Int. J. Anal. Appl. (2026), 24:156

International Journal of Analysis and Applications

# Tripolar Fuzzy $(m,n)$-Quasi-Ideals of Ordered Semigroups

**Anothai Phukhaengsi**[1], **Pannawit Khamrot**[2], **Thiti Gaketem**[1,*]

[1]*Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

[2]*Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University Technology Lanna Phitsanulok, Phitsanulok, Thailand*

[*]*Corresponding author:* thiti.ga@up.ac.th

**Abstract.** In 2009, M. A. Ansari et al. introduced the concept of $(m,n)$-quasi-ideals in semigroup theory. Nearly a decade later, in 2018, Rao proposed the framework of tripolar fuzzy sets, which generalizes the classical fuzzy, bipolar fuzzy, and intuitionistic fuzzy set theories. Building upon these developments, this paper introduces and studies the notion of tripolar fuzzy $(m,n)$-quasi-ideals in semigroups. We explore their fundamental characteristics, analyze their connections with conventional $(m,n)$-quasi-ideals, and establish relationships between these two structures. In addition, we investigate the notions of minimality, primeness, and semiprimeness within this newly defined framework.

## 1. Introduction

The study of ordered semigroups emerged as a natural generalization of classical semigroup theory in the early twentieth century, around 1905. Within this framework, the notion of $(m,n)$-ideals in ordered semigroups was introduced by T. Changphas [3], inspired by the earlier developments of $(m,n)$-ideals in ordinary semigroups. Subsequently, in 2009, M. A. Ansari et al. [1] extended these ideas by defining the concept of $(m,n)$-quasi-ideals in semigroup theory.

The theory of fuzzy sets, proposed by L. A. Zadeh in 1965 [24], provides a powerful mathematical approach for modeling vagueness and uncertainty in real-world phenomena. Since its inception, fuzzy set theory has found wide-ranging applications in areas such as medical diagnosis, robotics, computer science, information systems, control engineering, logic, and topology. Later, in 1986, K. T. Atanassov [2] extended Zadeh's framework by introducing *intuitionistic fuzzy sets*, which capture both membership and non-membership degrees within the unit interval $[0,1]$.

---

Received: Feb. 23, 2026.

2020 *Mathematics Subject Classification.* 20M10, 20N99.

*Key words and phrases.* tripoar fuzzy ideals; tripolar fuzzy $(m,n)$-quasi-ideals; semigroups; ordered semigroups.

https://doi.org/10.28924/2291-8639-24-2026-156

ISSN: 2291-8639

In the context of algebraic structures, N. Kuroki [8] initiated the study of fuzzy subsemigroups and fuzzy ideals, laying a foundation for subsequent advancements. W. Zhang [25] later introduced *bipolar fuzzy sets*, in which the degrees of membership range over the interval $[-1, 1]$, thereby accommodating both positive and negative information. Building upon these concepts, Kang [6] extended bipolar fuzzy theory to semigroups. Following this, several researchers contributed to the development of fuzzy and bipolar fuzzy ideals: Chinnadurai and Arulmozhi [4] investigated bipolar fuzzy ideals in ordered semigroups; P. Khamrot and M. Siripitukdet [7] examined generalized bipolar fuzzy subsemigroups; and T. Gaketem and P. Khamrot [5] explored bipolar fuzzy weakly interior ideals.

To further enrich these theoretical frameworks, Rao [19, 20] proposed the concept of tripolar fuzzy sets, which unify the notions of fuzzy, bipolar fuzzy, and intuitionistic fuzzy sets by representing information through three distinct components—positive, neutral, and negative. This idea has inspired extensive research [9,10,21–23]. Notably, Wattansiripong *et al.* [21] studied tripolar fuzzy pure ideals in ordered semigroups and subsequently characterized semisimple ordered semigroups using tripolar fuzzy interior ideals [23]. More recently, T. Promai et al. [11, 12] have investigated tripolar fuzzy ideals in semigroups, while P. Khamrot et al. [9] provided classifications and structural properties of such ideals. T. Gaketem and P. Khamrot [13] introduced the notion of interval-valued fuzzy $(m, n)$-ideals in semigroups. In the same period, A. Mahboob et al. [17, 18] developed the concept of fuzzy $(m, n)$-ideals and $(m, n)$-quasi-ideals in ordered semigroups. Further studies have addressed related topics, including fuzzy $(m, n)$-ideals [14], bipolar fuzzy $(m, m)$-ideals [15], and fuzzy ordered almost $(m, n)$-ideals and $(m, n)$-quasi-ideals [9, 16].

The primary objective of this paper is to extend the theory of tripolar fuzzy $(m, n)$-quasi-ideals from semigroups to the broader setting of ordered semigroups. We establish and prove several fundamental properties of tripolar fuzzy $(m, n)$-quasi-ideals within this context. Furthermore, we explore their essential characteristics, examine their relationships with classical $(m, n)$-quasi-ideals, and discuss the connections between these two theoretical frameworks. Finally, we investigate notions of minimality, primeness, and semiprimeness under the tripolar fuzzy setting.

## 2. PRELIMINARIES

In this section, we recall and extend certain fundamental concepts that will be employed in the subsequent developments.

Let $(\check{\mathcal{E}}, \cdot)$ be a semigroup and $(\check{\mathcal{E}}, \leq)$ a partially ordered set. We say that $(\check{\mathcal{E}}, \cdot, \leq)$ is an *ordered semigroup* provided that, for all $\check{\mathfrak{a}}, \check{\mathfrak{b}}, \check{\mathfrak{c}} \in \check{\mathcal{E}}$,

$$\check{\mathfrak{a}} \leq \check{\mathfrak{b}} \implies \check{\mathfrak{a}}\check{\mathfrak{c}} \leq \check{\mathfrak{b}}\check{\mathfrak{c}} \quad \text{and} \quad \check{\mathfrak{c}}\check{\mathfrak{a}} \leq \check{\mathfrak{c}}\check{\mathfrak{b}}.$$

For nonempty subsets $\check{\mathcal{X}}, \check{\mathcal{Y}} \subseteq \check{\mathcal{E}}$, we denote

$$(\check{\mathcal{X}}] := \{\check{\mathfrak{s}} \in \check{\mathcal{E}} \mid \check{\mathfrak{s}} \leq \check{\mathfrak{x}} \text{ for some } \check{\mathfrak{x}} \in \check{\mathcal{X}}, \},$$

and

$$\check{\mathcal{X}}\check{\mathcal{Y}} := \{\check{\mathfrak{x}}\check{\mathfrak{y}} \mid \check{\mathfrak{x}} \in \check{\mathcal{X}}, \check{\mathfrak{y}} \in \check{\mathcal{Y}}, \}.$$

The following facts are immediate:

(i) $\check{\mathcal{X}} \subseteq (\check{\mathcal{X}}]$,
(ii) if $\check{\mathcal{X}} \subseteq \check{\mathcal{Y}}$ then $(\check{\mathcal{X}}] \subseteq (\check{\mathcal{Y}}]$,
(iii) $((\check{\mathcal{X}}] = (\check{\mathcal{X}}]$,
(iv) $(\check{\mathcal{X}}](\check{\mathcal{Y}}] \subseteq (\mathcal{X}\mathcal{Y}]$,
(v) $((\check{\mathcal{X}}](\check{\mathcal{Y}}]] = (\mathcal{X}\mathcal{Y}]$,
(vi) $(\check{\mathcal{X}} \cup \check{\mathcal{Y}}] \subseteq (\check{\mathcal{X}}] \cup (\check{\mathcal{Y}}]$,
(vii) $(\check{\mathcal{X}} \cap \check{\mathcal{Y}}] \subseteq (\check{\mathcal{X}}] \cap (\check{\mathcal{Y}}]$.

A nonempty set $\check{\mathcal{K}} \subseteq \check{\mathcal{E}}$ is called a *subsemigroup* (SSG) if $\check{\mathcal{K}}^2 \subseteq \check{\mathcal{K}}$. A subset $\check{\mathcal{K}} \subseteq \check{\mathcal{E}}$ is a *left ideal* if $\check{\mathcal{E}}\check{\mathcal{K}} \subseteq \check{\mathcal{K}}$ and $(\check{\mathcal{K}}] = \check{\mathcal{K}}$. Similarly, a *right ideal* satisfies $\check{\mathcal{K}}\check{\mathcal{E}} \subseteq \check{\mathcal{K}}$ and $(\check{\mathcal{K}}] = \check{\mathcal{K}}$. An *ideal* is both a left and right ideal. A nonempty set $\check{\mathcal{K}} \subseteq \check{\mathcal{E}}$ is called a *quasi-idesl* (QID) if $\check{\mathcal{E}}\check{\mathcal{K}} \cap \check{\mathcal{K}}\check{\mathcal{E}} \subseteq \check{\mathcal{K}}$.

**Definition 2.1.** *Let $m, n \in \mathbb{N} \cup \{0\}$. A non-empty set $\check{\mathcal{K}}$ of an ordered semigroup $(\check{\mathcal{E}}, \cdot, \leq)$ is called an* $(m,n)$-quas-ideal *($(m,n)$-QID) if*

*(1) $\check{\mathcal{E}}^m\check{\mathcal{K}} \cap \check{\mathcal{E}}^n\check{\mathcal{K}} \subseteq \check{\mathcal{K}}$,*
*(2) $\check{\mathcal{K}} = (\check{\mathcal{K}}]$.*

A *fuzzy set* (FS) on a nonempty set $\check{\mathcal{T}}$ is a mapping $\check{\delta} : \check{\mathcal{T}} \to [0,1]$.

For any two FSs $\check{\delta}$ and $\check{\nu}$ of a non-empty set $\check{\mathcal{T}}$, define $\geq, =, \wedge$ and $\vee$ as follows:

(1) $\check{\delta} \geq \check{\nu} \Leftrightarrow \check{\delta}(\mathfrak{e}) \geq \check{\nu}(\mathfrak{e})$ for all $\mathfrak{e} \in \check{\mathcal{T}}$,
(2) $\check{\delta} = \check{\nu} \Leftrightarrow \check{\delta} \geq (\check{\nu})$ and $\check{\nu} \geq \check{\delta}$,
(3) $(\check{\delta} \wedge \check{\nu})(\mathfrak{e}) = \min\{\check{\delta}(\mathfrak{e}), \check{\nu}(\mathfrak{e})\} = \check{\delta}(\mathfrak{e}) \wedge \check{\nu}(\mathfrak{e})$ for all $\mathfrak{e} \in \check{\mathcal{T}}$,
(4) $(\check{\delta} \vee \check{\nu})(\mathfrak{e}) = \max\{\check{\delta}(\mathfrak{e}), \check{\nu}(\mathfrak{e})\} = \check{\delta}(\mathfrak{e}) \vee \check{\nu}(\mathfrak{e})$ for all $\mathfrak{e} \in \check{\mathcal{T}}$.

