite number of $PNSS\ in\ \tau_1^{PNSS} \cap \tau_2^{PNSS}$, $then\ (ƒ_i, Ě) \in \tau_1^{PNSS}\ and\ (ƒ_i, Ě) \in \tau_2^{PNSS},\ \forall\ i \in 1,2,3 \dots n,$ so $\cap_{i\in I}^{n}(ƒ_i, Ě) \in \tau_1^{PNSS}$ and $\cap_{i\in I}^{n}(ƒ_i, Ě) \in \tau_2^{PNSS}$. Thus, $\cap_{i\in I}^{n}(ƒ_i, Ě) \in \tau_1^{PNSS} \cap \tau_2^{PNSS}$.

Remark 4.3. The union of two PNST over Ú may or may not always be PNST on Ú.

Example 4.4. Assume that set $Ú = \{§_1, §_2, §_3, §_4\}$ and $Ě = \{e_1, e_2\}$.

$\Im_1^{PNSS} = \{0_{(Ú,Ě)}, 1_{(Ú,Ě)}, (ƒ_1, Ě), (ƒ_2, Ě), (ƒ_3, Ě)\}$ and $\Im_2^{PNSS} = \{0_{(Ú,Ě)}, 1_{(Ú,Ě)}, (ƒ_2, Ě), (ƒ_4, Ě)\}$ be PNST over Ú, here $(ƒ_1, Ě), (ƒ_2, Ě), (ƒ_3, Ě)$ and $(ƒ_4, Ě)$ are defined:

$$(ƒ_1, Ě) = \left\{\begin{matrix} e_1 = \begin{Bmatrix} (§_1, 0.7, 0.8, 0.6, 0.4, 0.3), (§_2, 0.8, 0.7, 0.9, 0.5, 0.4), \\ (§_3, 0.6, 0.8, 0.9, 0.7, 0.4), (§_4, 0.9, 0.7, 0.6, 0.5, 0.4) \end{Bmatrix} \\ e_2 = \begin{Bmatrix} (§_1, 0.8, 0.9, 0.8, 0.6, 0.4), (§_2, 0.9, 0.7, 0.6, 0.5, 0.6), \\ (§_3, 0.8, 0.9, 0.5, 0.4, 0.5), (§_4, 0.8, 0.5, 0.5, 0.4, 0.6) \end{Bmatrix} \end{matrix}\right\},$$

$$(ƒ_2, Ě) = \left\{\begin{matrix} e_1 = \begin{Bmatrix} (§_1, 0.5, 0.6, 0.4, 0.4, 0.8), (§_2, 0.6, 0.7, 0.6, 0.8, 0.8), \\ (§_3, 0.6, 0.7, 0.9, 0.8, 0.6), (§_4, 0.8, 0.7, 0.6, 0.6, 0.8) \end{Bmatrix} \\ e_2 = \begin{Bmatrix} (§_1, 0.7, 0.7, 0.6, 0.8, 0.4), (§_2, 0.8, 0.6, 0.5, 0.8, 0.6), \\ (§_3, 0.6, 0.8, 0.4, 0.4, 0.8), (§_4, 0.7, 0.5, 0.4, 0.6, 0.8) \end{Bmatrix} \end{matrix}\right\},$$

$$(ƒ_3, Ě) = \left\{\begin{matrix} e_1 = \begin{Bmatrix} (§_1, 0.4, 0.5, 0.4, 0.5, 0.8), (§_2, 0.5, 0.6, 0.5, 0.9, 0.8), \\ (§_3, 0.5, 0.6, 0.5, 0.8, 0.7), (§_4, 0.7, 0.5, 0.4, 0.8, 0.9) \end{Bmatrix} \\ e_2 = \begin{Bmatrix} (§_1, 0.6, 0.7, 0.5, 0.8, 0.6), (§_2, 0.7, 0.5, 0.4, 0.8, 0.7), \\ (§_3, 0.6, 0.7, 0.4, 0.5, 0.8), (§_4, 0.5, 0.4, 0.3, 0.7, 0.8) \end{Bmatrix} \end{matrix}\right\},$$

$$(ʄ_4, Ě) = \left\{\begin{matrix} e_1 = \left\{\begin{matrix}(§_1, 0.4, 0.3, 0.2, 0.8, 0.9), (§_2, 0.4, 0.5, 0.3, 0.9, 0.9), \\ (§_3, 0.3, 0.1, 0.4, 0.9, 0.8), (§_4, 0.5, 0.3, 0.2, 0.8, 0.9)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(§_1, 0.5, 0.6, 0.4, 0.9, 0.8), (§_2, 0.5, 0.3, 0.2, 0.9, 0.8), \\ (§_3, 0.4, 0.5, 0.2, 0.8, 0.9), (§_4, 0.4, 0.2, 0.1, 0.7, 0.9)\end{matrix}\right\}\end{matrix}\right\},$$

$(ʄ_1, Ě) \cup (ʄ_2, Ě) = (ʄ_1, Ě)$, $(ʄ_1, Ě) \cup (ʄ_3, Ě) = (ʄ_1, Ě)$, $(ʄ_2, Ě) \cup (ʄ_3, Ě) = (ʄ_2, Ě)$,

$(ʄ_2, Ě) \cup (ʄ_4, Ě) = (ʄ_2, Ě)$, $(ʄ_1, Ě) \cap (ʄ_2, Ě) = (ʄ_2, Ě)$, $(ʄ_1, Ě) \cap (ʄ_3, Ě) = (ʄ_3, Ě)$,

$(ʄ_2, Ě) \cap (ʄ_3, Ě) = (ʄ_3, Ě)$, $(ʄ_2, Ě) \cap (ʄ_4, Ě) = (ʄ_4, Ě)$.

Hence, $\tau_1^{PNSS} \cup \tau_2^{PNSS}$ is a PNST over Ú. But, this is not possible always as given in the following example.

Example 4.5. Assume that set $Ú = \{§_1, §_2, §_3, §_4\}$ and $Ě = \{e_1, e_2\}$.

$\mathfrak{I}_1^{PNSS} = \left\{0_{(Ú,Ě)}, 1_{(Ú,Ě)}, (ʄ_1, Ě), (ʄ_2, Ě)\right\}$ and $\mathfrak{I}_2^{PNSS} = \left\{0_{(Ú,Ě)}, 1_{(Ú,Ě)}, (ʄ_2, Ě), (ʄ_3, Ě)\right\}$ be PNST over Ú, here $(ʄ_1, Ě), (ʄ_2, Ě), (ʄ_3, Ě)$ and $(ʄ_4, Ě)$ are defined:

$$(ʄ_1, Ě) = \left\{\begin{matrix} e_1 = \left\{\begin{matrix}(§_1, 0.7, 0.8, 0.6, 0.4, 0.3), (§_2, 0.8, 0.7, 0.9, 0.5, 0.4), \\ (§_3, 0.6, 0.8, 0.9, 0.7, 0.4), (§_4, 0.9, 0.7, 0.6, 0.5, 0.4)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(§_1, 0.8, 0.9, 0.8, 0.6, 0.4), (§_2, 0.9, 0.7, 0.6, 0.5, 0.6), \\ (§_3, 0.8, 0.9, 0.5, 0.4, 0.5), (§_4, 0.8, 0.5, 0.5, 0.4, 0.6)\end{matrix}\right\}\end{matrix}\right\},$$

$$(ʄ_2, Ě) = \left\{\begin{matrix} e_1 = \left\{\begin{matrix}(§_1, 0.5, 0.6, 0.4, 0.4, 0.8), (§_2, 0.6, 0.7, 0.6, 0.8, 0.8), \\ (§_3, 0.6, 0.7, 0.9, 0.8, 0.6), (§_4, 0.8, 0.7, 0.6, 0.6, 0.8)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(§_1, 0.7, 0.7, 0.6, 0.8, 0.4), (§_2, 0.8, 0.6, 0.5, 0.8, 0.6), \\ (§_3, 0.6, 0.8, 0.4, 0.4, 0.8), (§_4, 0.7, 0.5, 0.4, 0.6, 0.8)\end{matrix}\right\}\end{matrix}\right\},$$

$$(ʄ_3, Ě) = \left\{\begin{matrix} e_1 = \left\{\begin{matrix}(§_1, 0.8, 0.5, 0.4, 0.5, 0.7), (§_2, 0.5, 0.6, 0.4, 0.3, 0.8), \\ (§_3, 0.7, 0.6, 0.5, 0.4, 0.4), (§_4, 0.9, 0.3, 0.4, 0.6, 0.7)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(§_1, 0.4, 0.5, 0.9, 0.3, 0.4), (§_2, 0.3, 0.5, 0.7, 0.8, 0.4), \\ (§_3, 0.6, 0.3, 0.6, 0.8, 0.9), (§_4, 0.4, 0.5, 0.8, 0.4, 0.5)\end{matrix}\right\}\end{matrix}\right\},$$

$(ʄ_1, Ě) \cup (ʄ_2, Ě) = (ʄ_1, Ě)$, $(ʄ_2, Ě) \cup (ʄ_3, Ě) \neq (ʄ_2, Ě)$,

$(ʄ_1, Ě) \cap (ʄ_2, Ě) = (ʄ_2, Ě)$, $(ʄ_2, Ě) \cap (ʄ_3, Ě) \neq (ʄ_3, Ě)$,

Hence, $(ʄ_2, Ě) \cup (ʄ_3, Ě) \neq \tau_1^{PNSS} \cup \tau_2^{PNSS}$, then $\tau_1^{PNSS} \cup \tau_2^{PNSS}$ *is not a PNST over* Ú.

Proposition 4.6. *Let* $(Ú, \tau^{PNSS}, Ě)$ *be a PNST* space over Ú, *and*

$\tau^{PNSS} = \{(ʄ_i, Ě) : (ʄ_i, Ě) \in PNSS(Ú, Ě)\} = \{[e, ʄ_i(e)]_{e\in Ě} : (ʄ_i, Ě) \in PNSS(Ú, Ě)\}$,

$ʄ_i(e) = \left\{\left(AbT_{ʄ_i(e)}(§), RelT_{ʄ_i(e)}(§), H_{ʄ_i(e)}(§), Relʄ_{ʄ_i(e)}(§), Abʄ_{ʄ_i(e)}(§)\right) : § \in Ú\right\}$,

$\tau_1 = \left\{\left[AbT_{ʄ_i(e)}(§)\right]_{e\in Ě}\right\}$, $\tau_2 = \left\{\left[RelT_{ʄ_i(e)}(§)\right]_{e\in Ě}\right\}$, $\tau_3 = \left\{\left[H_{ʄ_i(e)}(§)\right]_{e\in Ě}\right\}$, $\tau_4 = \left\{\left[Relʄ_{ʄ_i(e)}(§)\right]^c_{e\in Ě}\right\}$,

$\tau_5 = \left\{\left[Abʄ_{ʄ_i(e)}(§)\right]^c_{e\in Ě}\right\}$ is PFST on Ú.

Proof.

1. $0_{(Ú,Ě)}, 1_{(Ú,Ě)} \in \tau^{PNSS} \Rightarrow 0, 1 \in \tau_1$, $0, 1 \in \tau_2$, $0, 1 \in \tau_3, 0, 1 \in \tau_4$ and $0, 1 \in \tau_5$.
2. Let $\{(ʄ_i, Ě) | i \in I\}$ *be a family of PNSS in* τ^{PNSS}, *then* $\left\{\left[AbT_{ʄ_i(e)}(§)\right]_{e\in Ě}\right\}_{i\in I}$ is a family of PFSS in τ_1, $\left\{\left[RelT_{ʄ_i(e)}(§)\right]_{e\in Ě}\right\}_{i\in I}$ *is a family of* PFSS in τ_2, $\left\{\left[H_{ʄ_i(e)}(§)\right]_{e\in Ě}\right\}_{i\in I}$ is a family

of PFSS in τ_3, $\left\{\left[Rel\digamma_{\digamma_i(e)}\,(\S)\right]^c_{e\in\check{E}}\right\}_{i\in I}$ *is a family* of PFSS in τ_4 and $\left\{\left[Ab\digamma_{\digamma_i(e)}\,(\S)\right]^c_{e\in\check{E}}\right\}_{i\in I}$ is a family of PFSS in τ_5. Since τ^{PNSS} is a PNST, then $\bigcup_{i\in I}(\digamma_i,\check{E}) \in \tau^{PNSS}$.

$$\bigcup_{i\in I}(\digamma_i,\check{E}) = \left\{\begin{pmatrix} \sup\left[AbT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}, \sup\left[RelT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}, \sup\left[H_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}, \\ \inf\left[Rel\digamma_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}, \inf\left[Ab\digamma_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}} \end{pmatrix}\right\}_{i\in I} \in \tau^{PNSS},$$

therefore $\left\{sup\left[AbT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}\right\}_{i\in I} \in \tau_1$, $\left\{sup\left[RelT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}\right\}_{i\in I} \in \tau_2$,

$\left\{sup\left[H_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}\right\}_{i\in I} \in \tau_3$, $\left\{inf\left[RelF_{\digamma_i(e)}\,(\S)\right]^c_{e\in\check{E}}\right\}_{i\in I} \in \tau_4$, $\left\{inf\left[AbF_{\digamma_i(e)}\,(\S)\right]^c_{e\in\check{E}}\right\}_{i\in I} \in \tau_5$.

3. Assume that $\{(\digamma_i,\check{E}) \mid i = 1 \to n\}$ be a family of finite PNSS in τ^{PNSS}, then $\left\{\left[AbT_{\digamma_i(e)}\,(\acute{U})\right]_{e\in\check{E}}\right\}_{i=1\to n}$ is a family of PFSS in τ_1, $\left\{\left[RelT_{\digamma_i(e)}\,(\acute{U})\right]_{e\in\check{E}}\right\}_{i=1\to n}$ is a family of PFSS in τ_2, $\left\{\left[H_{\digamma_i(e)}\,(\acute{U})\right]_{e\in\check{E}}\right\}_{i=1\to n}$ is a family of PFSS in τ_3, $\left\{\left[Rel\digamma_{\digamma_i(e)}\,(\S)\right]^c_{e\in\check{E}}\right\}_{i=1\to n}$ is a family of PFSS in τ_4 and $\left\{\left[Ab\digamma_{\digamma_i(e)}\,(\S)\right]^c_{e\in\check{E}}\right\}_{i=1\to n}$ is a family of PFSS in τ_5, since, τ^{PNSS} is a PNST then $\bigcap_{i=1}^{n}(\digamma_i,\check{E}) \in \tau^{PNSS}$, so

$$\bigcap_{i=1}^{n}(\digamma_i,\check{E}) = \left\{\begin{pmatrix} min\left[AbT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}, min\left[RelT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}, min\left[H_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}, \\ ma\S\left[Ab\digamma_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}, ma\S\left[Ab\digamma_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}} \end{pmatrix}\right\}_{i\in 1,n} \in$$

τ^{PNSS}, therefore

$$\left\{min\left[AbT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}\right\} \in \tau_1, \left\{min\left[RelT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}\right\} \in \tau_2, \left\{min\left[H_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}\right\} \in \tau_3,$$
$$\left\{ma\S\left[Rel\digamma_{\digamma_i(e)}\,(\S)\right]^c_{e\in\check{E}}\right\} \in \tau_4, \left\{ma\S\left[Ab\digamma_{\digamma_i(e)}\,(\S)\right]^c_{e\in\check{E}}\right\}_{i\in I} \in \tau_5.$$

Proposition 4.7. *Let* $(\acute{U},\tau^{PNSS},\check{E})$ *be a PNST space over* $\acute{U}$, *then*
$\tau_{1_e} = \{[AbT_{\digamma(e)}(\S)] : (\digamma,\check{E}) \in \tau^{PNSS}\}$, $\tau_{2_e} = \{[RelT_{\digamma(e)}(\S)] : (\digamma,\check{E}) \in \tau^{PNSS}\}$,
$\tau_{3_e} = \{[H_{\digamma(e)}(\S)] : (\digamma,\check{E}) \in \tau^{PNSS}\}$, $\tau_{4_e} = \{[Rel\digamma_{\digamma(e)}(\S)] : (\digamma,\check{E}) \in \tau^{PNSS}\}$,
$\tau_{5_e} = \{[Ab\digamma_{\digamma(e)}(\S)] : (\digamma,\check{E}) \in \tau^{PNSS}\}$, *for each* $e \in \check{E}$, *PFST on* $\acute{U}$.

