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Fixed Point Theorems of Caristi's Type in Quasi b-Metric Spaces

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Abstract. The famous Cartisti's fixed point theorem became popular after the Ph. D. thesis of James Caristi entitled "The Fixed Point Theory for Mappings Satisfying Inwardness Conditions" defended in 1975 at Iowa State University, under the supervision of William Arthur Kirk. However, the result was first reported in the conference proceedings of the conference titled "Geometry of Metric Linear Spaces" held in June 1974. Subsequently, Felix E. Browder suggested one reformulation of basic result. In our work we establish corresponding result for quasi b - metric spaces and supplement the derived result with suitable non trivial example.

1. Introduction

In a lecture presented by J. Caristi and W. A. Kirk, in the conference on Geometry of Metric Linear Spaces held at June 1974, announcement for Ph. D. thesis of James Caristi supervised by William Arthur Kirk [4] entitled "The Fixed Point Theory for Mappings Satisfying Inwardness Conditions" was given, which was defended in 1975 at Iowa State University. After this lecture, Felix E. Browder suggested the following reformulation of basic result, which is presented at first in conference proceeding [7]. This statement today is known as Caristi's theorem [7].

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Theorem 1.1. *(Caristi, Kirk [7]) Let (M, d) be a complete metric space and $f : M \to M$. If there exists a lower semicontinuous function $\phi : M \to [0, +\infty)$ such that for each $x \in M$:*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)),$$

then f has a fixed point in M.

Before this lecture, Caristi, submited manuscript of paper [5] in may 1974, which contained main results of thesis, to Transaction of American Mathematical Society where in the original Caristi's formulation of Theorem was presented1.1 and the same is presented below:

Theorem 1.2. *(Caristi [5]) Let (X, d) be a complete metric space and K closed subset of X. Suppose $f : K \to K$ is an arbitrary function and $T : K \to X$ continuous. If there exists a real number $r < 0$ such that*

$$d(f(x), T(f(x))) \leq d(x, T(x)) + rd(x, f(x)),$$

then f has a fixed point.

On metric spaces Caristi's theorem generalize Banach's contarction principle to some classes of non - continuous mappings. Further results from Caristi's dissertation was published in [19] and [4]. The original Caristi's proof used transfinite induction method that was rather complicted. Easier and simpler proof's was obtained by F. Browder [3], C. S. Wong [28], J. Siegel [26] and L. Pasicki [23]. Application of Caristi's theorem had been reported by many researchers, for example one can refer [4,5,7,17–19].

There are many generalizations of Caristi's theorem. First of them was obtained by S. Kasahara [16] which is also first common fixed point result and first result for non necessary metrizable spaces. T. L. Hicks [13] proved first result of Caristi's type for mappings defined on quasi metric spaces. Further results for this class of spaces was given in Lj. Ćirić [8–10], see also [1]. Recent results of Caristi's type for class of b - metric spaces, can be founded in [15,20,22,24].

While, Theorem 3.1 of [20] stiuplates gauge function ϕ need not be lower semicontinuous, but stipulates "f to be continuous". Theorem 2.5 of [24], f is not necessarily continuous, but it has additional condition "ϕ is 0 - lower semicontinuous". In this paper extensions of these two results for quasi b - metric spaces will be proved.

2. Preliminary notes

The followoing are required in the sequel.

2.1. **Quasi b - metric spaces.**

Unified approach to class of quasi metric spaces was presented by Wilson [27]. Concept of Wilson includes earlier definitions of Fréchet, Hausdorff, Nemytskii and Aleksandroff. Notion of quasi b - metric space was introduced by Shah and Hussain [25].