For the symbol $\check{\delta} \leq \check{\nu}$, we mean $\check{\nu} \geq \check{\delta}$.

**Definition 2.2.** *A FS $\check{\delta}$ of an ordered semigroup $\check{\mathcal{E}}$ is a* fuzzy subsemigroup *(FSSG) if*

(1) $\check{\delta}(\check{\mathfrak{u}}\check{\mathfrak{v}}) \geq \check{\delta}(\check{\mathfrak{u}}) \wedge \check{\delta}(\check{\mathfrak{v}})$, *for all* $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$.
(2) $\check{\mathfrak{u}} \leq \check{\mathfrak{v}}$ *implies* $\check{\delta}(\check{\mathfrak{u}}) \geq \check{\delta}(\check{\mathfrak{v}})$ *for all* $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$.

**Definition 2.3.** *A FS $\check{\delta}$ of an ordered semigroup $\check{\mathcal{E}}$ is a* fuzzy ideal *(FI) if*

(1) $\check{\delta}(\check{\mathfrak{u}}\check{\mathfrak{v}}) \geq \check{\delta}(\check{\mathfrak{u}}) \vee \check{\delta}(\check{\mathfrak{v}})$, *for all* $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$.
(2) $\check{\mathfrak{u}} \leq \check{\mathfrak{v}}$ *implies* $\check{\delta}(\check{\mathfrak{u}}) \geq \check{\delta}(\check{\mathfrak{v}})$ *for all* $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$.

For $\check{\mathcal{K}} \subseteq \check{\mathcal{E}}$, the *characteristic function* $\lambda_{\check{\mathcal{K}}}$ is defined by

$$\lambda_{\check{\mathcal{K}}}(x) = \begin{cases} 1 & x \in \check{\mathcal{K}}, \\ 0 & x \notin \check{\mathcal{K}} \end{cases}$$

For $\check{\rho}$ and $\check{\nu}$ be a FSs. Defined the product $\check{\rho} \circ \check{\nu}$ of an SG $\check{\mathcal{E}}$ as follows:

$$(\check{\rho} \circ \check{\nu})(\check{\mathfrak{u}}) = \begin{cases} \bigvee\limits_{(\check{\mathfrak{r}},\check{\mathfrak{b}}) \in \check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\rho}(\check{\mathfrak{r}}) \wedge \check{\nu}(\check{\mathfrak{b}})\} & \text{if} \quad \check{\mathcal{F}}_{\check{\mathfrak{u}}} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 2.4.** *A FS $\check{\delta}$ of an ordered semigroup $\check{\mathcal{E}}$ is called a* fuzzy $(m,n)$-quasi-ideal *of $\check{\mathcal{E}}$ if*

(1) $(\check{\delta}^m \circ \lambda_{\check{\mathcal{E}}}) \wedge (\lambda_{\check{\mathcal{E}}} \circ \check{\delta}^n) \subseteq \check{\delta}$.

(2) $\check{\mathfrak{u}} \leq \check{\mathfrak{v}}$ *implies* $\check{\delta}(\check{\mathfrak{u}}) \geq \check{\delta}(\check{\mathfrak{v}})$ *for all* $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$.

**Lemma 2.1.** *A subset $\check{\mathcal{K}}$ is an SSG of an ordered semigroup $\check{\mathcal{E}}$ iff $\lambda_{\check{\mathcal{K}}}$ is a FSSG of $\check{\mathcal{E}}$.*

**Theorem 2.1.** *A subset $\check{\mathcal{K}}$ is an $(m,n)$-ideal of an ordered semigroup $\check{\mathcal{E}}$ iff $\lambda_{\check{\mathcal{K}}}$ is a fuzzy $(m,n)$-ideal of $\check{\mathcal{E}}$.*

**Definition 2.5.** *A* tripolar fuzzy set *(TFS) on a nonempty set $\check{\mathcal{E}}$ is given by*

$$\mathcal{TF} = \{(\mathfrak{u}, \check{\rho}(\mathfrak{u}), \check{\nu}(\mathfrak{u}), \check{\delta}(\mathfrak{u})) \mid \mathfrak{u} \in \check{\mathcal{E}}\},$$

*where $\check{\rho}, \check{\nu} : \check{\mathcal{E}} \to [0,1]$, $\check{\delta} : \check{\mathcal{E}} \to [-1,0]$, subject to*

$$0 \leq \check{\rho}(\mathfrak{u}) + \check{\nu}(\mathfrak{u}) \leq 1, \quad \forall \mathfrak{u} \in \check{\mathcal{E}}.$$

*Here $\check{\rho}$ represents membership, $\check{\nu}$ represents indeterminacy (or irrelevance), and $\check{\delta}$ encodes a counter-property.*

**Definition 2.6.** *A TFS $(\check{\rho}, \check{\nu}, \check{\delta})$ is a* TF subsemigroup *(TFSSG) if*

(1) $\check{\rho}(\check{\mathfrak{u}}\check{\mathfrak{v}}) \geq \check{\rho}(\check{\mathfrak{u}}) \wedge \check{\rho}(\check{\mathfrak{v}})$,

(2) $\check{\nu}(\check{\mathfrak{u}}\check{\mathfrak{v}}) \leq \check{\nu}(\check{\mathfrak{u}}) \vee \check{\nu}(\check{\mathfrak{v}})$,

(3) $\check{\delta}(\check{\mathfrak{u}}\check{\mathfrak{v}}) \leq \check{\delta}(\check{\mathfrak{u}}) \vee \check{\delta}(\check{\mathfrak{v}})$.

*for all* $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$.

**Definition 2.7.** *A TFS $(\check{\rho}, \check{\nu}, \check{\delta})$ is a* TF left (resp. right) ideal *(TFLID, [TFRID]) if*

(1) $\check{\rho}(\check{\mathfrak{u}}\check{\mathfrak{v}}) \geq \check{\rho}(\check{\mathfrak{v}})$ (*resp.* $\check{\rho}(\check{\mathfrak{u}}\check{\mathfrak{v}}) \geq \check{\rho}(\check{\mathfrak{u}})$),

(2) $\check{\nu}(\check{\mathfrak{u}}\check{\mathfrak{v}}) \leq \check{\nu}(\check{\mathfrak{v}})$ (*resp.* $\check{\nu}(\check{\mathfrak{u}}\check{\mathfrak{v}}) \leq \check{\nu}(\check{\mathfrak{u}})$),

(3) $\check{\delta}(\check{\mathfrak{u}}\check{\mathfrak{v}}) \leq \check{\delta}(\check{\mathfrak{v}})$ (*resp.* $\check{\delta}(\check{\mathfrak{u}}\check{\mathfrak{v}}) \leq \check{\delta}(\check{\mathfrak{u}})$).

(4) $\check{\mathfrak{u}} \leq \check{\mathfrak{v}}$ *implies* $\check{\rho}(\check{\mathfrak{u}}) \geq \check{\rho}(\check{\mathfrak{v}})$, $\check{\nu}(\check{\mathfrak{u}}) \leq \check{\nu}(\check{\mathfrak{v}})$, $\check{\delta}(\check{\mathfrak{u}}) \leq \check{\delta}(\check{\mathfrak{v}})$ *for all* $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$.

**Definition 2.8.** *A TFS is a* TF ideal *(TFID) if it is both a left and right ideal.*

**Definition 2.9.** *A TF subsemigroup $(\check{\rho}, \check{\nu}, \check{\delta})$ is a* TF bi-ideal *(TFBID) if*

(1) $\check{\rho}(\check{\mathfrak{u}}\check{\mathfrak{a}}\check{\mathfrak{v}}) \geq \check{\rho}(\check{\mathfrak{u}}) \wedge \check{\rho}(\check{\mathfrak{v}})$,

(2) $\check{\nu}(\check{\mathfrak{u}}\check{\mathfrak{a}}\check{\mathfrak{v}}) \leq \check{\nu}(\check{\mathfrak{u}}) \vee \check{\nu}(\check{\mathfrak{v}})$,

(3) $\check{\delta}(\check{\mathfrak{u}}\check{\mathfrak{a}}\check{\mathfrak{v}}) \leq \check{\delta}(\check{\mathfrak{u}}) \vee \check{\delta}(\check{\mathfrak{v}})$.

(4) $\check{\mathfrak{u}} \leq \check{\mathfrak{v}}$ *implies* $\check{\rho}(\check{\mathfrak{u}}) \geq \check{\rho}(\check{\mathfrak{v}})$, $\check{\nu}(\check{\mathfrak{u}}) \leq \check{\nu}(\check{\mathfrak{v}})$, $\check{\delta}(\check{\mathfrak{u}}) \leq \check{\delta}(\check{\mathfrak{v}})$.

*for all* $\check{\mathfrak{u}}, \check{\mathfrak{a}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$.

The *characteristic TFS* (CTFS) $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ of a non-empty subset $\check{\check{\mathcal{K}}}$ of $\check{\mathcal{E}}$ is defined as follows:

$$\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) = \begin{cases} 1 & \text{if } \check{\mathfrak{u}} \in \check{\mathcal{K}}, \\ 0 & \text{if } \check{\mathfrak{u}} \notin \check{\mathcal{K}}, \end{cases}$$

$$\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) = \begin{cases} 0 & \text{if } \check{\mathfrak{u}} \in \check{\mathcal{K}}, \\ 1 & \text{if } \check{\mathfrak{u}} \notin \check{\mathcal{K}}, \end{cases}$$

$$\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) = \begin{cases} -1 & \text{if } \check{\mathfrak{u}} \in \check{\mathcal{K}}, \\ 0 & \text{if } \check{\mathfrak{u}} \notin \check{\mathcal{K}} \end{cases}$$

for all $\check{\mathfrak{u}} \in \check{\mathcal{E}}$. In this case of $\check{\mathcal{K}} = \check{\mathcal{E}}$ defined $\mathcal{TF}_{\check{\mathcal{K}}} = (1, 0, -1)$.