Proof.

1. (i). $0_{(\acute{U},\check{E})} \in \Im_{1e}^{PNSS} \Rightarrow 0 \in \Im_{1e}^{PNSS}$ and $1_{(\acute{U},\check{E})} \in \Im_{1e}^{PNSS} \Rightarrow 1 \in \Im_{1e}^{PNSS} \Rightarrow 0,1 \in \Im_{1e}^{PNSS}$.

(ii). Let $\{AbT_{\digamma_i(e)} : (\digamma_i,\check{E})\}_{i\in I}$ be the collection of $\Im_{1e}^{PNSS}$, *so* $(\digamma_{1,}\check{E}), (\digamma_{2,}\check{E}) \ldots \ldots \in \Im_{1e}^{PNSS}$.

$\Rightarrow (\digamma_{1,}\check{E}) \cup (\digamma_{2,}\check{E}) \cup \ldots \ldots \in \Im_{1e}^{PNSS} \Rightarrow \left\{\left[AbT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}\right\}_{i\in I}$ is PFSS in $\Im_{1e}^{PNSS}$.

$\Rightarrow \bigcup_{i=I}\left[AbT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}} = \left\{sup\left[AbT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}\right\}_{i\in I}$ in $\Im_{1e}^{PNSS}$.

(iii). Let $\{AbT_{\digamma_i(e)} : (\digamma_i,\check{E})\}_{i\in I}$ be finite PNSS in $\Im_{1e}^{PNSS}$, so $(\digamma_{1,}\check{E}), (\digamma_{2,}\check{E}) \ldots (\digamma_{n,}\check{E}) \in \Im_{1e}^{PNSS}$.

$\Rightarrow (\digamma_{1,}\check{E}) \cap (\digamma_{2,}\check{E}) \ldots \cap (\digamma_{n,}\check{E}) \in \Im_{1e}^{PNSS} \Rightarrow \left\{\left[AbT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}\right\}_{i\in I}$ are finite PFSS in $\Im_{1e}^{PNSS}$.

$\Rightarrow \bigcap_{i=I}\left[AbT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}} = \left\{inf\left[AbT_{\digamma_i(e)}\,(\S)\right]_{e\in\check{E}}\right\}_{i\in I}$ in $\Im_{1e}^{PNSS}$.

2. (i). $0_{(\acute{U},\check{E})} \in \Im_{2e}^{PNSS} \Rightarrow 0 \in \Im_{2e}^{PNSS}$ and $1_{(\acute{U},\check{E})} \in \Im_{2e}^{PNSS} \Rightarrow 1 \in \Im_{2e}^{PNSS} \Rightarrow 0,1 \in \Im_{2e}^{PNSS}$.

(ii). Let $\left\{RelT_{Ƒ_i(e)}: (Ƒ_i, Ě)\right\}_{i\in I}$ be the collection of $\mathfrak{I}_{2e}^{PNSS}, so\ (Ƒ_1,Ě),(Ƒ_2,Ě) \dots \dots \in \mathfrak{I}_{2e}^{PNSS}$.

$\Rightarrow (Ƒ_1,Ě) \cup (Ƒ_2,Ě) \cup \dots \dots \in \mathfrak{I}_{2e}^{PNSS} \Rightarrow \left\{\left[RelT_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ is PFSS in $\mathfrak{I}_{2e}^{PNSS}$.

$\Rightarrow \bigcup_{i=I}\left[RelT_{Ƒ_i(e)}\ (§)\right]_{e\in Ě} = \left\{sup\left[RelT_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ in $\mathfrak{I}_{2e}^{PNSS}$.

(iii). Let $\left\{RelT_{Ƒ_i(e)}: (Ƒ_i, Ě)\right\}_{i\in I}$ be finite PNSS in $\mathfrak{I}_{2e}^{PNSS}, so\ (Ƒ_1,Ě),(Ƒ_2,Ě) \dots \dots (Ƒ_n,Ě) \in \mathfrak{I}_{2e}^{PNSS}$.

$\Rightarrow (Ƒ_1,Ě) \cap (Ƒ_2,Ě) \dots \dots \cap (Ƒ_n,Ě) \in \mathfrak{I}_{2e}^{PNSS} \Rightarrow \left\{\left[RelT_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ is finite PFSS in $\mathfrak{I}_{2e}^{PNSS}$.

$\Rightarrow \bigcap_{i=I}\left[RelT_{Ƒ_i(e)}\ (§)\right]_{e\in Ě} = \left\{inf\left[RelT_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ in $\mathfrak{I}_{2e}^{PNSS}$.

3. (i). $0_{(Ú,Ě)} \in \mathfrak{I}_{3e}^{PNSS} \Rightarrow 0 \in \mathfrak{I}_{3e}^{PNSS}$ and $1_{(Ú,Ě)} \in \mathfrak{I}_{3e}^{PNSS} \Rightarrow 1 \in \mathfrak{I}_{3e}^{PNSS} \Rightarrow 0, 1 \in \mathfrak{I}_{3e}^{PNSS}$.

(ii). Let $\left\{H_{Ƒ_i(e)}: (Ƒ_i, Ě)\right\}_{i\in I}$ be the collection of $\mathfrak{I}_{3e}^{PNSS}$, $so\ (Ƒ_1,Ě),\ (Ƒ_2,Ě) \dots \dots \in \mathfrak{I}_{3e}^{PNSS}$.

$\Rightarrow (Ƒ_1,Ě) \cup (Ƒ_2,Ě) \cup \dots \dots \in \mathfrak{I}_{3e}^{PNSS} \Rightarrow \left\{\left[H_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ are PFSS in $\mathfrak{I}_{3e}^{PNSS}$.

$\Rightarrow \bigcup_{i=I}\left[H_{Ƒ_i(e)}\ (§)\right]_{e\in Ě} = \left\{sup\left[H_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ in $\mathfrak{I}_{3e}^{PNSS}$.

(iii). Let $\left\{H_{Ƒ_i(e)}: (Ƒ_i, Ě)\right\}_{i\in I}$ be finite PNSS in $\mathfrak{I}_{3e}^{PNSS}$ $so,$ $(Ƒ_1,Ě),\ (Ƒ_2,Ě) \dots \dots (Ƒ_n,Ě) \in \mathfrak{I}_{3e}^{PNSS}$.

$\Rightarrow (Ƒ_1,Ě) \cap (Ƒ_2,Ě) \dots \cap (Ƒ_n,Ě) \in \mathfrak{I}_{3e}^{PNSS} \Rightarrow \left\{\left[H_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ are finite PFSS in $\mathfrak{I}_{3e}^{PNSS}$.

$\Rightarrow \bigcap_{i=I}\left[H_{Ƒ_i(e)}\ (§)\right]_{e\in Ě} = \left\{inf\left[H_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ in $\mathfrak{I}_{3e}^{PNSS}$.

4. (i). $0_{(Ú,Ě)} \in \mathfrak{I}_{4e}^{PNSS} \Rightarrow 0 \in \mathfrak{I}_{4e}^{PNSS}$ and $1_{(Ú,Ě)} \in \mathfrak{I}_{4e}^{PNSS} \Rightarrow 1 \in \mathfrak{I}_{4e}^{PNSS} \Rightarrow 0, 1 \in \mathfrak{I}_{4e}^{PNSS}$.

(ii). Let $\left\{RelƑ_{Ƒ_i(e)}: (Ƒ_i, Ě)\right\}_{i\in I}$ be the collection of $\mathfrak{I}_{4e}^{PNSS}$, so $(Ƒ_1,Ě),(Ƒ_2,Ě) \dots \dots \in \mathfrak{I}_{4e}^{PNSS}$.

$\Rightarrow (Ƒ_1,Ě) \cup (Ƒ_2,Ě) \cup \dots \dots \in \mathfrak{I}_{4e}^{PNSS} \Rightarrow \left\{\left[RelƑ_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ are PFSS in $\mathfrak{I}_{4e}^{PNSS}$.

$\Rightarrow \bigcup_{i=I}\left[RelƑ_{Ƒ_i(e)}\ (§)\right]_{e\in Ě} = \left\{Sup\left[RelƑ_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ in $\mathfrak{I}_{4e}^{PNSS}$.

(iii). Let $\left\{RelF_{Ƒ_i(e)}: (Ƒ_i, Ě)\right\}_{i\in I}$ be finite PNSS in $\mathfrak{I}_{4e}^{PNSS}$, so $(Ƒ_1,Ě),(Ƒ_2,Ě) \dots \dots (Ƒ_n,Ě) \in \mathfrak{I}_{4e}^{PNSS}$.

$\Rightarrow (Ƒ_1,Ě) \cap (Ƒ_2,Ě) \dots \cap (Ƒ_n,Ě) \in \mathfrak{I}_{4e}^{PNSS} \Rightarrow \left\{\left[Ƒ_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ are finite PFSS in $\mathfrak{I}_{4e}^{PNSS}$.

$\Rightarrow \bigcap_{i=I}\left[RelƑ_{Ƒ_i(e)}\ (§)\right]_{e\in Ě} = \left\{inf\left[RelƑ_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ in $\mathfrak{I}_{4e}^{PNSS}$.

5. (i). $0_{(Ú,Ě)} \in \mathfrak{I}_{5e}^{PNSS} \Rightarrow 0 \in \mathfrak{I}_{5e}^{PNSS}$ and $1_{(Ú,Ě)} \in \mathfrak{I}_{5e}^{PNSS} \Rightarrow 1 \in \mathfrak{I}_{5e}^{PNSS} \Rightarrow 0, 1 \in \mathfrak{I}_{5e}^{PNSS}$.

(ii). Let $\left\{AbF_{Ƒ_i(e)}: (Ƒ_i, Ě)\right\}_{i\in I}$ be the collection of $\mathfrak{I}_{5e}^{PNSS}$, so $(Ƒ_1,Ě),(Ƒ_2,Ě) \dots \in \mathfrak{I}_{5e}^{PNSS}$.

$\Rightarrow (Ƒ_1,Ě) \cup (Ƒ_2,Ě) \dots \in \mathfrak{I}_{5e}^{PNSS} \Rightarrow \left\{\left[AbƑ_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ are PFSS in $\mathfrak{I}_{5e}^{PNSS}$.

$\Rightarrow \bigcup_{i=I}[AbƑ(§)]_{e\in Ě} = \left\{sup\left[AbƑ_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ in $\mathfrak{I}_{5e}^{PNSS}$.

(iii). Let $\left\{AbF_{Ƒ_i(e)}: (Ƒ_i, Ě)\right\}_{i\in I}$ be finite $PNSS$ in $\mathfrak{I}_{5e}^{PNSS}$, so $(Ƒ_1,Ě),(Ƒ_2,Ě) \dots \dots (Ƒ_n,Ě) \in \mathfrak{I}_{5e}^{PNSS}$.

$\Rightarrow (Ƒ_1,Ě) \cap (Ƒ_2,Ě) \dots \dots \cap (Ƒ_n,Ě) \in \mathfrak{I}_{5e}^{PNSS} \Rightarrow \left\{\left[AbF_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I}$ are finite PFSS in $\mathfrak{I}_{5e}^{PNSS}$.

$\Rightarrow \bigcap_{i=I}\left[AbƑ_{Ƒ_i(e)}\ (§)\right]_{e\in Ě} = \left\{inf\left[AbƑ_{Ƒ_i(e)}\ (§)\right]_{e\in Ě}\right\}_{i\in I} \in \mathfrak{I}_{5e}^{PNSS}$.

Example 4.8. Assume that Example 4.4 that is $\tau^{PNSS}=$ $\{0_{(Ú,Ě)},1_{(Ú,Ě)},(Ƒ_1,Ě),(Ƒ_2,Ě),(Ƒ_3,Ě),(Ƒ_4,Ě)\}$ be a family of PNSS over Ú, here

$$(Ƒ_1,Ě)=\left\{\begin{matrix} e_1=\left\{\begin{matrix}(§_1,0.8,0.9,0.6,0.5,0.4),(§_2,0.9,0.6,0.8,0.3,0.5),(§_3,0.7,0.8,0.8,0.5,0.3),\\(§_4,0.9,0.8,0.9,0.2,0.4),(§_5,0.8,0.9,0.8,0.3,0.4)\end{matrix}\right\}\\ e_2=\left\{\begin{matrix}(§_1,0.9,0.7,0.7,0.3,0.6),(§_2,0.8,0.8,0.7,0.4,0.2),(§_3,0.8,0.7,0.9,0.3,0.4),\\(§_4,0.8,0.6,0.5,0.5,0.3),(§_5,0.9,0.7,0.8,0.5,0.6)\end{matrix}\right\}\end{matrix}\right\},$$

$$(Ƒ_2,Ě)=\left\{\begin{matrix} e_1=\left\{\begin{matrix}(§_1,0.7,0.6,0.5,0.6,0.5),(§_2,0.8,0.6,0.5,0.5,0.6),(§_3,0.6,0.8,0.5,0.6,0.4),\\(§_4,0.7,0.6,0.8,0.4,0.5),(§_5,0.8,0.8,0.7,0.5,0.6)\end{matrix}\right\}\\ e_2=\left\{\begin{matrix}(§_1,0.8,0.7,0.6,0.3,0.6),(§_2,0.8,0.6,0.6,0.4,0.3),(§_3,0.6,0.5,0.8,0.4,0.5),\\(§_4,0.6,0.5,0.4,0.6,0.4),(§_5,0.7,0.6,0.7,0.6,0.7)\end{matrix}\right\}\end{matrix}\right\},$$

$$(Ƒ_3,Ě)=\left\{\begin{matrix} e_1=\left\{\begin{matrix}(§_1,0.5,0.4,0.5,0.7,0.8),(§_2,0.6,0.4,0.5,0.7,0.8),(§_3,0.5,0.6,0.5,0.8,0.6),\\(§_4,0.6,0.5,0.7,0.6,0.7),(§_5,0.7,0.6,0.6,0.7,0.8)\end{matrix}\right\}\\ e_2=\left\{\begin{matrix}(§_1,0.6,0.5,0.4,0.5,0.7),(§_2,0.5,0.4,0.5,0.7,0.8),(§_3,0.5,0.3,0.6,0.5,0.7),\\(§_4,0.4,0.5,0.4,0.7,0.5),(§_5,0.6,0.5,0.7,0.8,0.8)\end{matrix}\right\}\end{matrix}\right\},$$

$$(Ƒ_4,Ě)=\left\{\begin{matrix} e_1=\left\{\begin{matrix}(§_1,0.5,0.3,0.4,0.7,0.9),(§_2,0.6,0.2,0.4,0.8,0.9),(§_3,0.4,0.5,0.4,0.9,0.7),\\(§_4,0.5,0.4,0.6,0.9,0.7),(§_5,0.5,0.4,0.6,0.8,0.9)\end{matrix}\right\}\\ e_2=\left\{\begin{matrix}(§_1,0.5,0.4,0.3,0.6,0.8),(§_2,0.4,0.1,0.3,0.7,0.9),(§_3,0.3,0.2,0.5,0.7,0.8),\\(§_4,0.3,0.4,0.2,0.8,0.9),(§_5,0.4,0.3,0.6,0.9,0.9)\end{matrix}\right\}\end{matrix}\right\}.$$