Definition 2.1. *Let X be a nonempty set, $x, y, z \in X$, $d : X \times X \to [0, +\infty)$ and $s \in \mathbb{R}$. We define the following five properties:*
(A1) $d(x, y) = 0$ if and only if $x = y$;

(A2) $d(x,y) = d(y,x)$;
(A3) $d(x,y) \leq d(x,z) + d(z,y)$;
(A4) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

If for any $x, y, z \in X$, (X,d) *satisfies:*
(A1), (A2) and (A3), then (X,d) *is metric space;*
(A1) and (A3), then (X,d) *is quasi-metric space.*
If there exists $s > 0$ *such that for any* $x, y, z \in X$, (X,d,s) *satisfies (A1), (A2) and (A4) ((A1) and (A4)), then* (X,d,s) *is b-metric space (quasi b-metric space).*

Example 2.1. *Let* $X = \mathbb{R}$ *and* $d : X \times X \to [0,+\infty)$ *defined by*

$$d(x,y) = |y - x| + \frac{y - x}{2},$$

for $x, y \in X$. *Then* (X,d) *is a quasi-metric space. Note that (A2) does not hold for the mapping* d.

Remark 2.1. *It is to be noted that every (quasi) b-metric space* $(X,d,1)$ *is (quasi) metric space.*

Proposition 2.1. *Let* (X,d,s) *be a (quasi) b-metric space with b constant* s. *Then* $s \geq 1$.

Proof. Let $x, y \in X$, then

$$d(x,y) \leq s[d(x,y) + d(y,y)] = sd(x,y),$$

so, $s \geq 1$. □

Let (X,d,s) be a (quasi) b - metric space, $x \in X$. Then for $B(x,r) = \{y \in X : \; d(y,x) < r\}$ for all $x \in X, r \in (0,+\infty)$ is l - ball with center in point x with radius r. On every quasi b - metric spaces (X,d,s) one topology can be introduced by taking the collection $\{B\left(x, \frac{1}{n}\right) : n = 1,2,\ldots\}$ as a base of neighborhood filter of the point x, for any $x \in X$. This topology will be denoted by τ_l. If $d(x_n,x) \to 0$ then $x_n \to x$ in the topology τ_l. The convergence of a sequence (x_n) in the topology τ_l implies $d(x_n,x) \to 0$, because $x \in (B(x,r))$, for each $r > 0$.

Definition 2.2. *Let* (X,d,s) *be a quasi b - metric space. We said that it has property* $(W3)$ *if for any* $x, y \in X$ *and* $(x_n) \subseteq X$, $\lim d(x_n,x) = 0$ *and* $\lim d(x_n,y) = 0$ *implies* $x = y$.

Remark 2.2. *Every b - metric space has the property* $(W3)$ *(see [2]), but it is not true for quasi b - metric spaces.*

Example 2.2. *Let* $X = \{0\} \cup \{\frac{1}{n} \; : \; n = 1,2,3,\ldots\}$ *and* $d : X \times X \to [0,+\infty)$ *mapping defined by*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ |x-y|, & \text{if } x = \frac{1}{m},\ y = \frac{1}{n},\ 1 < m,n \\ \frac{1}{m}, & \text{if } x = \frac{1}{m},\ y = 0 \text{ or } x = \frac{1}{m},\ y = 1 \\ 1, & \text{in all other cases.} \end{cases}$$

We get that $(X,d,2)$ *is a quasi b - metric space, which not satisfied property (W3), because* $d(\dfrac{1}{n},0) \to 0$ *and* $d(\dfrac{1}{n},1) \to 0$ *but* $0 \neq 1$.

Definition 2.3. *Let (X, d, s) be a quasi b - metric space.*
(i) A sequence $(x_n) \subseteq X$ is said to be l - Cauchy sequence if for given $\varepsilon > 0$ there exists any $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$, for all $n > m \geq N$.
(ii) (X, d, s) is l - complete if and only if for every l - Cauchy sequence $(x_n) \subseteq X$ there exists $x \in X$ such that $d(x_n, x) \to 0$.

