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Noncompactness Technique to Hilfer-Katugampola Non-Instantaneous Impulses Model with Varying Lower Limit

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Abstract. The primary objective of this paper is to present some results about the existence of solutions for a class of initial value problem using Hilfer-Katugampola derivative with varying lower limit for fractional non-instantaneous impulse models. It is possible to establish the coincidence degree theory and obtain solvability by carefully constructing significant operators that include impulsive terms that are not instantaneous. We use inequalities and nonlinear analytic techniques to investigate the stability of solutions. In the simplest case, we automatically found the solvability and stability of the Hilfer-Katugampola equations for impulses that are non-instantaneously present. Measurement techniques are utilized to determine noncompactness using Sadovskii's fixed point theorem. Instead of using anything else, we use Krasnoselskii's fixed point theorem to construct a concise approach. An example of clarifying the results is provided by it.

1. Introduction

Real-world processes that react to external impulses over non-zero intervals and have non-local behaviors or interactions can be described by fractional non-instantaneous impulses systems or differential equations [1,2].

The area of fractional instantaneous impulses provides a trustee tool to explain abrupt variations or discontinuous jumps in the development of dynamic systems, including shocks, disturbances, and natural disasters. Compared to the total time of the activity, the impulsive influence in instantaneous impulses lasts for a relatively brief time. Generally speaking, traditional instantaneous impulses cannot provide a satisfactory description of certain dynamics of evolution processes. Let us take for example, when we examine a person's hemodynamic equilibrium, we can see

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that drug administration and the body's subsequent absorption are both progressive and continuous processes. As a matter of fact, the new non-instantaneous impulsive model might fit this situation [3–9] and the references cited therein.

There is no doubt that the study of fractional derivatives with varying lower limits for non-instantaneous impulses, enables us to investigate the phenomenon with a different differential equation that describes its behavior in each period independently. This corresponds to many real-life problem in several scientific fields.

Many academics have produced intriguing findings in the last few decades about the availability of fixed point theorem-based solutions for various boundary value issues. It might be difficult with contributions to give exact solutions to equations with nonlinear terms. Consequently, various approximation methods were presented in this context. Ulam established HU stability in 1940, which aids in differentiating between exact and approximate solutions. Numerous academics have studied the stability of solutions to fractional equations using this method. It is widely used in many different sectors [10–15].

By reformulating the problems as fixed point problems and solving them with a fixed point justification, it is common to establish the existence of an operator equation. The classification of compact operators in Banach spaces, and metric fixed point theory are just a few of their applications. Numerous formulas of optimization, nonlinear analysis, integral, differential and integro-differential equations, fixed point theory and other fields all heavily rely on measures of non-compactness [16,17].

The study of fractional order differential equations has acquired popularity in recent decades. This is vindicated by the truth that fractional order derivatives offer useful instruments for detecting inherited or particular characteristics in a variety of engineering and scientific fields.

Fractional derivatives abound and include Hadamard, Caputo, Hilfer, Riemann-Liouville, and many more. The Hilfer-Katugampola-type fractional derivative is an intriguing variation of the fractional derivative. It is a generalization of the widely recognized fractional derivatives of Caputo and R-L. The term "Hilfer-Katugampola" alludes to the significant contributions to fractional calculus made by Hilfer and Katugampola. A special formula that enables the characterization of complicated, non-local behavior in mathematical models defines this type of fractional derivative. This kind is unique in that it combines a power of the time variable with the fractional order, providing a more flexible method for modeling and evaluating complicated systems.

Numerous disciplines, such as fractional calculus, mathematical physics, and engineering, have found use for the Hilfer-Katugampola-type fractional derivative because it is an effective tool for characterizing and comprehending the behavior of systems with fractional order dynamics. Since its introduction, fractional calculus has gained further tools, and new mathematical methodologies for simulating and examining real-world processes have been developed [18–23].

Abbas [24] discussed the Hilfer-Katugampola model with the form

$$\begin{cases} {}^{\rho}\mathcal{D}_{0^+}^{\alpha,\beta} y(\xi) = h(\xi, y(\xi)) + By(\xi), & \xi \in [0,b], \\ {}^{\rho}\mathcal{I}_{0^+}^{1-\sigma} y(0) = y_0, & \sigma = \beta + \alpha(1-\beta), \end{cases}$$

where ${}^{\rho}\mathcal{D}_{0^+}^{\alpha,\beta}$ denotes the Hilfer-Katugampola derivative with $0 < \alpha < 1$ and $0 \leq \beta \leq 1$ and ${}^{\rho}\mathcal{I}_{0^+}^{1-\sigma}$ is left-sided Katugampola fractional integral. Combining Sadovskii's settled point theorem andKuratowski's noncompactness measure allowed them to establish necessary conditions for the controllability of Hilfer-Katugampola models in Banach spaces.

In [25], Li et al. analyzed the stability of a fractional model that contains a non-instantaneous integral impulse and multiple point conditions of the particular form

$$\begin{cases} {}^{c}D^{\beta} z(\xi)(D+\lambda) = g(\xi, z(\xi), {}^{c}D^{\alpha} z(\xi)), & \xi \in (\xi_j, s_j], j = 0,1,\ldots,n, \\ z(\xi) = I_{s_{j-1},\xi_j}^{\alpha}(\phi_j(\xi, z(\xi))), & \xi \in (s_{j-1}, \xi_j],\ j = 1,\ldots,n, \\ az(\xi_j) + cz(s_j) = b, z(0) = 0, \end{cases}$$

where ${}^{c}D^{\beta}$ is a reference to Caputo derivatives with order $\beta \in (0,1]$, D indicates the ordinary derivative. They studied the existence and uniqueness of the fractional model involving non-instantaneous integral impulse by applying fixed point theorem due to Diaz-Margolis. They also discussed the Ulam stability for their system.

In [26], Wang et al. explored the UHs, uniqueness and the existence of the following FFDEs

$$\begin{cases} {}^{c}D_{0^+}^{\alpha} T(\mu) = g(\mu, T(\mu), T(\mu - x)), & \mu \in Y, \\ T(\mu) = \Psi(\mu), & \mu \in (-x, 0], \end{cases}$$

where ${}^{c}D_{0^+}^{\alpha}$ is the Caputo derivative with order $0 < \alpha < 1$, $Y = [0,L]$, $T(0) = \Psi(0) = \Psi_0$, $g : [0,L] \times E_c \times E_c \to E_c$ and $\Psi : (-x,0] \to E_c$ where $E_c = \{\vartheta : \mathbb{R} \to [0,1]\}$. They looked into whether solutions could be found by utilizing Schauder's fixed point theorem and a presumptive condition. Furthermore, they demonstrated the system's uniqueness by employing the Banach contraction and also they discussed UHs using generalized Gronwall inequality.

Motivated by the former contributions, we examine the solutions' existence and UHR stability for fractional non-instantaneous impulses equation. Particularly, we scrutinize the next equation with changed lower limit at the end of each interval of acting of the impulse

$$\begin{cases} {}^{\rho}\mathcal{D}_{0}^{\beta,\alpha} y(\tau) = f(\tau, y(\tau), {}^{\rho}\mathcal{D}_{0}^{\beta,\alpha} y(\tau)), & \text{for} \quad \tau \in [s_0, \tau_1], \\ {}^{\rho}\mathcal{D}_{s_r}^{\beta,\alpha} y(\tau) = f(\tau, y(\tau), {}^{\rho}\mathcal{D}_{s_r}^{\beta,\alpha} y(\tau)), & \text{for} \quad \tau \in (s_r, \tau_{r+1}], r = 1,\ldots,k, \\ y(\tau) = h_r(\tau, y(\tau)), & \text{for} \quad \tau \in (\tau_r, s_r], r = 1,\ldots,k, \\ {}^{\rho}\mathcal{I}_{s_r}^{1-\nu} y(s_r) = \mu_r y(\eta_r) + \mathfrak{D}_r, & \text{for} \quad \eta_r \in (s_r, \tau_{r+1}), r = 0,1,\ldots,k,\ \mathfrak{D}_r \in \mathbb{R}. \end{cases} \tag{1.1}$$

where $k \in \mathbb{N}_0$, all subintervals are subset of $J = [s_0, \tau_1] \bigcup_{r=1}^{k} (\tau_r, s_r] \bigcup_{r=1}^{k} (s_r, \tau_{r+1}] = [0, a]$, ${}^{\rho}\mathcal{D}_{s_r}^{\beta,\alpha}$ is the Hilfer-Katugampola model of order $0 < \beta < 1$ and types $\alpha \in [0,1]$, $\rho > 0$ and ${}^{\rho}I_{s_r}^{1-\nu}$ is the Katugampola fractional integral with $0 < \nu = \beta + \alpha - \alpha\beta < 1$. The functions $f: J_r \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. Also, $J_0 = [s_0, \tau_1]$, $J_r = (s_r, \tau_{r+1}]$, $r = 1, \ldots, k$, $J_r^* = (\tau_r, s_r]$, $r = 1, 2, \ldots, k$ and $0 = s_0 < \tau_1 < s_1 < \cdots < \tau_k < s_k < \tau_{k+1} = a$ are real numbers. Here, $h_r : J_r^* \times \mathbb{R} \to \mathbb{R}$, $r = 1, \ldots, k$ are non-instantaneous impulses, $\mathfrak{D}_r \in \mathbb{R}$ and $\eta_r \in (s_r, \tau_{r+1})$ with $r = 0, 1, \ldots, k; k \in \mathbb{N}$.

Novelty and Motivations: Here, we offer a few innovations and explanations that highlight the significance of our work:

- Fractional derivatives can be approached in a variety of ways, including Caputo, Hilfer, Hadamard, Riemann-Liouville, Caputo-Fabrizio, Caputo-Hadamard, etc. We concentrate on examining the Caputo-Katugampola technique, which represents a generalization to several of them, rather than analyzing each one individually.
- As far as we are aware, very few studies address fractional models with varying lower limits for non-instantaneous impulses. This indicates that our notion is an unexplored subject in the literature, and our paper will be primarily motivated by this.
- The study of fractional models with varying lower limits for non-instantaneous impulses allows us to examine the phenomenon according to a different differential equation that describes its behavior in each period separately.
- A broad range of applications are covered by the study of these systems. Analyzing higher order fractional damped systems is more logical from a physical point of view.
- The criteria under which the same sort of result is valid for time-varying systems will be examined in relation to the defined stability results.
- Consequently, this work shows how certain pertinent references that are already well-known in the literature could be improved, both theoretically and statistically.

The sections listed below are the division of our article: In Section 2, fundamental concepts and learning outcomes are provided. Two existence results for the model (1.1) can be found in Section 3: The first is demonstrated using Sadovskii's fixed point theorems related to measuring noncompactness, while the other is demonstrated by Krasnoselskii's fixed point theorem. Section 4 discusses UHR stability of our model. Finally, numerical applications are provided for the major results.