As, $\tau_1=\left\{\begin{pmatrix}AbT_{Ƒ_{0_{(Ú,Ě)}(e)}}(§),AbT_{Ƒ_{1_{(Ú,Ě)}(e)}}(§),AbT_{Ƒ_{1(e)}}(§),AbT_{Ƒ_{2(e)}}(§),AbT_{Ƒ_{3(e)}}(§),\\AbT_{Ƒ_{4(e)}}(§),AbT_{Ƒ_{5(e)}}(§)\end{pmatrix}_{e\in Ě}\right\},$

$$\tau_2=\left\{\begin{pmatrix}RelT_{Ƒ_{0_{(Ú,Ě)}(e)}}(§),RelT_{Ƒ_{1_{(Ú,Ě)}(e)}}(§),RelT_{Ƒ_{1(e)}}(§),RelT_{Ƒ_{2(e)}}(§),RelT_{Ƒ_{3(e)}}(§),\\RelT_{Ƒ_{4(e)}}(§),RelT_{Ƒ_{5(e)}}(§)\end{pmatrix}_{e\in Ě}\right\},$$

$$\tau_3=\left\{\left(H_{Ƒ_{0_{(Ú,Ě)}(e)}}(§),H_{Ƒ_{1_{(Ú,Ě)}(e)}}(§),H_{Ƒ_{1(e)}}(§),H_{Ƒ_{2(e)}}(§),H_{Ƒ_{3(e)}}(§),H_{Ƒ_{4(e)}}(§),H_{Ƒ_{5(e)}}(§)\right)_{e\in Ě}\right\},$$

$$\tau_4=\left\{\begin{pmatrix}RelƑ_{Ƒ_{0_{(Ú,Ě)}(e)}}(§),RelƑ_{Ƒ_{1_{(Ú,Ě)}(e)}}(§),RelƑ_{Ƒ_{1(e)}}(§),RelƑ_{Ƒ_{2(e)}}(§),RelƑ_{Ƒ_{3(e)}}(§),\\RelƑ_{Ƒ_{4(e)}}(§),RelƑ_{Ƒ_{5(e)}}(§)\end{pmatrix}_{e\in Ě}\right\},$$

$$\tau_5=\left\{\begin{pmatrix}AbƑ_{Ƒ_{0_{(Ú,Ě)}(e)}}(§),AbƑ_{Ƒ_{1_{(Ú,Ě)}(e)}}(§),AbƑ_{Ƒ_{1(e)}}(§),AbƑ_{Ƒ_{2(e)}}(§),AbƑ_{Ƒ_{3(e)}}(§),\\AbƑ_{Ƒ_{4(e)}}(§),AbƑ_{Ƒ_{5(e)}}(§)\end{pmatrix}_{e\in Ě}\right\},$$

are PFSS on Ú i.e.,

$$\tau_{1_{e_1}}=\left\{\begin{matrix}(0,0,0,0,0),(1,1,1,1,1),(0.8,0.9,0.7,0.9,0.8),(0.7,0.8,0.6,0.7,0.8),\\(0.5,0.6,0.5,0.6,0.7),(0.5,0.6,0.4,0.5,0.5)\end{matrix}\right\},$$

$$\tau_{2_{e_1}}=\left\{\begin{matrix}(0,0,0,0,0),(1,1,1,1,1),(0.9,0.6,0.8,0.8,0.9),(0.6,0.6,0.8,0.6,0.8),\\(0.4,0.4,0.6,0.5,0.6),(0.3,0.2,0.5,0.4,0.4)\end{matrix}\right\},$$

$$\tau_{3_{e_1}}=\left\{\begin{matrix}(0,0,0,0,0),(1,1,1,1,1),(0.6,0.8,0.8,0.9,0.8),(0.5,0.5,0.5,0.8,0.7),\\(0.5,0.5,0.5,0.7,0.6),(0.4,0.4,0.4,0.6,0.6)\end{matrix}\right\},$$

$$\tau_{4_{e_1}}=\left\{\begin{matrix}(0,0,0,0,0),(1,1,1,1,1),(0.5,0.3,0.5,0.2,0.3),(0.6,0.5,0.6,0.4,0.5),\\(0.7,0.7,0.8,0.6,0.7),(0.7,0.8,0.9,0.9,0.8)\end{matrix}\right\},$$

$$\tau_{5_{e_1}} = \left\{\begin{matrix}(0,0,0,0,0),(1,1,1,1,1),(0.4,0.5,0.3,0.4,0.4),(0.5,0.6,0.4,0.5,0.6),\\(0.8,0.8,0.6,0.7,0.8),(0.9,0.9,0.7,0.7,0.9)\end{matrix}\right\},$$

$$\tau_{1_{e_2}} = \left\{\begin{matrix}(0,0,0,0,0),(1,1,1,1,1),(0.9,0.8,0.8,0.8,0.9),(0.8,0.8,0.6,0.6,0.7),\\(0.6,0.5,0.5,0.4,0.6),(0.5,0.4,0.3,0.3,0.4)\end{matrix}\right\},$$

$$\tau_{2_{e_2}} = \left\{\begin{matrix}(0,0,0,0,0),(1,1,1,1,1),(0.7,0.8,0.7,0.6,0.7),(0.7,0.6,0.5,0.5,0.6),\\(0.5,0.4,0.3,0.5,0.5),(0.4,0.1,0.2,0.4,0.3)\end{matrix}\right\},$$

$$\tau_{3_{e_2}} = \left\{\begin{matrix}(0,0,0,0,0),(1,1,1,1,1),(0.7,0.7,0.9,0.5,0.8),(0.6,0.6,0.8,0.4,0.7),\\(0.4,0.5,0.6,0.4,0.7),(0.3,0.3,0.5,0.2,0.6)\end{matrix}\right\},$$

$$\tau_{4_{e_2}} = \left\{\begin{matrix}(0,0,0,0,0),(1,1,1,1,1),(0.3,0.4,0.3,0.5,0.5),(0.3,0.4,0.4,0.6,0.6),\\(0.5,0.7,0.5,0.7,0.8),(0.6,0.7,0.7,0.8,0.9)\end{matrix}\right\},$$

$$\tau_{5_{e_2}} = \left\{\begin{matrix}(0,0,0,0,0),(1,1,1,1,1),(0.6,0.2,0.4,0.3,0.6),(0.6,0.3,0.5,0.4,0.7),\\(0.7,0.8,0.7,0.5,0.8),(0.8,0.9,0.8,0.9,0.9)\end{matrix}\right\}.$$

Hence, $\left\{\tau_{1_{e_1}},\tau_{2_{e_1}},\tau_{3_{e_1}},\tau_{4_{e_1}},\tau_{5_{e_1}}\right\}$ and $\left\{\tau_{1_{e_2}},\tau_{2_{e_2}},\tau_{3_{e_2}},\tau_{4_{e_2}},\tau_{5_{e_2}}\right\}$ are PFST on Ú thus τ^{PNSS} is PNS topologies on Ú.

Definition 4.9. If $(\text{ƒ},\text{Ě}) \in PNSS(\text{Ú},\text{Ě})$ is a PNSS and $(\text{Ú},\tau^{PNSS},\text{Ě})$ is a PNST on Ú, then the PNS interior of $(\text{ƒ},\text{Ě})$, denoted by $(\text{ƒ},\text{Ě})^{\circ}$ is the PNS union of all PNS open subset of $(\text{ƒ},\text{Ě})$. The greatest PNS open set contained by $(\text{ƒ},\text{Ě})$ is $(\text{ƒ},\text{Ě})^{\circ}$.

Example 4.10. Assume that the PNST τ^{PNSS} given in Example 4.4, let any $(\text{ƒ},\text{Ě}) \in PNSS(\text{Ú},\text{Ě})$ is defined by:

$$(\text{ƒ},\text{Ě}) = \left\{\begin{matrix} e_1 = \left\{\begin{matrix}(§_1,0.8,0.8,0.9,0.4\ 0.2),(§_2,0.9,0.8,0.9,0.4\ 0.3),\\(§_3,0.7,0.8,0.9\ 0.5,0.4),(§_4,0.9,0.8,0.7,0.4,0.3)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(§_1,0.9,0.9,0.8,0.5,0.2),(§_2,0.9,0.8,0.6,0.4,0.4),\\(§_3,0.8,0.9,0.9,0.3,0.4),(§_4,0.8,0.6,0.7,0.3,0.5)\end{matrix}\right\}\end{matrix}\right\},$$

then $0_{(\text{Ú},\text{Ě})},(\text{ƒ}_1,\text{Ě}),(\text{ƒ}_2,\text{Ě}),(\text{ƒ}_3,\text{Ě}),(\text{ƒ}_4,\text{Ě}) \subseteq (\text{ƒ},\text{Ě})$ and $(\text{ƒ},\text{Ě})^{\circ} = 0_{(\text{Ú},\text{Ě})} \cup (\text{ƒ}_1,\text{Ě}) \cup (\text{ƒ}_2,\text{Ě}) \cup (\text{ƒ}_3,\text{Ě}) \cup (\text{ƒ}_4,\text{Ě}) = (\text{ƒ}_1,\text{Ě})$, where

$$(\text{ƒ}_1,\text{Ě}) = \left\{\begin{matrix} e_1 = \left\{\begin{matrix}(§_1,0.7,0.8,0.6,0.4,0.3),(§_2,0.8,0.7,0.9,0.5,0.4),\\(§_3,0.6,0.8,0.9,0.7,0.4),(§_4,0.9,0.7,0.6,0.5,0.4)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(§_1,0.8,0.9,0.8,0.6,0.4),(§_2,0.9,0.7,0.6,0.5,0.6),\\(§_3,0.8,0.9,0.5,0.4,0.5),(§_4,0.8,0.5,0.5,0.4,0.6)\end{matrix}\right\}\end{matrix}\right\},$$

$$(\text{ƒ}_2,\text{Ě}) = \left\{\begin{matrix} e_1 = \left\{\begin{matrix}(§_1,0.5,0.6,0.4,0.4,0.8),(§_2,0.6,0.7,0.6,0.8,0.8),\\(§_3,0.6,0.7,0.9,0.8,0.6),(§_4,0.8,0.7,0.6,0.6,0.8)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(§_1,0.7,0.7,0.6,0.8,0.4),(§_2,0.8,0.6,0.5,0.8,0.6),\\(§_3,0.6,0.8,0.4,0.4,0.8),(§_4,0.7,0.5,0.4,0.6,0.8)\end{matrix}\right\}\end{matrix}\right\},$$

$$(\text{ƒ}_3,\text{Ě}) = \left\{\begin{matrix} e_1 = \left\{\begin{matrix}(§_1,0.4,0.5,0.4,0.5,0.8),(§_2,0.5,0.6,0.5,0.9,0.8),\\(§_3,0.5,0.6,0.5,0.8,0.7),(§_4,0.7,0.5,0.4,0.8,0.9)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(§_1,0.6,0.7,0.5,0.8,0.6),(§_2,0.7,0.5,0.4,0.8,0.7),\\(§_3,0.6,0.7,0.4,0.5,0.8),(§_4,0.5,0.4,0.3,0.7,0.8)\end{matrix}\right\}\end{matrix}\right\},$$

$$(\text{ƒ}_4,\text{Ě}) = \left\{\begin{matrix} e_1 = \left\{\begin{matrix}(§_1,0.4,0.3,0.2,0.8,0.9),(§_2,0.4,0.5,0.3,0.9,0.9),\\(§_3,0.3,0.1,0.4,0.9,0.8),(§_4,0.5,0.3,0.2,0.8,0.9)\end{matrix}\right\} \\ e_2 = \left\{\begin{matrix}(§_1,0.5,0.6,0.4,0.9,0.8),(§_2,0.5,0.3,0.2,0.9,0.8),\\(§_3,0.4,0.5,0.2,0.8,0.9),(§_4,0.4,0.2,0.1,0.7,0.9)\end{matrix}\right\}\end{matrix}\right\}.$$

$\Rightarrow (\digamma_1, \check{E}) \cup (\digamma_2, \check{E}) = (\digamma_1, \check{E}), (\digamma_1, \check{E}) \cup (\digamma_3, \check{E}) = (\digamma_1, \check{E}), (\digamma_1, \check{E}) \cup (\digamma_4, \check{E}) = (\digamma_1, \check{E}),$
$(\digamma_2, \check{E}) \cup (\digamma_3, \check{E}) = (\digamma_2, \check{E}), (\digamma_2, \check{E}) \cup (\digamma_4, \check{E}) = (\digamma_2, \check{E}), (\digamma_3, \check{E}) \cup (\digamma_4, \check{E}) = (\digamma_3, \check{E}).$
Hence, $0_{(\acute{U},\check{E})}, (\digamma_1, \check{E}), (\digamma_2, \check{E}), (\digamma_3, \check{E}), (\digamma_4, \check{E}) \subseteq (\digamma, \check{E})$ and
$(\digamma, \check{E})^\circ = 0_{(\acute{U},\check{E})} \cup (\digamma_1, \check{E}) \cup (\digamma_2, \check{E}) \cup (\digamma_3, \check{E}) \cup (\digamma_4, \check{E}) = (\digamma_1, \check{E}).$

Remark 4.11. Let $(\acute{U}, \tau^{PNSS}, \check{E})$ be a PNST space over Ú. A $PNSS(\digamma, \check{E}) \in PNSS(\acute{U}, \check{E})$ is said to be PNSS-open set iff $(\digamma, \check{E}) = (\digamma, \check{E})^\circ$.

Theorem 4.12. Let $(\acute{U}, \tau^{PNSS}, \check{E})$ $be\ PNST\ and$ $(\digamma_1, \check{E}), (\digamma_2, \check{E}) \in PNSS(\acute{U}, \check{E})$, then

1. $[(\digamma_1, \check{E})^\circ]^\circ = (\digamma_1, \check{E})^\circ$
2. $(0_{(\acute{U},\check{E})})^\circ = 0_{(\acute{U},\check{E})}$ and $(1_{(\acute{U},\check{E})})^\circ = 1_{(\acute{U},\check{E})}$
3. $(\digamma_1, \check{E}) \subseteq (\digamma_2, \check{E}) \Rightarrow (\digamma_1, \check{E})^\circ \subseteq (\digamma_2, \check{E})^\circ$
4. $[(\digamma_1, \check{E}) \cap (\digamma_2, \check{E})]^\circ = (\digamma_1, \check{E})^\circ \cap (\digamma_2, \check{E})^\circ$
5. $(\digamma_1, \check{E})^\circ \cup (\digamma_2, \check{E})^\circ \subseteq [(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})]^\circ$.

Proof.