**Theorem 2.2.** *Let $\check{\mathcal{I}} \subseteq \check{\mathcal{E}}$. Then $\check{\mathcal{K}}$ is an SSG (resp. left ideal, right ideal, ideal) iff its CTFS $\mathcal{TF}_{\check{\mathcal{K}}} = (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a TFSSG (resp. TFLID, TFRID, TFID).*

The *support* of $\mathcal{TF} := (\check{\rho}, \check{\nu}, \check{\delta})$ TFS instead of $\operatorname{supp}(\mathcal{TF}) = \{\check{\mathfrak{u}} \in \check{\check{\mathcal{E}}} \mid \check{\rho}(\check{\mathfrak{u}}) \neq 0, \text{ or } \check{\nu}(\check{\mathfrak{u}}) \neq 0 \text{ or } \check{\delta}(\check{\mathfrak{u}}) \neq 0\}$.

**Example 2.1.** *Let $\check{\mathcal{K}} = \{\check{\mathfrak{u}}, \check{\mathfrak{v}}, \check{\mathfrak{z}}, \check{\mathfrak{w}}\}$. Define $\mathcal{TF} = \{(\check{\mathfrak{u}}, 0.4, 0.5, -0.1), (\check{\mathfrak{v}}, 0, 0, 0), (\check{\mathfrak{z}}, 0, 0.5, -0.3), (\check{\mathfrak{e}}, 0.4, 0, -0.5)\}$. Then $\operatorname{supp}(\mathcal{TF}) = \{\check{\mathfrak{u}}, \check{\mathfrak{z}}, \check{\mathfrak{w}}\}$.*

**Theorem 2.3.** *[11] Let $\check{\rho}$, $\check{\nu}$ and $\check{\delta}$ be nonzero fuzzy sets of an SG $\check{\mathcal{E}}$. Then $\mathcal{TF} := (\check{\rho}, \check{\nu}, \check{\delta})$ is a TFSSG of $\check{\mathcal{E}}$ if and only if $\operatorname{supp}(\mathcal{TF})$ is an SG of $\check{\mathcal{E}}$.*

For $\mathcal{TF}_1 = (\check{\rho}, \check{\nu}, \check{\delta})$ and $\mathcal{TF}_2 = (\check{\lambda}, \check{\mu}, \check{\omega})$ be a TFSs. Defined the product $\mathcal{TF}_1 \circ \mathcal{TF}_2$ of an SG $\check{\mathcal{E}}$ as follows:

$$(\check{\rho} \circ \check{\lambda})(\check{\mathfrak{u}}) = \begin{cases} \bigvee_{(\check{\mathfrak{r}}, \check{\mathfrak{b}}) \in \check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\rho}(\check{\mathfrak{r}}) \wedge \check{\lambda}(\check{\mathfrak{b}})\} & \text{if } \check{\mathcal{F}}_{\check{\mathfrak{u}}} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\check{\nu} \circ \check{\mu})(\check{\mathfrak{u}}) = \begin{cases} \bigwedge_{(\check{\mathfrak{r}}, \check{\mathfrak{b}}) \in \check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\nu}(\check{\mathfrak{r}}) \vee \check{\mu}(\check{\mathfrak{b}})\} & \text{if } \check{\mathcal{F}}_{\check{\mathfrak{u}}} \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

$$(\check{\delta} \circ \check{\omega})(\check{\mathfrak{u}}) = \begin{cases} \bigwedge_{(\check{\mathfrak{r}}, \check{\mathfrak{b}}) \in \check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\delta}(\check{\mathfrak{r}}) \vee \check{\omega}(\check{\mathfrak{b}})\} & \text{if } \check{\mathcal{F}}_{\check{\mathfrak{u}}} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

for all $\check{\mathfrak{u}} \in \check{\mathcal{E}}$. For $\check{\mathfrak{u}} \in \check{\mathcal{E}}$, define $F_{\hat{\mathfrak{z}}} = \{(\check{\mathfrak{r}}, \check{\mathfrak{b}}) \in \check{\mathcal{E}} \times \check{\mathcal{E}} \mid \check{\mathfrak{u}} \leq \check{\mathfrak{r}}\check{\mathfrak{b}}\}$. It is easy to verify that the structure $(\mathcal{TF}_1, \circ)$ is an SG. In the set of all TFSs of $\check{\mathcal{E}}$ we define the order relation as follows: $\mathcal{TF}_1 \sqsubseteq \mathcal{TF}_2$ if and only if $\check{\rho}(\check{\mathfrak{u}}) \leq \check{\lambda}(\check{\mathfrak{u}})$, $\check{\nu}(\check{\mathfrak{u}}) \geq \check{\mu}(\check{\mathfrak{u}})$ and $\check{\delta}(\check{\mathfrak{u}}) \geq \check{\omega}(\check{\mathfrak{u}})$ for all $h \in \check{\mathcal{E}}$. Finally, we define a binary operation $\sqcap$ on $\mathcal{TF}$ as follows:

$$\mathcal{TF}_1 \sqcap \mathcal{TF}_2 := (\check{\rho} \wedge \check{\lambda}, \check{\nu} \vee \check{\mu}, \check{\delta} \vee \check{\omega}),$$

where $(\check{\rho} \wedge \check{\lambda})(\check{\mathfrak{u}}) := \check{\rho}(\check{\mathfrak{u}}) \wedge \check{\lambda}(\check{\mathfrak{u}}), (\check{\nu} \vee \check{\mu})(\check{\mathfrak{u}}) := \check{\nu}(\check{\mathfrak{u}}) \vee \check{\mu}(\check{\mathfrak{u}})$ and $(\check{\delta} \vee \check{\omega})(\check{\mathfrak{u}}) := \check{\delta}(\check{\mathfrak{u}}) \vee \check{\omega}(\check{\mathfrak{u}})$ for all $\check{\mathfrak{u}} \in \check{\mathcal{E}}$.

**Definition 2.10.** *[11] A TFS is called a $\mathcal{T}\mathcal{F} = (\rho, \nu, \delta)$ of an SG $\mathcal{S}$ is called a tripolar fuzzy quasi-ideal (TFQI) of $\mathcal{S}$ if*

(1) $\check{\rho}(\check{\mathfrak{u}}) \geq (\check{\rho}_{\check{\mathcal{E}}} \circ \check{\rho})(\check{\mathfrak{u}}) \wedge (\check{\rho} \circ \rho_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$,
(2) $\check{\nu}(\check{\mathfrak{u}}) \leq (\check{\nu}_{\check{\mathcal{E}}} \circ \check{\nu})(\check{\mathfrak{u}}) \vee (\check{\nu} \circ \nu_{\check{\mathcal{E}}})(\ddot{u})$
(3) $\check{\delta}(\check{\mathfrak{u}}) \leq (\check{\delta}_{\check{\mathcal{E}}} \circ \check{\delta})(\check{\mathfrak{u}}) \vee (\check{\delta} \circ \check{\delta}_{\check{\mathcal{E}}})(\ddot{u})$
(4) $\check{\mathfrak{u}} \leq \check{\mathfrak{v}}$ *implies* $\check{\rho}(\check{\mathfrak{u}}) \geq \check{\rho}(\check{\mathfrak{v}}),\ \check{\nu}(\check{\mathfrak{u}}) \leq \check{\nu}(\check{\mathfrak{v}}), \check{\delta}(\check{\mathfrak{u}}) \leq \check{\delta}(\check{\mathfrak{v}})$. *for all* $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$.

**Theorem 2.4.** *The intersection of two TFQIDs is again a TFQID.*

## 3. Tripolar Fuzzy $(m,n)$-Quasi-ideals in ordered semigroups

In this work, we define the concept of tripolar fuzzy (m,n)-quasi-ideals in ordered semigroups and establish several fundamental theorems characterizing their structure and properties

**Definition 3.1.** *A TFSSG $\mathcal{T}\mathcal{F} := (\check{\rho}, \check{\nu}, \check{\delta})$ on $\check{\mathcal{E}}$ is called a* TF $(m,n)$-quasi-ideal *(TF $(m,n)$-QID) if*

(1) $\check{\rho}(\check{\mathfrak{u}}) \geq (\check{\rho}^{m}_{\check{\mathcal{E}}} \circ \check{\rho})(\check{\mathfrak{u}}) \wedge (\check{\rho} \circ \rho^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$,
(2) $\check{\nu}(\check{\mathfrak{u}}) \leq (\check{\nu}^{m}_{\check{\mathcal{E}}} \circ \check{\nu})(\check{\mathfrak{u}}) \vee (\check{\nu} \circ \check{\nu}^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$
(3) $\check{\delta}(\check{\mathfrak{u}}) \leq (\check{\delta}^{m}_{\check{\mathcal{E}}} \circ \check{\delta})(\check{\mathfrak{u}}) \vee (\check{\delta} \circ \check{\delta}^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$ *for all* $\check{\mathfrak{u}} \in \check{\mathcal{E}}$ *and* $m, n \in \mathbb{N}$.
(4) $\check{\mathfrak{u}} \leq \check{\mathfrak{v}}$ *implies* $\check{\rho}(\check{\mathfrak{u}}) \geq \check{\rho}(\check{\mathfrak{v}}),\ \check{\nu}(\check{\mathfrak{u}}) \leq \check{\nu}(\check{\mathfrak{v}}), \check{\delta}(\check{\mathfrak{u}}) \leq \check{\delta}(\check{\mathfrak{v}})$ *for all* $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$.

