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Riemann–Stieltjes Operators on $F(p,q,s)$ and $H(p,q,\phi)$ Spaces of the Unit Ball

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Abstract. Let g, f be holomorphic functions on the unit ball $\mathbb{B}_n$ of the $\mathbb{C}^n$. The Riemann–Stieltjes operator is defined by

$$L_g f(z) = \int_0^1 \mathcal{R}f(tz)g(tz)\frac{dt}{t}, \quad z \in \mathbb{B}_n,$$

where $\mathcal{R}f$ is the radial derivative of f. The objective of this paper is to find an estimation for the essential norm of this operator from the spaces $F(p,q,s)$ (general function space) and $H(p,q,\phi)$ (mixed–norm space) into Zygmund–type space in the unit ball of $\mathbb{C}^n$. As applications we find compactness characterization for the above operator.

1. Introduction and Preliminaries

Let $\mathbb{B}_n$ be the unit ball in $\mathbb{C}^n$ and $H(\mathbb{B}_n)$ be the class of all holomorphic functions on $\mathbb{B}_n$. For $f \in H(\mathbb{B}_n)$, with the Taylor expansion $f(z) = \sum_{|\beta|\geq 0} a_\beta z^\beta$, let $\mathcal{R}f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta$ be the radial derivative of f. Here $z = (z_1, z_2, \cdots, z_n) \in \mathbb{B}_n$, $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ is a multi–index, $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$ and $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$. It is easy to see that

$$\mathcal{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

For more information one can see [23]. The iterates of $\mathcal{R}$ is defined by $\mathcal{R}^m f(z) = \mathcal{R}(\mathcal{R}^{m-1} f(z))$, for $m \in \mathbb{N}\backslash\{1\}$. If $a(\neq 0) \in \mathbb{B}_n$, φ_a is denoted the Möbius transformation on $\mathbb{B}_n$ which interchanges the points 0 and a,

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad z \in \mathbb{B}_n.$$

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Here $s_a = 1 - |a|^2$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[a]$ generated by a, and Q_a is the orthogonal projection from $\mathbb{C}^n$ onto the orthogonal complement of $[a]$.

Let $0 < p < \infty$, $0 \le s < \infty$, $-n-1 < q < \infty$ and $-1 < q+s < \infty$. The space $F(p,q,s)$ is the space of all holomorphic functions in $\mathbb{B}_n$ for which

$$\|f\|^p_{F(p,q,s)} = |f(0)|^p + \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\mathcal{R}f(z)|^p (1-|z|^2)^q g^s(z,a)\, dV(z) < \infty.$$

Here dV is normalized volume measure on $\mathbb{B}_n$, $g(z,a) = \log|\frac{1}{\varphi_a(z)}|$. If $q+s \le -1$, then $F(p,q,s)$ is the space of constant functions. Using complex gradient ∇ and complex invariant gradient $\widetilde{\nabla}$, there are some equivalent noms for the functions in this space, see [19] for more details. The above space, introduced in [21], is called general function space which include some classes of functions like Bergman, Bloch, Hardy, BMOA and Q_p spaces, see [21] for some basic properties and recent results on $F(p,q,s)$ in the case $n = 1$.

Beside the general space $F(p,q,s)$, we use another general class of holomorphic functions $H(p,q,\phi)$ which is called mixed–norm space.

The function ϕ on $[0,1)$ is called normal if it is a positive continuous function for which there are two constants $b > a > 0$ such that

(i) $\frac{\phi(r)}{(1-r)^a}$ is non–increasing in $[0,1)$ and $\frac{\phi(r)}{(1-r)^a} \downarrow 0$
(ii) $\frac{\phi(r)}{(1-r)^b}$ is non–decreasing in $[0,1)$ and $\frac{\phi(r)}{(1-r)^b} \uparrow \infty$

as $r \to 1^-$. By a normal function $\phi : \mathbb{B}_n \to [0,\infty)$ we mean that $\phi(|z|)$ is normal and also ϕ is radial, $\phi(z) = \phi(|z|)$.

For $0 < p, q < \infty$ and normal function ϕ on $[0,1)$, the mixed–norm space $H(p,q,\phi)$ consists of all holomorphic functions in $\mathbb{B}_n$ for which

$$\|f\|^p_{p,q,\phi} = \int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}\,dr < \infty,$$

where

$$M_q(f,r) = \left(\int_{S_n} |f(r\xi)|^q d\sigma(\xi)\right)^{1/q},$$

where $d\sigma$ is the normalized surface measure on the unit sphere S_n. If $1 \le p < \infty$, then $H(p,q,\phi)$ becomes a Banach space equipped with the norm $\|.\|_{p,q,\phi}$. In the case $0 < p < 1$, $\|.\|_{p,q,\phi}$ is a quasi–norm on $H(p,q,\phi)$ and this space is a Fréchet space but is not a Banach space. For $\alpha > -1$, $\phi(r) = (1-r^2)^{(\alpha+1)/p}$ is a normal function. If $p = q$, then $H(p,p,(1-r^2)^{(\alpha+1)/p})$ is weighted Bergman space A^p_α which is defined by

$$A^p_\alpha = \{f \in H(\mathbb{B}_n) : \|f\|^p_{A^p_\alpha} = \int_{\mathbb{B}_n} |f(z)|^p (1-|z|^2)^\alpha dV(z) < \infty\}.$$