1. As $(\digamma, \check{E})$ is open iff $(\digamma, \check{E})^\circ = (\digamma, \check{E})$, since $(\digamma, \check{E}) \in (PNST) \Rightarrow (\digamma, \check{E})$ is open iff $(\digamma, \check{E})^\circ = (\digamma, \check{E})$. Let $(\digamma, \check{E})^\circ = (G, \check{E})$ then $(G, \check{E})^\circ = (G, \check{E})$, since $(G, \check{E})$ is open, so $((\digamma_1, \check{E})^\circ)^\circ = (\digamma_1, \check{E})^\circ$.
2. As $\phi_{(\acute{U},\check{E})}$ is open so, $(\phi_{(\acute{U},\check{E})})^\circ = \phi_{(\acute{U},\check{E})}$. Similarly $(1_{(\acute{U},\check{E})})^\circ = 1_{(\acute{U},\check{E})}$.
3. As $(\digamma_1, \check{E})^\circ \subseteq (\digamma_1, \check{E})$, but $(\digamma_1, \check{E}) \subseteq (\digamma_2, \check{E}) \Rightarrow (\digamma_1, \check{E})^\circ \subseteq (\digamma_2, \check{E})$.
   Since $(\digamma_1, \check{E})^\circ$ is single open set in $(\digamma_2, \check{E})$, but $(\digamma_2, \check{E})^\circ$ is the union of all open sets that contains $(\digamma_2, \check{E})$, so $(\digamma_1, \check{E})^\circ$ is one of the above set. Clearly, $(\digamma_1, \check{E})^\circ \subseteq (\digamma_2, \check{E})^\circ$.
4. $Sinc\check{e}$ $(\digamma_1, \check{E}) \cap (\digamma_2, \check{E}) \subseteq (\digamma_1, \check{E})$ and $(\digamma_1, \check{E}) \cap (\digamma_2, \check{E}) \subseteq (\digamma_2, \check{E}), then$ $[(\digamma_1, \check{E}) \cap (\digamma_2, \check{E})]^\circ \subseteq (\digamma_1, \check{E})^\circ and$ $[(\digamma_1, \check{E}) \cap (\digamma_2, \check{E})]^\circ \subseteq (\digamma_2, \check{E})^\circ \Rightarrow [(\digamma_1, \check{E}) \cap (\digamma_2, \check{E})]^\circ \subseteq (\digamma_1, \check{E})^\circ \cap (\digamma_2, \check{E})^\circ$,
   $since$ $(\digamma_1, \check{E})^\circ \subseteq (\digamma_1, \check{E})$ and $(\digamma_2, \check{E})^\circ \subseteq (\digamma_2, \check{E})$ $then$ $(\digamma_1, \check{E})^\circ \cap (\digamma_2, \check{E})^\circ \subseteq (\digamma_1, \check{E}) \cap (\digamma_2, \check{E})$.
   $\Rightarrow [(\digamma_1, \check{E}) \cap (\digamma_2, \check{E})]^\circ \subseteq (\digamma_1, \check{E}) \cap (\digamma_2, \check{E})$ is largest PNS open sets, therefore
   $(\digamma_1, \check{E})^\circ \cap (\digamma_2, \check{E})^\circ \subseteq [(\digamma_1, \check{E}) \cap (\digamma_2, \check{E})]^\circ$. Thus $[(\digamma_1, \check{E}) \cap (\digamma_2, \check{E})]^\circ = (\digamma_1, \check{E})^\circ \cap (\digamma_2, \check{E})^\circ$.
5. Since $(\digamma_1, \check{E}) \subseteq (\digamma_1, \check{E}) \cup (\digamma_2, \check{E})$, $(\digamma_2, \check{E}) \subseteq (\digamma_1, \check{E}) \cup (\digamma_2, \check{E}) \Rightarrow (\digamma_1, \check{E})^\circ \subseteq [(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})]^\circ and$ $(\digamma_2, \check{E})^\circ \subseteq [(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})]^\circ$. Therefore $(\digamma_1, \check{E})^\circ \cup (\digamma_2, \check{E})^\circ \subseteq [(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})]^\circ$.

Definition 4.13. If $(\digamma, \check{E}) \in PNSS(\acute{U}, \check{E})$is a PNSS and $(\acute{U}, \tau^{PNSS}, \check{E})$ is a PNST space over Ú, then the PNS closure of (Ϝ, Ě) represented by $\overline{(\digamma, \check{E})}$ is defined as the PNS intersection of all PNS closed supersets of (Ϝ, Ě).

Mathematically $\overline{(\digamma, \check{E})} = (\digamma_1, \check{E})^c \cap (\digamma_2, \check{E})^c \cap (\digamma_3, \check{E})^c, where$ $(\digamma_1, \check{E})^c, (\digamma_2, \check{E})^c, (\digamma_3, \check{E})^c \supseteq (\digamma, \check{E})$.

Remark 4.14. Let $(\acute{U}, \tau^{PNSS}, \check{E})$ $be\ PNST\ space\ over$ Ú $and\ let$ $(\digamma, \check{E}) \in PNSS(\acute{U}, \check{E}). Then$ $(\digamma, \check{E})$ is

$said\ to\ be\ a\ PNSC\ set\ iff\ (\digamma, \check{E}) = \overline{(\digamma, \check{E})}$.

Theorem 4.15. $Let\ (\acute{U}, \tau^{PNSS}, \check{E})\ be\ a\ PNST\ space\ over\ \acute{U}\ and\ (\digamma_1, \check{E}), (\digamma_2, \check{E}) \in PNSS(\acute{U}, \check{E}), then$

1. $\left[\overline{\overline{(\digamma_1, \check{E})}}\right] = \overline{(\digamma_1, \check{E})}$
2. $\overline{0_{(\acute{U},\check{E})}} = 0_{(\acute{U},\check{E})}$ and $\overline{1_{(\acute{U},\check{E})}} = 1_{(\acute{U},\check{E})}$
3. $(\digamma_1, \check{E}) \subseteq (\digamma_2, \check{E}) \Rightarrow \overline{(\digamma_1, \check{E})} \subseteq \overline{(\digamma_2, \check{E})}$
4. $\left[\overline{(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})}\right] = \overline{(\digamma_1, \check{E})} \cup \overline{(\digamma_2, \check{E})}$
5. $\left[\overline{(\digamma_1, \check{E}) \cap (\digamma_2, \check{E})}\right] \subseteq \overline{(\digamma_1, \check{E})} \cap \overline{(\digamma_2, \check{E})}$.

Proof.

1. $Let\ \overline{(\digamma_1, \check{E})} = (\digamma_2, \check{E})\ then\ (\digamma_2, \check{E})\ is\ a\ PNSC\ set, so\ (\digamma_2, \check{E}) = \overline{(\digamma_2, \check{E})} \Rightarrow \left[\overline{\overline{(\digamma_1, \check{E})}}\right] = \overline{(\digamma_1, \check{E})}$.
2. Obvious.
3. $As\ \overline{(\digamma, \check{E})} = (\digamma, \check{E}) \cup (\digamma, \check{E})^c$,since $\overline{(\digamma_1, \check{E})}$ is the smallest PNSC set containing $(\digamma_1, \check{E})$.
   $So, (\digamma_1, \check{E}) \subseteq (\digamma_2, \check{E})\ and\ (\digamma_1, \check{E})^c \subseteq (\digamma_2, \check{E})^c \Rightarrow (\digamma_1, \check{E}) \cup (\digamma_1, \check{E})^c \subseteq (\digamma_2, \check{E}) \cup (\digamma_2, \check{E})^c$.
   $$Thus\ \overline{(\digamma_1, \check{E})} \subseteq \overline{(\digamma_2, \check{E})}.$$
4. $(\digamma_1, \check{E}) \subseteq (\digamma_1, \check{E}) \cup (\digamma_2, \check{E})\ and\ (\digamma_2, \check{E}) \subseteq (\digamma_1, \check{E}) \cup (\digamma_2, \check{E}), then\ \overline{(\digamma_1, \check{E})} \subseteq \overline{(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})}$ and $\overline{(\digamma_2, \check{E})} \subseteq \overline{(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})} \Rightarrow \overline{(\digamma_1, \check{E})} \cup \overline{(\digamma_2, \check{E})} \subseteq [\overline{(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})}]$.
   Conversely, $(\digamma_1, \check{E}) \subseteq \overline{(\digamma_1, \check{E})}$ and $(\digamma_2, \check{E}) \subseteq \overline{(\digamma_2, \check{E})}\ then\ (\digamma_1, \check{E}) \cup (\digamma_2, \check{E}) \subseteq \overline{(\digamma_1, \check{E})} \cup \overline{(\digamma_2, \check{E})}$.
   $\overline{(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})}$ is the smallest set such that $\overline{(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})} \supseteq (\digamma_1, \check{E}) \cup (\digamma_2, \check{E})$,
   $\overline{(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})} \subseteq \overline{(\digamma_1, \check{E})} \cup \overline{(\digamma_2, \check{E})}$ Thus $\overline{(\digamma_1, \check{E}) \cup (\digamma_2, \check{E})} = \overline{(\digamma_1, \check{E})} \cup \overline{(\digamma_2, \check{E})}$.
5. $(\digamma_1, \check{E}) \cap (\digamma_2, \check{E}) \subseteq \overline{(\digamma_1, \check{E})} \cap \overline{(\digamma_2, \check{E})}\ and\ \overline{[(\digamma_1, \check{E}) \cap (\digamma_2, \check{E})]} \supseteq (\digamma_1, \check{E}) \cap (\digamma_2, \check{E})$. So $\overline{[(\digamma_1, \check{E}) \cap (\digamma_2, \check{E})]} \subseteq \overline{(\digamma_1, \check{E})} \cap \overline{(\digamma_2, \check{E})}$.

Theorem 4.16. Assume that $(\acute{U}, \tau^{PNSS}, \check{E})$ be a PNST, $(\digamma, \check{E}) \in PNSS(\acute{U}, \check{E})\ then$

1. $\left[\overline{(\digamma, \check{E})}\right]^c = \left[(\digamma, \check{E})^c\right]^\circ$
2. $[(\digamma, \check{E})^\circ]^c = \left[\overline{(\digamma, \check{E})^c}\right]$.

Proof.

1. As, $\overline{(\digamma, \check{E})} = \left((G_1, \check{E}) \cap (G_2, \check{E}) \dots \dots \cap (G_n, \check{E})\right)$.
   $\Rightarrow \left[\overline{(\digamma, \check{E})}\right]^c = \left[(G_1, \check{E}) \cap (G_2, \check{E}) \dots \dots \cap (G_n, \check{E})\right]^c = (G_1, \check{E})^c \cup (G_2, \check{E})^c \dots \cup (G_n, \check{E})^c$.
   $\Rightarrow \cup \left\{(G_i, \check{E})^c \in \tau^{PNSS} : (G, \check{E})^c \subseteq (\digamma, \check{E})^c\right\}, \forall\ i = 1, 2, \dots n \Rightarrow \left[\overline{(\digamma, \check{E})}\right]^c = [(\digamma, \check{E})^c]^\circ$.
2. As $(\digamma, \check{E})^\circ = [(G_1, \check{E}) \cup (G_2, \check{E}) \dots \cup (G_n, \check{E})]$.
   $\Rightarrow [(\digamma, \check{E})^\circ]^c = \left[(G_1, \check{E}) \cup (G_2, \check{E}) \dots \cup (G_n, \check{E})\right]^c = (G_1, \check{E})^c \cap (G_2, \check{E})^c \dots \cap (G_n, \check{E})^c$

$$\Rightarrow \cap \left\{ \left(G_i, \check{\mathrm{E}}\right)^c \in \tau^{PNSS} : \left(G, \check{\mathrm{E}}\right)^c \supseteq \left(\text{Ӻ}, \check{\mathrm{E}}\right)^c \right\}, \forall\, i = 1, 2, \dots n \Rightarrow \left[\left(\text{Ӻ}, \check{\mathrm{E}}\right)^{\circ}\right]^c = \left[\, \overline{\left(\text{Ӻ}, \check{\mathrm{E}}\right)^c}\,\right].$$

5. Some Similarities Measures

Definition 5.1. Let $X = \{x_1, x_2, x_3, \dots, x_n\}$ be a finite universe of discourse and let

$$A = \left\{ \left\langle \langle T_A^j(x_i), ReT_A^j(x_i), C_A^j(xi), ReF_A^j(x_i), F_A^j(x_i) \rangle \middle| x_i \in X. j = 1,2, \dots, p \right\rangle \right\}$$

and

$$B = \left\{ \left\langle \langle T_B^j(x_i), ReT_B^j(x_i), C_B^j(xi), ReF_B^j(x_i), F_B^j(x_i) \rangle \middle| x_i \in X. j = 1,2, \dots, p \right\rangle \right\}$$

Be two five-valued neutrosophic refined sets, where:

- T denotes the degree of Truth membership,
- ReT denotes the degree of Relative truth membership,
- C denotes the degree of Contradiction,
- ReF denotes the degree of Relative truth membership,
- F denotes the degree of False membership.

Then the correlation between A and B is define as:

$$\rho FVNRS(A, B) = \frac{c_{FVNRS(A,\ B)}}{\sqrt{c_{FVNRS(A,\ A)} \times c_{FVNRS(B,\ B)}}} \tag{1}$$

Where

$$c_{FVNRS(A,\ B)} = \frac{1}{p} \sum_{j=1}^{p} \sum_{i=1}^{n} \left[T_A^j(x_i) T_B^j(x_i) + ReT_A^j(x_i) ReT_B^j(x_i) + C_A^j(x_i) H_B^j(x_i) + ReF_A^j(x_i) ReF_B^j(x_i) + F_A^j(x_i) F_B^j(x_i) \right]$$

$$c_{FVNRS(A,\ A)} = \frac{1}{p} \sum_{j=1}^{p} \sum_{i=1}^{n} \left[\left(T_A^j(x_i)\right)^2 + \left(ReT_A^j(x_i)\right)^2 + \left(C_A^j(x_i)\right)^2 + \left(ReF_A^j(x_i)\right)^2 + \left(F_A^j(x_i)\right)^2 \right]$$

And

$$c_{FVNRS(A,\ B)} = \frac{1}{p} \sum_{j=1}^{p} \sum_{i=1}^{n} \left[\left(T_B^j(x_i)\right)^2 + \left(ReT_B^j(x_i)\right)^2 + \left(C_B^j(x_i)\right)^2 + \left(ReF_B^j(x_i)\right)^2 + \left(F_B^j(x_i)\right)^2 \right]$$

Definition 5.2 Let $T_i = (AbT_{ik}, RelT_{ik}, H_{ik}, RelF_{ik}, AbF_{ik})$, $C_j = \left(AbT_{jk}, RelT_{jk}, H_{jk}, RelF_{jk}, AbF_{jk}\right)$ are two PNSSs. Each set is represented by four components for each feature k:

AbT_{ik}: Absolute truth membership of feature k for T_i.

$RelT_{ik}$: Relative truth value of feature k for T_i.

H_{ik}: Hesitation value of feature k for T_i.

$RelF_{ik}$: Relative false value of feature k for T_i.

AbF_{ik}: Absolute false value of feature k for T_i.

The values typically satisfy: $AbT_{ik}, RelT_{ik}, H_{ik}, RelF_{ik}, AbF_{ik} \in [0,1]$.

Similarly, for C_j:

The values typically satisfy: $AbT_{ik}, RelT_{ik}, H_{ik}, RelF_{ik}, AbF_{ik} \in [0,1]$.

The TSM between T_i and C_j is defined as: $Tangent_{PNSS}(T_i, C_j) =$

$$\frac{1}{n}\sum_{k=1}^{n}\left\{1 - tan\frac{\pi}{12}\left(\begin{array}{c}|AbT_{ik} - AbT_{jk}| + |RelT_{ik} - RelT_{jk}| + |H_{ik} - H_{jk}| + \\ |RelF_{ik} - RelF_{jk}| + |AbF_{ik} - AbF_{jk}|\end{array}\right)\right\} \tag{2}$$

This calculation compares the comparable five-dimensional neutrosophic representation of T_i and C_i on feature k, calculating the average cosine similarity of all features.

6. Proposed Methodology

A thorough examination of the symptoms, which can vary over time and include some degree of uncertainty, is necessary for the accurate identification of life-threatening malignancies. Conventional diagnostic techniques typically rely on a single observation, which may not be sufficient to document changes in the symptoms' expression. To overcome this limitation, we propose a multi-time inspection model based on five-term diagnostic vectors that enable systematic handling of time variability and clinical fluctuation in medical data.