**Example 3.1.** *Let $\check{\mathcal{E}} = \{\mathfrak{w}, \mathfrak{x}\}$ be an ordered semigroup with $\mathfrak{w} \leq \mathfrak{x}$. Define the binary operation on $\check{\mathcal{E}}$*

| $\cdot$ | $\mathfrak{w}$ | $\mathfrak{x}$ |
|---|---|---|
| $\mathfrak{w}$ | $\mathfrak{w}$ | $\mathfrak{w}$ |
| $\mathfrak{x}$ | $\mathfrak{w}$ | $\mathfrak{x}$ |

*Define $\mathcal{T}\mathcal{F} := (\check{\rho}, \check{\nu}, \check{\delta})$ by $\check{\rho}(\mathfrak{w}) = 0.9, \check{\rho}(\mathfrak{x}) = 0.6$; $\check{\nu}(\mathfrak{w}) = 0.1, \check{\nu}(\mathfrak{x}) = 0.3$, and $\check{\delta}(\mathfrak{w}) = -0.5, \check{\delta}(\mathfrak{x}) = -0.1,$.*

*Let $TF_{\check{\mathcal{E}}}$ be the full TFSS on $\check{\mathcal{E}}$. For all $\mathfrak{w} \in \check{\mathcal{E}}$, we have*

$$\check{\rho}(\mathfrak{w}) \geq (\check{\rho}_{\check{\mathcal{E}}} \circ \check{\rho})(\mathfrak{w}) \wedge (\check{\rho} \circ \check{\rho}_{\check{\mathcal{E}}})(\mathfrak{w}),$$
$$\check{\nu}(\mathfrak{w}) \leq (\check{\nu}_{\check{\mathcal{E}}} \circ \check{\nu})(\mathfrak{w}) \vee (\check{\nu} \circ \check{\nu}_{\check{\mathcal{E}}})(\mathfrak{w}),$$
$$\check{\delta}(\mathfrak{w}) \leq (\check{\delta}_{\check{\mathcal{E}}} \circ \check{\delta})(\mathfrak{w}) \vee (\check{\delta} \circ \check{\delta}_{\check{\mathcal{E}}})(\mathfrak{w}).$$

*Moreover, since $a \leq b$, it follows that*

$$\check{\rho}(\mathfrak{w}) \geq \check{\rho}(\mathfrak{x}), \quad \check{\nu}(\mathfrak{w}) \leq \check{\nu}(\mathfrak{x}), \quad \check{\delta}(\mathfrak{w}) \leq \check{\delta}(\mathfrak{x}).$$

*Hence, $\mathcal{T}\mathcal{F} := (\check{\rho}, \check{\nu}, \check{\delta})$ is a TF $(1,1)$-QID of $\check{\mathcal{E}}$.*

**Theorem 3.1.** *Let $\{\mathcal{T}\mathcal{F}_i \mid i \in \check{\mathcal{J}}\}$ be a family of TF $(m,n)$-QIDs of $\check{\mathcal{E}}$. Then a TF Set $\bigwedge\limits_{i\in\check{\mathcal{F}}} \mathcal{T}\mathcal{F}_i := (\bigwedge\limits_{i\in\check{\mathcal{F}}} \check{\rho}_i, \bigwedge\limits_{i\in\check{\mathcal{F}}} \check{\mu}_i, \bigvee\limits_{i\in\check{\mathcal{F}}} \check{\delta}_i)$ is a TF $(m,n)$-QID of $\check{\mathcal{E}}$.*

*Proof.* Let $\check{\mathfrak{u}} \in \check{\mathcal{E}}$. Then,

$$\bigwedge_{i\in\mathcal{J}} \check{\rho}_i(\check{\mathfrak{u}}) \geq \bigwedge_{i\in\mathcal{J}} \{(\check{\rho}^m_{\check{\mathcal{E}}} \circ \check{\rho}_i)(\check{\mathfrak{u}}) \wedge (\check{\rho}_i \circ \rho^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}}),$$

$$\bigvee_{i\in\mathcal{J}} \check{\nu}_i(\check{\mathfrak{u}}) \leq \bigvee_{i\in\mathcal{J}} \{(\check{\nu}^m_{\check{\mathcal{E}}} \circ \check{\nu}_i)(\check{\mathfrak{u}}) \vee (\check{\nu}_i \circ \check{\nu}^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}})\}$$

and

$$\bigvee_{i\in\mathcal{J}} \check{\delta}_i(\check{\mathfrak{u}}) \leq \bigvee_{i\in\mathcal{J}} \{(\check{\delta}^m_{\check{\mathcal{E}}} \circ \check{\delta}_i)(\check{\mathfrak{u}}) \vee (\check{\delta}_i \circ \check{\delta}^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}}))\}$$

Thus, $\bigwedge_{i\in\mathcal{J}} \mathcal{T}\mathcal{F}_i$ is a TF $(m,n)$-ID of $\check{\mathcal{E}}$. $\square$

**Theorem 3.2.** *Let $\check{\mathcal{K}}$ be a nonempty subset of an ordered semigroup $\check{\mathcal{E}}$ and $m,n$ are positive integers. Then $\check{\mathcal{K}}$ is an $(m,n)$-ideal of $\check{\mathcal{E}}$ if and only if the CTFS $\mathcal{T}\mathcal{F}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a TF $(m,n)$-QID of $\check{\mathcal{E}}$.*

*Proof.* Suppose that $\check{\mathcal{K}}$ is an $(m,n)$-QID of $\check{\mathcal{E}}$ and let $\check{\mathfrak{u}} \in \check{\mathcal{E}}$.

If $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) \leq (\check{\rho}^m_{\check{\mathcal{E}}} \circ \check{\rho}_{\check{\mathcal{K}}})(\ddot{u}) \wedge (\check{\rho}_{\check{\mathcal{K}}} \circ \rho^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$ or $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) \geq (\check{\nu}^m_{\check{\mathcal{E}}} \circ \check{\nu}_{\check{\mathcal{K}}})(\check{\mathfrak{u}}) \vee (\check{\nu}_{\check{\mathcal{K}}} \circ \check{\nu}^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$ or $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) \geq (\delta^m_{\check{\mathcal{E}}} \circ \check{\delta}_{\check{\mathcal{K}}})(\check{\mathfrak{u}}) \vee (\check{\delta}_{\check{\mathcal{K}}} \circ \check{\delta}^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$, then $(\check{\rho}^m_{\check{\mathcal{E}}} \circ \check{\rho}_{\check{\mathcal{K}}})(\ddot{u}) \neq 1$ or $(\check{\nu}^m_{\check{\mathcal{E}}} \circ \check{\nu}_{\check{\mathcal{K}}})(\check{\mathfrak{u}}) \neq 0$ or $(\check{\delta}^m_{\check{\mathcal{E}}} \circ \check{\delta}_{\check{\mathcal{K}}})(\check{\mathfrak{u}}) \neq -1$. Thus, there exists $(\check{\mathfrak{a}}_1, \check{\mathfrak{b}}_1) \in \check{\mathcal{F}}_{\check{\mathfrak{u}}_1}$ such that $\check{\mathfrak{u}}_1 \leq \check{\mathfrak{a}}_1\check{\mathfrak{b}}_1$ implies that $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}_1) \geq 1$ or $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}_1) \leq 0$ or $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}_1) \leq -1$. Without losing general meaning there exists $(\check{\mathfrak{a}}_m, \check{\mathfrak{b}}_m) \in \check{\mathcal{F}}_{\check{\mathfrak{u}}_m}$ such that $\check{\mathfrak{u}}_m \leq \check{\mathfrak{a}}_m\check{\mathfrak{b}}_m$ implies that $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}_m) \geq 1$ or $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}_m) \leq 0$ or $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}_m) \leq -1$. So, $\check{\mathfrak{u}}_1 \leq \check{\mathfrak{a}}_1\check{\mathfrak{b}}_1$, $\check{\mathfrak{u}}_2 \leq \check{\mathfrak{a}}_2\check{\mathfrak{b}}_2, \ldots, \check{\mathfrak{u}}_m \leq \check{\mathfrak{a}}_m\check{\mathfrak{b}}_m$ such that $\check{\mathfrak{u}}_1, \check{\mathfrak{u}}_2, \ldots \check{\mathfrak{u}}_m \in \check{\mathcal{K}}$. Hence, $\check{\mathfrak{u}}_1, \check{\mathfrak{u}}_2, \ldots \check{\mathfrak{u}}_m \in \check{\mathcal{E}}^m\check{\mathcal{K}}$. Similarly, $(\check{\rho}_{\check{\mathcal{K}}} \circ \check{\rho}^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}}) \neq 1$ or $(\check{\nu}_{\check{\mathcal{K}}} \circ \check{\nu}^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}}) \neq 0$ or $(\check{\delta}_{\check{\mathcal{K}}} \circ \check{\delta}^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}}) \neq -1$ we can show that $\check{\mathfrak{u}}_1, \check{\mathfrak{u}}_2, \ldots \check{\mathfrak{u}}_m \in \check{\mathcal{K}}\check{\mathcal{E}}^n$. Therefore, $\check{\mathcal{K}}$ is an $(m,n)$-QID of $\check{\mathcal{E}}$. It is a contradiction, so $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) \geq (\check{\rho}^m_{\check{\mathcal{E}}} \circ \check{\rho}_{\check{\mathcal{K}}})(\check{\mathfrak{u}}) \wedge (\check{\rho}_{\check{\mathcal{K}}} \circ \check{\rho}^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$, $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) \leq (\check{\nu}^m_{\check{\mathcal{E}}} \circ \check{\nu}_{\check{\mathcal{K}}})(\check{\mathfrak{u}}) \vee (\check{\nu}_{\check{\mathcal{K}}} \circ \check{\nu}^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$ $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) \leq (\check{\delta}^m_{\check{\mathcal{E}}} \circ \check{\delta}_{\check{\mathcal{K}}})(\check{\mathfrak{u}}) \vee (\check{\delta}_{\check{\mathcal{K}}} \circ \delta^n_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$ for all $\check{\mathfrak{u}} \in \check{\mathcal{E}}$.

Let $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$ such that $\check{\mathfrak{u}} \leq \check{\mathfrak{v}}$ and $\check{\mathfrak{u}} \in \check{\mathcal{K}}$. Then $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) \geq \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{v}})$, $\check{\nu}_{\ddot{\mathcal{K}}}(\check{\mathfrak{u}}) \leq \check{\nu}_{\ddot{\mathcal{K}}}(\check{\mathfrak{v}}), \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) \leq \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{v}})$.

Therefore, $\mathcal{T}\mathcal{F}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a TF $(m,n)$-QID of $\check{\mathcal{E}}$.