Let μ be a positive continuous function on $[0,1)$, such a function is called weight. The Bloch–type space $\mathcal{B}_\mu$ and Zygmund–type space $\mathcal{Z}_\mu$ are defined as

$$\begin{aligned}\mathcal{B}_\mu =& \{f \in H(\mathbb{B}_n) : \|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{B}_n} \mu(|z|)|\mathcal{R}f(z)| < \infty\},\\ \mathcal{Z}_\mu =& \{f \in H(\mathbb{B}_n) : \|f\|_{\mathcal{Z}_\mu} = |f(0)| + \sup_{z\in\mathbb{B}_n} \mu(|z|)|\mathcal{R}^2 f(z)| < \infty\}.\end{aligned}$$

When $\mu(r) = (1 - r^2)$ then we have classical Bloch and Zygmund space, $\mathcal{B}$ and $\mathcal{Z}$. If $\mu(r) = (1 - r^2)^\alpha$, $\alpha > 0$, then α–Bloch and Zygmund spaces are obtained, $\mathcal{B}^\alpha$ and $\mathcal{Z}^\alpha$. Let $g \in H(\mathbb{B}_n)$. The Riemann–Stieltjes operator L_g is defined by

$$L_g f(z) = \int_0^1 \mathcal{R}f(tz)g(tz)\frac{dt}{t}, \ \ f \in H(\mathbb{B}_n), \ \ z \in \mathbb{B}_n.$$

Essential norm of Riemann–Stieltjes operator on weighted Bergman spaces with doubling weights is estimated in [2]. Volterra–type operators from $F(p,q,s)$ space to Bloch–Orlicz and Zygmund–Orlicz spaces are investigated in [7]. Boundedness and compactness of this operator between Zygmund–type spaces are studied in [1]. For more results on integral operators or generalized integral operators on some spaces of holomorphic functions one can see [4–6, 8, 9, 12–14, 16–18, 18, 21, 22] and references therein.

Bounded and compact Riemann–Stieltjes operator from general space of holomorphic functions to Zygmund–type spaces and little Zygmund–type spaces on the unit ball are characterized in [11]. The compactness characterization is that $L_g : F(p,q,s) \to \mathcal{Z}_\mu$ is compact if and only if

$$\limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{(1 - |z|^2)^{\alpha+1}} = 0, \ \ \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1 - |z|^2)^{\alpha}} = 0.$$

Also Liu and Yu have investigated boundedness and compactness of L_g from mixed–norm spaces to Zygmund–type spaces in [10] and they prove that $L_g : H(p,q,\phi) \to \mathcal{Z}_\mu$ is compact if and only if

$$\limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{\phi(|z|)(1 - |z|^2)^{2+\frac{n}{q}}} = 0, \ \ \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{\phi(|z|)(1 - |z|^2)^{1+\frac{n}{q}}} = 0.$$

Motivated by the above results, we are going to find an estimation of the essential norm of the Riemann–Stieltjes operator from the general space of holomorphic functions, $F(p,q,s)$, and mixed–norm space, $H(p,q,\phi)$, into Zygmund–type spaces in the unit ball.

In this paper, for real scalars A and B, the notation $\preceq$ means $A \leq CB$ for some positive constant C. Also, the notation $A \approx B$ means $A \preceq B$ and $B \preceq A$. The essential norm of an operator T, $\|T\|_e$, is the distance of T from the space of all compact operators.

2. MAIN RESULTS

In this section, first we recall some basic lemmas from the literature for later use in the paper. Then we estimate the essential norm of Riemann–Stieltjes operator L_g from the general function space and mixed-norm space into Zygmund–type space in the unit ball of $\mathbb{C}^n$.

Lemma 2.1. *[15] For every $f, g \in H(\mathbb{B}_n)$,*

$$\mathcal{R}L_g(f)(z) = \mathcal{R}f(z)g(z), \qquad z \in \mathbb{B}_n.$$

Lemma 2.2. *[18] Let $0 < p, s < \infty$, $\max\{-n-1, -s-1\} < q < \infty$ and $f \in F(p,q,s)$. Then $f \in \mathcal{B}^{\frac{n+1+q}{p}}$ and*

$$\|f\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \leq C\|f\|_{F(p,q,s)}.$$

According to the Proposition 2.1 of [3], for $\alpha > 0$ and $m \in \mathbb{N}$, we have $f \in \mathcal{B}^{\alpha}$ if and only if

$$\sup_{z \in \mathbb{B}_n} |\mathcal{R}^m f(z)|(1-|z|^2)^{\alpha+m-1} < \infty. \tag{2.1}$$

Lemma 2.3. *[16] Assume that $m \in \mathbb{N}$, $p, q \in (0, \infty)$, ϕ is normal and $f \in H(p,q,\phi)$. Then there exists a positive constant C independent of f such that*

$$|\mathcal{R}^m f(z)| \leq \frac{C|z|}{\phi(|z|)(1-|z|^2)^{m+n/q}}\|f\|_{p,q,\phi}, \quad z \in \mathbb{B}_n.$$