Proposition 2.2. *Let (X, d, s) be a quasi b - metric space which has the property $(W3)$, $q \in X$ and $(x_n) \subseteq X$. If $\lim d(x_n, q) = 0$, then for any $(x_{n_j}) \subseteq (x_n)$, we have $\lim d(x_{n_j}, q) = 0$.*

Proof. From $\lim d(x_n, q) = 0$ it follows that for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $n > n_0$ implies $d(x_n, q) < \varepsilon$. Hence, there exist $j_0 \in \mathbb{N}$ such that $j > j_0$ implies $d(x_{n_j}, q) < \varepsilon$. So $\lim d(x_{n_j}, q) = 0$. q is unique because (X, d, s) has the property $(W3)$. □

Proposition 2.3. *Let (X, d, s) be a quasi b-metric space which has the property $(W3)$ and $(x_n) \subseteq X$ be l - Cauchy sequence. If there exists $p \in X$ and $(x_{n_j}) \subseteq (x_n)$ such that $\lim d(x_{n_j}, p) = 0$, then $\lim d(x_n, p) = 0$.*

Proof. For every $\epsilon > 0$ there exists l such that $n > m > l$ implies $d(x_m, x_n) < \frac{\varepsilon}{2s}$. From $\lim d(x_{n_j}, p) = 0$ follows that there exists i such that $n_i > l$ and $j > i$ implies $d(x_{n_j}, p) < \frac{\varepsilon}{2s}$. Then for all $n > n_j > l$ we have

$$d(x_n, p) \leq s(d(x_n + x_{n_j}) + d(x_{n_j}, p)) < \varepsilon.$$

□

The following Lemma was proved in [21]. It generalizes the result (Lemma 2.5)formulated and proved by R. Miculescu and A. Mihail [20] for b-metric spaces.

Lemma 2.1. *Let (X, d, s) be a quasi b - metric spaces and $(x_n) \subseteq X$. Then*

$$d(x_0, x_k) \leq s^n \sum_{i=0}^{k-1} d(x_i, x_{i+1}),$$

for any $n = 1, 2, 3, \ldots$ and every $k = 1, 2, \ldots, 2^n - 1, 2^n$.

Definition 2.4. *Let (X, d, s) be a quasi b - metric spaces and $x \in X$.*

$f : X \to X$ is l - continuous if for each $x \in X$ and for any $(x_n) \subseteq X$ such that $d(x_n, x) = 0$ implies $\lim d(f(x_n), f(x)) = 0$.

$\phi : X \to [0, +\infty)$ is 0 - lower semicontinuous if for each $x \in X$ and any $(x_n) \subseteq X$ such that from $d(x_n, x) = 0$ and $\lim \phi(x_n) = 0$ implies $\phi(x) = 0$.

2.2. d_l **- completeness.** Let X be a Hausdorff topological space and $d : X \times X \to [0, \infty)$. We shall consider the following two properties.

(α) for any $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$;

(β) there exists $\lambda \geq 1$ such that for each sequences $(x_n) \subseteq X$, $\sum_{n=0}^{\infty} \lambda^n d(x_n, x_{n+1}) < \infty$ implies that (x_n) is convergent.

The pair (X,d) is d_l - complete topological space if it satisfies α and β. If β is true for $\lambda = 1$ we said that (X,d) is d - complete topological space. The notion of d - complete spaces, was introduced by S. Kasahara [16] for L - spaces and T. L. Hicks [14] for topological spaces.

Its clearly that any d_l - complete topological space (X,d) is d - complete, but converse is not true.