Conversely, suppose that $\mathcal{T}\mathcal{F}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a TF $(m,n)$-QID of $\check{\mathcal{E}}$ and let $\check{\mathfrak{u}} \in \check{\mathcal{E}}^m\check{\mathcal{K}} \cap \check{\mathcal{K}}\check{\mathcal{E}}^n$. Then $\check{\mathfrak{u}} \in \check{\mathcal{E}}^m\check{\mathcal{K}}$ and $\check{\mathfrak{u}} \in \check{\mathcal{K}}\check{\mathcal{E}}^n$. Thus, there exists $\check{\mathfrak{a}}_1, \check{\mathfrak{b}}_1, \ldots, \check{\mathfrak{a}}_m, \check{\mathfrak{b}}_m \in \check{\mathcal{K}}$ with $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}_1) \geq 1, \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{b}}_1) \geq 1, \ldots \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}_m) \geq 1, \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{b}}_m) \geq 1$, $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}_1) \leq 0, \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{b}}_1) \leq 0, \ldots \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}_m) \leq 0, \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{b}}_m) \leq 0$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}_1) \leq -1, \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{b}}_1) \leq -1, \ldots \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}_m) \leq -1, \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{b}}_m) \leq -1$ suth that $\check{\mathfrak{u}}_1 \leq \check{\mathfrak{a}}_1\check{\mathfrak{b}}_1$, $\check{\mathfrak{u}}_2 \leq \check{\mathfrak{a}}_2\check{\mathfrak{b}}_2, \ldots, \check{\mathfrak{u}}_m \leq \check{\mathfrak{a}}_m\check{\mathfrak{b}}_m$. It implies that

$$\begin{aligned}(\check{\rho}^m_{\check{\mathcal{E}}} \circ \check{\rho}_{\check{\mathcal{K}}})(\check{\mathfrak{u}}) &= \bigvee_{(\check{\mathfrak{a}},\check{\mathfrak{b}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\rho}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}) \vee \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{b}})\} = \check{\rho}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\ldots\check{\mathfrak{a}}_m) \wedge \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{b}}) \\ &= \check{\rho}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\ldots\check{\mathfrak{a}}_m) \wedge 1 = \check{\rho}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\ldots\check{\mathfrak{a}}_m).\end{aligned}$$

Similarly

$$\begin{aligned}(\check{\rho}_{\check{\mathcal{K}}} \circ \rho^m_{\check{\mathcal{E}}})(\check{\mathfrak{u}}) &= \bigvee_{(\check{\mathfrak{a}},\check{\mathfrak{b}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}) \vee \check{\rho}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}})\} = \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}) \vee \check{\rho}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\ldots\check{\mathfrak{b}}_m) \\ &= 1 \vee \check{\rho}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\ldots\check{\mathfrak{b}}_m) = \check{\rho}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\ldots\check{\mathfrak{b}}_m) \neq 0,\end{aligned}$$

$$\begin{aligned}(\check{\nu}^m_{\check{\mathcal{E}}} \circ \check{\nu}_{\check{\mathcal{K}}})(\check{\mathfrak{u}}) &= \bigwedge_{(\check{\mathfrak{r}},\check{\mathfrak{b}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\nu}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}) \vee \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{b}})\} = \check{\nu}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\ldots\check{\mathfrak{a}}_m) \vee \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{b}}) \\ &= \check{\nu}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\ldots\check{\mathfrak{a}}_m) \vee 0 = \check{\nu}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\ldots\check{\mathfrak{a}}_m) \neq 0.\end{aligned}$$

Similarly

$$\begin{aligned}(\check{\nu}_{\check{\mathcal{K}}} \circ \check{\nu}^m_{\check{\mathcal{E}}})(\check{\mathfrak{u}}) &= \bigwedge_{(\check{\mathfrak{r}},\check{\mathfrak{b}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}) \vee \check{\nu}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}})\} = \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}) \vee \check{\nu}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\ldots\check{\mathfrak{b}}_m) \\ &= 0 \vee \check{\nu}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\ldots\check{\mathfrak{b}}_m) = \check{\nu}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\ldots\check{\mathfrak{b}}_m) \neq 0\end{aligned}$$

and

$$\begin{aligned}(\check{\delta}^m_{\check{\mathcal{E}}} \circ \check{\delta}_{\check{\mathcal{K}}})(\check{\mathfrak{u}}) &= \bigwedge_{(\check{\mathfrak{r}},\check{\mathfrak{b}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\delta}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}) \vee \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{b}})\} = \check{\delta}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\ldots\check{\mathfrak{a}}_m) \vee \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{b}}) \\ &= \check{\delta}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\ldots\check{\mathfrak{a}}_m) \vee -1 = \check{\delta}^m_{\check{\mathcal{E}}}(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\ldots\check{\mathfrak{a}}_m) \neq 0.\end{aligned}$$

Similarly

$$\begin{aligned}(\check{\delta}_{\check{\mathcal{K}}} \circ \check{\delta}^m_{\check{\mathcal{E}}})(\check{\mathfrak{u}}) &= \bigwedge_{(\check{\mathfrak{r}},\check{\mathfrak{b}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}) \vee \check{\delta}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}})\} = \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{a}}) \vee \check{\delta}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\ldots\check{\mathfrak{b}}_m) \\ &= -1 \vee \check{\delta}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\ldots\check{\mathfrak{b}}_m) = \check{\delta}^n_{\check{\mathcal{E}}}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\ldots\check{\mathfrak{b}}_m)\end{aligned}$$

Thus, $\check{\mathfrak{u}} \in \check{\mathcal{K}}$. Hence, $\check{\mathcal{E}}^m\check{\mathcal{K}} \cap \check{\mathcal{K}}\check{\mathcal{E}}^n \subseteq \check{\mathcal{K}}$.

Let $\check{\mathfrak{u}} \in \check{\mathcal{K}}$ such that $\check{\mathfrak{v}} \leq \check{\mathfrak{u}}$ and $\check{\mathfrak{v}} \in \check{\mathcal{E}}$. Then $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) \geq \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{v}}) = 1$, $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) \leq \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{v}}) = 0, \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{u}}) \leq \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{v}}) = -1$. Thus, $\check{\mathfrak{v}} \in \check{\mathcal{K}}$. Therefore, $\check{\mathcal{K}}$ is an $(m,n)$-QID of $\check{\mathcal{E}}$. □

**Theorem 3.3.** *Let $\mathcal{TF} := (\check{\rho}, \check{\nu}, \check{\delta})$ be a nonzero TFS of an SG $\check{\mathcal{E}}$. Then $\mathcal{TF} := (\check{\rho}, \check{\nu}, \check{\delta})$ is a TF $(m,n)$-QID of $\mathcal{E}$ if and only if* $\operatorname{supp}(\mathcal{TF})$ *is an $(m,n)$-QID of $\check{\mathcal{E}}$.*

*Proof.* Suppose that $\mathcal{TF} := (\check{\rho}, \check{\nu}, \check{\delta})$ is a TF $(m,n)$-QID of $\check{\mathcal{E}}$ and let $\check{\mathfrak{u}} \in \operatorname{supp}(\mathcal{TF})^m\mathcal{E} \cap \mathcal{E}\operatorname{supp}(\mathcal{TF})^n$. Then $\check{\mathfrak{u}} \in \operatorname{supp}(\mathcal{TF})^m\mathcal{E}$ and $\check{\mathfrak{u}} \in \mathcal{E}\operatorname{supp}(\mathcal{TF})^n$. Hence, there exist $\check{\mathfrak{a}}_1, \check{\mathfrak{a}}_2, \ldots, \check{\mathfrak{a}}_m, \check{\mathfrak{x}} \in \mathcal{E}$ such that $\check{\mathfrak{a}}_1, \check{\mathfrak{a}}_2, \ldots, \check{\mathfrak{a}}_m \in \operatorname{supp}(\mathcal{TF})$ and $\check{\mathfrak{u}} \leq \check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\cdots\check{\mathfrak{a}}_m x$. Similarly, there exist $\check{\mathfrak{b}}_1, \check{\mathfrak{b}}_2, \ldots, \check{\mathfrak{b}}_n, \check{\mathfrak{y}} \in \check{\mathcal{E}}$ such that $\check{\mathfrak{b}}_1, \check{\mathfrak{b}}_2, \ldots, \check{\mathfrak{b}}_n \in \operatorname{supp}(\mathcal{TF})$ and $\check{\mathfrak{u}} \leq \check{\mathfrak{y}}\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\cdots\check{\mathfrak{b}}_n$. Since $\check{\mathfrak{a}}_1, \check{\mathfrak{a}}_2, \ldots, \check{\mathfrak{a}}_m, \check{\mathfrak{b}}_1, \check{\mathfrak{b}}_2, \ldots, \check{\mathfrak{b}}_n \in \operatorname{supp}(\mathcal{TF})$ we have $\check{\rho}(\check{\mathfrak{a}}_i) \neq 0, \quad \check{\nu}(\check{\mathfrak{a}}_i) \neq 0, \quad \check{\delta}(\check{\mathfrak{a}}_i) \neq 0$ for all $i = 1, 2, \ldots, m$, and $\check{\rho}(\check{\mathfrak{b}}_j) \neq 0, \quad \check{\nu}(\check{\mathfrak{b}}_j) \neq 0, \quad \check{\delta}(\check{\mathfrak{b}}_j) \neq 0$ for all $j = 1, 2, \ldots, n$. Thus,

$$\begin{aligned}(\check{\rho}^m \circ \check{\rho}_{\mathcal{E}})(\check{\mathfrak{u}}) &= \bigvee_{(\check{\mathfrak{r}},\check{\mathfrak{x}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\rho}^m(\check{\mathfrak{r}}) \wedge \check{\rho}_{\mathcal{E}}(\check{\mathfrak{x}})\} \geq \check{\rho}^m(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\cdots\check{\mathfrak{a}}_m) \wedge 1_{\mathcal{E}} = \check{\rho}^m(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\cdots\check{\mathfrak{a}}_m) \\ &\geq \check{\rho}(\check{\mathfrak{a}}_1) \wedge \check{\rho}(\check{\mathfrak{a}}_2) \wedge \cdots \wedge \check{\rho}(\check{\mathfrak{a}}_m) \neq 0.\end{aligned}$$

or

$$\begin{aligned}(\check{\nu}^m \circ \check{\nu}_{\mathcal{E}})(\check{\mathfrak{u}}) &= \bigwedge_{(\check{\mathfrak{r}},\check{\mathfrak{x}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\nu}^m(\check{\mathfrak{r}}) \vee \check{\nu}_{\mathcal{E}}(\check{\mathfrak{x}})\} \leq \check{\nu}^m(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\cdots\check{\mathfrak{a}}_m) \vee 0_{\mathcal{E}} = \check{\nu}^m(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\cdots\check{\mathfrak{a}}_m) \\ &\leq \check{\nu}(\check{\mathfrak{a}}_1) \vee \check{\nu}(\check{\mathfrak{a}}_2) \vee \cdots \vee \check{\nu}(\check{\mathfrak{a}}_m) \neq 0.\end{aligned}$$

or

$$\begin{aligned}(\check{\delta}^m \circ \check{\delta}_{\mathcal{E}})(\check{\mathfrak{u}}) &= \bigwedge_{(\check{\mathfrak{r}},\check{\mathfrak{x}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \left\{\check{\delta}^m(\check{\mathfrak{r}}) \vee \check{\delta}_{\mathcal{E}}(\check{\mathfrak{x}})\right\} \leq \check{\delta}^m(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\cdots\check{\mathfrak{a}}_m) \vee -1_{\mathcal{E}} = \check{\delta}^m(\check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\cdots\check{\mathfrak{a}}_m) \\ &\leq \check{\delta}(\check{\mathfrak{a}}_1) \vee \check{\delta}(\check{\mathfrak{a}}_2) \vee \cdots \vee \check{\delta}(\check{\mathfrak{a}}_m) \neq 0.\end{aligned}$$