2. Preliminaries

To support the reader, we offer some fundamental definitions and lemmas that will be utilized throughout this investigation. Take $J = [0, a]$. Let the Banach space $C(J, \mathbb{R})$ contains all continuous functions on J equipped with the norm $\|z\|_C = \sup_{\tau \in J} |z(\tau)|$. The weighted Banach space can be defined as

$$C_{\nu,\rho}(J_r,\mathbb{R})=\left\{y:J_r\to\mathbb{R}:y(\tau)\left(\frac{\tau^\rho-s_r^\rho}{\rho}\right)^\nu\in C(J_r,\mathbb{R})\right\},\quad 0\le\nu<1,$$

equipped with the norm

$$\|y\|_{C_{\nu,\rho}}=\sup_{\tau\in J_r}\left|\left(\frac{\tau^\rho-s_r^\rho}{\rho}\right)^\nu y(\tau)\right|.$$

Take into consideration the space $X_d^p(a,b)$ (for $d\in\mathbb{R}$, $1\le p\le\infty$, $0\le a<b\le\infty$) which contains Lebesgue measurable functions x on $[a,b]$ with complex-valued with $\|x\|_{X_d^p}<\infty$, and

$$\|x\|_{X_d^p}=\left(\int_a^b\left|\xi^d x(\xi)\right|^p\frac{d\xi}{\xi}\right)^{\frac{1}{p}},\quad(1\le p<\infty,d\in\mathbb{R}).$$

Definition 2.1 ([27] Katugampola fractional integral)**.** *Let $\beta\in\mathbb{R}_+$, $d\in\mathbb{R}$, $0\le a<b\le\infty$ and $z\in X_d^p(a,b)$. The Katugampola integral of order β is given as*

$$\left({}^\rho\mathcal{I}_{a^+}^\beta z\right)(x)=\int_a^x s^{\rho-1}\left(\frac{x^\rho-s^\rho}{\rho}\right)^{\beta-1}\frac{z(s)}{\Gamma(\beta)}ds,\quad x>a,\rho>0.$$

Theorem 2.1 ([27])**.** *Suppose that $\beta>0$, $\alpha>0$, $0<a<b<\infty$, $1\le p\le\infty$, and $\rho,d\in\mathbb{R}$ with $\rho\ge d$. Then, the semigroup property is valid for $z\in X_c^p(a,b)$, i.e.*

$$\left({}^\rho\mathcal{I}_{a^+}^\alpha\,{}^\rho\mathcal{I}_{a^+}^\beta z\right)(x)=\left({}^\rho\mathcal{I}_{a^+}^{\alpha+\beta}z\right)(x).$$

Definition 2.2 ([27] Katugampola fractional derivative)**.** *Let $\rho>0$, $\beta\in\mathbb{R}_+$ with $\beta\notin\mathbb{N}$, and $z\in C_{\nu,\rho}(J)$. The Katugampola derivative ${}^\rho\mathcal{D}_{a^+}^\beta$ of order β is presented as*

$$\begin{aligned}\left({}^\rho\mathcal{D}_{a^+}^\beta z\right)(x)&=\delta_\rho^n\left({}^\rho I_{a^+}^{n-\beta}z\right)(x)\\&=\left(x^{1-\rho}\frac{d}{dx}\right)^n\int_a^x\xi^{\rho-1}\left(\frac{x^\rho-\xi^\rho}{\rho}\right)^{n-\beta-1}\frac{z(\xi)}{\Gamma(n-\beta)}d\xi,\quad x>a,\rho>0,\end{aligned}$$

where $n=[\beta]+1$ and $\delta_\rho=x^{1-\rho}\frac{d}{dx}$.

Lemma 2.1 ([27])**.** *Consider $\beta>0$, $0\le\nu<1$ and $z\in C_{\nu,\rho}[a,b]$. Then,*

$$\left({}^\rho\mathcal{D}_{a^+}^\beta\,{}^\rho\mathcal{I}_{a^+}^\beta z\right)(x)=z(x),\qquad x\in(a,b].$$

Lemma 2.2 ([27])**.** *Let $0<\beta<1$ and $0\le\nu<1$. If $z\in C_{\nu,\rho}[a,b]$ and ${}^\rho\mathcal{I}_{a^+}^{1-\beta}z\in C_{\nu,\rho}[a,b]$. Then,*

$$\left({}^\rho\mathcal{I}_{a^+}^\beta\,{}^\rho\mathcal{D}_{a^+}^\beta z\right)(x)=z(x)-\frac{\left({}^\rho\mathcal{I}_{a^+}^{1-\beta}z\right)(a)}{\Gamma(\beta)}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\beta-1},\qquad x\in(a,b].$$

Lemma 2.3 ([28,29]). *Let $x > a$. Then, for $\beta > 0$ and $\mu > 0$, we have*

$$\left[{}^{\rho}\mathcal{I}^{\beta}_{a^+}\left(\frac{s^\rho - a^\rho}{\rho}\right)^{\mu-1}\right](x) = \frac{\Gamma(\mu)}{\Gamma(\mu+\beta)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\mu+\beta-1},$$

$$\left[{}^{\rho}\mathcal{D}^{\beta}_{a^+}\left(\frac{s^\rho - a^\rho}{\rho}\right)^{\mu-1}\right](x) = \frac{\Gamma(\mu)}{\Gamma(\mu-\beta)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\mu-\beta-1},$$

$$\left[{}^{\rho}\mathcal{D}^{\beta}_{a^+}\left(\frac{s^\rho - a^\rho}{\rho}\right)^{\beta-1}\right](x) = 0.$$

Lemma 2.4 ([30]). *Consider $\beta > 0$ and $0 \le \nu < 1$. Then, ${}^{\rho}\mathcal{I}^{\beta}_{a^+}$ is bounded from $C_{\nu,\rho}(J)$ into $C_{\nu,\rho}(J)$.*

Lemma 2.5 ([30]). *Let $0 \le a < b \le \infty$, $\beta > 0$, $0 \le \nu < 1$ and $y \in C_{\nu,\rho}(J)$. If $\beta > \nu$, then ${}^{\rho}\mathcal{I}^{\beta}_{a^+}y$ is continuous on J and*

$$\left({}^{\rho}\mathcal{I}^{\beta}_{a^+}y\right)(a) = \lim_{x\to a^+}\left({}^{\rho}\mathcal{I}^{\beta}_{a^+}y\right)(x) = 0$$

Definition 2.3 (Hilfer-Katugampola fractional derivative [31]). *Consider $n-1 < \beta < n$ and $0 \le \alpha \le 1$ with $n \in \mathbb{N}$. The Hilfer-Katugampola derivative of a function $z \in C_{1-\nu,\rho}[a,b]$ is provided as*

$$\begin{aligned}\left({}^{\rho}\mathcal{D}^{\beta,\alpha}_{a^+}z\right)(x) &= \left({}^{\rho}\mathcal{I}^{\alpha(n-\beta)}_{a^+}\left(x^{\rho-1}\frac{d}{dx}\right)^{n}{}^{\rho}\mathcal{I}^{(1-\alpha)(n-\beta)}_{a^+}z\right)(x)\\ &= \left({}^{\rho}\mathcal{I}^{\alpha(n-\beta)}_{a^+}\delta^{n}_{\rho}{}^{\rho}\mathcal{I}^{(1-\alpha)(n-\beta)}_{a^+}z\right)(x).\end{aligned}$$

Property 2.1 ([31]). *The operator ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_r^+}, r = 0,1,\ldots,k$ with order $0 < \alpha < 1$ can be written as*

$${}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+} = {}^{\rho}\mathcal{I}^{\alpha(1-\beta)}_{s_r^+}\delta_\rho{}^{\rho}\mathcal{I}^{1-\nu}_{s_r^+} = {}^{\rho}\mathcal{I}^{\alpha(1-\beta)}_{s_r^+}{}^{\rho}\mathcal{D}^{\nu}_{s_r^+}, \quad \nu = \beta + \alpha(1-\beta).$$

Definition 2.4. *Let the parameters β, α and ν be related as*

$$\nu = \beta + \alpha(1-\beta), \quad 0 < \alpha, \beta, \nu < 1.$$

Therefore, we provide the spaces

$$C^{\beta,\alpha}_{1-\nu,\rho}(J_r) = \left\{y \in C_{1-\nu,\rho}(J_r), {}^{\rho}\mathcal{D}^{\beta,\alpha}_{a^+}y \in C_{1-\nu,\rho}(J_r)\right\},$$

$$C^{\nu}_{1-\nu,\rho}(J_r) = \left\{y \in C_{1-\nu,\rho}(J_r), {}^{\rho}\mathcal{D}^{\nu}_{s_r^+}y \in C_{1-\nu,\rho}(J_r)\right\}.$$

Since ${}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+}y = {}^{\rho}\mathcal{I}^{\alpha(1-\beta)}_{s_r^+}{}^{\rho}\mathcal{D}^{\nu}_{s_r^+}y$, by Lemma 2.1, it follows that

$$C^{\nu}_{1-\nu,\rho}(J) \subset C^{\beta,\alpha}_{1-\nu,\rho}(J) \subset C_{1-\nu,\rho}(J).$$

It is clear that the space $C^{\nu}_{1-\nu,\rho}(J)$ is a Banach space equipped with the norm

$$\|y\|_{C^{\nu}_{1-\nu,\rho}} = \max_{r=0,\ldots,k}\left\{\sup_{\tau\in J_r}\left|\left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{1-\nu}y(\tau)\right|\right\}.$$

Finally, take into consideration the space

$$\mathcal{PC}_{\nu,\rho}(J,\mathbb{R}) = \left\{y : J \to \mathbb{R} \mid y \in C^{\nu}_{1-\nu,\rho}(\cup_{r=0}^{k}J_r) \cap C(\cup_{r=1}^{k}J_r^*)\right\}.$$

which can be easily to prove it is a Banach space equipped with the norm

$$\|y\|_{\mathcal{PC}_{\nu,\rho}} = \max\left\{\|y\|_{C^{\nu}_{1-\nu,\rho}}, \max_{r=1,\dots,k}\left\{\sup_{\tau\in J^*_r}\left|y(\tau)\right|\right\}\right\}.$$

Lemma 2.6 ([31]). *Suppose* $0<\beta<1$, $0\le\alpha\le 1$ *and* $\nu=\beta+\alpha(1-\beta)$. *If* $y\in C^{\nu}_{1-\nu,\rho}(J_r)$. *Then,*

$${}^{\rho}\mathcal{I}^{\nu}_{s_r^+}\,{}^{\rho}\mathcal{D}^{\nu}_{s_r^+}y = {}^{\rho}\mathcal{I}^{\beta}_{s_r^+}\,{}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+}y,$$

and

$${}^{\rho}\mathcal{D}^{\nu}_{s_r^+}\,{}^{\rho}\mathcal{I}^{\beta}_{s_r^+}y = {}^{\rho}\mathcal{D}^{\alpha(1-\beta)}_{s_r^+}y.$$

It is common knowledge that the Kuratowski measure of noncompactness has these properties.

Definition 2.5 ([32]). *Assume* $\mathscr{S}$ *is a bounded subset of a Banach space* $\mathscr{E}$. *The Kuratowski noncompactness' measure* $\mu(\cdot)$ *is introduced on bounded* $\mathscr{S}$ *as*

$$\mu(\mathscr{S}) = \inf\left\{\iota>0:\ \mathscr{S}\subset\bigcup_{j=0}^{k}\mathscr{S}_j,\ \mathscr{S}_j\subset\mathscr{E},\ diam(\mathscr{S}_j)<\iota,\ j=1,\dots,n;\ n\in\mathbb{N}\right\}.$$

where

$$diam(\mathscr{S}_j) = \sup\{\|y_1-y_2\|, y_1,y_2\in\mathscr{S}_j\}.$$

Lemma 2.7 ([32]). *Let* $\mathscr{S}$ *and* $\mathscr{U}$ *be bounded in a Banach space* $\mathscr{E}$. *The following properties hold:*

- : *(1)* $\mu(\mathscr{S})=0$, *iff* $\bar{\mathscr{S}}$ *is compact, (*$\bar{\mathscr{S}}$ *the closure set);*
- : *(2)* $\mu(\mathscr{S})=\mu(\bar{\mathscr{S}})=\mu(conv\mathscr{S})$, *(*$conv\mathscr{S}$ *the convex set);*
- : *(3)* $\mu(\varrho\mathscr{S}))=|\varrho|\mu(\mathscr{S})$ *for any* $\sigma\in\mathbb{R}$;
- : *(4)* $\mathscr{S}\subset\mathscr{U}$ *leads to* $\mu(\mathscr{S})\le\mu(\mathscr{U})$;
- : *(5)* $\mu(\mathscr{S}+\mathscr{U})\le\mu(\mathscr{S})+\mu(\mathscr{U})$, *where* $\mathscr{S}+\mathscr{U}=\{x|x=y+z, y\in\mathscr{S}, z\in\mathscr{U}\}$;
- : *(6)* $\mu(\mathscr{S}\cup\mathscr{U})=\max\{\mu(\mathscr{S}),\mu(\mathscr{U})\}$;
- : *(7) Assume the map* $G: D(G)\subset\mathscr{E}\to\mathcal{Y}$ *is continuous Lipschitz with constant* κ. *Then,* $\mu(G(\mathcal{N}))\le\kappa\mu(\mathcal{N})$ *where* $\mathcal{N}\in D(G)$ *is bounded subset and* $\mathcal{Y}$ *is Banach space.*

Lemma 2.8 (Sadovskii fixed point theorem [32]). *Let* Υ *be closed and convex and bounded subset of a Banach space* $\mathcal{E}$. *Assume that* $\Phi:\Upsilon\to\Upsilon$ *is continuous* μ*-condensing operator, in other words* $\mu(\Phi(\Upsilon))<\mu(\Upsilon)$. *Then,* Φ *has at least one fixed point in* Υ.