6.1 Multi-Time Symptom Inspection

In the suggested framework, patients' symptoms are tracked multiple times over a monitoring period rather than just once. Doctors are better able to identify variations in the severity of symptoms and their presence over time when they check patients repeatedly. Each observation contributes fresh data to the patient's diagnostic profile, which can be utilized to develop a more reliable diagnosis. In this study, each patient is examined five times at Day 0, Day 3, Day 7, Day 10, and Day 14.

6.2 Five-Term Diagnostic Representation of Symptoms

Each symptom is therefore represented by five-term diagnostic values, which characterize five possible states:

- AbT (Abnormal Test Value): Degree of deviation from the normal range.
- ReT (Relevant Threshold): Proximity to the clinical decision threshold.
- H (Historical Value): Baseline or previous value of the symptom.
- ReF (Relative Fluctuation): Variation relative to the baseline.
- AbF (Abnormal Fluctuation): Degree of unexpected variation.

In this form, AbT indicates the extent of abnormality in test results, RelT indicates how close the measurement is to the clinical warning level, H provides the historical baseline for comparison, RelF captures the relative change over time, and AbF quantifies unusual fluctuations that may indicate disease progression. This five-term paradigm enables a nuanced comprehension of symptoms that may change over time and vary across different patients.

6.3 Diagnostic Process Correlation

Correlation metrics are used in the diagnostic process to quantify the relationships between patients, symptoms, and diseases using Tangent Similarity Measure (TSM). Two primary correlations are examined:

1. **Patient-Symptom Correlation (Q):**
   This is used to gauge the relationship between the symptoms that patients report and their measurements at various examination times. Table I (Q) represents this correlation, where each patient's symptoms are recorded across five time points using the five-term vectors (AbT, ReT, H, ReF, AbF).
2. **Symptom-Disease Correlation (R):**
   This is the degree to which particular symptoms are associated with particular diseases. Table II (R) represents this correlation, where each symptom's association with breast cancer, lung cancer, prostate cancer, and liver cancer is quantified using the same five-term diagnostic values. The framework evaluates the overall relationship between each patient and the possible diseases by combining these correlations using TSM. The most likely diagnosis is the one with the highest correlation value, as shown in Table III (Q ∘ R).

7. Problem Formulation of Multi-Time Triple-Valued Neutrosophic Framework for Cancer Diagnosis

A medical case involving patients, symptoms, and cancer types could serve as an illustration of the proposed TSM-based diagnostic platform. Assume that D = {Breast Cancer, Lung Cancer, Prostate Cancer, Liver Cancer} is the set of diseases, S = {Tumor Size, Weight Loss, Fatigue, Pain Severity, Biomarker Level} is the set of symptoms, and P = {P1, P2, P3, P4, P5} is the set of patients. In order to reflect the ambiguity, uncertainty, and variability of medical diagnosis, this framework allows us to record changes in symptoms over time in a variety of individuals and disorders. Each patient is examined five times at Day 0, Day 3, Day 7, Day 10, and Day 14. During each examination, the symptoms are evaluated using five-term vectors, and the existence and severity of each symptom are recorded. These values show different degrees of abnormality and fluctuation regarding the existence of a symptom: AbT (Abnormal Test Value) indicates the degree of deviation from normal range, RelT (Relevant Threshold) indicates how close the value is to clinical decision threshold, H (Historical Value) indicates the baseline or previous value, RelF (Relative Fluctuation) indicates the variation relative to baseline, and AbF (Abnormal Fluctuation) indicates the degree of unexpected variation. In order to capture the dynamic nature of symptoms throughout the monitoring period while biasing toward the variability and fluctuation typically encountered in medical assessment, the framework can record such observations at several time points. A more precise and trustworthy diagnosis of life-threatening

tumors will result from the use of this time-dependent complete data to calculate correlation parameters between patients, symptoms, and diseases using Tangent Similarity Measure (TSM).
Table 7.1: Q (The Relation between Cancer Patients and Symptoms)

Q	Tumor Size	Weight Loss	Fatigue	Pain Severity	Biomarker Level
P1	(0.6,0.4,0.8,0.3,0.8)	(0.6,0.5,0.8,0.2,0.8)	(0.8,0.2,0.8,0.3,0.6)	(0.6,0.5,0.8,0.9,0.8)	(0.2,0.4,0.6,0.3,0.8)
	(0.3,0.8,0.9,0.4,0.4)	(0.3,0.6,0.7,0.5,0.4)	(0.3,0.4,0.9,0.5,0.4)	(0.7,0.2,0.9,0.4,0.2)	(0.6,0.2,0.9,0.5,0.4)
	(0.8,0.7,0.3,0.6,0.6)	(0.8,0.7,0.3,0.3,0.6)	(0.3,0.5,0.3,0.7,0.6)	(0.7,0.7,0.4,0.6,0.7)	(0.8,0.7,0.3,0.5,0.5)
	(0.4,0.3,0.6,0.2,0.3)	(0.3,0.3,0.2,0.2,0.6)	(0.4,0.1,0.6,0.3,0.7)	(0.4,0.7,0.5,0.2,0.3)	(0.3,0.3,0.9,0.2,0.8)
	(0.8,0.2,0.4,0.8,0.2)	(0.4,0.2,0.5,0.8,0.2)	(0.50.2,0.4,0.4,0.2)	(0.9,0.2,0.2,0.8,0.2)	(0.3,0.2,0.9,0.8,0.1s)
P2	(0.4,0.5,0.5,0.4,0.4)	(0.8,0.6,0.5,0.9,0.4)	(0.7,0.5,0.1,0.8,0.4)	(0.4,0.8,0.5,0.2,0.4)	(0.3,0.5,0.8,0.4,0.5)
	(0.6,0.4,0.6,0.8,0.8)	(0.7,0.4,0.6,0.5,0.8)	(0.6,0.7,0.6,0.9,0.8)	(0.3,0.5,0.6,0.3,0.8)	(0.6,0.3,0.6,0.7,0.8)
	(0.4,0.8,0.8,0.9,0.9)	(0.3,0.8,0.5,0.9,0.2)	(0.4,0.4,0.8,0.3,0.9)	(0.2,0.8,0.7,0.1,0.7)	(0.4,0.5,0.8,0.4,0.9)
	(0.8,0.2,0.3,0.3,0.3)	(0.7,0.2,0.3,0.6,0.3)	(0.5,0.2,0.3,0.2,0.3)	(0.8,0.6,0.3,0.3,0.8)	(0.8,0.2,0.7,0.9,0.3)
	(0.3,0.6,0.9,0.4,0.6)	(0.8,0.6,0.9,0.3,0.6)	(0.3,0.7,0.9,0.2,0.6)	(0.7,0.6,0.9,0.3,0.9)	(0.3,0.3,0.8,0.4,0.5)
P3	(0.5,0.8,0.4,0.6,0.6)	(0.5,0.8,0.4,0.4,0.6)	(0.5,0.8,0.7,0.6,0.3)	(0.4,0.8,0.7,0.2,0.6)	(0.5,0.4,0.4,0.5,0.3)
	(0.7,0.6,0.6,0.8,0.3)	(0.4,0.6,0.6,0.3,0.3)	(0.7,0.5,0.6,0.4,0.3)	(0.4,0.6,0.4,0.8,0.5)	(0.4,0.6,0.6,0.7,0.3)
	(0.3,0.4,0.3,0.3,0.4)	(0.3,0.8,0.3,0.9,0.4)	(0.8,0.4,0.3,0.6,0.4)	(0.7,0.4,0.2,0.3,0.4)	(0.3,0.7,0.3,0.4,0.7)
	(0.4,0.6,0.2,0.3,0.6)	(0.5,0.7,0.2,0.5,0.6)	(0.4,0.7,0.5,0.4,0.8)	(0.4,0.1,0.2,0.4,0.8)	(0.5,0.7,0.9,0.4,0.9)
	(0.8,0.5,0.6,0.2,0.7)	(0.8,0.7,0.6,0.2,0.8)	(0.6,0.5,0.1,0.2,0.4)	(0.7,0.5,0.1,0.2,0.7)	(0.7,0.5,0.1,0.2,0.9)
P4	(0.3,0.2,0.8,0.2,0.8)	(0.4,0.7,0.8,0.8,0.1)	(0.4,0.8,0.8,0.2,0.3)	(0.3,0.2,0.1,0.2,0.9	(0.4,0.1,0.8,0.6,0.8)
	(0.2,0.8,0.6,0.6,0.7)	(0.2,0.7,0.6,0.1,0.7)	(0.2,0.7,0.6,0.8,0.7)	(0.4,0.8,0.9,0.6,0.1)	(0.2,0.6,0.6,0.7,0.7)
	(0.3,0.3,0.4,0.4,0.3)	(0.7,0.3,0.1,0.4,0.3)	(0.3,0.6,0.4,0.9,0.1)	(0.3,0.7,0.4,0.6,0.3)	(0.7,0.3,0.4,0.9,0.3)
	(0.5,0.4,0.3,0.3,0.6)	(0.2,0.4,0.3,0.8,0.6)	(0.5,0.3,0.3,0.6,0.6)	(0.5,0.8,0.4,0.3,0.6)	(0.5,0.4,0.8,0.3,0.1)
	(0.9,0.4,0.6,0.2,0.4)	(0.8,0.4,0.6,0.1,0.4)	(0.3,0.4,0.6,0.6,0.4)	(0.5,0.4,0.3,0.2,0.4)	(0.3,0.8,0.6,0.2,0.3s)
P5	(0.7,0.3,0.9,0.8,0.2)	(0.7,0.4,0.9,0.8,0.7)	(0.6,0.3,0.2,0.8,0.2)	(0.6,0.3,0.3,0.8,0.8)	(0.7,0.4,0.9,0.1,0.2)
	(0.6,0.8,0.3,0.9,0.6)	(0.4,0.8,0.2,0.9,0.9)	(0.6,0.7,0.3,0.1,0.6)	(0.7,0.8,0.3,0.3,0.6)	(0.6,0.8,0.8,0.9,0.9)
	(0.8,0.9,0.2,0.3,0.5)	(0.3,0.9,0.2,0.1,0.5)	(0.8,0.9,0.6,0.3,0.3)	(0.8,0.7,0.2,0.3,0.2)	(0.8,0.7,0.2,0.1,0.5)
	(0.3,0.5,0.6,0.4,0.7)	(0.3,0.4,0.6,0.8,0.7)	(0.6,0.5,0.6,0.1,0.7)	(0.3,0.2,0.6,0.8,0.7)	(0.4,0.5,0.8,0.4,0.2)
	(0.1,0.4,0.8,0.3,0.8)	(0.1,0.6,0.8,0.3,0.6)	(0.1,0.5,0.8,0.8,0.8)	(0.3,0.4,0.8,0.2,0.8)	(0.1,0.3,0.8,0.3,0.7)

Table 7.1 reveals the usefulness of the suggested multi-time five-term diagnostic model; it is possible to record the effect of the patient symptoms at various times (Day 0, Day 3, Day 7, Day 10, and Day 14). It is a more realistic description of the presence of symptoms, as it is a way to express abnormality and fluctuation with the help of five components (AbT, ReT, H, ReF, AbF). The difference between patients is also emphasized in the table, which gives evidence to personalized analysis. Also, it offers systematic information to calculate the association between patients, symptoms, and diseases to help in precise diagnosis. All in all, it incorporates the time variation, measurement variability, and diversity of patients in one framework to make reliable medical decisions.

Table 7.2 is significant in that it establishes the correlation between the symptoms and various cancer diseases with five-term diagnostic values. It demonstrates the level of association of each symptom with such types of diseases as breast cancer, lung cancer, prostate cancer, and liver cancer. The table enhances abnormality, fluctuation, and historical trends in medical observations by use of five components (AbT, ReT, H, ReF, AbF), thus giving it a more realistic view than simple yes/no relationships. This table is an indispensable bridge between the observed symptoms and potential diseases, which assists in determining which disease has a more likely occurrence because of the pattern of the symptom. It also aids in making right decisions as it enables one to compute the correlations between symptoms and diseases using Tangent Similarity Measure (TSM). On the whole, Table II makes the diagnosis more reliable as it adds multi-dimensional clinical data and offers the systematic means of analysing the connections between diseases and symptoms.

Table 7.2: R (The Relation among Cancer Symptoms and Diseases)

R	**Breast Cancer**	**Lung Cancer**	**Prostate Cancer**	**Liver Cancer**
Tumor Size	(0.2,0.3,0.2,0.6,0.3)	(0.4,0.4,0.2,0.5,0.2)	(0.6,0.3,0.1,0.5,0.7)	(0.3,0.4,0.2,0.8,0.2)
Weight Loss	(0.6,0.2,0.2,0.6,0.4)	(0.8,0.1,0.1,0.3,0.2)	(0.3,0.1,0.1,0.6,0.2)	(0.2,0.2,0.2,0.1,0.0)
Fatigue	(0.5,0.1,0.1,0.3,0.3)	(0.4,0.3,0.2,0.3,0.1)	(0.4,0.3,0.2,0.5,0.2)	(0.2,0.4,0.2,0.2,0.2)
Pain Severity	(0.6,0.5,0.3,0.2,0.2)	(0.2,0.2,0.1,0.6,0.1)	(0.1,0.3,0.3,0.3,0.1)	(0.2,0.3,0.2,0.5,0.3)
Biomarker Level	(0.7,0.3,0.2,0.4,0.3)	(0.2,0.2,0.1,0.4,0.2)	(0.2,0.2,0.1,0.4,0.3)	(0.2,0.1,0.1,0.3,0.2)

By applying the TSM formula (Equation 1) to the patient-symptom matrix (Table 7.1: Q) and the symptom-disease matrix (Table 7.2: R), we obtain Table 7.3: Correlation Measure between Cancer Patients and Cancer Types (Q ∘ R) , which represents the overall similarity between each patient and each cancer type.

Table 7.3: Correlation Measure between Cancer Patients and Cancer Types (Q ∘ R)

Table 7.3 presents the correlation measures obtained between the cancer patients Pi and different cancer disease profiles, namely Breast Cancer, Lung Cancer, Prostate Cancer, and Liver Cancer. The correlation values range between 0 and 1, where values closer to 1 indicate a stronger similarity or association between the patient's symptom profile and the corresponding cancer type.

Table 7.3: Correlation Measure between Cancer Patients and Cancer Types (Q ∘ R)

Correlation Measure	**Breast Cancer**	**Lung Cancer**	**Prostate Cancer**	**Liver Cancer**
P1	0.872	0.901	0.864	0.869
P2	0.845	0.743	0.878	0.787
P3	0.792	0.839	0.869	0.811
P4	0.803	0.821	0.845	0.792
P5	0.889	0.856	0.903	0.874

By applying Similarity formula (Equation 2) between Table 7.1 and Table 7.2 to get Table 7.4

Table 7.4: Correlation Measure between Cancer Patients and Cancer Types (Q ∘ R)

Correlation Measure/Tangent Formula	Breast Cancer	Lung Cancer	Prostate Cancer	Liver Cancer
P_1	0.496	0.468	0.512	0.455
P_2	0.523	0.549	0.501	0.472
P_3	0.441	0.487	0.463	0.421
P_4	0.412	0.439	0.428	0.401
P_5	0.558	0.541	0.572	0.510

Combined Table 7.5: Correlation and Tangent Correlation Measures Between Cancer Patients and Cancer Types Q∘R

Table 7.5: Joint Correlation–Tangent Measure for Cancer Patient Analysis

Correlation Measure/Tangent Formula	Breast Cancer	Lung Cancer	Prostate Cancer	Liver Cancer
P_1	0.872 / 0.496	0.901/0.468	0.864 / 0.512	0.869 / 0.455
P_2	0.845 / 0.523	0.743 /0.549	0.878 / 0.501	0.787 / 0.472
P_3	0.792 / 0.441	0.839 /0.487	0.869 / 0.463	0.811 / 0.421
P_4	0.803 / 0.412	0.821 /0.439	0.845 / 0.428	0.792 / 0.401
P_5	0.889 / 0.558	0.856 /0.541	0.903 / 0.572	0.874 / 0.510

8. Techniques

In this section, you will find several visualization and analytics tools that facilitate a comprehensive understanding of the proposed data. They include Heatmaps, graph analysis, Principal Component Analysis (PCA), and t-distributed Stochastic Neighbor Embedding (t-SNE). Such methods are useful for uncovering hidden patterns, similarities, and relationships among the components of the dataset. PCA and t-SNE can be seen as effective methods for dimensionality reduction, transforming high-dimensional data into low-dimensional form while preserving its essential characteristics.