And,

$$\begin{aligned}(\check{\rho}_{\mathcal{E}} \circ \check{\rho}^{n})(\check{\mathfrak{u}}) &= \bigvee_{(\check{\mathfrak{y}},\check{\mathfrak{r}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\rho}_{\mathcal{E}}(\check{\mathfrak{r}}) \wedge \check{\rho}^{n}(\check{\mathfrak{x}})\} \geq 1 \wedge \check{\rho}^{n}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\cdots\check{\mathfrak{b}}_n) = \check{\rho}^{n}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\cdots\check{\mathfrak{b}}_n)\\ &\geq \check{\rho}(\check{\mathfrak{b}}_1) \wedge \check{\rho}(\check{\mathfrak{b}}_2) \wedge \cdots \wedge \check{\rho}(\check{\mathfrak{b}}_m) \neq 0.\end{aligned}$$

or

$$\begin{aligned}(\check{\nu}_{\mathcal{E}} \circ \check{\nu}^{n})(\check{\mathfrak{u}}) &= \bigwedge_{(\check{\mathfrak{y}},\check{\mathfrak{x}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \{\check{\nu}_{\mathcal{E}}(\check{\mathfrak{y}}) \vee \check{\nu}^{n}(\check{\mathfrak{x}})\} \leq 0 \vee \check{\nu}^{n}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\cdots\check{\mathfrak{b}}_n) = \check{\nu}^{n}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\cdots\check{\mathfrak{b}}_n)\\ &\leq \check{\nu}(\check{\mathfrak{b}}_1) \vee \check{\nu}(\check{\mathfrak{b}}_2) \vee \cdots \vee \check{\nu}(\check{\mathfrak{b}}_m) \neq 0.\end{aligned}$$

or

$$\begin{aligned}(\check{\delta}_{\mathcal{E}} \circ \check{\delta}^{n})(\check{\mathfrak{u}}) &= \bigwedge_{(\check{\mathfrak{y}},\check{\mathfrak{x}})\in\check{\mathcal{F}}_{\check{\mathfrak{u}}}} \left\{\check{\delta}_{\mathcal{E}}(\check{\mathfrak{y}}) \vee \check{\delta}^{n}(\check{\mathfrak{x}})\right\} \leq -1 \vee \check{\delta}^{n}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\cdots\check{\mathfrak{b}}_n) = \check{\delta}^{n}(\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\cdots\check{\mathfrak{b}}_n)\\ &\leq \check{\delta}(\check{\mathfrak{b}}_1) \vee \check{\delta}(\check{\mathfrak{b}}_2) \vee \cdots \vee \check{\delta}(\check{\mathfrak{b}}_m) \neq 0.\end{aligned}$$

Hence, $\check{\mathfrak{u}} \in \mathrm{supp}(\mathcal{TF})$ so, $\mathrm{supp}(\mathcal{TF})^{m}\mathcal{E} \cap \mathcal{E}\,\mathrm{supp}(\mathcal{TF})^{n} \subseteq \mathrm{supp}(\mathcal{TF})$. Next, let $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \mathcal{E}$ such that $\check{\mathfrak{u}} \leq \check{\mathfrak{v}}$ and $\check{\mathfrak{u}} \in \mathrm{supp}(\mathcal{TF})$. Then $\check{\rho}(\check{\mathfrak{u}}) \neq 0$ or $\check{\nu}(\check{\mathfrak{u}}) \neq 0$ or $\check{\delta}(\check{\mathfrak{u}}) \neq 0$. Thus, $\check{\rho}(\check{\mathfrak{v}}) \geq \check{\rho}(\check{\mathfrak{u}})$, $\check{\nu}(\check{\mathfrak{v}}) \leq \check{\nu}(\check{\mathfrak{u}})$, $\check{\delta}(\check{\mathfrak{v}}) \leq \check{\delta}(\check{\mathfrak{u}})$. Hence, $\check{\rho}(\check{\mathfrak{v}}) \neq 0$ or $\check{\nu}(\check{\mathfrak{v}}) \neq 0$ or $\check{\delta}(\check{\mathfrak{v}}) \neq 0$. Therefore, $\check{\mathfrak{v}} \in \mathrm{supp}(\mathcal{TF})$. Consequently, $\mathrm{supp}(\mathcal{TF})$ is an $(m,n)$-QID of $\mathcal{E}$.

Conversely, suppose that $\mathrm{supp}(\mathcal{TF})$ is an $(m,n)$-QID of $\mathcal{E}$ and let $\check{\mathfrak{u}} \in \mathcal{E}$.

If $\check{\rho}(\check{\mathfrak{u}}) < (\check{\rho}^{m}_{\check{\mathcal{E}}} \circ \check{\rho})(\check{\mathfrak{u}}) \wedge (\check{\rho} \circ \rho^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$, or $\check{\nu}(\check{\mathfrak{u}}) > (\check{\nu}^{m}_{\check{\mathcal{E}}} \circ \check{\nu})(\check{\mathfrak{u}}) \vee (\check{\nu} \circ \check{\nu}^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$ or $\check{\delta}(\check{\mathfrak{u}}) > (\check{\delta}^{m}_{\check{\mathcal{E}}} \circ \check{\delta})(\check{\mathfrak{u}}) \vee (\check{\delta} \circ \check{\delta}^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$, then $(\check{\rho}^{m}_{\check{\mathcal{E}}} \circ \check{\rho})(\check{\mathfrak{u}}) \neq 0$ or $(\check{\nu}^{m}_{\check{\mathcal{E}}} \circ \check{\nu})(\check{\mathfrak{u}}) \neq 0$ or $(\check{\delta}^{m}_{\check{\mathcal{E}}} \circ \check{\delta})(\check{\mathfrak{u}}) \neq 0$. Thus, there exist $\check{\mathfrak{a}}_1, \check{\mathfrak{a}}_2, \ldots, \check{\mathfrak{a}}_m, \check{\mathfrak{x}} \in \check{\mathcal{E}}$ such that $\check{\mathfrak{u}} \leq \check{\mathfrak{a}}_1\check{\mathfrak{a}}_2\cdots\check{\mathfrak{a}}_m\check{\mathfrak{x}}$ where $\check{\mathfrak{a}}_1, \check{\mathfrak{a}}_2, \ldots, \check{\mathfrak{a}}_m \in \mathrm{supp}(\mathcal{TF})$.

Similarly, $(\check{\rho} \circ \check{\rho}^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}}) \neq 0$ or $(\check{\nu} \circ \check{\nu}^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}}) \neq 0$ or $(\check{\delta} \circ \check{\delta}^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}}) \neq 0$ implies that there exist $\check{\mathfrak{b}}_1, \check{\mathfrak{b}}_2, \ldots, \check{\mathfrak{b}}_n, \check{\mathfrak{y}} \in \check{\mathcal{E}}$ such that $\check{\mathfrak{u}} \leq \check{\mathfrak{y}}\check{\mathfrak{b}}_1\check{\mathfrak{b}}_2\cdots\check{\mathfrak{b}}_n$, where $\check{\mathfrak{b}}_1, \check{\mathfrak{b}}_2, \ldots, \check{\mathfrak{b}}_n \in \mathrm{supp}(\mathcal{TF})$. Hence, $\check{\mathfrak{u}} \in \mathrm{supp}(\mathcal{TF})^{m}\check{\mathcal{E}} \cap \check{\mathcal{E}}\,\mathrm{supp}(\mathcal{TF})^{n}$. By assumption, $\check{\mathfrak{u}} \in \mathrm{supp}(\mathcal{TF})$. Thus, $\check{\rho}(\check{\mathfrak{u}}) \neq 0$, $\check{\nu}(\check{\mathfrak{u}}) \neq 0$, $\check{\delta}(\check{\mathfrak{u}}) \neq 0$. This contradicts the assumption. Therefore, $\check{\rho}(\check{\mathfrak{u}}) \geq (\check{\rho}^{m}_{\check{\mathcal{E}}} \circ \check{\rho})(\check{\mathfrak{u}}) \wedge (\check{\rho} \circ \check{\rho}^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$, $\check{\nu}(\check{\mathfrak{u}}) \leq (\check{\nu}^{m}_{\check{\mathcal{E}}} \circ \check{\nu})(\check{\mathfrak{u}}) \vee (\check{\nu} \circ \check{\nu}^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$, and $\check{\delta}(\check{\mathfrak{u}}) \leq (\check{\delta}^{m}_{\check{\mathcal{E}}} \circ \check{\delta})(\check{\mathfrak{u}}) \vee (\check{\delta} \circ \check{\delta}^{n}_{\check{\mathcal{E}}})(\check{\mathfrak{u}})$. Finally, let $\check{\mathfrak{u}}, \check{\mathfrak{v}} \in \check{\mathcal{E}}$ such that $\check{\mathfrak{u}} \leq \check{\mathfrak{v}}$ and $\check{\mathfrak{u}} \in \mathrm{supp}(\mathcal{TF})$. Then $\check{\rho}(\check{\mathfrak{u}}) \neq 0$, or $\check{\nu}(\check{\mathfrak{u}}) \neq 0$, or $\check{\delta}(\check{\mathfrak{u}}) \neq 0$. Thus, $\check{\rho}(\check{\mathfrak{u}}) \geq \check{\rho}(\check{\mathfrak{v}})$, $\check{\nu}(\check{\mathfrak{u}}) \leq \check{\nu}(\check{\mathfrak{v}})$, $\check{\delta}(\check{\mathfrak{u}}) \leq \check{\delta}(\check{\mathfrak{v}})$. Hence, $\check{\rho}(\check{\mathfrak{v}}) \neq 0$, $\check{\nu}(\check{\mathfrak{v}}) \neq 0$, $\check{\delta}(\check{\mathfrak{v}}) \neq 0$. Therefore, $\check{\mathfrak{v}} \in \mathrm{supp}(\mathcal{TF})$. consequently, $\mathcal{TF} := (\check{\rho}, \check{\nu}, \check{\delta})$ is a TF $(m,n)$-QID of $\check{\mathcal{E}}$. □

**Definition 3.2.** *An $(m,n)$-QID $\check{\mathcal{K}}$ of an ordered semigroup $\check{\mathcal{E}}$ is called a* minimal *if for every $(m,n)$-QID of $\check{\mathcal{J}}$ of $\check{\mathcal{E}}$ such that $\check{\mathcal{J}} \subseteq \check{\mathcal{K}}$, we have $\check{\mathcal{J}} = \check{\mathcal{K}}$.*

**Definition 3.3.** *A TF $(m,n)$-QID $\mathcal{TF}_1 := (\check{\rho},\ \check{\nu},\ \check{\delta})$ of an ordered semigroup $\check{\mathcal{E}}$ is a* minimal *if for all TF $(m,n)$-QID $\mathcal{TF}_2 := (\check{\lambda},\ \check{\mu},\ \check{\omega})$ of $\check{\mathcal{E}}$ such that $\mathcal{TF}_2 \sqsubseteq \mathcal{TF}_1$, then* $\mathrm{supp}(\mathcal{TF}_2) = \mathrm{supp}(\mathcal{TF}_1)$.