3. Characterization of l - completeness of quasi b - metric spaces

Lemma 3.1. *Let (X,d,s) be a quasi b - metric space and $(x_n) \subseteq X$. If (x_n) is l - Cauchy sequence then for any $\lambda \geq 1$ there exists $(x_{n_j}) \subseteq (x_n)$ such that*

$$\sum_{j=0}^{\infty} \lambda^j d(x_{n_j}, x_{n_{j+1}}) < \infty.$$

Proof. Let (X,d,s) be a quasi b - metric space, $(x_n) \subseteq X$ and $\lambda > 1$. For $(n_j) \subseteq \mathbb{N}$ defined by

$$\begin{aligned} n_0 &= \min\{n \in \mathbb{N} \, : \, m > n \text{ implies } d(x_n, x_m) < 1\} \\ n_k &= \min\{n \in \mathbb{N} \, : \, m > n > n_{k-1} \text{ implies } \lambda^k d(x_n, x_m) < 2^{-k}\}, \end{aligned}$$

we get that

$$\sum_{j=0}^{\infty} \lambda^j d(x_{n_j}, x_{n_{j+1}}) < 2.$$

□

Lemma 3.2. *Let (X,d,s) be a quasi b - metric space, $(x_n) \subseteq X$ and $\sum_{n=0}^{\infty} \lambda^n d(x_n, x_{n+1}) < \infty$ for some $\lambda \geq 1$, where where $\lambda > 1$ when $s > 1$. Then (x_n) is a l - Cauchy sequence.*

Proof. Suppose that (X,d,s) is l - complete and $s = 1$. Then for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0+1}^{\infty} d(x_n, x_{n+1}) < \varepsilon$. Hence, for every $m > l > n_0$ we have

$$d(x_l, x_m) \leq \sum_{n=l}^{m-1} d(x_n, x_{n+1}) \leq \sum_{n=n_0+1}^{\infty} d(x_n, x_{n+1}) < \varepsilon$$

which implies that (x_n) is l - Cauchy sequence.

Suppose that (X,d,s) is l - complete and $s > 1$. Then $\sum_{n=0}^{\infty} \lambda^n d(x_n, x_{n+1}) < \infty$ for some $\lambda > 1$. By

$$\lim(n+1)(n+2) - \lambda n^2 = -\infty,$$

we get that

$$\lim_{n\to\infty} s^{(n+1)(n+2)-\lambda n^2} = 0,$$

which implies that sequence

$$s^{(n+1)(n+2)-\lambda n^2} = \frac{s^{(n+1)(n+2)}}{2^{\gamma n^2}},$$

where $\alpha = \gamma \log_2 s$, is bounded. From

$$\sup_{x\in\mathbb{R}}(n+1)(n+2) - \lambda n^2 = 2 + \frac{9}{4\lambda - 1}$$

it follows

$$\sup \frac{s^{(n+1)(n+2)}}{2^{\gamma n^2}} \leq 2 + \frac{9}{4\lambda - 1} := M.$$

Let $l = \sqrt{\log_2(m+1)}$. We get that

$$2^{l^2} - 1 \leq m < 2^{l^2+1} - 1$$

which implies

$$\begin{aligned}
& d(x_n, x_{n+m}) \leq s \cdot d(x_n, x_{n+1}) + s \cdot d(x_{n+1}, x_{n+m}) \leq \\
\leq \; & sd(x_{n+2^{0^2}-1}, x_{n+2^{0+1^2}-1}) + s^2 d(x_{n+2^{1^2}-1}, x_{n+2^{2^2}-1}) + \\
+ \; & s^2 d(x_{n+2^{2^2}} - 1), x_{n+m}) \leq \\
\leq \; & \sum_{i=0}^{l-1} s^{i+1} d(x_{n+2^{i^2}-1}, x_{n+2^{(i+1)^2}-1}) + s^l d(x_{n+2^{l^2}-1}, x_{n+m})) \leq \\
\leq \; & \sum_{i=0}^{l-1} s^{i+1} s^{(i+1)^2} \left(\sum_{j=2^{i^2}}^{2^{(i+1)^2}-1} d(x_{n+j-1}, x_{n+j}) \right) + \\
+ \; & s^{l+1} s^{(l+1)^2} \left(\sum_{j=2^{l^2}}^{2^{(l+1)^2}-1} d(x_{n+j-1}, x_{n+j}) \right) = \\
= \; & \sum_{i=0}^{l} s^{i+1} s^{(i+1)^2} \left(\sum_{j=2^{i^2}}^{2^{(i+1)^2}-1} d(x_{n+j-1}, x_{n+j}) \right) \leq \\
\leq \; & \sum_{i=0}^{\infty} \frac{s^{i+1} s^{i+2}}{2^{\gamma i^2}} \left(\sum_{j=2^{i^2-1}}^{2^{(i+1)^2}-1} d(x_{n+j-1}, x_{n+j+1}) \right) \leq \\
\leq \; & M \sum_{i=0}^{\infty} (i+1)^\gamma d(x_{n+i}, x_{n+i+1}) \leq M \sum_{i=0}^{\infty} (n+i)^\gamma d(x_{n+i}, x_{n+i+1}) \leq \\
\leq \; & M \sum_{i=n}^{\infty} i^\gamma d(x_i, x_{i+1}).
\end{aligned}$$