We, also, present some definitions and properties of Ulam-Hyers-Rassias stability taking from [33].

Definition 2.6. *Problem (1.1) is UHR stable with respect to* $\psi\in C(J,\mathbb{R}_+)$ *if there is a positive constant* $\mathfrak{c}_u>0$ *where for all* $\varepsilon>0$, $\varphi>0$ *and for all solution* $w\in\mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$ *of the inequality*

$$\begin{cases}\left|{}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+}w(\tau)-f\left(\tau,w(\tau),{}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+}w(\tau)\right)\right|\le\varepsilon\psi(\tau), & t\in J_r, r=0,1,\dots,k,\\ \left|w(\tau)-h_r(\tau,w(\tau))\right|\le\varepsilon\varphi, & t\in J^*_r, r=1,\dots,k.\end{cases} \tag{2.1}$$

Then, there is a solution $y \in \mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$ of equation (1.1) such that

$$\|w - y\|_{\mathcal{PC}_{\nu,\rho}} \leq \mathfrak{c}_u \varepsilon(\psi(\tau) + \varphi), \quad \tau \in J.$$

Definition 2.7. *Problem (1.1) is GUHR stable with respect to $\psi \in C(J,\mathbb{R}_+), \varphi > 0$ if there is a positive constant $\mathfrak{c}_u > 0$ where for all solution $w \in \mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$ of the inequality*

$$\begin{cases} \left|{}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+} w(\tau) - f\left(\tau, w(\tau), {}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+} w(\tau)\right)\right| \leq \psi(\tau), & t \in J_r, r = 0,1,\ldots,k, \\ \left|y(\tau) - h_r(\tau, y(\tau))\right| \leq \varphi, & t \in J_r^*, r = 1,\ldots,k. \end{cases} \tag{2.2}$$

Then, there is a solution $y \in \mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$ of equation (1.1) such that

$$\|w - y\|_{\mathcal{PC}_{\nu,\rho}} \leq \mathfrak{c}_u(\psi(\tau) + \varphi), \quad \tau \in J.$$

Remark 2.1. *A map $w \in \mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$ is a solution of (2.1) if and only if there exists $\mathcal{Q} \in \mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$ (depending on w) where*

1. $|\mathcal{Q}(\tau)| \leq \varepsilon\psi(\tau), \quad \tau \in J_r, r = 0,\ldots,k, |\mathcal{Q}_r| \leq \varepsilon\varphi, \qquad \tau \in J_r^*, r = 1,\ldots,k,$
2. ${}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+} w(\tau) = f\left(\tau, w(\tau), {}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+} w(\tau)\right) + \mathcal{Q}(\tau), \qquad \tau \in J_r, r = 0,\ldots,k,$
3. $y(\tau) = h_r(\tau, y(\tau)) + \mathcal{Q}_r, \qquad \tau \in J_r^* r = 1,\ldots,k.$

It's possible to make a similar comment about inequality (2.2).

Lemma 2.9. *Assume $\nu = \beta + \alpha - \beta\alpha, 0 < \beta < 1$ and $\alpha \in [0,1]$. If $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function where $f(\cdot, y(\cdot), {}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+} y(\cdot)) \in C^{\nu}_{1-\nu,\rho}(J_r)$, for any $y \in C^{\nu}_{1-\nu,\rho}(J_r)$, then $y(\tau)$ verifies the equation*

$${}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+} y(\tau) = f(\tau, y(\tau), {}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+} y(\tau)), \ \tau \in (s_r, \tau_{r+1}], r = 0,1,\ldots,k; k \in \mathbb{N}, \tag{2.3}$$

if and only if it verifies the Volterra integral equation

$$y(\tau) = \frac{\left({}^{\rho}\mathcal{I}^{1-\nu}_{s_r^+} y\right)(s_r)}{\Gamma(\nu)} \left(\frac{\tau^{\rho} - s_r^{\rho}}{\rho}\right)^{\nu-1} + \left({}^{\rho}\mathcal{I}^{\beta}_{s_r^+} f(t, y(t), {}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+} y(t))\right)(\tau). \tag{2.4}$$

Proof. ($\Rightarrow$) Assume that $y \in C^{\nu}_{1-\nu,\rho}(J_r)$ is a solution of the equation (2.3). We show that y is too a solution of (2.4). Using Definition 2.3, Lemma 2.4 and from the definition of $C_{1-\nu,\rho}(J_r)$, we obtain

$$\left({}^{\rho}\mathcal{I}^{1-\nu}_{s_r^+} y\right)(\tau) \in C_{1-\nu,\rho}(J_r) \quad \text{and} \quad ({}^{\rho}\mathcal{D}^{\nu}_{s_r^+} y)(\tau) = \left(\delta_{\rho} {}^{\rho}I^{1-\nu}_{s_r^+} y\right)(\tau) \in C_{1-\nu,\rho}(J_r).$$

From Lemma 2.2, we can write

$$\left({}^{\rho}\mathcal{I}^{\nu}_{s_r^+} {}^{\rho}\mathcal{D}^{\nu}_{s_r^+} y\right)(\tau) = y(\tau) - \frac{\left({}^{\rho}\mathcal{I}^{1-\nu}_{s_r^+} y\right)(s_r)}{\Gamma(\nu)} \left(\frac{\tau^{\rho} - s_r^{\rho}}{\rho}\right)^{\nu-1}. \tag{2.5}$$

In light of Lemma 2.6, we can write

$$\left({}^{\rho}\mathcal{I}^{\nu}_{s_r^+} {}^{\rho}\mathcal{D}^{\nu}_{s_r^+} y\right)(\tau) = \left({}^{\rho}\mathcal{I}^{\beta}_{s_r^+} {}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+} y\right)(\tau) = \left({}^{\rho}\mathcal{I}^{\beta}_{s_r^+} f(t, y(t), {}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r} y(t))\right)(\tau). \tag{2.6}$$

When (2.5) and (2.6) are compared, we find that

$$y(\tau)=\frac{\left({}^{\rho}\mathcal{I}^{1-\nu}_{s_r^+}y\right)(s_r)}{\Gamma(\nu)}\left(\frac{t^{\rho}-s_r^{\rho}}{\rho}\right)^{\nu-1}+\left({}^{\rho}\mathcal{I}^{\beta}_{s_r^+}f(t,y(t),{}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r}y(t))\right)(\tau).$$

($\Leftarrow$) Assume that $y\in C^{\nu}_{1-\nu,\rho}(J_r)$ verifies (2.4), and we show that it also meets (2.3). By operating by ${}^{\rho}\mathcal{D}^{\nu}_{s_r^+}$ on both sides of (2.4). Then, from Lemmas 2.3 and 2.6 and Definition 2.3, we obtain

$$\left({}^{\rho}\mathcal{D}^{\nu}_{s_r^+}y\right)(\tau)=\left({}^{\rho}\mathcal{D}^{\alpha(1-\beta)}_{s_r^+}f(t,y(t),{}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r}y(t))\right)(\tau). \tag{2.7}$$

According to hypothesis ${}^{\rho}\mathcal{D}^{\nu}_{s_r^+}y\in C^{\nu}_{\nu,\rho}(J_r)$, (2.7) implies that

$$\left({}^{\rho}\mathcal{D}^{\nu}_{s_r^+}y\right)(t)=\left(\delta_{\rho}{}^{\rho}\mathcal{I}^{1-\alpha(1-\beta)}_{s_r^+}f\right)(\tau)=\left({}^{\rho}\mathcal{D}^{\alpha(1-\beta)}_{s_r^+}f\right)(\tau)\in C_{1-\nu,\rho}(J_r). \tag{2.8}$$

Since $f(\cdot,y(\cdot),{}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r^+}y(\cdot))\in C_{1-\nu,\rho}(J_r)$ and by Lemma 2.4, we get

$$\left({}^{\rho}\mathcal{I}^{1-\alpha(1-\beta)}_{s_r^+}f\right)\in C_{1-\nu,\rho}(J_r). \tag{2.9}$$

The relations (2.8) and (2.9) and the definition of $C_{1-\nu,\rho}(J_r)$ yield

$$\left({}^{\rho}\mathcal{I}^{1-\alpha(1-\beta)}_{s_r^+}f\right)\in C_{1-\nu,\rho}(J_r).$$

Applying the operator ${}^{\rho}\mathcal{I}^{\alpha(1-\beta)}_{s_r^+}$ on both sides of (2.8) with applying Lemmas 2.4 and 2.2 and Property 2.1, we obtain

$$\begin{aligned}\left({}^{\rho}\mathcal{I}^{\alpha(1-\beta)}_{s_r^+}{}^{\rho}\mathcal{D}^{\nu}_{s_r^+}y\right)(\tau)&=f(\tau,y(\tau),{}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r}y(\tau))-\frac{\left({}^{\rho}\mathcal{I}^{1-\alpha(1-\beta)}_{s_r^+}f\right)(s_r)}{\Gamma(\alpha(1-\beta))}\left(\frac{\tau^{\rho}-s_r^{\rho}}{\rho}\right)^{\alpha(1-\beta)-1}\\&=({}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r}y)(\tau)=f(\tau,y(\tau),{}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r}y(\tau))\end{aligned}$$

which implies that (2.3) holds. □

3. Existence Results

The current section presents our existence results for the problem (1.1). To demonstrate the main theorems, we begin with a significant results.