**Lemma 3.1.** *For any non-empty subsets $\check{\mathcal{I}}$ and $\check{\mathcal{K}}$ of an ordered semigroup $\check{\mathcal{E}}$, we have $\check{\mathcal{I}} \subseteq \check{\mathcal{K}}$ if and only if $\mathcal{TF}_{\check{\mathcal{I}}} \sqsubseteq \mathcal{TF}_{\check{\mathcal{K}}}$ where $\mathcal{TF}_{\check{\mathcal{I}}} := (\check{\rho}_{\check{\mathcal{I}}},\ \check{\nu}_{\check{\mathcal{I}}},\ \check{\delta}_{\check{\mathcal{I}}})$ and $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}},\ \check{\nu}_{\check{\mathcal{K}}},\ \check{\delta}_{\check{\mathcal{K}}})$ are CTFS $\check{\mathcal{I}}$ and $\check{\mathcal{K}}$ respective.*

**Theorem 3.4.** *Let $\emptyset \neq \check{\mathcal{K}} \subseteq \check{\mathcal{E}}$. Then $\check{\mathcal{K}}$ is a minimal $(m,n)$-QID if and only if $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}},\ \check{\nu}_{\check{\mathcal{K}}},\ \check{\delta}_{\check{\mathcal{K}}})$ is a minimal TF $(m,n)$-QID of $\check{\mathcal{E}}$.*

*Proof.* Let $\check{\mathcal{K}}$ be a minimal $(m,n)$-ID of $\check{\mathcal{E}}$. Then $\check{\mathcal{K}}$ is an $(m,n)$-ID. Thus, by Theorem 3.2, $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a TF $(m,n)$-ID of $\check{\mathcal{E}}$. Let $\check{\mathcal{J}}$ be an $(m,n)$-ID of $\check{\mathcal{E}}$ such that $\check{\mathcal{J}} \subseteq \check{\mathcal{K}}$. Then by Theorem 3.2, $\mathcal{TF}_{\check{\mathcal{J}}} = (\check{\rho}_{\check{\mathcal{J}}}, \check{\nu}_{\check{\mathcal{J}}}, \check{\delta}_{\check{\mathcal{J}}})$ is a TF $(m,n)$-ID of $\check{\mathcal{E}}$ and $\mathcal{TF}_{\check{\mathcal{J}}} \sqsubseteq \mathcal{TF}_{\check{\mathcal{K}}}$. Thus, $\operatorname{supp}(\mathcal{TF}_{\check{\mathcal{J}}}) \sqsubseteq \operatorname{supp}(\mathcal{TF}_{\check{\mathcal{K}}}) = \check{\mathcal{K}}$. It implies that $\operatorname{supp}(\mathcal{TF}_{\check{\mathcal{J}}}) \sqsubseteq \check{\mathcal{K}}$. By Theorem 3.3, $\operatorname{supp}(\mathcal{TF}_{\check{\mathcal{J}}})$ is an $(m,n)$-ID of $\check{\mathcal{E}}$. Since $\check{\mathcal{K}}$ is a minimal $(m,n)$-ID of $\check{\mathcal{E}}$ we have $\operatorname{supp}(\mathcal{TF}_{\check{\mathcal{J}}}) = \check{\mathcal{K}}$. Thus, $\operatorname{supp}(\mathcal{TF}_{\check{\mathcal{J}}}) = \operatorname{supp}(\mathcal{TF}_{\check{\mathcal{K}}})$. Hence, $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is minimal TF $(m,n)$-ID of $\check{\mathcal{E}}$.

Conversely, $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is minimal TF $(m,n)$-ID of $\check{\mathcal{E}}$. Then $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a TF $(m,n)$-ID of $\check{\mathcal{E}}$. Thus, by Theorem 3.2, $\check{\mathcal{K}}$ is an $(m,n)$-ID of $\check{\mathcal{E}}$. Let $\mathcal{TF}_{\check{\mathcal{J}}} = (\check{\rho}_{\check{\mathcal{J}}}, \check{\nu}_{\check{\mathcal{J}}}, \check{\delta}_{\check{\mathcal{J}}})$ be a TF $(m,n)$-ID of $\check{\mathcal{E}}$ such that $\mathcal{TF}_{\check{\mathcal{J}}} \sqsubseteq \mathcal{TF}_{\check{\mathcal{K}}}$. Then by Theorem 3.2, $\check{\mathcal{J}}$ is an $(m,n)$-ID of $\check{\mathcal{E}}$ such that $\check{\mathcal{J}} \subseteq \check{\mathcal{K}}$. By assumption, we have $\operatorname{supp}(\mathcal{TF}_{\check{\mathcal{J}}}) = \operatorname{supp}(\mathcal{TF}_{\check{\mathcal{K}}})$. Thus, $\check{\mathcal{J}} = \check{\mathcal{K}}$. Hence, $\check{\mathcal{K}}$ is a minimal $(m,n)$-ID of $\check{\mathcal{E}}$. □

Next, we give the relationship between prime, semiprime $(m,n)$-quasi-ideals and prime, semiprime TF $(m,n)$-quasi-ideals.

**Definition 3.4.** *Let $\check{\mathcal{K}}$ be an $(m,n)$-QID of an ordred semigroup $\check{\mathcal{E}}$ is called*

(1) prime *if $\check{\mathfrak{e}}\check{\mathfrak{h}} \in \check{\mathcal{K}}$ implies $\check{\mathfrak{e}} \in \check{\mathcal{K}}$ or $\check{\mathfrak{h}} \in \check{\mathcal{K}}$ for all $\check{\mathfrak{e}}, \check{\mathfrak{h}} \in \check{\mathcal{E}}$,*

(2) semiprime *if $\check{\mathfrak{e}}^2 \in \check{\mathcal{K}}$ implies $\check{\mathfrak{e}} \in \check{\mathcal{K}}$ for all $\check{\mathfrak{e}} \in \check{\mathcal{E}}$.*

**Definition 3.5.** *Let $\mathcal{TF} := (\check{\rho}, \check{\nu}, \check{\delta})$ be a TF $(m,n)$-QID of an ordered semigroup $\check{\mathcal{E}}$ is called*

(1) prime *if $\check{\rho}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \leq \check{\rho}(\check{\mathfrak{e}}) \vee \check{\rho}(\check{\mathfrak{h}})$, $\check{\nu}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \geq \check{\nu}(\check{\mathfrak{e}}) \wedge \check{\nu}(\check{\mathfrak{h}})$ and $\check{\delta}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \geq \check{\delta}(\check{\mathfrak{e}}) \wedge \check{\delta}(\check{\mathfrak{h}})$ for all $\check{\mathfrak{e}}, \check{\mathfrak{h}} \in \check{\mathcal{E}}$,*

(2) semiprime *if $\check{\rho}(\check{\mathfrak{e}}^2) \leq \check{\rho}(\check{\mathfrak{e}})$ $\check{\nu}(\check{\mathfrak{e}}^2) \geq \check{\nu}(\check{\mathfrak{e}})$ and $\check{\delta}(\check{\mathfrak{e}}^2) \geq \check{\delta}(\check{\mathfrak{e}})$ for all $\check{\mathfrak{e}} \in \check{\mathcal{E}}$.*

**Remark 3.1.** *Every prime $(m,n)$-QID is a semiprime $(m,n)$-QID in an ordered semigroup.*

**Theorem 3.5.** *Let $\check{\mathcal{K}}$ be a non-empty subset of an ordered semigroup $\check{\mathcal{E}}$. Then*

(1) *$\check{\mathcal{K}}$ is a prime $(m,n)$-QID of $\check{\mathcal{E}}$ if and only if $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a prime TF $(m,n)$-QID of $\check{\mathcal{E}}$.*

(2) *$\check{\mathcal{K}}$ is a semiprime $(m,n)$-QID of $\check{\mathcal{E}}$ if and only if $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a semiprime TF $(m,n)$-QID of $\check{\mathcal{E}}$.*

*Proof.* (1) Suppose that $\check{\mathcal{K}}$ is a prime $(m,n)$-QID of $\check{\mathcal{E}}$. Then $\check{\mathcal{K}}$ is an $(m,n)$-QID of $\check{\mathcal{E}}$. Thus, by Theorem 3.2 $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a TF $(m,n)$-QID of $\check{\mathcal{E}}$. Let $\check{\mathfrak{e}}, \check{\mathfrak{h}} \in \check{\mathcal{E}}$.

Case 1: If $\check{\mathfrak{e}}\check{\mathfrak{h}} \in \check{\mathcal{K}}$, then $\check{\mathfrak{e}} \in \check{\mathcal{K}}$ or $\check{\mathfrak{h}} \in \mathcal{K}$. Thus, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = 1 = \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) = \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$, $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = 0 = \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) = \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = -1 = \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) = \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$. Hence, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \leq \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) \vee \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$, $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \geq \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) \wedge \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \geq \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) \wedge \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$.

Case 2: If $\check{\mathfrak{e}}\check{\mathfrak{h}} \notin \mathcal{K}$, then $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = 0$, $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = 1$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = 0$. Thus, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \leq \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) \vee \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$, $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \geq \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) \wedge \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \geq \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) \wedge \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$.

Therefore, $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a prime TF $(m,n)$-QID of $\check{\mathcal{E}}$.