Hence, series $\sum_{n=0}^{\infty} n^\gamma d(x_i, x_{i+1})$ is convergent, because series $\sum_{i=0}^{\infty} \lambda^n d(x_n, x_{n+1})$ is convergent and $n^\gamma < \alpha^n$. So (x_n) is l - Cauchy sequence. □

Theorem 3.1. *A Quasi b - metric space (X, d, s) which has the property $(W3)$ is l complete if and only if it is d_λ - complete, where $\lambda = 1$ if and only if $s = 1$.*

Proof. Let (X, d, s) be a d_l - complete quasi b - metric space which has the property $(W3)$ and $(x_n) \subseteq X$ be l - Cauchy sequence. Then, by Lemma 3.1 for every $\lambda \geq 1$ there exists $(x_{n_j}) \subseteq (x_n)$ such that

$$\sum_{j=0}^{\infty} d(x_{n_j}, x_{n_{j+1}}) < \infty.$$

So, there exists $p \in X$ such that $\lim x_{n_j} = p$ because (X, d, s) is d - complete. Hence by Proposition 2.3 we obtain $\lim x_n = p$.

Suppose that (X, d, s) is l - complete quasi b - metric space. Let $\sum_{n=0}^{\infty} \lambda^n d(x_n, x_{n+1}) < \infty$ for some $\lambda \geq 1$. By Lemma 3.2 follows that (x_n) is l - Cauchy sequence. It is convergent because (X, d, s) is l - complete. □

Next statement follows directly from Theorem 3.1 for $\lambda = s$. Also, it can be proved independently.

Corollary 3.1. *Quasi b - metric space (X, d, s) is l - complete if and only if each sequences $(x_n) \subseteq X$ such that*
$\sum_{n=0}^{\infty} s^n d(x_n, x_{n+1}) < \infty$ *is convergent.*

Proof. We get that

$$\begin{aligned} d(x_n, x_{n+k}) &\leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+k}) \leq \sum_{i=0}^{k-1} s^i d(x_{n+i}, x_{n+i+1}) \leq \\ &\leq \cdots \leq \sum_{i=0}^{\infty} s^i d(x_{n+i}, x_{n+i+1}). \end{aligned}$$

Hence, (x_n) is Cauchy sequences. It is convergent because (X, d, s) is l - complete. □

For $s = 1$ statement directly from Theorem 3.1 follows next statement.

Corollary 3.2. *Quasi metric space is l - complete if and only if it is d - complete.*

4. Fixed point result

We apply the method of Pasicki [23] in the proof to the following theorem to obtain extension of result of Wong [28] to quasi metric space.

Theorem 4.1. *Let $(X, d, 1)$ be a l - complete quasi metric space and $\phi : X \to [0, +\infty)$ sequentially lower semi - continuous function. If for any $x \in X$ there exists $y \in X$ such that $y \neq x$ and*

$$d(x, y) < \phi(x) - \phi(y),$$

then f has a fixed point.