Theorem 3.1. *Assume that $k\in\mathbb{N}$ and $\nu=\beta+\alpha-\beta\alpha$ such that $0<\beta<1$ and $0\le\alpha\le1$. Let $v_r:J_r\to\mathbb{R}$ be a function where $v_r\in C^{\nu}_{1-\nu,\rho}(J_r)$ with $r=0,1,\dots,k$ and for any $y\in\mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$. Then, there exists a solution for the linear case*

$$\begin{aligned}{}^{\rho}\mathcal{D}^{\beta,\alpha}_{s_r}y(\tau)&=v_r(\tau),\quad \textit{for}\quad \tau\in(s_r,\tau_{r+1}],r=0,1,\dots,k\\ y(\tau)&=h_r(\tau,y(\tau)),\quad \textit{for}\quad \tau\in(\tau_r,s_r],r=1,2,\dots,k\\ {}^{\rho}\mathcal{I}^{1-\nu}_{s_r}y(s_r)&=\mu_r y(\eta_r)+\mathfrak{D}_r,\ s_r<\eta_r<\tau_{r+1},r=0,1,\dots,k\end{aligned} \tag{3.1}$$

given by

$$y(\tau)=\begin{cases}\Lambda_r\left(\dfrac{\tau^\rho-s_r^\rho}{\rho}\right)^{\nu-1}\left(\mu_r({}^\rho\mathcal{I}^\beta_{s_r}v_r)(\eta_r)+\mathfrak{D}_r\right)\\ +({}^\rho\mathcal{I}^\beta_{s_r}v_r)(\tau) & \tau\in(s_r,\tau_{r+1}], r=0,1,\dots,k,\\ h_r(\tau,y(\tau)), & \tau\in(\tau_r,s_r), r=1,\dots,k\end{cases}\tag{3.2}$$

where

$$\Lambda_r=\frac{1}{\Gamma(\nu)-\mu_r\left(\dfrac{\eta_r^\rho-s_r^\rho}{\rho}\right)^{\nu-1}},\qquad r=0,1,\dots,k,\tag{3.3}$$

and

$$\Gamma(\nu)\neq\mu_r\left(\frac{\eta_r^\rho-s_r^\rho}{\rho}\right)^{\nu-1},\qquad r=0,1,\dots,k.$$

Proof. ($\Rightarrow$) Assume that $y\in\mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$ is a solution of the problem (3.1). Then, by using Lemma 2.9 for $\tau\in(s_r,\tau_{r+1}]$ and $r=0,1,\dots,k$, we have

$$y(\tau)=\frac{\left({}^\rho\mathcal{I}^{1-\nu}_{s_r^+}y\right)(s_r)}{\Gamma(\nu)}\left(\frac{\tau^\rho-s_r^\rho}{\rho}\right)^{\nu-1}+\frac{1}{\Gamma(\beta)}\int_{s_r^+}^{\tau}\left(\frac{\tau^\rho-t^\rho}{\rho}\right)^{\beta-1}t^{\rho-1}v_r(t)dt.$$

Next, we substitute $\tau=\eta_r\in(s_r,\tau_{r+1}], r=0,1,\dots,k$ into the above equation

$$y(\eta_r)=\frac{\left({}^\rho\mathcal{I}^{1-\nu}_{s_r^+}y\right)(s_r)}{\Gamma(\nu)}\left(\frac{\eta_r^\rho-s_r^\rho}{\rho}\right)^{\nu-1}+\frac{1}{\Gamma(\beta)}\int_{s_r^+}^{\eta_r}\left(\frac{\eta_r^\rho-t^\rho}{\rho}\right)^{\beta-1}t^{\rho-1}v_r(t)dt.$$

Thus, we obtain

$$\begin{aligned}{}^\rho\mathcal{I}^{1-\nu}_{s_r}y(s_r)&=\mu_r y(\eta_r)+\mathfrak{D}_r\\&=\frac{\left({}^\rho\mathcal{I}^{1-\nu}_{s_r^+}y\right)(s_r)}{\Gamma(\nu)}\mu_r\left(\frac{\eta_r^\rho-s_r^\rho}{\rho}\right)^{\nu-1}+\frac{1}{\Gamma(\beta)}\mu_r\int_{s_r^+}^{\eta_r}\left(\frac{\eta_r^\rho-t^\rho}{\rho}\right)^{\beta-1}t^{\rho-1}v_r(t)dt+\mathfrak{D}_r\end{aligned}$$

which implies that

$${}^\rho I^{1-\nu}_{s_r}y(s_r)=\Lambda_r\Gamma(\nu)\mu_r({}^\rho\mathcal{I}^\beta_{s_r}v_r)(\eta_r)+\Lambda_r\Gamma(\nu)\mathfrak{D}_r.$$

Hence, $y(\tau)$ can be rewritten as (3.2).

($\Leftarrow$) For $\tau\in J_r$, by applying operator ${}^\rho\mathcal{I}^{1-\nu}_{s_r^+}$ on both sides of (3.2) with applying Lemma 2.3 followed by the Theorem 2.1, we get

$$({}^\rho\mathcal{I}^{1-\nu}_{s_r^+}y)(\tau)=\Lambda_r\Gamma(\nu)\mu_r({}^\rho\mathcal{I}^\beta_{s_r}v_r)(\eta_r)+({}^\rho\mathcal{I}^{1-\alpha(1-\beta)}_{s_r^+}v_r)(\tau)+\Lambda_r\mathfrak{D}_r\Gamma(\nu).$$

Taking $\tau \to s_r^+$ and applying Lemma 2.5 with noting $1-\nu < 1-\alpha(1-\beta)$, we get

$$\lim_{\tau\to s_r}({}^\rho\mathcal{I}^{1-\nu}_{s_r^+}y)(\tau) = \Lambda_r\Gamma(\nu)\left(\mu_r({}^\rho\mathcal{I}^{\beta}_{s_r}v_r)(\eta_r)+\mathfrak{D}_r\right).$$

Now, by substituting $\tau = \eta_r$ in (3.2), we have

$$y(\eta_r) = \Lambda_r\left(\frac{\eta_r^\rho - s_r^\rho}{\rho}\right)^{\nu-1}\mu_r({}^\rho\mathcal{I}^{\beta}_{s_r}v_r)(\eta_r) + ({}^\rho I^{\beta}_{s_r}v_r)(\eta_r) + \Lambda_r\mathfrak{D}_r\left(\frac{\eta_r^\rho - s_r^\rho}{\rho}\right)^{\nu-1}.$$

Then, we get

$$\begin{aligned}
\mu_r y(\eta_r)+\mathfrak{D}_r =& \Lambda_r\mu_r\left(\frac{\eta_r^\rho - s_r^\rho}{\rho}\right)^{\nu-1}\mu_r({}^\rho\mathcal{I}^{\beta}_{s_r}v_r)(\eta_r)\\
&+\mu_r({}^\rho\mathcal{I}^{\beta}_{s_r}v_r)(\eta_r)+\Lambda_r\mathfrak{D}_r\mu_r\left(\frac{\eta_r^\rho - s_r^\rho}{\rho}\right)^{\nu-1}+\mathfrak{D}_r\\
&=\left(\Lambda_r\mu_r\left(\frac{\eta_r^\rho - s_r^\rho}{\rho}\right)^{\nu-1}+1\right)\left(\mu_r({}^\rho\mathcal{I}^{\beta}_{s_r}v_r)(\eta_r)+\mathfrak{D}_r\right)\\
&=\Lambda_r\Gamma(\nu)\left(\mu_r({}^\rho\mathcal{I}^{\beta}_{s_r}v_r)(\eta_r)+\mathfrak{D}_r\right)
\end{aligned}$$

which implies that

$$\lim_{\tau\to s_r}({}^\rho\mathcal{I}^{1-\nu}_{s_r^+}y)(\tau) = \mu_r y(\eta_r)+\mathfrak{D}_r.$$

Using Lemmas 2.3 and 2.6 and Definition 2.3 with operating by ${}^\rho\mathcal{D}^{\nu}_{s_r^+}$ on both sides of (3.2) for $\tau\in J_r$, we acquire

$$\left({}^\rho\mathcal{D}^{\nu}_{s_r^+}y\right)(\tau) = \left({}^\rho D^{\alpha(1-\beta)}_{s_r^+}v\right)(\tau). \tag{3.4}$$

By hypothesis ${}^\rho\mathcal{D}^{\nu}_{s_r^+}y \in C_{1-\nu,\rho}(J_r)$. Then, (3.4) implies that

$$\left({}^\rho\mathcal{D}^{\nu}_{s_r^+}y\right)(\tau) = \left(\delta_\rho{}^\rho\mathcal{I}^{1-\alpha(1-\beta)}_{s_r^+}v\right)(\tau) = \left({}^\rho\mathcal{D}^{\alpha(1-\beta)}_{s_r^+}v\right)(\tau)\in C_{1-\nu,\rho}(J_r). \tag{3.5}$$

As $v_r\in C_{1-\nu,\rho}(J_r)$ and from Lemma 2.4, we acquire

$$\left({}^\rho\mathcal{I}^{1-\alpha(1-\beta)}_{s_r^+}v_r\right)\in C_{1-\nu,\rho}(J_r). \tag{3.6}$$

From (3.5) and (3.6) and with using the definition of $C_{1-\nu,\rho}(J_r)$, we have

$$\left({}^\rho\mathcal{I}^{1-\alpha(1-\beta)}_{s_r^+}v_r\right)\in C_{1-\nu,\rho}(J_r).$$

Using Lemmas 2.3 and 2.2 and Definition 2.3 with operating by ${}^\rho\mathcal{I}^{\alpha(1-\beta)}_{s_r^+}$ on both sides of (3.5), we get

$$\begin{aligned}
\left({}^\rho\mathcal{I}^{\alpha(1-\beta)}_{s_r^+}{}^\rho\mathcal{D}^{\nu}_{s_r^+}y\right)(\tau) &= v_r(\tau)+\frac{\left({}^\rho\mathcal{I}^{1-\alpha(1-\beta)}_{s_r^+}v_r\right)(s_r)}{\Gamma(\alpha(1-\beta))}\left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{\alpha(1-\beta)-1}\\
&=\left({}^\rho\mathcal{D}^{\beta,\alpha}_{s_r^+}y\right)(\tau) = v_r(\tau).
\end{aligned}$$

The proof is complete. □

As a result of Theorem 3.1, we acquire lemma below.

Lemma 3.1. *Assume that* $\nu = \beta + \alpha(1-\beta)$ *such that* $0 < \beta < 1$ *and* $\alpha \in [0,1]$. *Assume also, that* $f : J_r \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a function verifies* $f(\cdot, y(\cdot), v_r(\cdot)) \in C_{1-\nu,\rho}(J,\mathbb{R})$ *for each* $v_r \in C_{1-\nu,\rho}(J,\mathbb{R})$ *and* $y \in \mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$. *Then, y verifies the model (1.1) if and only if y is a fixed point of* $\mathcal{M} : \mathcal{PC}_{\nu,\rho}(J,\mathbb{R}) \to \mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$ *which is given by*

$$(\mathcal{M}y)(\tau) = \begin{cases} \Lambda_r \left(\dfrac{\tau^\rho - s_r^\rho}{\rho}\right)^{\nu-1} \left(\mu_r ({}^\rho\mathcal{I}^\beta_{s_r} v_r)(\eta_r) + \mathfrak{D}_r\right) & \\ +({}^\rho\mathcal{I}^\beta_{s_r} v_r)(\tau), & \tau \in (s_r, \tau_{r+1}], r = 0,1,\dots,k, \\ h_r(\tau, y(\tau)), & \tau \in (\tau_r, s_r), r = 1,\dots,k \end{cases} \tag{3.7}$$

where $v_r(\tau) = f(\tau, y(\tau), v_r(\tau))$.

Our main results are explained and proved through the introduction of these hypotheses. Consider the assumptions:

($\mathfrak{B}_1$): The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous such that $f(\cdot, y(\cdot), v_r(\cdot)) \in C^\nu_{1-\nu,\rho}$ for each $r = 0,1,\dots,k; k \in \mathbb{N}$, $v_r \in C^\nu_{1-\nu,\rho}(J_r,\mathbb{R})$ and $y \in \mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$;

($\mathfrak{B}_2$): There exist positive constants $\mathcal{L}$ and $\mathcal{P}_r$ for all $r = 1,\dots,k; k \in \mathbb{N}$, satisfy

$$\left\|f(\tau,y,v_r) - f(\tau,y^*,v_r^*)\right\|_{C^\nu_{1-\nu,\rho}} \le \mathcal{L}\left\|y-y^*\right\|_{C^\nu_{1-\nu,\rho}} + \mathcal{P}_r\left\|v_r - v_r^*\right\|_{C^\nu_{1-\nu,\rho}}$$

for all $y, y^* \in \mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$ and $v_r, v_r^* \in C_{1-\nu,\rho}(J)$. Furthermore, we have

$$\|v_r\|_{C^\nu_{1-\nu,\rho}} = \left\|f(\tau,y,v_r)\right\|_{C^\nu_{1-\nu,\rho}} \le \mathcal{L}\|y\|_{C^\nu_{1-\nu,\rho}} + \mathcal{P}_r\|v_r\|_{C^\nu_{1-\nu,\rho}} + \mathcal{V}_0$$

which leads to

$$\left\|f(\tau,y,v_r)\right\|_{C^\nu_{1-\nu,\rho}} \le \frac{\mathcal{L}\left\|y\right\|_{C^\nu_{1-\nu,\rho}} + \mathcal{V}_0}{1-\mathcal{P}_r}$$

where $\mathcal{V}_0 = \sup_{\tau\in J_r}|f(\tau,0,0)|$ and $0 < \mathcal{P}_r < 1$ for all $r = 0,1,\dots,k$;

($\mathfrak{B}_3$): The functions $h_r, : J_r^* \times \mathbb{R} \to \mathbb{R}$ are continuous for all $r = 1,\dots,k; k \in \mathbb{N}$;

($\mathfrak{B}_4$): There exist positive constants $\Re_r$, $r = 1,\dots,k; k \in \mathbb{N}$ such that

$$\left|h_r(\tau,y) - h_r(\tau,y')\right| \le \Re_r|y-y'|$$

for each $\tau \in J_r^*$, $r = 1,\dots,k; k \in \mathbb{N}$ and $y, y' \in C(J,\mathbb{R})$. These signify

$$\left\|h_r(\tau,y)\right\|_C \le \Re_r\|y\|_C + h_r^0, \quad \text{where} \quad h_r^0 = \sup_{\tau\in J_r^*}|h_r(\tau,0)|;$$

($\mathfrak{B}_5$): For each $\tau \in J_r$, $r = 0,1,\dots,k; k \in \mathbb{N}$, there exists $\chi_r > 0$ satisfies

$$({}^\rho\mathcal{I}^\beta_{s_r^+}\psi)(\tau) \le \chi_r\psi(\tau)$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function.