Lemma 2.4. *[4] Let $0 < p, s < \infty$, $\max\{-n-1, -s-1\} < q < \infty$, μ be a weight and $g \in H(\mathbb{B}_n)$. Then $L_g : F(p,q,s) \to \mathcal{Z}_\mu$ is compact if and only if $L_g : F(p,q,s) \to \mathcal{Z}_\mu$ is bounded and for any bounded sequence $\{f_k\}$ in $F(p,q,s)$, which converges uniformly to zero on compact subsets of $\mathbb{B}_n$, as $k \to \infty$, we have $\|L_g f_k\|_{\mathcal{Z}_\mu} \to 0$, as $k \to \infty$.*

Lemma 2.5. *[5] Let $0 < p, q < \infty$, μ be a weight, ϕ be a normal function and $g \in H(\mathbb{B}_n)$. Then $L_g : H(p,q,\phi) \to \mathcal{Z}_\mu$ is compact if and only if $L_g : H(p,q,\phi) \to \mathcal{Z}_\mu$ is bounded and for any bounded sequence $\{f_k\}$ in $H(p,q,\phi)$ which converges uniformly to zero on compact subsets of $\mathbb{B}_n$ as $k \to \infty$, we have $\|L_g f_k\|_{\mathcal{Z}_\mu} \to 0$ as $k \to \infty$.*

By the above mentioned lemmas, in the next theorem we find an estimation of the essential norm of the Riemann–Stieltjes operator $L_g : F(p,q,s) \to \mathcal{Z}_\mu$. For simplicity, put $\alpha = \frac{n+1+q}{p}$.

Theorem 2.1. *Let $0 < p, s < \infty$, $\max\{-n-1, -s-1\} < q < \infty$, μ be a weight and $g \in H(\mathbb{B}_n)$. If $L_g : F(p,q,s) \to \mathcal{Z}_\mu$ is bounded, then*

$$\|L_g\|_{e, F(p,q,s) \to \mathcal{Z}_\mu} \approx \max\left\{\limsup_{|z| \to 1} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^{\alpha+1}}, \limsup_{|z| \to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^{\alpha}}\right\}.$$

Proof. Since L_g is bounded, then there exists $C > 0$ such that

$$\|L_g f\|_{\mathcal{Z}_\mu} \leq C\|f\|_{F(p,q,s)}, \qquad \text{for all} \quad f \in F(p,q,s).$$

By applying this inequality to $f_1(z) = z$, we obtain

$$\sup_{z \in \mathbb{B}_n} \mu(|z|)|\mathcal{R}g(z)| < \infty.$$

Also, by applying the inequality to $f_2(z) = z^2$, we have

$$\sup_{z \in \mathbb{B}_n} \mu(|z|)|g(z)| < \infty.$$

Let $\{z_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathbb{D}$, such that $|z_i| \to 1$. We define the sequence of functions as follows

$$f_i(z) = \frac{(1-|z_i|^2)^{\alpha+1}}{(1-\overline{z_i}z)^{2\alpha}} - 2\frac{1-|z_i|^2}{(1-\overline{z_i}z)^{\alpha}}, \quad i \in \mathbb{N}.$$

Direct calculations shows that for every $i \in \mathbb{N}$, $f_i \in F(p,q,s)$, $\sup_i \|f_i\|_{F(p,q,s)} \leq C$ and $f_i \to 0$, uniformly on compact subsets of $\mathbb{B}_n$. Also,

$$\mathcal{R}f_i(z_i) = 0 \quad \text{and} \quad \mathcal{R}^2 f_i(z_i) = 2\alpha^2 \frac{|z_i|^4}{(1-|z_i|^2)^{\alpha+1}}.$$

So for any compact operator $K : F(p,q,s) \to \mathcal{Z}_\mu$ we have

$$\lim_{i\to\infty} \|Kf_i\|_{\mathcal{Z}_\mu} = 0.$$

By the above observations we have

$$\begin{aligned}\|L_g - K\|_{F(p,q,s)\to\mathcal{Z}_\mu} &\geq \limsup_{i\to\infty} \|L_g f_i - Kf_i\|_{\mathcal{Z}_\mu} \\ &\geq \limsup_{i\to\infty} \|L_g f_i\|_{\mathcal{Z}_\mu} - \limsup_{i\to\infty} \|Kf_i\|_{\mathcal{Z}_\mu} \\ &= \limsup_{i\to\infty} \mu(|z_i|)|\mathcal{R}^2(L_g f_i)(z_j)| \\ &\geq \limsup_{i\to\infty} \frac{\mu(|z_i|)|g(z_i)||z_i|^4}{(1-|z_i|^2)^{\alpha+1}}.\end{aligned}$$

Therefore,

$$\|L_g\|_{e,F(p,q,s)\to\mathcal{Z}_\mu} = \inf_K \|L_g - K\|_{F(p,q,s)\to\mathcal{Z}_\mu} \geq \limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^{\alpha+1}}. \tag{2.2}$$

Now we define the sequence $\{h_i\}$ as follows

$$h_i(z) = \frac{(1-|z_i|^2)^{\alpha+1}}{(1-\overline{z_i}z)^{2\alpha}} - \frac{2+4\alpha}{\alpha+1}\frac{1-|z_i|^2}{(1-\overline{z_i}z)^{\alpha}}.$$