8.1 Heatmap Analysis

The Correlation–Tangent line Heatmaps perfectly illustrate the similarity of relationships between different patients and cancer types. Heatmaps can effectively establish dominant and weaker relationships, and row-wise and column-wise normalizations will enhance patient- and cancer-specific comparisons. The lack of consistency in recognizing Patient P5-Prostate Cancer as the primary relationship suggests that the proposed similarity approach is robust.

8.2 Graphical Analysis

The proposed joint correlation-tangent similarity measure is illustrated through complementary graphical representations, including two-dimensional bar charts, three-dimensional bar charts, and three-dimensional surface charts. Bar charts facilitate the comparison of the quantitative values of both Pearson correlation and tangent similarity measures, whereas the surface chart illustrates the overall variation and trend in similarity.

8.3 PCA-Based Analysis

The Principal Component Analysis (PCA) reduces the dimensionality of the similarities data while preserving its greatest variance. By effectively clustering the patients, the combination of PCA and K-means clustering demonstrates that the proposed correlation-tangent features possess significant discriminative power for cancer classification.

8.4 t-SNE-Based Analysis

With the 3D t-Distributed Stochastic Neighbor Embedding (t-SNE) method, non-linear dimensionality reduction is achieved by preserving the data's neighborhood structure. Using the K-Means clustering algorithm, t-SNE produces clearly distinct patient clusters, showcasing the similarity framework's efficacy in creating highly informative features for cancer classification.

9. Comparative Analysis

The tangent and correlation the Heatmap illustrated in Fig. 9.1 shows the relationship between cancer patients and cancer types using two analogous parameters: the correlation coefficient and the tangent similarity transformation. With the Heatmap, an overall comparative analysis is presented. Here, the color intensity indicates the correlation coefficient, while the cells display the correlated values of correlation and tangent similarity formatted as "Correlation / Tangent." The outcome indicates that patient P5 exhibits the highest resemblance to the prostate cancer type, with a correlation coefficient peak of 0.903 and a corresponding tangent value of 0.572. The consistent results from both methods applied to the case in question confirm the effectiveness of the proposed framework in identifying the most significant relationship. Additionally, greater correlation coefficients have been identified for other patient-cancer combinations, specifically P1-lung cancer (0.901), P5-breast cancer (0.889), and P5-liver cancer (0.874). As a result, while many major relationships have been identified, the link between P5 and prostate cancer emerges as the most crucial. Results based on tangents show a comparable correlation pattern, thereby validating the suggested ranking order and demonstrating the stability of the proposed transformation method. Lower correlations and tangents regarding P4 for any cancer type indicate relatively weak relationships. This clearly demonstrates the ability of the presented technique to differentiate between high-risk and low-risk patients. To summarize, the strong correlation between the correlation and tangent metrics serves as additional evidence of the proposed similarity technique's robustness and efficiency. The results demonstrate that the model can identify important trends in the uncertain data.

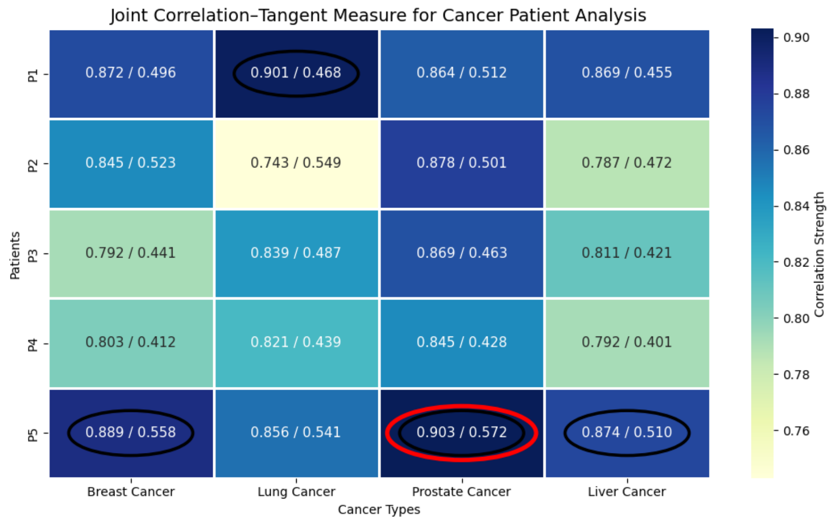

Fig 9.1: Joint correlation-Tangent measure for cancer patient analysis

The row-wise normalized correlation-tangent measure Heatmap in Fig. 9.2 demonstrates that the introduced correlation measure consistently yields high similarity values across all patients and cancer types, indicating a strong agreement among the cases studied. The cells marked in dark red signify the highest normalized correlations for each patient, thus representing the most pertinent type of cancer for that patient via row-wise normalization. It is observed that Prostate Cancer exhibits the highest normalized correlation for Patients P2, P3, P4, and P5, whereas Lung Cancer shows the highest normalized correlation for Patient P1, indicating that these cancers are most closely associated with the respective patients.

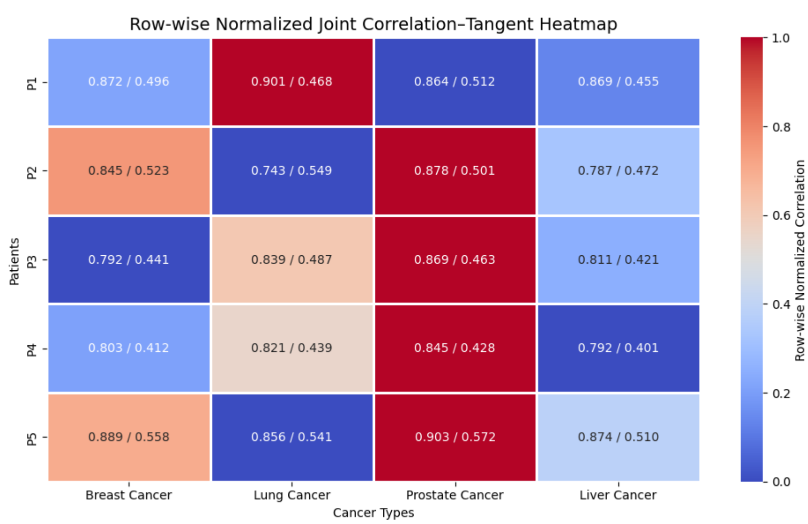

Fig 9.2: Row-wise normalized correlation-tangent Heatmap

Fig. 9.3 presents a column-wise normalized joint correlation-tangent Heatmap that illustrates the distribution of the proposed correlation measure among all patients for each cancer type case.

The cells with the darkest red hues indicate the highest normalized correlations across the various cancer categories, signifying the patients with the strongest correlation after column-wise normalization. For example, P5 exhibited the highest normalized correlation for Breast Cancer, Prostate Cancer, and Liver Cancer, whereas P1 had the highest normalized correlation for Lung Cancer. This indicates that the highest relative correlations occurred in these specific instances.

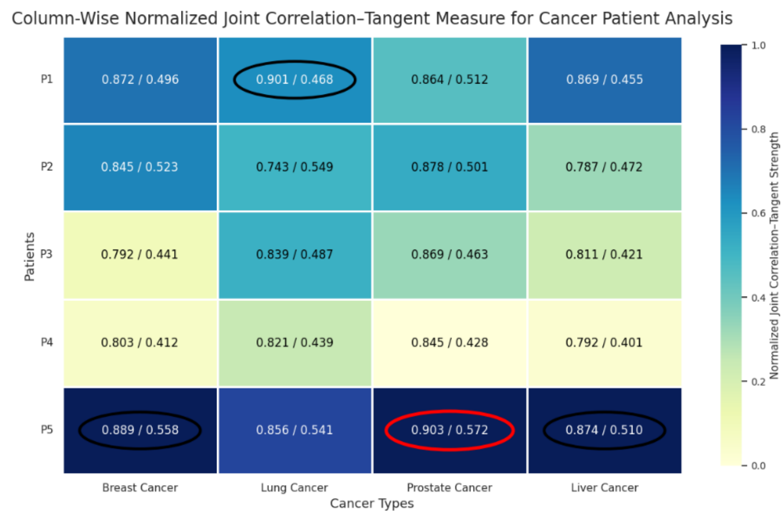

Fig 9.3: Column-wise normalized correlation-tangent Heatmap

Below, Figure 9.4 shows a two-dimensional bar chart illustrating the joint values of correlation-tangent pairs for every possible combination of patients and cancers. Each pair of bars corresponds to the Pearson correlation and tangent value of that pair. The correlation values remain constant across all cancers, as illustrated, while the tangent values provide further information that reflects the variability of the transformed similarity metric. The selected bars indicate that the combination of Patient P5-Prostate Cancer has the highest Pearson correlation of 0.903 and the highest tangent value of 0.572, demonstrating that this is the most significant relationship among the patients and cancers in all analyzed combinations. Conversely, the combination of Patient P4-Liver Cancer yields the lowest tangent value of 0.401, while Patient P2-Lung Cancer has the lowest Pearson correlation of 0.743, signifying weaker associations.

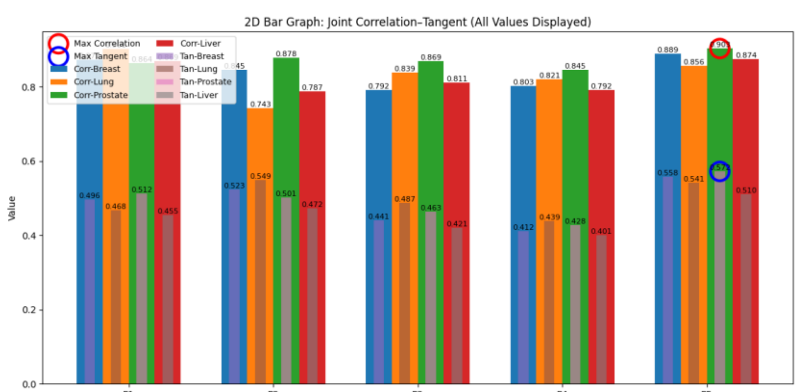

Fig 9.4:2D Bar graph: Joint correlation-Tangent graph

All patient-cancer pairs' correlation and tangent measures joint values are illustrated in a 3-D format in the 3-D bar chart in Fig. 9.5. This enables a detailed comparison of correlation strength across different cancers. With increasing bar height, the Pearson correlations also increase. Furthermore, the tangent values provide additional insights into the transformed similarity measure. The indicated point denotes the patient-cancer pair with the highest Pearson correlation and tangent values of (0.903) and (0.572), respectively. In contrast, lower bars with respect to height, particularly when the correlation is between Patient P2 and Lung Cancer (Pearson = 0.743) and Patient P4 and Liver Cancer (Tangent = 401), indicate that the correlations are relatively weak. In general, the 3D graph has proven effective in facilitating the interpretation of multidimensional correlations due to its inclusion of both Pearson and Tangent measurements.

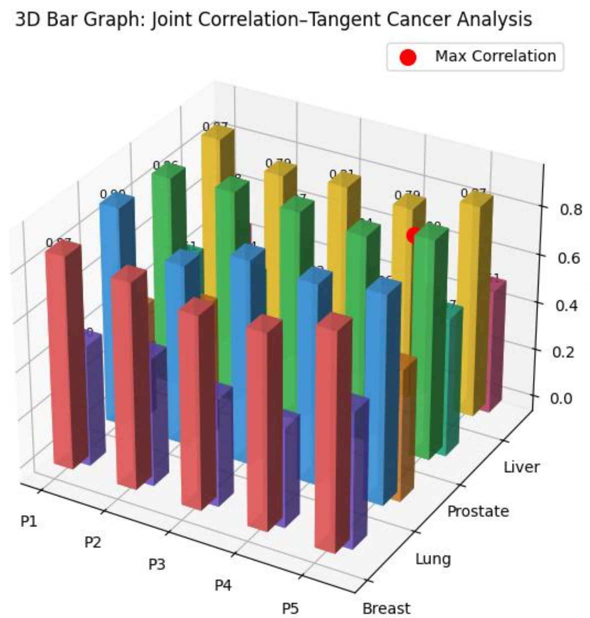

Fig 9.5: 3D Bar graph: Joint correlation-tangent graph

All patient-cancer combinations' correlation-tangent values are illustrated in a complete view in Figure 9.6, which employs a 3D surface plot. The surface plot on top displays the Pearson correlation values, whereas the surface plot below displays the corresponding tangent values. The two plots' smooth curves indicate high correlation values, with moderate differences among the various patient-cancer combinations. It is remarkable that the Patient P5-Prostate Cancer combination boasts the highest surface values, as well as the maximum Pearson correlation (0.903) and the maximum tangent (0.572) value, establishing it as the strongest combination overall. In contrast, the combination of Patient P2-Lung Cancer has the lowest Pearson correlation value (0.743), and the combination of Patient P4-Liver Cancer has the lowest tangent value (0.401). The 3D surface serves as an effective tool for visualizing the patterns of the suggested correlation-tangent relationship between patient and cancer types.

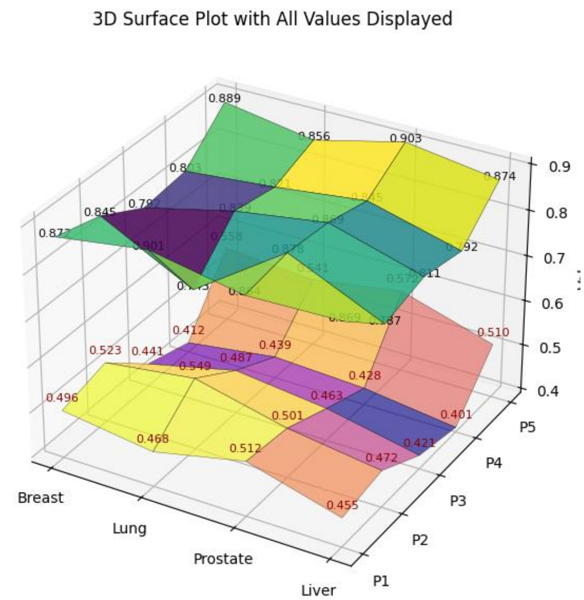

Fig 9.6: 3D Surface graph

Fig. 9.7 below illustrates the K-Means clustering of cancer patients based on the joint correlation-tangent measure we proposed, following dimensional reduction through PCA. The figure clearly illustrates that the patients are organized into two clusters, indicating that our correlation-tangent features can differentiate between various patients. P1, P2, and P5 are part of a cluster characterized by a high joint correlation, whereas P3 and P4 belong to a different cluster with low correlation. Cluster centers (black × symbols) indicate the centroids of both clusters and validate the tightness and distinctiveness of the clusters. In general, the figure illustrates the fact that the joint correlation-tangent measure is a good method to represent the features for the purpose of clustering.