3.1. **Compactness Case.** To convenience the readers, we present the following symbols for big quantities: For all $r = 0, 1, \ldots, k$ with $k \in \mathbb{N}$, we consider

$$\Omega_r = \frac{\Gamma(\nu)}{(1 - \mathcal{P}_r)\Gamma(\beta + \nu)},$$
$$S_r = |\mu_r| \left(\frac{\eta_r^{\rho} - s_r^{\rho}}{\rho} \right)^{\beta + \nu - 1},$$
$$T_r = \left(\frac{\tau_{r+1}^{\rho} - s_r^{\rho}}{\rho} \right)^{\beta}.$$

Theorem 3.2. *Consider Hypotheses ($\mathfrak{B}_1$)-($\mathfrak{B}_4$) are satisfied. If*

$$\max_r \{\mathcal{L}\Omega_r |\Lambda_r| S_r, \mathfrak{K}_r\} < 1.$$

Then, the fractional model (1.1) possesses a minimum of one fixed solution in the space $\mathcal{PC}_{\nu,\rho}(J, \mathbb{R})$.

Proof. Create

$$B_x = \left\{ y \colon y \in \mathcal{PC}_{\nu,\rho}(J, \mathbb{R}),\ \|y\|_{\mathcal{PC}_{\nu,\rho}} \le x \right\}$$

with radius

$$x \geqslant \max_r \left\{ \frac{\Omega_r \mathcal{V}_0 \left(|\Lambda_r| S_r + T_r \right) + |\Lambda_r| |\mathfrak{D}_r|}{|1 - \mathcal{L}\Omega_r (|\Lambda_r| S_r + T_r)|}, \frac{h_0^r}{|1 - \mathfrak{K}_r|} \right\}.$$

It is possible to think of the operator $\mathcal{M}$ as the sum of two operators $\mathcal{M}_1$ and $\mathcal{M}_2$ which are defined as

$$(\mathcal{M}_1 y)(\tau) = \begin{cases} ({}^{\rho}\mathcal{I}_{s_r}^{\beta} v_r)(\tau) + \Lambda_r \mathfrak{D}_r \left(\frac{\tau^{\rho} - s_r^{\rho}}{\rho} \right)^{\nu - 1}, & \tau \in (s_r, \tau_{r+1}], r = 0, 1, \ldots, k \\ 0, & \tau \in (\tau_r, s_r), r = 1, \ldots, k, \end{cases}$$

$$(\mathcal{M}_2 y)(\tau) = \begin{cases} \Lambda_r \mu_r \left(\frac{\tau^{\rho} - s_r^{\rho}}{\rho} \right)^{\nu - 1} ({}^{\rho}\mathcal{I}_{s_r}^{\beta} v_r)(\eta_r), & \tau \in (s_r, \tau_{r+1}], r = 0, 1, \ldots, k, \\ h_r(\tau, y(\tau)), & \tau \in (\tau_r, s_r), r = 1, \ldots, k. \end{cases}$$

Then, for $y, y^* \in \mathcal{PC}_{\nu,\rho}(J, \mathbb{R})$ it follows that $\|\mathcal{M}_1 y + \mathcal{M}_2 y^*\|_{\mathcal{PC}_{\nu,\rho}} \le x$, which concludes that $\mathcal{M}_1 y + \mathcal{M}_2 y^* \in \mathcal{B}_x$. Now, we want to show that $\mathcal{M}$ maps bounded sets into the bounded set.

- For each $\tau \in J_r,\ r = 0, 1, \ldots, k; k \in \mathbb{N}$ and $y \in B_x$, we obtain

$$\begin{aligned} &\left| \left(\frac{\tau^{\rho} - s_r^{\rho}}{\rho} \right)^{1 - \nu} (\mathcal{M}_1 y)(\tau) \right| \\ &\le \left(\frac{\tau^{\rho} - s_r^{\rho}}{\rho} \right)^{1 - \nu} ({}^{\rho}\mathcal{I}_{s_r}^{\beta} |v_r|)(\tau) + |\Lambda_r| |\mathfrak{D}_r| \\ &= \left(\frac{\tau^{\rho} - s_r^{\rho}}{\rho} \right)^{1 - \nu} \left({}^{\rho}\mathcal{I}_{s_r}^{\beta} \left(\frac{t^{\rho} - s_r^{\rho}}{\rho} \right)^{\nu - 1} \left| \left(\frac{t^{\rho} - s_r^{\rho}}{\rho} \right)^{1 - \nu} v_r \right| \right)(\tau) + |\Lambda_r| |\mathfrak{D}_r| \end{aligned}$$

$$\leq (\mathcal{L}x + \mathcal{V}_0)\left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{1-\nu}\left({}^\rho\mathcal{I}_{s_r}^\beta\left(\frac{t^\rho - s_r^\rho}{\rho}\right)^{\nu-1}\right)(\tau) + |\Lambda_r||\mathfrak{D}_r|$$

$$\leq (\mathcal{L}x + \mathcal{V}_0)\,\Omega_r T_r + |\Lambda_r||\mathfrak{D}_r|$$

and similarly, we deduce that

$$\left|\left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{1-\nu}(\mathcal{M}_2 y)(\tau)\right| \leq |\Lambda_r||\mu_r|({}^\rho\mathcal{I}_{s_r}^\beta|v_r|)(\eta_r) \leq (\mathcal{L}x + \mathcal{V}_0)\,\Omega_r|\Lambda_r|S_r$$

which imply that

$$\|\mathcal{M}y\|_{C^\nu_{1-\nu,\rho}} = \max_r\left\{\sup_{\tau\in J_r}\left|\left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{1-\nu}((\mathcal{M}_1 + \mathcal{M}_2)y)(\tau)\right|\right\}$$

$$\leq \max_r\{(\mathcal{L}x + \mathcal{V}_0)\,\Omega_r\,(|\Lambda_r|S_r + T_r) + |\Lambda_r||\mathfrak{D}_r|\} \leq x.$$

- For each $\tau \in J_r^*,\ r = 1, 2, \ldots, k; k \in \mathbb{N}$ and $y \in B_x$, we get

$$\|\mathcal{M}y\|_C \leq \Re_r\|y\|_C + h_0^r \leq \Re_r x + h_0^r \leq x$$

Therefore, $\|\mathcal{M}y\|_{\mathcal{P}C_{\nu,\rho}} \leq x$. As a result, a bounded subset in $\mathcal{B}_x$ is mapped into another bounded subset via the operator $\mathcal{M}$.

The following fulfillment is to confirm that the operator $\mathcal{M}_1$ is equicontinuous. As a result of $(\mathfrak{B}_1)$ and $(\mathfrak{B}_3)$, $\mathcal{M}_1$ is continuous. For each $0 \leqslant \xi_1 < \xi_2 \leqslant \tau_1$ and $y \in \mathcal{B}_x$, we obtain that

$$I \triangleq \left|\left(\frac{\xi_2^\rho - s_r^\rho}{\rho}\right)^{1-\nu}(\mathcal{M}_1 y)(\xi_2) - \left(\frac{\xi_1^\rho - s_r^\rho}{\rho}\right)^{1-\nu}(\mathcal{M}_1 y)(\xi_1)\right|$$

$$\leq \frac{1}{\Gamma(\beta)}\int_{s_r}^{\xi_1}\left|\left(\frac{\xi_2^\rho - s_r^\rho}{\rho}\right)^{1-\nu}\left(\frac{\xi_2^\rho - t^\rho}{\rho}\right)^{\beta-1} - \left(\frac{\xi_1^\rho - s_r^\rho}{\rho}\right)^{1-\nu}\left(\frac{\xi_1^\rho - t^\rho}{\rho}\right)^{\beta-1}\right||v_r(t)|\frac{dt}{t^{1-\rho}}$$

$$+ \left(\frac{\xi_2^\rho - s_r^\rho}{\rho}\right)^{1-\nu}\frac{1}{\Gamma(\beta)}\int_{\xi_1}^{\xi_2}\left(\frac{\xi_2^\rho - t^\rho}{\rho}\right)^{\beta-1}|v_r(t)|\frac{dt}{t^{1-\rho}}$$

$$\leq \frac{\left(\frac{\xi_2^\rho - s_r^\rho}{\rho}\right)^{1-\nu} - \left(\frac{\xi_1^\rho - s_r^\rho}{\rho}\right)^{1-\nu}}{\Gamma(\beta)}\int_{s_r}^{\xi_1}\left(\frac{\xi_2^\rho - t^\rho}{\rho}\right)^{\beta-1}|v_r(t)|\frac{dt}{t^{1-\rho}}$$

$$+ \frac{\left(\frac{\xi_1^\rho - s_r^\rho}{\rho}\right)^{1-\nu}}{\Gamma(\beta)}\int_{s_r}^{\xi_1}\left[\left(\frac{\xi_1^\rho - t^\rho}{\rho}\right)^{\beta-1} - \left(\frac{\xi_2^\rho - t^\rho}{\rho}\right)^{\beta-1}\right]|v_r(t)|\frac{dt}{t^{1-\rho}}$$

$$+ \left(\frac{\xi_2^\rho - s_r^\rho}{\rho}\right)^{1-\nu}\frac{1}{\Gamma(\beta)}\int_{\xi_1}^{\xi_2}\left(\frac{\xi_2^\rho - t^\rho}{\rho}\right)^{\beta-1}|v_r(t)|\frac{dt}{t^{1-\rho}}.$$