Then $\{h_i\}$ is a bounded sequence in $F(p,q,s)$ which converges to 0 uniformly on compact subsets of $\mathbb{B}_n$. Also

$$\begin{aligned}\mathcal{R}h_i(z) =& 2\alpha\frac{\overline{z_i}z(1-|z_i|^2)^{\alpha+1}}{(1-\overline{z_i}z)^{2\alpha+1}} - \alpha\frac{2+4\alpha}{\alpha+1}\frac{\overline{z_i}z(1-|z_i|^2)}{(1-\overline{z_i}z)^{\alpha+1}} \\ \mathcal{R}^2 h_i(z) =& 2\alpha\frac{\overline{z_i}z(1-|z_i|^2)^{\alpha+1}}{(1-\overline{z_i}z)^{2\alpha+1}} + 2\alpha(2\alpha+1)\frac{(\overline{z_i}z)^2(1-|z_i|^2)^{\alpha+1}}{(1-\overline{z_i}z)^{2\alpha+2}} \\ &- \alpha\frac{2+4\alpha}{\alpha+1}\frac{\overline{z_i}z(1-|z_i|^2)}{(1-\overline{z_i}z)^{\alpha+1}} - \alpha(2+4\alpha)\frac{(\overline{z_i}z)^2(1-|z_i|^2)}{(1-\overline{z_i}z)^{\alpha+2}}.\end{aligned}$$

Hence

$$\mathcal{R}h_i(z_i) = \mathcal{R}^2 h_i(z_i) = -\frac{2\alpha^2}{\alpha+1}\frac{|z_j|^2}{(1-|z_i|^2)^{\alpha}}.$$

For any compact operator $K : F(p,q,s) \to \mathcal{Z}_\mu$, using the definition of the operator norm and the norm in Zygmund-type space and (2.2), we have

$$\begin{aligned}
\|L_g - K\|_{F(p,q,s)\to\mathcal{Z}_\mu} &\geq \limsup_{i\to\infty} \|L_g h_i - K h_i\|_{\mathcal{Z}_\mu} \\
&\geq \limsup_{i\to\infty} \|L_g h_i\|_{\mathcal{Z}_\mu} - \limsup_{i\to\infty} \|K h_i\|_{\mathcal{Z}_\mu} \\
&= \limsup_{i\to\infty} \mu(|z_i|)|\mathcal{R}^2(L_g h_i)(z_i)| \\
&= \limsup_{i\to\infty} \mu(|z_i|)|\mathcal{R}^2 h_i(z_i) g(z_i) + \mathcal{R} h_i(z_i)\mathcal{R} g(z_i)| \\
&= \limsup_{i\to\infty} \mu(|z_i|)\left| -\frac{2\alpha^2}{\alpha+1}\frac{g(z_i)|z_i|^2}{(1-|z_i|^2)^\alpha} - \frac{2\alpha^2}{\alpha+1}\frac{\mathcal{R}g(z_i)|z_i|^2}{(1-|z_i|^2)^\alpha}\right| \\
&\geq \limsup_{i\to\infty} \frac{2\alpha^2}{\alpha+1}\frac{\mu(|z_i|)|\mathcal{R}g(z_i)||z_i|^2}{(1-|z_i|^2)^\alpha} - \limsup_{i\to\infty} \frac{2\alpha^2}{\alpha+1}\frac{\mu(|z_i|)|g(z_i)||z_i|^2}{(1-|z_i|^2)^\alpha} \\
&= \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^\alpha} - \limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^\alpha} \\
&\geq \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^\alpha} - \limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^{\alpha+1}} \\
&\geq \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^\alpha} - \|L_g\|_{e,F(p,q,s)\to\mathcal{Z}_\mu}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|L_g\|_{e,F(p,q,s)\to\mathcal{Z}_\mu} &= \inf_K \|L_g - K\|_{F(p,q,s)\to\mathcal{Z}_\mu} \\
&\geq \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^\alpha} - \|L_g\|_{e,F(p,q,s)\to\mathcal{Z}_\mu},
\end{aligned}$$

and so

$$\|L_g\|_{e,F(p,q,s)\to\mathcal{Z}_\mu} \geq \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^\alpha}. \tag{2.3}$$

From (2.2) and (2.3), the lower estimate of the essential norm is achieved.

For the upper estimate, let $\{r_i\} \subset (0,1)$ be a sequence such that $r_i \to 1$, as $i \to \infty$. Then as $f_{r_i} \to f$ uniformly on compact subsets of $\mathbb{B}_n$, as $i \to \infty$, in which $f_r(z) = f(rz)$. Define the operators K_r on $F(p,q,s)$, $K_r f(z) = f_r(z)$, where $0 < r < 1$. Then it is clear that K_r is a compact operator on $F(p,q,s)$. Hence for any positive integer i, the operator $L_g K_{r_i} : F(p,q,s) \to \mathcal{Z}_\mu$ is compact and so

$$\|L_g\|_{e,F(p,q,s)\to\mathcal{Z}_\mu} \leq \limsup_{i\to\infty} \|L_g - L_g K_{r_i}\|_{F(p,q,s)\to\mathcal{Z}_\mu}. \tag{2.4}$$