Conversely, suppose that $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{\nu}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a prime TF $(m,n)$-QID of $\check{\mathcal{E}}$. Thus, by Theorem 3.2, $\mathcal{K}$ is an $(m,n)$-QID of $\check{\mathcal{E}}$. Let $\check{\mathfrak{e}}, \check{\mathfrak{h}} \in \check{\mathcal{E}}$ with $\check{\mathfrak{e}}\check{\mathfrak{h}} \in \check{\mathcal{K}}$. Then, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = 1$, $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = 0$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = -1$. If $\check{\mathfrak{e}} \notin \check{\mathcal{K}}$ and $\check{\mathfrak{h}} \notin \mathcal{K}$, then $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) = 0 = \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$, $\check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) = 1 = \check{\nu}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$ and

$\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) = 0 = \check{\delta}_{\check{\mathcal{K}}}(\mathfrak{h})$. By assumption, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \leq \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) \vee \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{h}})$, $\check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \geq \check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) \wedge \check{v}_{\check{\mathcal{K}}}(\mathfrak{h})$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) \geq \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) \wedge \check{\delta}_{\check{\mathcal{K}}}(\mathfrak{h})$. Thus, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = 0$, $\check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = 1$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}\check{\mathfrak{h}}) = 0$. It is a contradiction, so $\check{\mathfrak{e}} \in \check{\mathcal{K}}$ or $\mathfrak{h} \in \mathcal{K}$. Hence, $\check{\mathcal{K}}$ is a prime $(m,n)$-QID of $\check{\mathcal{E}}$.

(2) Suppose that $\check{\mathcal{K}}$ is a semiprime $(m,n)$-QID of $\check{\mathcal{E}}$. Then $\check{\mathcal{K}}$ is an $(m,n)$-QID of $\check{\mathcal{E}}$. Thus, by Theorem 3.2 $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{v}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a TF $(m,n)$-QID of $\check{\mathcal{E}}$. Let $\check{\mathfrak{e}} \in \check{\mathcal{E}}$.

Case 1: If $\check{\mathfrak{e}}^2 \in \check{\mathcal{K}}$, then $\check{\mathfrak{e}} \in \check{\mathcal{K}}$. Thus, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = 1 = \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}})$, $\check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = 0 = \check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}})$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = -1 = \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}})$. Hence, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) \leq \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}})$, $\check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) \geq \check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}})$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) \geq \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}})$.

Case 2: If $\check{\mathfrak{e}}^2 \notin \mathcal{K}$, then $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = 0$, $\check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = 1$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = 0$. Thus, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) \leq \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}})$, $\check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) \geq \check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}})$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) \geq \check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}})$.

Therefore, $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{v}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a semiprime TF $(m,n)$-QID of $\check{\mathcal{E}}$.

Conversely, suppose that $\mathcal{TF}_{\check{\mathcal{K}}} := (\check{\rho}_{\check{\mathcal{K}}}, \check{v}_{\check{\mathcal{K}}}, \check{\delta}_{\check{\mathcal{K}}})$ is a semiprime TF $(m,n)$-QID of $\check{\mathcal{E}}$. Thus, by Theorem 3.2, $\mathcal{K}$ is an $(m,n)$-QID of $\check{\mathcal{E}}$. Let $\check{\mathfrak{e}} \in \check{\mathcal{E}}$ with $\check{\mathfrak{e}}^2 \in \check{\mathcal{K}}$. Then, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = 1$, $\check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = 0$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = -1$. If $\check{\mathfrak{e}} \notin \check{\mathcal{K}}$, then $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) = 0$, $\check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) = 1$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}) = 0$. By assumption, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) \leq \check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}})$, $\check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) \geq \check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}})$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) \geq \check{\delta}(\check{\mathfrak{e}})$. Thus, $\check{\rho}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = 0$, $\check{v}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = 1$ and $\check{\delta}_{\check{\mathcal{K}}}(\check{\mathfrak{e}}^2) = 0$. It is a contradiction, so $\check{\mathfrak{e}} \in \check{\mathcal{K}}$. Hence, $\check{\mathcal{K}}$ is a semiprime $(m,n)$-QID of $\check{\mathcal{E}}$.

□

## 4. CONCLUSION

In this paper, we introduce the concept of tripolar fuzzy $(m,n)$-quasi-ideals in semigroups and investigate their properties. Additionally, we establish the relationship between $(m,n)$-quasi-ideals and tripolar fuzzy $(m,n)$-quasi-ideals. In the future, we plan to explore hybrid almost $(m,n)$-ideals and $n$-interior ideals in semigroups or within the algebraic context.

**Acknowledgments.** This research was supported by the Rajamangala University Technology Lanna, Phitsanulok, Thailand (Fundamental Fund 2026, Grant No. FF2569075).

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

## REFERENCES

[1] M.A. Ansari, M.R. Khan, J.P. Kaushik, A Note on $(m,n)$-Quasi-Ideals in Semigroups, Int. J. Math. Anal. 3 (2009), 1853–1858.

[2] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 20 (1986), 87–96. https://doi.org/10.1016/s0165-0114(86)80034-3.

[3] T. Changphas, On $(m,n)$-Ideals of an Ordered Semigroup, Int. J. Pure Appl. Math. 100 (2015), 1–5. https://doi.org/10.12732/ijpam.v100i1.1.

[4] V. Chinnadurai, K. Arulmozhi, Characterization of Bipolar Fuzzy Ideals in Ordered Gamma Semigroups, J. Int. Math. Virtual Inst. 8 (2018), 141–156.

[5] T. Gaketem, P. Khamrot, On Some Semigroups Characterized in Terms of Bipolar Fuzzy Weakly Interior Ideals, IAENG Int. J. Comput. Sci. 48 (2021), 250–256.

[6] C.S. Kim, J.G. Kang, J.M. Kang, Ideal Theory of Semigroups Based on the Bipolar Valued Fuzzy Set Theory, Ann. Fuzzy Math. Inform. 2 (2012), 193–206.

[7] P. Khamrot, M. Siripitukdet, On Properties of Generalized Bipolar Fuzzy Semigroups, Songklanakarin J. Sci. Technol. 41 (2019), 405–413.

[8] N. Kuroki, On Fuzzy Ideals and Fuzzy Bi-Ideals in Semigroups, Fuzzy Sets Syst. 5 (1981), 203–215. https://doi.org/10.1016/0165-0114(81)90018-x.

[9] P. Khamrot, A. Iampun, T. Gaketem, Tripolar Fuzzy Ideals Which Coincide in Semigroups, IAENG Int. J. Appl. Math. 55 (2025), 361–368.

[10] P. Khamrot, S. Saekuai, T. Phongthai, T. Gaketem, Ordered Almost $(m, n)$-Ideal and Fuzzy Ordered Almost $(m, n)$-Ideals in Ordered Semigroups, IAENG Int. J. Appl. Math. 55 (2025), 2342–2348.

[11] T. Promai, A. Iampun, T. Gaketem, Tripolar Fuzzy Ideals in Semigroups, IAENG Int. J. Appl. Math. 54 (2024), 2775–2782.

[12] P. Khamrot, A. Iampun, T. Gaketem, Tripolar Fuzzy Bi-Ideals Which Coincide in Semigroups, IAENG Int. J. Appl. Math. 55 (2025), 1972–1979.

[13] T. Gaketem, P. Khamrot, Generalized Interval Valued Fuzzy Ideals in Semigroups, J. Math. Comput. Sci. 34 (2024), 116–127. https://doi.org/10.22436/jmcs.034.02.02.

[14] P. Khramrot, A. Iampan, T. Gaketem, Fuzzy $(m, n)$-Ideals and $n$-Interior Ideals in Ordered Semigroups, Eur. J. Pure Appl. Math. 18 (2025), 5596. https://doi.org/10.29020/nybg.ejpam.v18i1.5596.

[15] P. Khamrot, A. Iampan, T. Gaketem, Bipolar Fuzzy $(m, n)$-Ideals and $n$-Interior Ideals of Semigroups, IAENG Int. J. Comput. Sci. 52 (2025), 598–605.

[16] P. Khramrot, P. Chaisuwan, P. Keawton, C. Wangsamphao, T. Gaketem, Almost $(m, n)$-Quasi-Ideals and Fuzzy Almost $(m, n)$-Quasi-Ideals in Ordered Semigroups, Eur. J. Pure Appl. Math. 18 (2025), 5837. https://doi.org/10.29020/nybg.ejpam.v18i2.5837.

[17] A. Mahboob, M. Al-Tahan, G. Muhiuddin, Characterizations of Ordered Semigroups in Terms of Fuzzy $(m, n)$-Substructures, Soft Comput. 28 (2024), 10827–10834. https://doi.org/10.1007/s00500-024-09880-z.

[18] G. Muhiuddin, A. Mahboob, N.M. Khan, D. Al-Kadi, New Types of Fuzzy $(m, n)$-Quasi-Ideals in Ordered Semigroups, J. Intell. Fuzzy Syst. 41 (2021), 6561–6574.

[19] M.M.K. Rao, Tripolar Fuzzy Interior Ideals of Γ-Semigroup, Ann. Fuzzy Math. Inform. 15 (2018), 199–206. https://doi.org/10.30948/afmi.2018.15.2.199.

[20] M.M.K. Rao, Tripolar Fuzzy Ideals of a Γ-Semiring, Asia Pac. J. Math. 15 (2018), 192–207.

[21] N. Wattanasiripong, J. Mekwian, H. Sanpan, S. Lekkoksung, On Tripolar Fuzzy Pure Ideals in Ordered Semigroups, Int. J. Anal. Appl. 20 (2022), 49. https://doi.org/10.28924/2291-8639-20-2022-49.

[22] N. Wattanasiripong, N. Lekkoksung, S. Lekkoksung, On Tripolar Fuzzy Interior Ideals in Ordered Semigroups, Int. J. Innov. Comput. Inf. Control 18 (2022), 1291–1304.

[23] N. Wattanasiripong, N. Lekkoksung, S. Lekkoksung, On Tripolar Fuzzy Interior Ideals in Ordered Semigroups, Int. J. Innov. Comput. Inf. Control 18 (2022), 1291–1304.

[24] L.A. Zadeh, Fuzzy Sets, Inf. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.

[25] W.R. Zhang, Bipolar Fuzzy Sets and Relations: A Computational Framework for Cognitive Modeling and Multiagent Decision Analysis, in: Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference, IEEE, pp. 305–309. https://doi.org/10.1109/ijcf.1994.375115.