With suitable transformations, we can deduce the following needed integral

$$\int_{\ell}^{\eta}\left(\frac{\xi^{\rho}-t^{\rho}}{\rho}\right)^{\beta-1}\left(\frac{t^{\rho}-s_r^{\rho}}{\rho}\right)^{\nu-1}\frac{dt}{t^{1-\rho}}=\left(\frac{\xi^{\rho}-s_r^{\rho}}{\rho}\right)^{\beta+\nu-1}\left(\mathbf{B}_{r(\eta,\xi)}(\nu,\beta)-\mathbf{B}_{r(\ell,\xi)}(\nu,\beta)\right)$$

where $\mathbf{B}_x(\cdot,\cdot)$ is the complete Beta function with noting that $\mathbf{B}_1(\cdot,\cdot)=\mathbf{B}(\cdot,\cdot)$ and $\mathbf{B}_0(\cdot,\cdot)=0$ and $r(q,\xi)=(q^{\rho}-s_r^{\rho})/(\xi^{\rho}-s_r^{\rho})$. By calculating the integrals with simplifying, we get

$$\begin{aligned}
I &\leq \frac{\mathcal{L}x+\mathcal{V}_0}{(1-\mathcal{P}_r)\Gamma(\beta)}\left[\left(\frac{\xi_1^{\rho}-s_r^{\rho}}{\rho}\right)^{\beta}+\left(\frac{\xi_2^{\rho}-s_r^{\rho}}{\rho}\right)^{\beta}\right]\mathbf{B}(\nu,\beta)\\
&\quad-2\frac{\mathcal{L}x+\mathcal{V}_0}{(1-\mathcal{P}_r)\Gamma(\beta)}\left(\frac{\xi_1^{\rho}-s_r^{\rho}}{\rho}\right)^{1-\nu}\left(\frac{\xi_2^{\rho}-s_r^{\rho}}{\rho}\right)^{\beta+\nu-1}\mathbf{B}_{r(\xi_1,\xi_2)}(\nu,\beta)\\
&=\frac{\mathcal{L}x+\mathcal{V}_0}{(1-\mathcal{P}_r)\Gamma(\beta)}\left(\frac{\xi_2^{\rho}-s_r^{\rho}}{\rho}\right)^{\beta}\left[\left(\frac{\xi_1^{\rho}-s_r^{\rho}}{\xi_2^{\rho}-s_r^{\rho}}\right)^{\beta}\mathbf{B}(\nu,\beta)+\mathbf{B}(\nu,\beta)-2\left(\frac{\xi_1^{\rho}-s_r^{\rho}}{\xi_2^{\rho}-s_r^{\rho}}\right)^{1-\nu}\mathbf{B}_{r(\xi_1,\xi_2)}(\nu,\beta)\right]\\
&\leq\frac{2(\mathcal{L}x+\mathcal{V}_0)}{(1-\mathcal{P}_r)\Gamma(\beta)}\left(\frac{\xi_2^{\rho}-s_r^{\rho}}{\rho}\right)^{\beta}\left[\mathbf{B}(\nu,\beta)-\left(\frac{\xi_1^{\rho}-s_r^{\rho}}{\xi_2^{\rho}-s_r^{\rho}}\right)^{1-\nu}\mathbf{B}_{r(\xi_1,\xi_2)}(\nu,\beta)\right]
\end{aligned}$$

which approaches zero as $\xi_1\to\xi_2$. As a result, $\mathcal{M}_1$ is a relatively compact on $\mathcal{B}_x$. On $\mathcal{B}_x$, the operator $\mathcal{M}_1$ is completely continuous, as per the Arezela Ascoli theorem. The only thing remaining to accomplish is to demonstrate that $\mathcal{M}_2$ is a contraction mapping. based on the assumptions $\mathfrak{B}_2$ and $\mathfrak{B}_4$, we obtain

- For each $\tau\in J_r, r=0,1,\dots,k; k\in\mathbb{N}$ and $y,y^*\in\mathcal{B}_x$, we have

$$\left|\left(\frac{\tau^{\rho}-s_r^{\rho}}{\rho}\right)^{1-\nu}\left((\mathcal{M}_2y)-(\mathcal{M}_2y^*)\right)(\tau)\right|\leq\mathcal{L}\Omega_r|\Lambda_r|S_r\left\|y-y^*\right\|_{C_{1-\nu,\rho}^{\nu}}$$

- For each $\tau\in J_r^*,\ r=1,2,\dots,k$ and $y,y^*\in C$, we have

$$\left\|\mathcal{M}_2y-\mathcal{M}_2y^*\right\|_{C}\leq\mathfrak{K}_r\left\|y-y^*\right\|_{C}.$$

Thus, it follows

$$\left\|(\mathcal{M}y(\tau)-\mathcal{M}y^*(\tau))\right\|_{\mathcal{P}C_{\nu,\rho}}\leq\max_r\left\{\mathcal{L}\Omega_r|\Lambda_r|S_r,\mathfrak{K}_r\right\}\left\|y-y^*\right\|_{\mathcal{P}C_{\nu,\rho}}.$$

According to the Krasnoselskii theorem, nonlinear fractional Hilfer-katogumpola model with non-instantaneous impulses (1.1) possesses a minimum of one fixed solution in the space $\mathcal{P}C_{\nu,\rho}(J,\mathbb{R})$.

□

3.2. **Noncompactness Case.** Kuratowski's noncompactness measure and Sadovski's fixed point theorem can help investigate whether there are solutions in noncompactness cases. To discuss this issue, we must take into account the following result of existence.

Theorem 3.3. *Consider the Hypotheses ($\mathfrak{B}_1$)-($\mathfrak{B}_4$) are satisfied. If*

$$F_{\mathcal{PC}_{\geq,\subset}} = \max_r \{\mathcal{L}\Omega_r [|\Lambda_r| S_r + T_r], \mathfrak{R}_r\} < 1$$

Then, the problem in Equation (1.1) possesses a minimum of one fixed solution in the space $\mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$.

Proof. First, we demonstrate that $\mathcal{M}\colon \mathcal{B}_x \to \mathcal{B}_x$ is continuous where $\mathcal{B}_x \subset \mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$ is defined in the proof of Theorem 3.2. Simply put, $\mathcal{B}_x$ is a nonempty, convex, bounded and closed subset of a Banach space.

Assume that $\{y_n\}_{n\in\mathbb{N}}$ is a sequence where $y_n \to y$ as $n \to \infty$ in $\mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$. Then, we get

- For each $\tau \in J_r, r = 0,1,\ldots,k$, we have

$$\begin{aligned}\left|\left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{1-\nu} (\mathcal{M}y_n(\tau) - \mathcal{M}y(\tau))\right| &\leq |\Lambda_r||\mu_r|({}^\rho\mathcal{I}^\beta_{s_r}|v_r^n - v_r|)(\eta_r) \\ &+ \left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{1-\nu} ({}^\rho\mathcal{I}^\beta_{s_r}|v_r^n - v_r|)(\tau) \\ &\leq \mathcal{L}\Omega_r [|\Lambda_r|S_r + T_r] \left\|y_n - y\right\|_{C^\nu_{1-\nu,\rho}} \Rightarrow 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

- For each $\tau \in J_r^*, r = 1,2,\ldots,k$ and $y \in \mathcal{B}_x$, we have

$$\left|((\mathcal{M}y_n(\tau) - (\mathcal{M}y(\tau))\right| \leq \mathfrak{R}_r \left\|y_n - y\right\|_C \Rightarrow 0 \quad \text{as} \quad n \to \infty.$$

Consequently,

$$\left\|(\mathcal{M}y_n(\tau) - \mathcal{M}y(\tau))\right\|_{\mathcal{PC}_{\nu,\rho}} \Rightarrow 0 \quad \text{as} \quad n \to \infty.$$

The next step is to show that the operator $\mathcal{M}$ maps $\mathcal{B}_x$ onto $\mathcal{B}_x$. As in the Theorem 3.2, it is verified. The operator $\mathcal{M}$ must be shown to satisfy the Kuratowski measure of noncompactness inequality. In light of the assumptions $\mathfrak{B}_2$, $\mathfrak{B}_4$, we get

- For each $\tau \in J_r, r = 0,1,\ldots,k$ and $y, y^* \in \mathcal{B}_x$, as above, we obtain

$$\left|\left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{1-\nu} ((\mathcal{M}y(\tau) - (\mathcal{M}y^*(\tau))\right| \leq \mathcal{L}\Omega_r [|\Lambda_r|S_r + T_r] \left\|y - y^*\right\|_{C^\nu_{1-\nu,\rho}}.$$

- For each $\tau \in J_r^*, r = 1,2,\ldots,k$ and $y, y^* \in C$, we have

$$\left|((\mathcal{M}y(\tau) - (\mathcal{M}y^*(\tau))\right| \leq \mathfrak{R}_r \left\|y - y^*\right\|_C.$$

Thus, it follows

$$\left\|(\mathcal{M}y(\tau) - \mathcal{M}y^*(\tau))\right\|_{\mathcal{PC}_{\nu,\rho}} \leq F_{\mathcal{PC}_{\geq,\subset}} \left\|y - y^*\right\|_{\mathcal{PC}_{\geq,\subset}}.$$

Assume that $\mathscr{S} \subset \mathcal{B}_x$ is a closed subset where there exist $\mathscr{S}_j,\ j = 1, 2, \ldots, n, n \in \mathbb{N}, \mathscr{S} \subseteq \cup_{j=1}^{n} \mathscr{S}_j$

$$\begin{aligned}
\mu(\mathcal{M}\mathscr{S}) &= \inf\left\{\vartheta : diam(\mathcal{M}\mathscr{S}_j) \leq \vartheta, \mathscr{S} \subseteq \cup_{j=1}^{n} \mathscr{S}_j\right\} \\
&= \inf\left\{\vartheta : \sup\left\{\|(\mathcal{M}y)(\tau) - (\mathcal{M}y^*)(\tau)\|_{\mathcal{PC}_{\geq,c}}\right\} \leq \vartheta, y, y^* \in \mathscr{S}_j, \mathscr{S} \subseteq \cup_{j=1}^{n} \mathscr{S}_j\right\} \\
&\leq F_{\mathcal{PC}_{\geq,c}} \inf\left\{\vartheta : \sup\left\{\|y(\tau) - y^*(\tau)\|_{\mathcal{PC}_{\geq,c}}\right\} \leq \vartheta, y, y^* \in \mathscr{S}_j, \mathscr{S} \subseteq \cup_{j=1}^{n} \mathscr{S}_j\right\} \\
&\leq F_{\mathcal{PC}_{\geq,c}} \inf\left\{\vartheta : diam(\mathcal{M}\mathscr{S}_j) \leq \vartheta, y, y^* \in \mathscr{S}_j, \mathscr{S} \subseteq \cup_{j=1}^{n} \mathscr{S}_j\right\} \\
&= F_{\mathcal{PC}_{\geq,c}} \mu(\mathscr{S}).
\end{aligned}$$

Regarding to the statement (7) in Lemma 2.7, the operator $\mathcal{M} : \mathcal{B}_x \to \mathcal{B}_x$ is μ-condensing. The operator $\mathcal{M}$: $\mathcal{B}_x \to \mathcal{B}_x$ is able to solve the problem (1.1), and Sadovskii's fixed point theorem indicates that at least one fixed point $y \in \mathcal{B}_x$ exists. □

4. Stability Analysis

This section is dedicated to presenting the stability of our Ulam-Hyers-Rassias problem (1.1) results.

Theorem 4.1. *Assume that the Hypotheses ($\mathfrak{B}_1$)- ($\mathfrak{B}_5$) are satisfied. If* $\max_r \{\mathcal{L}\Omega_r T_r, \Re\} < 1$. *Then, the fractional non-instantaneous impulsive model (1.1) is UHR stable with respect to* (φ, ψ).