Therefore, for every $f \in F(p,q,s)$, with $\|f\|_{F(p,q,s)} \leq 1$ we have

$$\|L_g f - L_g K_{r_i} f\|_{\mathcal{Z}_\mu} = \sup_{z\in\mathbb{B}_n} \mu(|z|)|\mathcal{R}^2(L_g f)(z) - \mathcal{R}^2(L_g K_{r_i} f)(z)|$$

$$
\begin{aligned}
&= \sup_{z\in\mathbb{B}_n} \mu(|z|)|\mathcal{R}^2 f(z)g(z) + \mathcal{R}f(z)\mathcal{R}g(z) \\
&\quad - r_i\mathcal{R}^2 f_{r_i}(z)g(z) - \mathcal{R}f_{r_i}(z)\mathcal{R}g(z)| \\
&\le \sup_{z\in\mathbb{B}_n} \mu(|z|)|\mathcal{R}^2 f(z) - r_i\mathcal{R}^2 f_{r_i}(z)||g(z)| \\
&\quad + \sup_{z\in\mathbb{B}_n} \mu(|z|)|\mathcal{R}f(z) - \mathcal{R}f_{r_i}(z)||\mathcal{R}g(z)| \\
&\le \sup_{|z|\le\delta} \mu(|z|)|\mathcal{R}^2 f(z) - r_i\mathcal{R}^2 f_{r_i}(z)||g(z)| \\
&\quad + \sup_{|z|>\delta} \mu(|z|)|\mathcal{R}^2 f(z) - r_i\mathcal{R}^2 f_{r_i}(z)||g(z)| \\
&\quad + \sup_{|z|\le\delta} \mu(|z|)|\mathcal{R}f(z) - \mathcal{R}f_{r_i}(z)||\mathcal{R}g(z)| \\
&\quad + \sup_{|z|>\delta} \mu(|z|)|\mathcal{R}f(z) - \mathcal{R}f_{r_i}(z)||\mathcal{R}g(z)|,
\end{aligned}
$$

where $\delta \in (0,1)$ is fixed. By Weierstrass theorem, $\mathcal{R}f_{r_i} \to \mathcal{R}f$ and $\mathcal{R}^2 f_{r_i} \to \mathcal{R}^2 f$ uniformly on compact subsets of $\mathbb{B}_n$, as $i \to \infty$. Since the ball $\{z : |z| \le \delta\}$ is a compact subset of $\mathbb{B}_n$, we have

$$
\begin{aligned}
\limsup_{i\to\infty} \sup_{|z|\le\delta} \mu(|z|)|\mathcal{R}^2 f(z) - r_i\mathcal{R}^2 f_{r_i}(z)||g(z)| &= 0 \\
\limsup_{i\to\infty} \sup_{|z|\le\delta} \mu(|z|)|\mathcal{R}f(z) - \mathcal{R}f_{r_i}(z)||\mathcal{R}g(z)| &= 0.
\end{aligned}
$$

Therefore, using Lemma 2.2 and (2.1) we have

$$
\begin{aligned}
\|L_g f - L_g K_{r_i} f\|_{\mathcal{Z}_\mu} &\le \sup_{|z|>\delta} \mu(|z|)|\mathcal{R}^2 f(z) - r_i\mathcal{R}^2 f_{r_i}(z)||g(z)| \\
&\quad + \sup_{|z|>\delta} \mu(|z|)|\mathcal{R}f(z) - \mathcal{R}f_{r_i}(z)||\mathcal{R}g(z)| \\
&\le \sup_{|z|>\delta} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^{\alpha+1}} + \sup_{|z|>\delta} \frac{r_i\mu(|z|)|g(z)|}{(1-|r_i z|^2)^{\alpha+1}} \\
&\quad + \sup_{|z|>\delta} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^{\alpha}}\|f\|_{\mathcal{B}^\alpha} + \sup_{|z|>\delta} \frac{r_i\mu(|z|)|\mathcal{R}g(z)|}{(1-|r_i z|^2)^{\alpha}}\|f\|_{\mathcal{B}^\alpha} \\
&\le \sup_{|z|>\delta} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^{\alpha+1}} + \sup_{|z|>\delta} \frac{r_i\mu(|z|)|g(z)|}{(1-|r_i z|^2)^{\alpha+1}} \\
&\quad + \sup_{|z|>\delta} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^{\alpha}}\|f\|_{F(p,q,s)} + \sup_{|z|>\delta} \frac{r_i\mu(|z|)|\mathcal{R}g(z)|}{(1-|r_i z|^2)^{\alpha}}\|f\|_{F(p,q,s)},
\end{aligned}
$$

letting $i \to \infty$, we obtain

$$
\|L_g f - L_g K_{r_i} f\|_{\mathcal{Z}_\mu} \le 2\sup_{|z|>\delta} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^{\alpha+1}} + 2\sup_{|z|>\delta} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^{\alpha}}.
$$

When $\delta \to 1$, from (2.4) we have

$$
\|L_g\|_{e,F(p,q,s)\to\mathcal{Z}_\mu} \le \limsup_{|z|\to1} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^{\alpha+1}} + \limsup_{|z|\to1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^{\alpha}},
$$

and consequently we have an upper estimate of the essential norm. ☐

In the sequel we find an estimate for the essential norm of the Riemann–Stieltjes operator $L_g : H(p,q,\phi) \to \mathcal{Z}_\mu$.