Proof. Let $\mathcal{A}$ be set of all nonempty $A \subseteq X$ such that for any $x, y \in A$ such that $x \neq y$ we have $d(x, y) < |\phi(x) - \phi(y)|$. We get that for every $x \in X$ there exists $y \in X$ such that $\{x, y\} \in \mathcal{A}$. So $\mathcal{A} \neq \emptyset$. Binary relation $'' \subseteq ''$ be a partial ordering on $\mathcal{A}$, in which $\cup_{i \in I} A_i$ is upper bound for chain $(A_i)_{i \in I}$.

By Zorn's lemma, there exists $Z \in \mathcal{A}$, which is maximal element in $(\mathcal{A}, \subseteq)$, i.e. every $x \in Z$ satisfies $d(x,y) > |\phi(x) - \phi(y)|$ for any $y \in X \backslash Z$.

For $p = \inf \phi(Z)$, there exists sequence $(z_n) \subseteq Z$ such that $(\phi(z_n)) \subseteq \mathbb{R}$ is monotne non - increasing and $\lim \phi(z_n) = p$. For $n > m$ we have $d(z_m, z_n) < \phi(x_m) - \phi(x_n)$ which implies that (x_n) is l - Cauchy sequence. Let $z = \lim z_n$.

For each $x \in Z$ and sufficiently large i, we have

$$d(x,z) \leq d(x,z_i) + d(z_i,z) \leq |\phi(x) - \phi(z_i)| + d(z_i,z),$$

for $\phi(x) \neq p$, which implies $d(x,z) \leq |\phi(x) - \phi(z)|$, because ϕ is sequentially lower semi - continuous function. Hence $z \in Z$. If $\phi(x) \neq p$ when $i \to +\infty$ we obtained $d(x,z) \leq |p - \phi(z)| = 0$. So $d(x,z) = 0$. So $z \in Z$. For any $y \in X$ such that $y \notin Z$ we have

$$d(z,y) \nleq \phi(z) - \phi(y),$$

because Z is maximal. So we get that $z = y = f(z)$. □

Let (X,d,s) be a quasi b - metric space which has the property $(W3)$ and $f : X \to X$. $\phi : \mathbb{R} \to \mathbb{R}$ is called gauge function of Caristi's type if there exists $\lambda \geq 1$ such that

$$d(x, f(x)) \leq \phi(x) - \lambda \cdot \phi(f(x)).$$

Necessary and sufficient condition for existence of gauge function of Caristi's type ϕ which was considered in 1.1 will be given in next resust, which extended Theorem of J. Eisenfeld and V. Lakshmikantham [12] to class of quasi b - metric space.

Lemma 4.1. *Let (X,d,s) be a quasi b - metric space which has the property $(W3)$ and $f : X \to X$. If there exists $\lambda \geq 1$ such that $\lambda = 1$ if and only if $s = 1$, then there exists $\phi : X \to [0,+\infty)$*

$$d(x, f(x)) \leq \phi(x) - \lambda \cdot \phi(f(x))$$

if and only if for any $X \in X$ series

$$\sum_{n=0}^{\infty} \lambda^n d(f^n(x), f^{n+1}(x))$$

is convergent.

Proof. Let there exists $\phi : X \to [0,+\infty)$

$$d(x, f(x)) \leq \phi(x) - \lambda \cdot \phi(f(x)).$$

$$\begin{aligned} \sum_{n=0}^{m} \lambda^n d(f^n(x), f^{n+1}(x) &\leq \sum_{n=0}^{m} \lambda^n (\phi(x) - \lambda \cdot \phi(f(x))) \leq \\ &\leq \phi(x) - \lambda^{n+1} \phi(f^{n+1}(x)) \leq \phi(x). \end{aligned}$$

So, the series is convergent, for each $x \in X$. Suppose that this series is convergent for every $x \in X$. Then we can take

$$\phi(x) = \sum_{n=0}^{\infty} \lambda^n d(f^n(x), f^{n+1}(x)),$$

for any $x \in X$. □

The following theorem which is the main result of this paper extends Theorem 3.1 of [20] in the setting of quasi b - metric space.