Proof. Assuming a unique solution $y \in \mathcal{PC}_{\nu,\rho}(J, \mathbb{R})$ to the problem (1.1) corresponds to any solution $x \in \mathcal{PC}_{\nu,\rho}(J, \mathbb{R})$ of (2.1). Then, in view of Lemma 2.9, we acquire

$$y(\tau) = \begin{cases} \dfrac{({}^{\rho}I_{s_r}^{1-\nu} y)(s_r)}{\Gamma(\nu)} + ({}^{\rho}I_{s_r}^{\beta} v_r)(\tau), & \tau \in (s_r, \tau_{r+1}], r = 0, 1, \ldots, k \\ h_r(\tau, y(\tau)), & \tau \in (\tau_r, s_r), r = 1, 2, \ldots, k. \end{cases}$$

Further, if x is the solution of (2.1) and using Remark 2.1, we obtain

$$\begin{cases} {}^{\rho}\mathcal{D}_{s_r}^{\beta,\alpha} x(\tau) = \widetilde{v}_r(\tau) + \mathcal{Q}(\tau), & \tau \in (s_r, \tau_{r+1}], r = 0, 1, \ldots, k, \\ x(\tau) = h_r(\tau, x(\tau)) + \mathcal{Q}_r, & \tau \in (\tau_r, s_r), r = 1, 2, \ldots, k, \\ ({}^{\rho}I_{s_r}^{1-\nu} x)(s_r) = ({}^{\rho}I_{s_r}^{1-\nu} y)(s_r) \end{cases}$$

where

$$\widetilde{v}_r(\tau) = f(\tau, x(\tau), \widetilde{v}_r(\tau))$$

and

$$x(\tau) = \begin{cases} \dfrac{({}^{\rho}I_{s_r}^{1-\nu} x)(s_r)}{\Gamma(\nu)} + ({}^{\rho}I_{s_r}^{\beta} \widetilde{v}_r)(\tau) + ({}^{\rho}I_{a}^{\beta} \mathcal{Q})(\tau), & \tau \in (s_r, \tau_{r+1}], , r = 0, 1, \ldots, k \\ h_r(\tau, x(\tau)) + \mathcal{Q}_r, & \tau \in (\tau_r, s_r), r = 1, 2, \ldots, k. \end{cases}$$

For each $\tau \in J_r, r = 0, 1, 2, \ldots, k$, we acquire

$$\begin{aligned}\left\|x - y\right\|_{C^{\nu}_{1-\nu,\rho}} &\le \left(\frac{\tau^{\rho} - s_r^{\rho}}{\rho}\right)^{1-\nu} \left[({}^{\rho}I^{\beta}_{s_r}|\widetilde{v}_r - v_r|)(\tau) + ({}^{\rho}I^{\beta}_{a}|\mathcal{Q}|)(\tau)\right] \\ &\le \mathcal{L}\Omega_r T_r \|x - y\|_{C^{\nu}_{1-\nu,\rho}} + \varepsilon \left(\frac{\tau^{\rho}_{r+1} - s_r^{\rho}}{\rho}\right)^{1-\nu} ({}^{\rho}I^{\beta}_{a}\psi)(\tau)\end{aligned}$$

which reveals

$$\left\|x - y\right\|_{C^{\nu}_{1-\nu,\rho}} \le \frac{\varepsilon \chi_{\psi}}{1 - \mathcal{L}\Omega_r T_r} \left(\frac{\tau^{\rho}_{r+1} - s_r^{\rho}}{\rho}\right)^{1-\nu} \psi(\tau), \qquad \mathcal{L}\Omega_r T_r < 1.$$

For each $\tau \in J^*_r, r = 1, 2, \cdots, k$, we acquire

$$\begin{aligned}\|x - y\|_{C} &\le \sup_{\tau \in J^*} \left|h_r(\tau, x(\tau)) - h_r(\tau, y(\tau))\right| + |\mathcal{Q}_r| \\ &\le \Re_r \left\|x - y\right\|_{C} + \epsilon\varphi.\end{aligned}$$

Consequently, we have

$$\left\|x - y\right\|_{C} \le \frac{\epsilon\varphi}{1 - \Re_r}, \qquad \Re_r < 1.$$

Then, for each $\tau \in J$, we obtain

$$\left\|x - y\right\|_{\mathcal{PC}_{\nu,\rho}} \le \mathfrak{c}_u \varepsilon(\varphi + \psi(\tau))$$

where

$$\mathfrak{c}_u = \max_r \left\{\frac{\chi_{\psi}}{1 - \mathcal{L}\Omega_r T_r} \left(\frac{\tau^{\rho}_{r+1} - s_r^{\rho}}{\rho}\right)^{1-\nu}, \frac{1}{1 - \Re_r}\right\}.$$

Therefore, the solution of fractional non-instantaneous impulsive model (1.1) is UHR stable with respect to (φ, ψ). Furthermore, by using definition 2.7 and $\varepsilon = 1$, we can see that the solution of the fractional non-instantaneous impulsive model (1.1) is GUHR stable with respect to (φ, ψ). □

5. An Application

The purpose of this section is to illustrate how our main results can be applied by describing an application.

Example 5.1. *Take into account the fractional non-instantaneous impulsive model listed below:*

$$\begin{aligned}{}^{\rho}D^{\beta,\alpha}_{s_r} y(\tau) &= f(\tau, y(\tau), {}^{\rho}D^{\beta,\alpha}_{s_r} y(\tau)), \; \tau \in (s_r, \tau_r], \; r = 0, 1, 2, \\ y(\tau) &= h_r(\tau, y(\tau)), \qquad \tau \in (\tau_r, s_r], \; r = 1, 2, \\ {}^{\rho}\mathcal{I}^{\frac{1}{3}}_{s_r} y(s_r) &= \mu_r y(\mu_r), \qquad r = 0, 1, 2\end{aligned} \tag{5.1}$$

where $J = [0,1], 0 = s_0 < \tau_1 = \frac{1}{5} < s_1 = \frac{2}{5} < \tau_2 = \frac{3}{5} < s_2 = \frac{4}{5} < \tau_3 = 1, \rho = \frac{1}{2}, \beta = \frac{1}{2}, \alpha = \frac{1}{3}, \nu = \frac{2}{3}, \mu_0 = -1, \mu_1 = 1, \mu_2 = 2, \eta_0 = \frac{1}{6}, \eta_1 = \frac{1}{2}, \eta_2 = \frac{9}{10}, \mathfrak{D}_1 = \frac{2}{9}$ *and* $\mathfrak{D}_2 = 1$. *In our example, we take*

$$h_r(\tau, y(\tau)) = \frac{|y(\tau)|}{7e^{\tau-\tau_r}(1+|y(\tau)|)}, \qquad \tau \in (\tau_r, s_r],\ r = 1,2.$$

It is obvious that it is continuous on the intervals $\tau \in (\tau_r, s_r],\ r = 1,2$ *which meets the assumption* $(\mathfrak{B}_3)$ *and satisfies*

$$\begin{aligned}|h_r(\tau, y(\tau)) - h_r(\tau, y'(\tau)| &= \left|\frac{|y(\tau)|}{7e^{\tau-\tau_r}(1+|y(\tau)|)} - \frac{|y'(\tau)|}{7e^{\tau-\tau_r}(1+|y'(\tau)|)}\right| \\ &\le \frac{\big||y(\tau)| - |y'(\tau)|\big|}{7e^{\tau-\tau_r}\left|(1+|y(\tau)|)(1+|y'(\tau)|)\right|} \\ &\le \frac{1}{7}\left|y - y'\right|\end{aligned}$$

for all $y, y' \in \mathbb{R}$. *These imply that the assumption* $(\mathfrak{B}_4)$ *hold with* $\mathfrak{K}_1 = \mathfrak{K}_2 = \frac{1}{7}$. *Additionally, we take* $v_r(\tau) = {}^{\rho}D^{\beta,\alpha}_{s_r} y(\tau))$ *for all* $\tau \in (s_r, \tau_r]$ *and* $r = 0,1,2$. *Based on this, we consider*

$$f(\tau, y(\tau), v_r(\tau)) = \frac{2}{\sqrt{(\tau^2+100)}}\left(\frac{|y(\tau)| + |v_r(\tau)|}{1 + |y(\tau)| + |v_r(\tau)|}\right), \qquad \tau \in (s_r, \tau_r],\ r = 0,1,2.$$

It is evident that the function $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and it holds the assumption* $(\mathfrak{B}_2)$, *it follows*

$$\begin{aligned}\left|f(\tau, y, v_r) - f(\tau, y^*, v_r^*)\right| &\le \frac{2\left|(|y| + |v_r|) - (|y^*| + |v_r^*|)\right|}{(\sqrt{(\tau^2+100)})\left|(1+|y|+|v_r|)\,(1+|y^*|+|v_r^*|)\right|} \\ &\le \frac{1}{5}\left||y| - |y^*|\right| + \left||v_r| - |v_r^*|\right| \\ &\le \frac{1}{5}\left(|y - y^*| + |v_r - v_r^*|\right).\end{aligned}$$

with $\mathcal{L} = \mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}_3 = 1/5$. *Using the given data, we can find that*

$$\begin{array}{lll} \Lambda_0 \approx 0.357607, & \Lambda_1 \approx -2.81011, & \Lambda_2 \approx -0.484066 \\ \Omega_0 \approx 1.82453, & \Omega_1 \approx 1.82453, & \Omega_2 \approx 1.82453 \\ S_0 \approx 0.966776, & S_1 \approx 0.728357, & S_2 \approx 1.38126 \\ T_0 \approx 0.945742, & T_1 \approx 0.533181, & T_2 \approx 0.459506. \end{array}$$

- **Krasnoselskii fixed point theorem:**

  To test Theorem 3.2 assumption, we have

$$\begin{aligned}\max_r \{\mathcal{L}\Omega_r|\Lambda_r|S_r, \mathfrak{K}_r\} &= \max\{\mathcal{L}\Omega_0|\Lambda_0|S_0, \mathcal{L}\Omega_1|\Lambda_1|S_1, \mathcal{L}\Omega_2|\Lambda_2|S_2, \mathfrak{K}_1, \mathfrak{K}_2\} \\ &\approx \max\left\{0.126157, 0.746875, 0.243984, \frac{1}{7}, \frac{1}{7}\right\} = 0.746875 < 1.\end{aligned}$$

  Therefore, the assumptions of the theorem hold and so, on $[0,1]$, there is minimum one fixed solution to the model (5.1).

- **Sadovskii fixed point theorem:**
To test Theorem 3.3 assumption, we have

$$F_{\mathcal{PC}_{\geq,\subset}} = \max_{r=0,1,2}\left\{\mathcal{L}\Omega_r\left[|\Lambda_r|S_r+T_r\right],\Re_r\right\}$$
$$\approx \max\left\{0.471263, 0.39822, 0.41166, \frac{1}{7}, \frac{1}{7}\right\} = 0.471263 < 1$$

Hence, the model (5.1) possesses a minimum of one fixed solution in the space $\mathcal{PC}_{\nu,\rho}(J,\mathbb{R})$.

- **Stability** To demonstrate Theorem 4.1, we take

$$\psi(\tau) = \left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{\sigma-1}, \qquad \tau \in (s_r, \tau_{r+1}],\ r = 0,1,2.$$

Plainly, the function $\psi(\tau)$ is nondecreasing on $(s_r, \tau_{r+1}]$, $r = 0,1,2$ if $\sigma \geq 1$. Also, we can see that

$$({}^\rho\mathcal{I}^{\frac{1}{2}}_{s_r^+}\psi)(\tau) = \frac{\Gamma(\sigma)}{\Gamma(\sigma+\frac{1}{2})}\left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{\sigma-\frac{1}{2}}$$
$$= \frac{\Gamma(\sigma)}{\Gamma(\sigma+\frac{1}{2})}\left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{\frac{1}{2}}\psi(\tau).$$

Since $\rho = \frac{1}{2} < 1$, then

$$({}^\rho\mathcal{I}^{\frac{1}{2}}_{s_r^+}\psi)(\tau) \leq \frac{\Gamma(\sigma)}{\sqrt{\rho}\,\Gamma(\sigma+\frac{1}{2})}\left(\tau_{r+1} - s_r\right)^{\frac{1}{2}\rho}\psi(\tau)$$
$$= \frac{\sqrt{2}\,\Gamma(\sigma)}{\sqrt[4]{5}\,\Gamma(\sigma+\frac{1}{2})}\psi(\tau)$$

which implies that the assumption $(\mathfrak{B}_5)$ holds by taking $\chi_r = \frac{\sqrt{2}\Gamma(\sigma)}{\sqrt[4]{5}\Gamma(\sigma+\frac{1}{2})}$ for all $\sigma \geq 1$. If we put $\sigma = 2$, we get $\chi_r \approx 0.711437$.