Theorem 2.2. *Let* $0 < p, q < \infty$, μ *be a weight,* ϕ *is a normal function and* $g \in H(\mathbb{B}_n)$*. If* $L_g : H(p,q,\phi) \to \mathcal{Z}_\mu$ *is bounded, then*

$$\|L_g\|_{e,H(p,q,\phi)\to\mathcal{Z}_\mu} \approx \max\left\{\limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{\phi(|z|)(1-|z|^2)^{2+\frac{n}{q}}},\ \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}}\right\}.$$

Proof. Let $\{z_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|z_i| \to 1$. Define the sequence of functions as follows

$$f_i(z) = (\gamma+1)\frac{(1-|z_i|^2)^{b+1}}{\phi(|z_i|)(1-\overline{z_i}z)^{\gamma}} - \gamma\frac{(1-|z_i|^2)^{b+2}}{\phi(|z_i|)(1-\overline{z_i}z)^{\gamma+1}},\quad i \in \mathbb{N},$$

where $\gamma = b + 1 + \frac{n}{q}$ and b comes from the definition of the normal function ϕ. It is clear that for each $i \in \mathbb{N}$, $f_i \in H(p,q,\phi)$, $\sup_i \|f_i\|_{p,q,\phi} \leq C$ and $f_i \to 0$ uniformly on compact subsets of $\mathbb{B}_n$. Also

$$\mathcal{R}f_i(z_i) = 0,\quad \mathcal{R}^2 f_i(z_i) = -\gamma(\gamma+1)\frac{|z_i|^4}{\phi(|z_i|)(1-|z_i|^2)^{2+\frac{n}{q}}}\quad i \in \mathbb{N}.$$

Hence for any compact operator $K : H(p,q,\phi) \to \mathcal{Z}_\mu$ we have $\lim_{n\to\infty} \|Kf_i\|_{\mathcal{Z}_\mu} = 0$ and so

$$\begin{aligned}
\|L_g - K\|_{H(p,q,\phi)\to\mathcal{Z}_\mu} &\geq \limsup_{i\to\infty} \|L_g f_i - K f_i\|_{\mathcal{Z}_\mu} \\
&\geq \limsup_{i\to\infty} \|L_g f_i\|_{\mathcal{Z}_\mu} - \limsup_{i\to\infty} \|K f_i\|_{\mathcal{Z}_\mu} \\
&\geq \limsup_{i\to\infty} \mu(|z_i|)|\mathcal{R}^2(L_g f_i)(z_i)| \\
&\geq \limsup_{i\to\infty} \frac{\mu(|z_i|)|g(z_i)||z_i|^4}{\phi(|z_i|)(1-|z_i|^2)^{2+\frac{n}{q}}}.
\end{aligned}$$

Therefore, we get that

$$\|L_g\|_{e,H(p,q,\phi)\to\mathcal{Z}_\mu} = \inf_K \|L_g - K\|_{H(p,q,\phi)\to\mathcal{Z}_\mu} \geq \limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{\phi(|z|)(1-|z|^2)^{2+\frac{n}{q}}}. \tag{2.5}$$

Now we define the sequence $\{h_j\}$ as follows

$$h_i(z) = (\beta+2)\frac{(1-|z_i|^2)^{b+1}}{\phi(|z_i|)(1-\overline{z_i}z)^{\beta}} - \beta\frac{(1-|z_i|^2)^{b+2}}{(1-\overline{z_i}z)^{\beta+1}}.$$

It is easy to see that $\{h_i\}$ is a bounded sequence in $H(p,q,\phi)$, which converges to 0, uniformly on compact subsets of $\mathbb{B}_n$. Also

$$\mathcal{R}h_i(z_i) = \mathcal{R}^2 h_i(z_i) = \beta\frac{|z_i|^2}{\phi(|z_i|)(1-|z_i|^2)^{1+\frac{n}{q}}}.$$

Using (2.5) we get that

$$-\limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}} \geq -\limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{\phi(|z|)(1-|z|^2)^{2+\frac{n}{q}}} \geq -\|L_g\|_{e,H(p,q,\phi)\to\mathcal{Z}_\mu},$$

and then for any compact operator $K : H(p,q,\phi) \to \mathcal{Z}_\mu$ we have

$$\begin{aligned}
\|L_g - K\|_{H(p,q,\phi)\to\mathcal{Z}_\mu} &\geq \limsup_{i\to\infty} \|L_g h_i - K h_i\|_{\mathcal{Z}_\mu} \\
&\geq \limsup_{i\to\infty} \|L_g h_i\|_{\mathcal{Z}_\mu} - \limsup_{i\to\infty} \|K h_i\|_{\mathcal{Z}_\mu} \\
&= \limsup_{i\to\infty} \mu(|z_i|)|\mathcal{R}^2(L_g h_i)(z_i)| \\
&= \limsup_{i\to\infty} \mu(|z_i|)|\mathcal{R}^2 h_i(z_i) g(z_i) + \mathcal{R} h_i(z_i)\mathcal{R} g(z_i)| \\
&= \limsup_{i\to\infty} \mu(|z_i|)\left|\gamma \frac{g(z_i)|z_i|^2}{\phi(|z_i|)(1-|z_i|^2)^{1+\frac{n}{q}}} + \gamma \frac{\mathcal{R}g(z_i)|z_i|^2}{\phi(|z_i|)(1-|z_i|^2)^{1+\frac{n}{q}}}\right| \\
&\geq \limsup_{i\to\infty} \gamma \frac{\mu(|z_i|)|\mathcal{R}g(z_i)||z_i|^2}{\phi(|z_i|)(1-|z_i|^2)^{1+\frac{n}{q}}} - \limsup_{i\to\infty} \gamma \frac{\mu(|z_i|)|g(z_i)||z_i|^2}{\phi(|z_i|)(1-|z_i|^2)^{1+\frac{n}{q}}} \\
&\geq \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}} - \limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}} \\
&\geq \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}} - \|L_g\|_{e,H(p,q,\phi)\to\mathcal{Z}_\mu}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|L_g\|_{e,H(p,q,\phi)\to\mathcal{Z}_\mu} &= \inf_K \|L_g - K\|_{H(p,q,\phi)\to\mathcal{Z}_\mu} \\
&\geq \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}} - \|L_g\|_{e,H(p,q,\phi)\to\mathcal{Z}_\mu},
\end{aligned}$$

and so

$$\|L_g\|_{e,H(p,q,\phi)\to\mathcal{Z}_\mu} \geq \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}}. \tag{2.6}$$