Theorem 4.2. *Let (X, d, s) be a l - complete quasi b - metric space which has the property $(W3)$ and $f : X \to X, \phi : X \to [0, +\infty)$. Suppose that there exists $\lambda \geq 1$, which satisfies $\lambda = 1$ if and only if $s = 1$, such that for any $x \in X$*

$$d(x, f(x)) \leq \phi(x) - \lambda \cdot \phi(f(x)).$$

If for arbitrary $x \in X$ and any $(x_n) \subseteq X$ $\lim d(x_n, x) = 0$ and $\lim d(x_n, f(x_n)) = 0$ implies $\lim d(f(x_n), f(x)) = 0$, then f has a fixed point.

Proof. Let $x \in X$ be arbitrary. From Lemma 4.1, we obtain that the series

$$\sum_{n=0}^{\infty} \lambda^n d(f^n(x), f^{n+1}(x))$$

is convergent, which implies by Lemma 3.2 that $(f^n(x))$ is l - Cauchy sequence. Then there exist $w \in X$ such that $\lim d(f^n(x), w) = 0$ because (X, d, s) is l - complete. This limit is unique because (X, d, s) satisfies property $(W3)$. Further, convergence of series implies $\lim d(f^i(x), f^{i+1}(x)) = 0$. Hence, from $\lim d(f^n(x), f(w)) = 0$ we obtained $f(w) = \lim f^n(x)$, which implies $w = f(w)$, because limit of Cauchy sequence is unique. □

When f is sequentially continuous, we obtain the following result.

Corollary 4.1. *Let (X, d, s) be a l - complete quasi b - metric space which has the property $(W3)$and $f : X \to X, \phi : X \to [0, +\infty)$. Suppose that there exists $\lambda \geq 1$, which satisfies $\lambda = 1$ if and only if $s = 1$, such that for any $x \in X$*

$$d(x, f(x)) \leq \phi(x) - \lambda \cdot \phi(f(x)).$$

If for each $x \in X$ and $(x_n) \subseteq X$ such that $\lim d(x_n, x) = 0$ we have $\lim d(f(x_n), f(x)) = 0$, then f has a fixed point.

Proof. Let $x \in X$, $(x_n) \subseteq X$ and $\lim d(x_n, x) = 0$. If $\lim d(x_n, f(x_n)) = 0$ then by

$$d(x_n, f(x)) \leq s \cdot (d(x_n, f(x_n)) + d(f(x_n), f(x))),$$

we get that $\lim d(x_n, f(x)) = 0$. Now statement follows from Theorem $T4.3$. □

Example 4.1. *Let (X, d, s) be the quasi b - metric space where $X = [0, +\infty)$, $s = 2$ and d is the quasi b - metric on X given by $d(x, y) = |y - x|^2$ for all $x, y \in X$. Let $f : X \to X$ be defined by $f(x) = \frac{1}{2}x$. Let and $\phi : X \to [0, +\infty]$ defined by $\phi(x) = 4d(x, f(x))$, for all $x \in X$. We obtain*

$$d(x, f(x)) \leq \phi(x) - \frac{3}{2}\phi(f(x))$$

for all $x \in X$. Therefore, by Collorary 4.1, T has a unique fixed point.

We now present an extension of the results reported in [24] to quasi b - metric space.

Theorem 4.3. *Let (X, d, s) be a l - complete quasi b - metric space which has the property $(W3)$, $f : X \to X$ and $\phi : X \to [0, +\infty)$. Suppose that there exists $\lambda \geq 1$, which satisfies $\lambda = 1$ if and only if $s = 1$, such that for any $x \in X$*

$$d(x, f(x)) \leq \phi(x) - \lambda \cdot \phi(f(x)).$$

If ϕ is 0 - lower semicontinuous, then f has a fixed point.