In conclusion, we have

$$\left\|x - y\right\|_{\mathcal{PC}_{\nu,\rho}} \leq \mathfrak{c}_u\varepsilon(\varphi + \psi(\tau))$$

where

$$\mathfrak{c}_u = \max_{r=0,1,2}\left\{\frac{\chi_\psi}{1-\mathcal{L}\Omega_r T_r}\left(\frac{\tau_{r+1}^\rho - s_r^\rho}{\rho}\right)^{1-\nu}, \frac{1}{1-\Re_r}\right\}$$
$$\approx \max\left\{1.04668, 0.580791, 0.508985, \frac{7}{6}, \frac{7}{6}\right\} = \frac{7}{6}.$$

and ε and φ are any positive real constants.

Now, the main condition in Theorem 4.1 is

$$\max_{r=0,1,2}\left\{\mathcal{L}\Omega_r T_r, \Re_r\right\} = \max\left\{\mathcal{L}\Omega_0 T_0, \mathcal{L}\Omega_1 T_1, \mathcal{L}\Omega_2 T_2, \Re_1, \Re_2\right\}$$
$$\approx \max\left\{0.345106, 0.194561, 0.167676, \frac{1}{7}, \frac{1}{7}\right\} = 0.345106.$$

Thus, the model (5.1) is UHR stable with respect to (φ, ψ).

Example 5.2. *The main goal of this example to introduce the graph of the solution. Consider the next fractional non-instantaneous impulsive model:*

$$\begin{aligned} {}^{\rho}D^{\beta,\alpha}_{s_r} y(\tau) &= \frac{3}{2}\left(\frac{\tau^{\rho} - s_r^{\rho}}{\rho}\right)^{\sigma-1} + \frac{3}{2}\sin(y(\tau)) - \frac{1}{2}\,{}^{\rho}D^{\beta,\alpha}_{s_r} y(\tau), \ \tau \in (s_r, \tau_r],\ r = 0,1,2, \\ y(\tau) &= \frac{1}{10}e^{\tau-\tau_r}\sin(\pi(\tau-\tau_r)) + y^2(\tau), \qquad \tau \in (\tau_r, s_r],\ r = 1,2, \\ {}^{\rho}\mathcal{I}^{\frac{1}{3}}_{s_r} y(s_r) &= \mu_r y(\mu_r) + \mathfrak{D}_r, \qquad r = 0,1,2 \end{aligned}$$

where $J = [0,1], 0 = s_0 < \tau_1 = \frac{1}{5} < s_1 = \frac{2}{5} < \tau_2 = \frac{3}{5} < s_2 = \frac{4}{5} < \tau_3 = 1, \mu_0 = -1, \mu_1 = 1, \mu_2 = 2, \eta_0 = \frac{1}{6}, \eta_1 = \frac{1}{2}, \eta_2 = \frac{9}{10}, \mathfrak{D}_0 = -2, \mathfrak{D}_1 = \frac{2}{5}$ *and* $\mathfrak{D}_2 = 1$. *In our example, we consider*

$$h_r(\tau, y(\tau)) = \frac{1}{10}e^{\tau-\tau_r}\sin(\pi(\tau-\tau_r)) + y^2(\tau), \qquad \tau \in (\tau_r, s_r],\ r = 1,2.$$

Obviously, it is continuous on the intervals $\tau \in (\tau_r, s_r]$, $r = 1,2$ *which meets the assumption* $(\mathfrak{B}_3)$ *and satisfies*

$$|h_r(\tau, y(\tau)) - h_r(\tau, y'(\tau)| \le \left|y + y'\right| \left|y - y'\right| \le \mathfrak{K}_r|y - y'|$$

which is true by the continuity of $y, y' \in \mathbb{R}$ *for all* $\tau \in (\tau_r, s_r]$, $r = 1,2$. *Additionally, we take* $v_r(\tau) = {}^{\rho}D^{\beta,\alpha}_{s_r} y(\tau))$ *for all* $\tau \in (s_r, \tau_r]$ *and* $r = 0,1,2$. *Based on this, we consider*

$$f(\tau, y(\tau), v_r(\tau)) = \frac{3}{2}\left(\frac{\tau^{\rho} - s_r^{\rho}}{\rho}\right)^{\sigma-1} + \frac{3}{2}\sin(y(\tau)) - \frac{1}{2}v_r(\tau), \qquad \tau \in (s_r, \tau_r],\ r = 0,1,2.$$

Clearly, the function $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and that it verifies the assumption* $(\mathfrak{B}_2)$

$$\left|f(\tau, y, v_r) - f(\tau, y^*, v_r^*)\right| \le \frac{3}{2}|y - y^*| + \frac{1}{2}|v_r - v_r^*|.$$

with $\mathcal{L} = \frac{3}{2}$ *and* $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}_3 = \frac{1}{2} < 1$. *Using the given data, we can see that all assumptions of Theorem 3.1 hold.*

Now, our example can rewrite as

$$\begin{aligned} {}^{\rho}D^{\beta,\alpha}_{s_r} y(\tau) &= \left(\frac{\tau^{\rho} - s_r^{\rho}}{\rho}\right)^{\sigma-1} + \sin(y(\tau)), \qquad \tau \in (s_r, \tau_r],\ r = 0,1,2, \\ y(\tau) &= \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{2}{5}e^{\tau-\tau_r}\sin(\pi(\tau-\tau_r))}\right), \qquad \tau \in (\tau_r, s_r],\ r = 1,2, \\ {}^{\rho}\mathcal{I}^{\frac{1}{3}}_{s_r} y(s_r) &= \mu_r y(\mu_r), \qquad r = 0,1,2 \end{aligned}$$

which has the solution

$$y(\tau) = \Lambda_r \left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{\nu-1} \left[\frac{\mu_r \Gamma(\sigma)}{\Gamma(\sigma+\beta)}\left(\frac{\eta_r^\rho - s_r^\rho}{\rho}\right)^{\beta+\sigma-1} + \mu_r ({}^\rho \mathcal{I}_{s_r}^\beta \sin y)(\eta_r) + \mathfrak{D}_r\right] + \frac{\Gamma(\sigma)}{\Gamma(\sigma+\beta)}\left(\frac{\tau^\rho - s_r^\rho}{\rho}\right)^{\nu+\beta+\sigma-2} + ({}^\rho \mathcal{I}_{s_r}^\beta \sin y)(\tau) \tag{5.2}$$

for all $\tau \in (s_r, \tau_{r+1}]$, $r = 0, 1, 2$, and

$$y(\tau) = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{2}{5} e^{\tau - \tau_r} \sin(\pi(\tau - \tau_r))}\right) \tag{5.3}$$

for all $\tau \in (\tau_r, s_r)$, $r = 1, 2$.

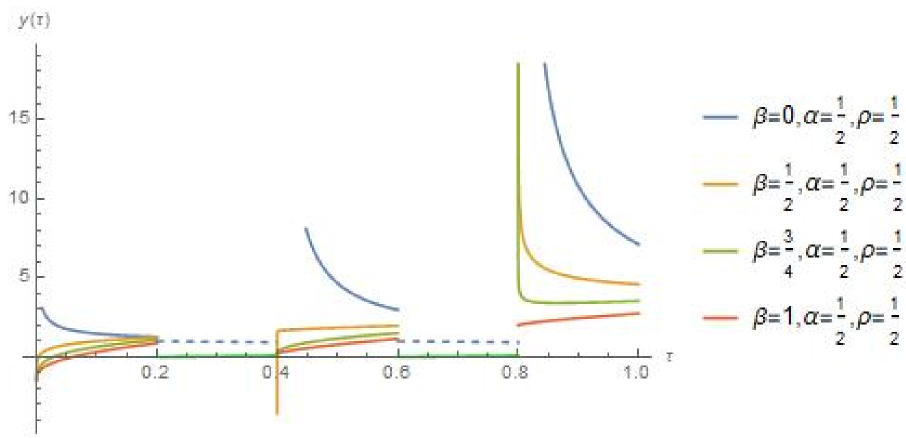

FIGURE 1. Graph of the state function $y(\tau)$ for $\alpha = \frac{1}{2}$, $\rho = \frac{1}{2}$ and various values of β as indicated in the legend.

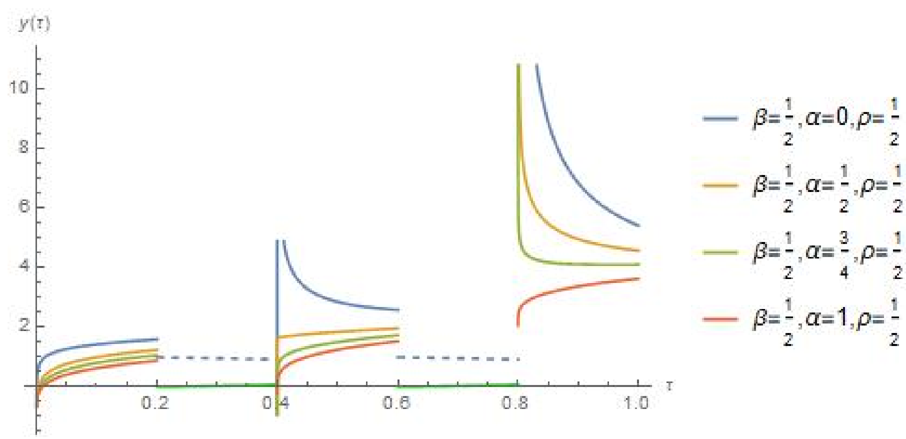

FIGURE 2. Graph of the state function $y(\tau)$ for $\beta = \frac{1}{2}$, $\rho = \frac{1}{2}$ and various values of α as indicated in the legend.

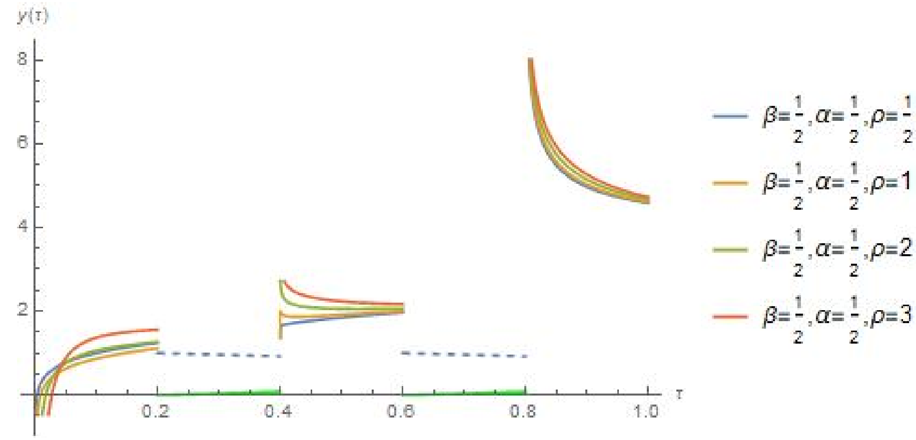

Figure 3. Graph of the state function $y(\tau)$ for $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$ and various values of ρ as indicated in the legend.

Note that the dash and green lines represent the state function for positive and negative signs of the root, respectively. Here, we used the iteration method for plotting the state function in various intervals by drawing y_1 with taking $y(s_r) = \pi/2$ for $r = 0, 1, 2$.

6. Conclusion

The study of fractional models with non-instantaneous impulses has received increased attention in recent years due to its implementations in several fields of applied mathematics, physical sciences, and many engineering fields. Motivated by the importance of these equations, we employed a pair of approaches to investigate the theory and stability of nonlinear implicit Hilfer-Katugampola fractional models with non-instantaneous impulses of order $\alpha \in (0, 1)$, which were supplemented by a multi-point boundary condition. The first argument is due to Krasnoselskii's theorem, which states that F can behave like $\left|f(\tau, y, v) - f(\tau, y^*, v^*)\right| \leq \mathcal{L}\left|y - y^*\right| + \mathcal{P}\left|v - v^*\right|$ (see($\mathfrak{B}_2$)). The tools we chose to our model are simplified to use and can expand the scope of the results acquired with simple assumptions. The second result is determined by employing the Kuratowski measure, Sadovskii's theory, and the technique of measuring non-compactness.

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