Therefore, we have a lower estimate. In order to prove the upper estimate, let $\{r_i\} \subset (0,1)$ be a sequence such that $r_i \to 1$ as $i \to \infty$. Then $f_{r_i} \to f$ uniformly on compact subsets of $\mathbb{B}_n$, as $i \to \infty$. We define the operators T_r on $H(p,q,\phi)$, $T_r f(z) = f_r(z)$, where $0 < r < 1$. Then T_r is a compact operator on $H(p,q,\phi)$. So, for any positive integer i, the operator $L_g T_{r_i} : H(p,q,\phi) \to \mathcal{Z}_\mu$ is compact. Thus

$$\|L_g\|_{e,H(p,q,\phi)\to\mathcal{Z}_\mu} \leq \limsup_{j\to\infty} \|L_g - L_g T_{r_i}\|_{H(p,q,\phi)\to\mathcal{Z}_\mu}. \tag{2.7}$$

It will be sufficient to compute $\|L_g - L_g T_{r_i}\|_{H(p,q,\phi)\to\mathcal{Z}_\mu}$. Let $f \in H(p,q,\phi)$, with $\|f\|_{p,q,\phi} \leq 1$. Then similar to the proof of Theorem 2.1 we have

$$\begin{aligned}
\|L_g f - L_g T_{r_i} f\|_{\mathcal{Z}_\mu} \leq& \sup_{|z|\leq\delta} \mu(|z|)|\mathcal{R}^2 f(z) - r_i \mathcal{R}^2 f_{r_j}(z)||g(z)| \\
&+ \sup_{|z|>\delta} \mu(|z|)|\mathcal{R}^2 f(z) - r_i \mathcal{R}^2 f_{r_i}(z)||g(z)|
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{|z|\le\delta}\mu(|z|)|\mathcal{R}f(z)-\mathcal{R}f_{r_i}(z)||\mathcal{R}g(z)|\\
&+\sup_{|z|>\delta}\mu(|z|)|\mathcal{R}f(z)-\mathcal{R}f_{r_i}(z)||\mathcal{R}g(z)|,
\end{aligned}
$$

where $\delta \in (0,1)$ is fixed. By Weierstrass theorem, $\mathcal{R}f_{r_i} \to \mathcal{R}f$ and $\mathcal{R}^2 f_{r_i} \to \mathcal{R}^2 f$ uniformly on compact subsets of $\mathbb{B}_n$, as $i \to \infty$. So

$$
\begin{aligned}
\limsup_{i\to\infty}\sup_{|z|\le\delta}\mu(|z|)|\mathcal{R}^2 f(z)-r_j\mathcal{R}^2 f_{r_i}(z)||g(z)| &=0,\\
\limsup_{i\to\infty}\sup_{|z|\le\delta}\mu(|z|)|\mathcal{R}f(z)-\mathcal{R}f_{r_i}(z)||\mathcal{R}g(z)| &=0.
\end{aligned}
$$

Here we use the fact that $\sup_{z\in\mathbb{B}_n}\mu(|z|)|\mathcal{R}g(z)| < \infty$ and $\sup_{z\in\mathbb{B}_n}\mu(|z|)|g(z)| < \infty$ which is obtained from boundedness of the operator. Using Lemma 2.3 we get that

$$
\begin{aligned}
\|L_g f - L_g T_{r_i} f\|_{\mathcal{Z}_\mu} &\le \sup_{|z|>\delta}\mu(|z|)|\mathcal{R}^2 f(z)-r_i\mathcal{R}^2 f_{r_i}(z)||g(z)|\\
&\quad+\sup_{|z|>\delta}\mu(|z|)|\mathcal{R}f(z)-\mathcal{R}f_{r_i}(z)||\mathcal{R}g(z)|\\
&\le \sup_{|z|>\delta}\frac{\mu(|z|)|g(z)||z|}{\phi(|z|)(1-|z|^2)^{2+\frac{n}{q}}}\|f\|_{p,q,\phi}+\sup_{|z|>\delta}\frac{r_i\mu(|z|)|g(z)||r_i z|}{\phi(|r_i z|)(1-|r_i z|^2)^{2+\frac{n}{q}}}\|f\|_{p,q,\phi}\\
&\quad+\sup_{|z|>\delta}\frac{\mu(|z|)|\mathcal{R}g(z)||z|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}}\|f\|_{p,q,\phi}+\sup_{|z|>\delta}\frac{r_i\mu(|z|)|\mathcal{R}g(z)||r_i z|}{\phi(|r_i z|)(1-|r_i z|^2)^{1+\frac{n}{q}}}\|f\|_{p,q,\phi}\\
&\le \sup_{|z|>\delta}\frac{\mu(|z|)|g(z)||z|}{\phi(|z|)(1-|z|^2)^{2+\frac{n}{q}}}+\sup_{|z|>\delta}\frac{r_i\mu(|z|)|g(z)||r_i z|}{\phi(|r_i z|)(1-|r_i z|^2)^{2+\frac{n}{q}}}\\
&\quad+\sup_{|z|>\delta}\frac{\mu(|z|)|\mathcal{R}g(z)||z|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}}+\sup_{|z|>\delta}\frac{r_i\mu(|z|)|\mathcal{R}g(z)||r_i z|}{\phi(|r_i z|)(1-|r_i z|^2)^{1+\frac{n}{q}}}.
\end{aligned}
$$