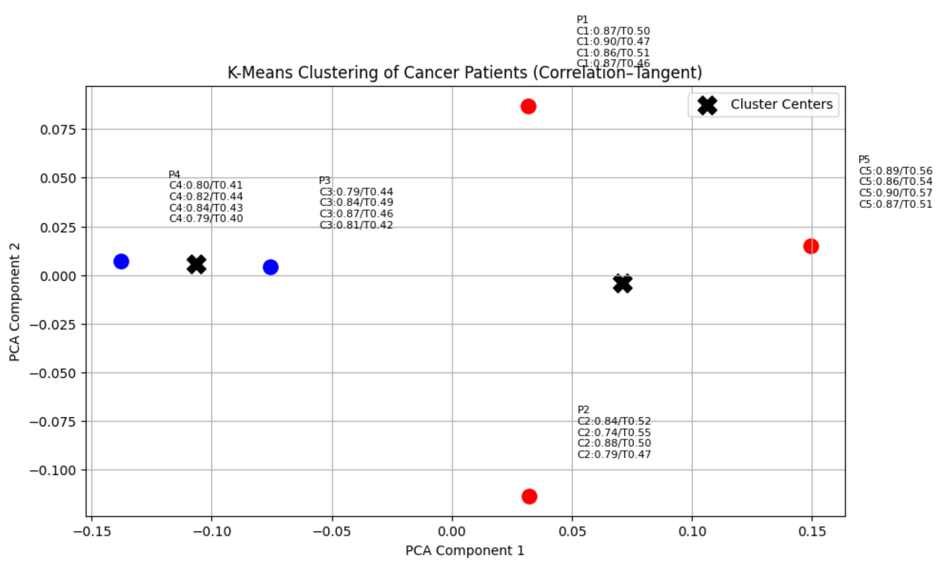

Fig 9.7: K-Mean clustering graph

The 3D PCA and K-Means clustering plot in Fig. 9.8 illustrates the distribution of cancer patients within a three-dimensional PCA space, utilizing the proposed joint correlation-tangent measure method. The illustration demonstrates the emergence of two separate clusters, signifying that the chosen features can represent the similarities and differences found in the patients' data sets. The first cluster includes P1, P2, and P5, whereas the second cluster is made up of P3 and P4. The black × symbols indicate the centroids of the two clusters, demonstrating that the clusters are highly compact and separable. To summarize, the PCA representation in three-dimensional space affirms the effectiveness of the joint correlation-tangent measure approach for clustering and dimensionality reduction.

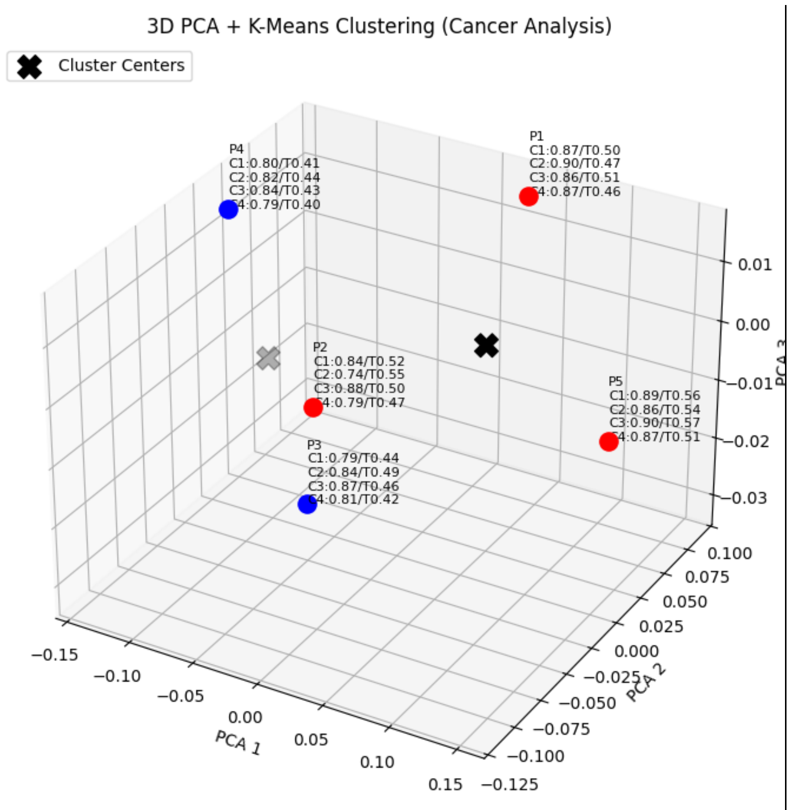

Fig 9.8: 3D PCA and K-Mean clustering

Figure 9.9 presents a 3D t-SNE + K-means clustering visualization that illustrates the nonlinear arrangement of cancer patients using the proposed joint correlation-tangent metric. Utilizing t-SNE to maintain the local neighborhood structure, the method identifies two clusters; this indicates that patients with similar joint correlation-tangent metrics will be grouped together, while those with differing patient profiles will be clearly separated. Patients P1, P2, and P5 constitute one cluster with a relatively higher joint correlation, whereas P3 and P4 make up another cluster with a lower correlation.

The position of the cluster centroid (indicated by a black ×) demonstrates the centroid of the clustering and the separation of the clusters from one another. To summarize, the 3D t-SNE plot illustrates how effective the joint correlation-tangent method is for producing highly discriminating feature vectors for dimensionality reduction and clustering.

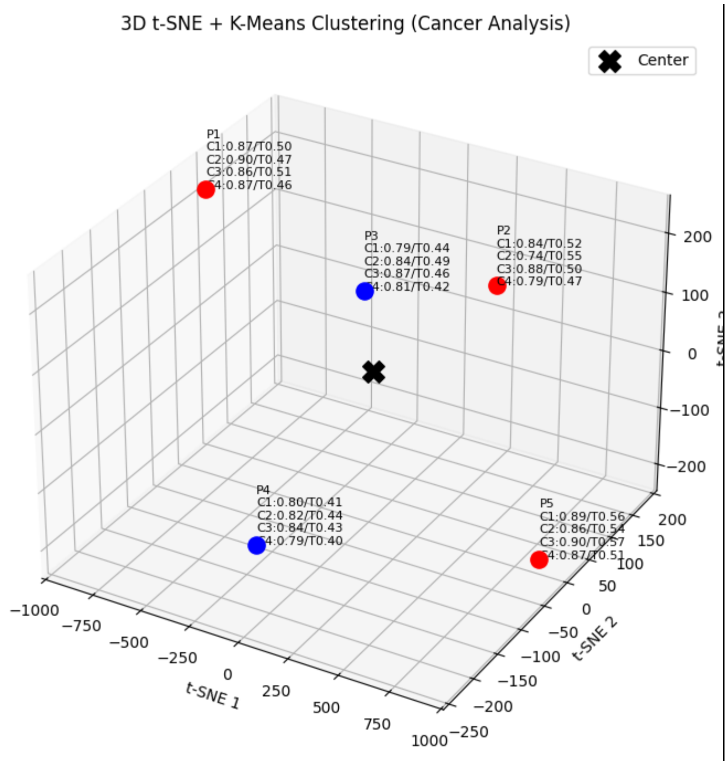

Fig 9.9: 2D t-SNE and K-Mean clustering

10. Comprehensive Analytical Evaluation

The current section offers a detailed assessment of the suggested framework using several types of analysis, such as aspect-based analysis, method-based analysis, cluster-based analysis, and others. The sensitivity analysis helps to verify the reliability of results received due to variations in key parameters.

10.1 Aspect-Based Analysis

To investigate different facets of cancer prognosis, the proposed model was scrutinized from multiple perspectives. Regarding similarities, the Pearson coefficient is effective for depicting the degree of association between patients and cancers, whereas the tangent transformation provides additional insights into nonlinearity while maintaining the order of similarity measures. This consistency suggests that the two measures portray the same diagnostic characteristics of cancer patients, albeit from different mathematical angles. With the patient-oriented approach, row-wise normalization facilitates the identification of each individual patient’s dominant cancer type by emphasizing the most significant correlation among the characteristics in each patient’s profile. On the other hand, normalization by columns aids in pinpointing patients who have the most robust connection to each type of cancer. The two methods of normalization contribute to a more comprehensive interpretation of the diagnostic characteristics of both patients and cancers. Visualization results demonstrate that the proposed similarity measure allows for the classification of strong, medium, and weak relationships in all graphs without changing their rankings. Consequently, the proposed system provides a powerful means of analyzing medical data.

10.2 Method-Based Analysis

The developed approach has been validated through the use of various visualization tools, including traditional Heatmaps, normalized Heatmaps, two-dimensional and three-dimensional bar charts, and three-dimensional surface charts. Every tool illustrates the same data from the new perspective while maintaining the consistency of the results. Heatmaps provide an easily interpretable visualization of similarity distributions, and row and column normalization enables a focus on local discriminations by emphasizing the prevailing relationships among patients and cancer types. Bar charts facilitate a direct comparison of Pearson correlation and tangent similarities, whereas surface charts exhibit multidimensional smoothing of the entire dataset.The key point to emphasize is that regardless of the visualization method employed, it consistently identifies Patient P5–Prostate Cancer as the best match and Patient P2–Lung Cancer as one of the worst Pearson correlation pairs. The fact that different visualization methods yield consistent results demonstrates the robustness of the proposed framework and its lack of any visualization bias. Consequently, the approach is characterized by a high degree of interpretability in addition to numerical accuracy.

10.3 Cluster-Based Analysis

The similarity measure's discriminative power was examined based on clustering results from K-Means after dimensionality reduction via PCA and three-dimensional t-SNE analysis. The analysis demonstrated that both methods result in a clear division of patients into two distinct clusters, indicating that the similarity features reflect the natural structure of the dataset. Patients P1, P2, and P5 are grouped together in a cluster with high joint similarity values, whereas patients P3 and P4 are placed in a different cluster with low similarity measures. The centers of the clusters are well-defined and distinct from one another. Moreover, the successful acquisition of clusters through both the linear PCA projection and the nonlinear t-SNE embedding demonstrates that the joint correlation-tangent features possess strong discriminative power, independent of the dimensionality reduction method used. This affirms that the suggested method is appropriate for applications involving intelligent patient stratification and disease classification.

10.4 Other Types of Analysis

Further observations that support the effectiveness of the proposed framework are as follows. To begin with, the sequence of similarity remains constant across all visualization methods, indicating that the tangent transformation preserves the sequence of Pearson correlations.

Secondly, the proposed framework produces visualizations that are highly comprehensible, with diagnostic patterns clearly visible at first glance and without the need for statistical computation. Moreover, the proposed framework encompasses not just quantitative methods for assessing similarity but also intuitive visual representations. This makes it applicable in the field of explainable artificial intelligence within healthcare. The similarity measure proposed is shown to be independent of the visualization method employed, as evidenced by the consistency of results across heat maps, bar charts, surfaces, PCA, and t-SNE. The presented framework successfully integrates statistics, visualization, and machine learning into a single decision support system capable of analyzing uncertain medical data.

10.5 Sensitivity Analysis

To evaluate the stability of the suggested joint correlation-tangent method, the produced similarity patterns were examined with various visualization, normalization, and dimensional reduction techniques. The results demonstrate that the primary diagnostic correlations remain consistent across all experiments, confirming the high stability of the proposed approach. The correlation between Patient P5 and prostate cancer is the most notable in all experiments, irrespective of whether the data is shown as Heatmaps, normalized Heatmaps, bar charts, surface plots, PCA, or t-SNE. Weaker correlations, like those of Patient P2 with Lung Cancer and Patient P4 with Liver Cancer, can be regarded similarly. In addition, the clustering pattern remains consistent across both PCA and t-SNE-based dimensionality reductions, demonstrating that the proposed feature representation is invariant to the choice of dimensionality reduction method. Moreover, normalizing the data both row-wise and column-wise aids in preserving the ranking of the predominant patients-cancer associations while rendering them locally interpretable. This shows robustness regarding the scaling of the data. This confirms that the suggested joint correlation-tangent similarity method is numerically stable, consistent in its clustering outcomes, and offers good visualization performance. Thus, the suggested method is a strong contender for use in cancer prediction and medical decision support systems.

Conclusion: A new model named Pentapartitioned Neutrosophic Soft Topological Space (PNSTS) has been introduced in the work, which enhances expressivity in modeling uncertainty by dividing the indeterminate component of the neutrosophic soft set into multiple categories. A tangent-based similarity measure has been proposed to assess the similarity between two pentapartitioned neutrosophic soft sets, and basic topological notions have been defined. This suggested approach has been applied in the realm of cancer diagnosis. Techniques for visualization, including heat maps, 3D surface charts, PCA, t-SNE, and DBSCAN clustering algorithms, have been employed to illustrate patterns of similarities and patient clusters. The experimental findings demonstrated the effectiveness and robustness of the proposed method, with the highest similarity coefficient of 0.7508 indicating a strong connection between the patient's profile and the selected decision. In summary, the proposed PNSTS algorithm offers a dependable mathematical and visualization-based approach for making intelligent decisions in the medical field, and it can be applied to address other instances of uncertain decision support challenges. Future research will focus on creating weighted and dynamic versions of PNSTS and employing machine learning techniques.

Acknowledgement: The authors also extend their appreciation to the Deanship of the Scientific Research at Northern Border University, Arar, KSA Saudi Arabia, for supporting this work through the project number (NBU-FFR-2026-2230-05).

Data availability: The dataset used in this study is available upon request to the corresponding author.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] L.A. Zadeh, Fuzzy Sets, Inf. Control 8 (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[2] I.B. Turksen, Interval Valued Fuzzy Sets Based on Normal Forms, Fuzzy Sets Syst. 20 (1986), 191-210. https://doi.org/10.1016/0165-0114(86)90077-1.

[3] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 20 (1986), 87-96. https://doi.org/10.1016/S0165-0114(86)80034-3.

[4] K. Atanassov, G. Gargov, Interval Valued Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 31 (1989), 343-349. https://doi.org/10.1016/0165-0114(89)90205-4.

[5] F. Smarandache, A Unifying Field in Logics: Neutrosophy, Neutrosophic Probability, Set and Logic, American Research Press, Rehoboth, 1998. https://doi.org/10.5281/zenodo.49174.

[6] F. Gallego Lupiáñez, On Neutrosophic Topology, Kybernetes 37 (2008), 797-800. https://doi.org/10.1108/03684920810876990.

[7] H. Wang, F. Smarandache, Y. Zhang, R. Sunderraman, Single Valued Neutrosophic Sets, Multispace Multistruct. 4 (2010), 410-413. https://fs.unm.edu/SingleValuedNeutrosophicSets.pdf.

[8] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Interval Neutrosophic Sets and Logic: Theory and Applications in Computing, Hexis, Phoenix, 2005. https://doi.org/10.5281/zenodo.8818.

[9] S. Broumi, F. Smarandache, Cosine Similarity Measure of Interval Valued Neutrosophic Sets, Neutrosophic Sets Syst. 5 (2014), 15-20. https://doi.org/10.6084/M9.FIGSHARE.1502645.

[10] S. Broumi, F. Smarandache, New Distance and Similarity Measures of Interval Neutrosophic Sets, in: 17th International Conference on Information Fusion, Salamanca, Spain, 2014, pp. 249-255. https://digitalrepository.unm.edu/math_fsp/487.

[11] J. Ye, Similarity Measures between Interval Neutrosophic Sets and Their Applications in Multicriteria Decision-Making, J. Intell. Fuzzy Syst. 26 (2014), 165-172. https://doi.org/10.3233/IFS-120724.