Proof. Let $x \in X$ be arbitrary. From Lemma 4.1 we obtained that series

$$\sum_{n=0}^{\infty} \lambda^n d(f^n(x), f^{n+1}(x))$$

is convergent, which implies by Lemma 3.2 that $(f^n(x))$ is l - Cauchy sequence. Then there exist $w \in X$ such that $\lim d(f^n(x), w) = 0$ because (X, d, s) is left complete.

Let $\lambda > 1$. We have $0 \leq \lambda^n \phi(f^n(x)) \leq \phi(x)$, which implies that $\lim \phi(f^n(x)) = 0$, because $\lim \lambda^n = \infty$. So, we get that $\phi(w) = 0$, which implies that

$$0 \leq d(w, f(w)) \leq \phi(w) - \lambda \cdot \phi(f(w)) = 0.$$

Hence $w = f(w)$.

Let $\lambda = 1$. Rest of the proof follows by Theorem 4.1. □

Example 4.2. *Let (X, d, s) be the quasi b - metric space where $X = [0, +\infty)$, $s = 2$ and d is the quasi b - metric on X given by $d(x, y) = \max\{x, y\}^2$ for all $x, y \in X$ with $x \neq y$. We first note that (X, d, s) is complete because the Cauchy sequences are those that converges to 0 in topology τ_l (equivalently, those that converges to 0 for the usual topology on X). Let f be the self mapping of X given by*

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } 0 < x < 4; \\ \sqrt{x}, & \text{if } x \geq 4 \end{cases}$$

Note that f has two fixed points: 0 and 1. Moreover it is not continuous at $x = 0$.

Let $\phi : X \to [0, +\infty)$ be mappings defined by

$$\phi(x) = \begin{cases} 2, & \text{if } x = 0; \\ 1, & \text{when } x \in (0, 1) \text{ and } x \in (1, 4) \\ 0, & \text{if } x = 1; \\ 2x^2, & \text{if } x \geq 4 \end{cases}$$

Note that ϕ *is not lower semi-continuous, because* $\frac{1}{n} \to 0$, $\phi(0) = 2$ *and* $\phi(\frac{1}{n}) = 1$ *for all* $n \in \{2, 3, 4, \ldots\}$. *But* ϕ *satisfies condition "*$d(x_n, x) = 0$ *and* $\lim \phi(x_n) = 0$ *implies* $\phi(x) = 0$*" of Theorem 4.3, because if* $x_n \to x$ *with* $\lim \phi(x_n) = 0$ *then* $x_n = 1$, *which implies* $x = 1$. *Further we have phi*$(1) = 0$.

Finally, take $x \in X$ *such that* $d(x, f(x)) > 0$. *If* $x \in (0, 4)\backslash\{1\}$ *we get,*

$$d(x, f(x)) = d(x, 1) = 1 = \phi(x) - 2\phi(1) = \phi(x) - 2\phi(f(x)),$$

and if $x \geq 4$ *we distinguish following two cases:*
Case 1. $f(x) \geq 4$. *Then*

$$d(x, f(x)) = d(x, \sqrt{x}) = x^2 \leq 2x^2 - 4x = \phi(x) - 2\phi(\sqrt{x}) = \phi(x) - 2\phi(f(x)).$$

Case 2. $f(x) < 4$. *Then* $f(x) = \sqrt{x} > 2$ *so phi*$(f(x)) = 1$ *and hence*

$$d(x, f(x)) = x^2 < 2x^2 - 2 = \phi(x) - 2\phi(f(x)).$$

Therefore, all conditions of Theorem 4.3 are satisfied. However, we cannot apply Collorary 4.1 because f *is not continuous.*

Conclusion

In this article we have established analogue of fixed point results of Caristi's type in quasi b - metric spaces. The results have been supported with non trivial example. We have also corolloraised some proven results of the past based on the derived results. It will be an open problem to establish fixed point results of Caristi's type in more generalised metric and metric like spaces.

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