If $i \to \infty$, then

$$
\|L_g f - L_g T_{r_i} f\|_{\mathcal{Z}_\mu} \le 2\sup_{|z|>\delta}\frac{\mu(|z|)|g(z)||z|}{\phi(|z|)(1-|z|^2)^{2+\frac{n}{q}}}+2\sup_{|z|>\delta}\frac{\mu(|z|)|\mathcal{R}g(z)||z|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}}.
$$

If $\delta \to 1$, then (2.7) implies that

$$
\|L_g\|_{e,H(p,q,\phi)\to\mathcal{Z}_\mu} \le \limsup_{|z|\to 1}\frac{\mu(|z|)|g(z)|}{\phi(|z|)(1-|z|^2)^{2+\frac{n}{q}}}+\limsup_{|z|\to 1}\frac{\mu(|z|)|\mathcal{R}g(z)|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}}.
$$

The proof is complete. □

We know that the operator is compact if and only if the essential norm equal to zero. So Theorems 2.1 and 2.2 imply the following criteria for the compactness of Riemann-Stieltjes operator which is the part of the results of [10] and [11].

Corollary 2.1. *Let $0 < p, s < \infty$, $\max\{-n-1, -s-1\} < q < \infty$, μ be a weight and $g \in H(\mathbb{B}_n)$. If $L_g : F(p,q,s) \to \mathcal{Z}_\mu$ is bounded, then is compact if and only if*

$$
\limsup_{|z|\to 1}\frac{\mu(|z|)|g(z)|}{(1-|z|^2)^{\alpha+1}}=0,\quad \limsup_{|z|\to 1}\frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^{\alpha}}=0.
$$

Corollary 2.2. *Let $0 < p, q < \infty$, μ be a weight, ϕ is a normal function and $g \in H(\mathbb{B}_n)$. If $L_g : H(p,q,\phi) \to \mathcal{Z}_\mu$ is bounded, then it is compact if and only if*

$$\limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{\phi(|z|)(1-|z|^2)^{2+\frac{n}{q}}} = 0, \quad \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}} = 0.$$

If $p = q$ and $\alpha > -1$, then $H(p,p,(1-r^2)^{(\alpha+1)/p})$ is weighted Bergman space A^p_α.

Corollary 2.3. *Let $\alpha > -1$ and $p > 0$. If $L_g : A^p_\alpha \to \mathcal{Z}_\mu$ is bounded, then*

$$\|L_g\|_{e,A^p_\alpha \to \mathcal{Z}_\mu} \approx \max\left\{\limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^{\frac{\alpha+n+1}{p}+2}}, \limsup_{|z|\to 1} \frac{\mu(|z|)|\mathcal{R}g(z)|}{(1-|z|^2)^{\frac{\alpha+n+1}{p}+1}}\right\}.$$

Example 2.1. *For $n = 1$, we obtain several classical spaces of analytic functions included in $F(p,q,s)$, [22]. We bring some of them here:*

- *F(2,0,1) = BMOA, consists of the analytic functions $f : \mathbb{D} \to \mathbb{C}$ whose boundary values have bounded mean oscillation on the unit circle $\mathbb{T}$.*
- *For $1 < p < \infty$, $F(p, p-2, 0) = B_p$, the analytic Besov space.*
- *$F(2,1,0) = H^2$, Hardy space.*

So from Theorem 2.1, we get

$$\|L_g\|_{e,BMOA \to \mathcal{Z}_\mu} \approx \max\left\{\limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^2}, \limsup_{|z|\to 1} \frac{\mu(|z|)|g'(z)|}{1-|z|^2}\right\},$$

$$\|L_g\|_{e,B_p \to \mathcal{Z}_\mu} \approx \max\left\{\limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^2}, \limsup_{|z|\to 1} \frac{\mu(|z|)|g'(z)|}{1-|z|^2}\right\},$$

$$\|L_g\|_{e,H^2 \to \mathcal{Z}_\mu} \approx \max\left\{\limsup_{|z|\to 1} \frac{\mu(|z|)|g(z)|}{(1-|z|^2)^{5/2}}, \limsup_{|z|\to 1} \frac{\mu(|z|)|g'(z)|}{(1-|z|^2)^{3/2}}\right\}.$$

3. CONCLUSION

By using test functions, we found an approximation for the essential norm of one the well–known integral operators, called Riemann-Stieltjes operator, from some general classes of holomorphic functions to Zygmund-type space. These classes include some of classical spaces of holomorphic functions. For future work, we can propose the problem of estimating the essential norm of this operator into n–th weighted type space instead of Zygmund-type space.

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