[12] S. Broumi, F. Smarandache, Correlation Coefficient of Interval Neutrosophic Set, in: Proceedings of the International Conference ICMERA, Bucharest, Romania, 2013. https://doi.org/10.5281/zenodo.48908.
[13] P. Majumdar, S.K. Samanta, On Similarity and Entropy of Neutrosophic Sets, J. Intell. Fuzzy Syst. 26 (2014), 1245-1252. https://doi.org/10.3233/IFS-130810.
[14] S.-M. Chen, A New Approach to Handling Fuzzy Decision-Making Problems, IEEE Trans. Syst. Man Cybern. 18 (1988), 1012-1016. https://doi.org/10.1109/21.23100.
[15] S.-M. Chen, M.-S. Yeh, P.-Y. Hsiao, A Comparison of Similarity Measures of Fuzzy Values, Fuzzy Sets Syst. 72 (1995), 79-89. https://doi.org/10.1016/0165-0114(94)00284-E.
[16] D. Molodtsov, Soft Set Theory—First Results, Comput. Math. Appl. 37 (1999), 19-31. https://doi.org/10.1016/S0898-1221(99)00056-5.
[17] H. Aktaş, N. Çağman, Soft Sets and Soft Groups, Inf. Sci. 177 (2007), 2726-2735. https://doi.org/10.1016/j.ins.2006.12.008.
[18] K.V. Babitha, J.J. Sunil, Soft Set Relations and Functions, Comput. Math. Appl. 60 (2010), 1840-1849. https://doi.org/10.1016/j.camwa.2010.07.014.
[19] D. Chen, E.C.C. Tsang, D.S. Yeung, Some Notes on the Parameterization Reduction of Soft Sets, in: Proceedings of the 2003 International Conference on Machine Learning and Cybernetics, IEEE, 2003, pp. 1442-1445. https://doi.org/10.1109/ICMLC.2003.1259720.
[20] D. Chen, E.C.C. Tsang, D.S. Yeung, X. Wang, The Parameterization Reduction of Soft Sets and Its Applications, Comput. Math. Appl. 49 (2005), 757-763. https://doi.org/10.1016/j.camwa.2004.10.036.
[21] N. Çağman, S. Enginoğlu, Soft Set Theory and Uni–Int Decision Making, Eur. J. Oper. Res. 207 (2010), 848-855. https://doi.org/10.1016/j.ejor.2010.05.004.
[22] F. Feng, Y.B. Jun, X. Liu, L. Li, An Adjustable Approach to Fuzzy Soft Set Based Decision Making, J. Comput. Appl. Math. 234 (2010), 10-20. https://doi.org/10.1016/j.cam.2009.11.055.
[23] A.R. Roy, P.K. Maji, A Fuzzy Soft Set Theoretic Approach to Decision Making Problems, J. Comput. Appl. Math. 203 (2007), 412-418. https://doi.org/10.1016/j.cam.2006.04.008.
[24] K.N. Singh, J.K. Mantri, A Clinical Decision Support System Using Rough Set Theory and Machine Learning for Disease Prediction, Intell. Med. 4 (2024), 200-208. https://doi.org/10.1016/j.imed.2023.08.002.
[25] B. Ahmad, A. Kharal, On Fuzzy Soft Sets, Adv. Fuzzy Syst. 2009 (2009), 586507. https://doi.org/10.1155/2009/586507.
[26] B.K. Tripathy, R.K. Mohanty, T.R. Sooraj, On Intuitionistic Fuzzy Soft Set and Its Application in Group Decision Making, in: 2016 International Conference on Emerging Trends in Engineering, Technology and Science (ICETETS), IEEE, 2016, pp. 1-5. https://doi.org/10.1109/ICETETS.2016.7603002.
[27] P.K. Maji, R. Biswas, A.R. Roy, Soft Set Theory, Comput. Math. Appl. 45 (2003), 555-562. https://doi.org/10.1016/S0898-1221(03)00016-6.
[28] N. Çağman, I. Deli, Means of FP-Soft Sets and Their Applications, Hacet. J. Math. Stat. 41 (2012), 615-625. https://dergipark.org.tr/tr/download/article-file/86337.
[29] N. Çağman, I. Deli, Products of FP-Soft Sets and Their Applications, Hacet. J. Math. Stat. 41 (2012), 365-374. https://zbmath.org/6242318.
[30] I. Deli, Interval-Valued Neutrosophic Soft Sets and Its Decision Making, arXiv:1402.3130, 2014. https://doi.org/10.48550/arXiv.1402.3130.

[31] P. Majumdar, S.K. Samanta, Similarity Measure of Soft Sets, New Math. Nat. Comput. 4 (2008), 1-12. https://doi.org/10.1142/S1793005708000908.

[32] A. Kharal, Distance and Similarity Measures for Soft Sets, New Math. Nat. Comput. 6 (2010), 321-334. https://doi.org/10.1142/S1793005710001724.

[33] P. Grzegorzewski, Distances between Intuitionistic Fuzzy Sets and/or Interval-Valued Fuzzy Sets Based on the Hausdorff Metric, Fuzzy Sets Syst. 148 (2004), 319-328. https://doi.org/10.1016/j.fss.2003.08.005.

[34] M. Arora, R. Biswas, U.S. Pandey, Neutrosophic Relational Database Decomposition, Int. J. Adv. Comput. Sci. Appl. 2 (2011), 121-125. https://doi.org/10.14569/IJACSA.2011.020822.

[35] M. Arora, R. Biswas, Deployment of Neutrosophic Technology to Retrieve Answer for Queries Posed in Natural Language, in: 2010 3rd International Conference on Computer Science and Information Technology, IEEE, 2010, pp. 435-439. https://doi.org/10.1109/ICCSIT.2010.5564125.

[36] S. Aggarwal, R. Biswas, A.Q. Ansari, Neutrosophic Modeling and Control, in: 2010 International Conference on Computer and Communication Technology (ICCCT), IEEE, 2010, pp. 718-723. https://doi.org/10.1109/ICCCT.2010.5640435.

[37] H.D. Cheng, Y. Guo, A New Neutrosophic Approach to Image Thresholding, New Math. Nat. Comput. 4 (2008), 291-308. https://doi.org/10.1142/S1793005708001082.

[38] Y. Guo, H.D. Cheng, New Neutrosophic Approach to Image Segmentation, Pattern Recognit. 42 (2009), 587-595. https://doi.org/10.1016/j.patcog.2008.10.002.

[39] M. Zhang, L. Zhang, H.D. Cheng, A Neutrosophic Approach to Image Segmentation Based on Watershed Method, Signal Process. 90 (2010), 1510-1517. https://doi.org/10.1016/j.sigpro.2009.10.021.

[40] A.Q. Ansari, R. Biswas, S. Aggarwal, Proposal for Applicability of Neutrosophic Set Theory in Medical AI, Int. J. Comput. Appl. 27 (2011), 5-11. https://doi.org/10.5120/3299-4505.

[41] M.I. Ali, F. Feng, X. Liu, W.K. Min, M. Shabir, On Some New Operations in Soft Set Theory, Comput. Math. Appl. 57 (2009), 1547-1553. https://doi.org/10.1016/j.camwa.2008.11.009.

[42] M.I. Ali, A Note on Soft Sets, Rough Soft Sets and Fuzzy Soft Sets, Appl. Soft Comput. 11 (2011), 3329-3332. https://doi.org/10.1016/j.asoc.2011.01.003.

[43] U. Acar, F. Koyuncu, B. Tanay, Soft Sets and Soft Rings, Comput. Math. Appl. 59 (2010), 3458-3463. https://doi.org/10.1016/j.camwa.2010.03.034.

[44] Y. Jiang, Y. Tang, Q. Chen, H. Liu, J. Tang, Interval-Valued Intuitionistic Fuzzy Soft Sets and Their Properties, Comput. Math. Appl. 60 (2010), 906-918. https://doi.org/10.1016/j.camwa.2010.05.036.

[45] N. Çağman, I. Deli, Similarity Measures of Intuitionistic Fuzzy Soft Sets and Their Decision Making, arXiv:1301.0456, 2013. https://doi.org/10.48550/arXiv.1301.0456.

[46] S. Broumi, Generalized Neutrosophic Soft Set, Int. J. Comput. Sci. Eng. Inf. Technol. 3 (2013), 17-30. https://doi.org/10.5121/ijcseit.2013.3202.

[47] S. Broumi, F. Smarandache, Several Similarity Measures of Neutrosophic Sets, Neutrosophic Sets Syst. 1 (2013), 54-62. https://digitalrepository.unm.edu/nss_journal/vol1/iss1/10.

[48] T. Bera, N.K. Mahapatra, Introduction to Neutrosophic Soft Topological Space, OPSEARCH 54 (2017), 841-867. https://doi.org/10.1007/s12597-017-0308-7.

[49] P.K. Maji, Neutrosophic Soft Set, Ann. Fuzzy Math. Inform. 5 (2013), 157-168.

[50] I. Deli, S. Broumi, Neutrosophic Soft Relations and Some Properties, Ann. Fuzzy Math. Inform. 9 (2015), 169-182. https://doi.org/10.5281/zenodo.23153.

[51] F. Smarandache, Neutrosophic Set – a Generalization of the Intuitionistic Fuzzy Set, in: 2006 IEEE International Conference on Granular Computing, IEEE, 2006, pp. 38-42. https://doi.org/10.1109/GRC.2006.1635754.

[52] C.D. Manning, P. Raghavan, H. Schütze, Introduction to Information Retrieval, Cambridge University Press, 2008. https://doi.org/10.1017/CBO9780511809071.

[53] L. Wilkinson, The Grammar of Graphics, Springer-Verlag, 2005. https://doi.org/10.1007/0-387-28695-0.

[54] J. Han, M. Kamber, J. Pei, Data Mining: Concepts and Techniques, 3rd ed., Elsevier, 2011. https://doi.org/10.1016/C2009-0-61819-5.

[55] E.R. Tufte, The Visual Display of Quantitative Information, Graphics Press, 2001.

[56] K. Pearson, On Lines and Planes of Closest Fit to Systems of Points in Space, Philos. Mag. J. Sci. 2 (1901), 559-572. https://doi.org/10.1080/14786440109462720.

[57] C. de Boor, A Practical Guide to Splines, Springer New York, 1978. https://doi.org/10.1007/978-1-4612-6333-3.

[58] MathWorks, MATLAB Documentation, The MathWorks Inc., Natick, 2026. https://www.mathworks.com/help/matlab/.

[59] S. Lloyd, Least Squares Quantization in PCM, IEEE Trans. Inf. Theory 28 (1982), 129-137. https://doi.org/10.1109/TIT.1982.1056489.

[60] R.A. Fisher, The Use of Multiple Measurements in Taxonomic Problems, Ann. Eugen. 7 (1936), 179-188. https://doi.org/10.1111/j.1469-1809.1936.tb02137.x.

[61] M.M. Saeed, R. Hatamleh, A.M. AbdEl-latif, A. Al-Husban, H.M. Attaalfadeel, T. Fujita, K.A. Aldwoah, A.M. Khattak, A Breakthrough Approach to Quadri-Partitioned Neutrosophic Soft Topological Spaces, Eur. J. Pure Appl. Math. 18 (2025), 5845. https://doi.org/10.29020/nybg.ejpam.v18i2.5845.

[62] A.A. Abubaker, R. Hatamleh, K. Matarneh, A. Al-Husban, On the Numerical Solutions for Some Neutrosophic Singular Boundary Value Problems by Using (LPM) Polynomials, Int. J. Neutrosophic Sci. 25 (2025), 197-205. https://doi.org/10.54216/IJNS.250217.

[63] A.A. Abubaker, R. Hatamleh, K. Matarneh, A. Al-Husban, On the Irreversible k-Threshold Conversion Number for Some Graph Products and Neutrosophic Graphs, Int. J. Neutrosophic Sci. 25 (2025), 183-196. https://doi.org/10.54216/IJNS.250216.

[64] R. Hatamleh, A. Al-Husban, K. Sundareswari, G. Balaj, M. Palanikumar, Complex Tangent Trigonometric Approach Applied to (γ, τ)-Rung Fuzzy Set Using Weighted Averaging, Geometric Operators and Its Extension, Commun. Appl. Nonlinear Anal. 32 (2025), 133-144. https://doi.org/10.52783/cana.v32.2978.

[65] R. Hatamleh, A. Hazaymeh, On Some Topological Spaces Based on Symbolic n-Plithogenic Intervals, Int. J. Neutrosophic Sci. 25 (2025), 23-37. https://doi.org/10.54216/IJNS.250102.

[66] H. Qawaqneh, Fractional Analytic Solutions and Fixed Point Results with Some Applications, Adv. Fixed Point Theory 14 (2024), 1. https://doi.org/10.28919/afpt/8279.

[67] H. Qawaqneh, M.S.M. Noorani, H. Aydi, A. Zraiqat, A.H. Ansari, On Fixed Point Results in Partial b-

Metric Spaces, J. Funct. Spaces 2021 (2021), 1-9. https://doi.org/10.1155/2021/8769190.

[68] H. Qawaqneh, M.S.M. Noorani, H. Aydi, Some New Characterizations and Results for Fuzzy Contractions in Fuzzy b-Metric Spaces and Applications, AIMS Math. 8 (2023), 6682-6696. https://doi.org/10.3934/math.2023338.

[69] H. Qawaqneh, J. Manafian, M. Alharthi, Y. Alrashedi, Stability Analysis, Modulation Instability, and Beta-Time Fractional Exact Soliton Solutions to the van der Waals Equation, Mathematics 12 (2024), 2257. https://doi.org/10.3390/math12142257.

[70] R. Hatamleh, On the Form of Correlation Function for a Class of Nonstationary Field with a Zero Spectrum, Rocky Mt. J. Math. 33 (2003), 159-173. https://doi.org/10.1216/rmjm/1181069991.

[71] R. Hatamleh, V.A. Zolotarev, On Two-Dimensional Model Representations of One Class of Commuting Operators, Ukr. Math. J. 66 (2014), 122-144. https://doi.org/10.1007/s11253-014-0916-9.

[72] R. Hatamleh, V.A. Zolotarev, On Model Representations of Non-Selfadjoint Operators with Infinitely Dimensional Imaginary Component, Zh. Mat. Fiz. Anal. Geom. 11 (2015), 174-186. https://doi.org/10.15407/mag11.02.174.

[73] R. Hatamleh, V.A. Zolotarev, Triangular Models of Commutative Systems of Linear Operators Close to Unitary Operators, Ukr. Math. J. 68 (2016), 791-811. https://doi.org/10.1007/s11253-016-1258-6.

[74] A.S. Heilat, H. Zureigat, R. Hatamleh, B. Batiha, New Spline Method for Solving Linear Two-Point Boundary Value Problems, Eur. J. Pure Appl. Math. 14 (2021), 1283-1294. https://doi.org/10.29020/nybg.ejpam.v14i4.4124.

[75] A. Qazza, I. Bendib, R. Hatamleh, R. Saadeh, A. Ouannas, Dynamics of the Gierer–Meinhardt Reaction–Diffusion System: Insights into Finite-Time Stability and Control Strategies, Partial Differ. Equ. Appl. Math. 14 (2025), 101142. https://doi.org/10.1016/j.padiff.2025.101142.

[76] T. Qawasmeh, A. Qazza, R. Hatamleh, M.W. Alomari, R. Saadeh, Further Accurate Numerical Radius Inequalities, Axioms 12 (2023), 801. https://doi.org/10.3390/axioms12080801.

[77] A. Shihadeh, K.A.M. Matarneh, R. Hatamleh, M.O. Al-Qadri, A. Al-Husban, On the Two-Fold Fuzzy n-Refined Neutrosophic Rings for $2 \leq n \leq 3$, Neutrosophic Sets Syst. 68 (2024), 8-25. https://doi.org/10.5281/zenodo.11406449.

[78] T.Y. Ozturk, C.G. Aras, S. Bayramov, A New Approach to Operations on Neutrosophic Soft Sets and to Neutrosophic Soft Topological Spaces, Commun. Math. Appl. 10 (2019), 481-493. https://doi.org/10.26713/cma.v10i3.1068.